Important Differentiation Rules and notes

$F'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}(f(x))$

If $y = ax$, $y' = a$

If $y = a$, $y' = 0$

If $y = ax^{n}$, $y' = n.ax^{n-1}$

If $y = f(x).g(x)$ $y' = f(x).g'(x) + f'(x).g(x)$

If $y = \frac{f(x)}{g(x)}$ $y' = \frac{f(x).g'(x) - f'(x).g(x)}{(g(x))}$

If $y = f(x)^{n}$ $y' = n\ f(x)^{n-1}.f'(x)$

Notes :

$\sqrt{x^{n}} = x^{n/2}$ $\frac{a}{x^{n}} = ax^{-n}$

$\text{Sin}2x = 2\sin x . \cos x$

1. $If\ y = sinx \Rightarrow \frac{dy}{dx} = cosx$

2. $If\ y = cos\ x \Rightarrow \frac{dy}{dx} = -sinx$

3. $If\ y = tanx \Rightarrow \frac{dy}{dx} = sec2x$

Corollaries:

1. $If\ y = si(ax+b) \Rightarrow \frac{dy}{dx} = a\ cos(ax+b)$

2. $If \quad = cos(ax+b) \quad \Rightarrow \dfrac{dy}{dx} = -a\ sin(ax+b)$

3. $If \quad = tan(ax+b) \quad \Rightarrow \dfrac{dy}{dx} = a\ sec2(ax+b)$

Exercise 1

1

Find the points on the curve $y = x^3 - 4x + 2$ at which the slope of the tangent = -1

$\frac{dy}{dx} = 3x^2 - 4$

slope of the tangent equal to -1:
$3x^2 - 4 = -1 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

For $x = 1$: $y = 1^3 - 4(1) + 2 = -1$

For $x = -1$: $y = (-1)^3 - 4(-1) + 2 = -1 + 4 + 2 = 5$

Answer: (1, -1) and (-1, 5)

2

Find the points on the curve $y = 2x^3 - x + 3$ at which the slope of the tangent = 5

$\frac{dy}{dx} = 6x^2 - 1$
slope of the tangent equal to 5:
$6x^2 - 1 = 5 \Rightarrow 6x^2 = 6 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

For $x = 1$: $y = 2(1)^3 - 1 + 3 = 4$

For $x = -1$: $y = 2(-1)^3 - (-1) + 3 = -2 + 1 + 3 = 2$

Answer: (1, 4) and (-1, 2)

3

a) . Find the first derivative of $y = x^5 - 3x^2 + 5x - 9$

$\frac{dy}{dx} = 5x^4 - 6x + 5$

b) . Find the first derivative of $y = 2x^2 + 5$

$\frac{dy}{dx} = 4x$

4

4. Find the slope of the tangent to the curve $y = x^2 - \frac{1}{X}$ at $x = 1$

$\frac{dy}{dx} = 2x + \frac{1}{X^2}$

slope of the tangent $= 2(1) + \frac{1}{1^2} = 3$

5

5. Find the measure of the angle which the tangent to the curve $y = 2x^3 + 3x$ makes with the positive direction of X-axis at the point (-1, 0)

$\frac{dy}{dx} = 6x^2 + 3$

At $x = -1$: $\frac{dy}{dx} = 6(1) + 3 = 9$

$\tan(\theta) = 9 \Rightarrow \theta = \tan{-1}(9) \approx 83.66°$

Answer: 83.66°

6

6. Find the points on the curve $y = x^3 - 4x + 1$ at which the tangent makes an angle of 135° with the positive direction of X-axis

$\tan(135°) = -1 \Rightarrow \frac{dy}{dx} = -1$

$\frac{dy}{dx} = 3x^2 - 4$

slope of the tangent equal to -1:

$3x^2 - 4 = -1 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

For $x = 1$: $y = 1 - 4 + 1 = -2$

For $x = -1$: $y = -1 + 4 + 1 = 4$

Answer: (1, -2) and (-1, 4)

7

7. If the slope of the tangent to the curve $y = x^2 + ax$ at the point (1, 3) is equal to 6 then, find the value of a

$\frac{dy}{dx} = 2x + a$

At $x = 1, \frac{dy}{dx} = 2(1) + a = 6 \Rightarrow a = 4$

Answer: a = 4

Exercise 2

1

1. Find the points on the curve $y=(x^2+5)(x-1)$ at which the slope of the tangent is equal to 10 .

$\frac{dy}{dx} = (x^2 + 5)(1) + (x-1)(2x) = 3x^2 - 2x + 5$

slope of the tangent equal to 10:

$3x^2 - 2x + 5 = 10$

$3x^2 - 2x - 5 = 0$

solve equation by factoring:

$(3x + 5)(x - 1) = 0$

$3x + 5 = 0,\ x1 = \frac{-5}{3}$

$x - 1 = 0,\ x2 = 1$

For $x = \frac{-5}{3}$: $y = ((\frac{-5}{3})^2 + 5)((\frac{-5}{3}) - 1) = \frac{140}{27}$

For x = 1: $y = (1^2 + 5)(1 - 1) = 0$

Answer: (1, 0) and $(\frac{-5}{3}, \frac{140}{27})$

2

Find the first derivative of:

(a) $y = (x^5 - 3x)(5x - 9)$

$\frac{dy}{dx} = (5x^4 - 3)(5x - 9) + (x^5 - 3x)(5)$
$= (5x^4 - 3)(5x - 9) + 5(x^5 - 3x)$

(b) $y = (x^2 + 5)(x^2 - 3)$

$\frac{dy}{dx} = (2x)(x^2 - 3) + (x^2 + 5)(2x) = 2x(x^2 - 3 + x^2 + 5)$
$= 2x(2x^2 + 2) = 4x(x^2 + 1)$

3

Find the slope of the tangent to the curve $y = x^2(5x - 1)$ at x = 1.

$\frac{dy}{dx} = 2x(5x - 1) + x^2(5) = 2x(5x - 1) + 5x^2$

At x = 1:
slope of the tangent = 2(1)(5 - 1) + 5 = 8 + 5 = 13

4

Find the measure of the angle which the tangent to the curve $y = (x^3 + 3x)(x - 5)$ makes with the positive direction of X-axis at the point (-1, 0).

$y = (x^3 + 3x)(x - 5)$

$\frac{dy}{dx} = (3x^2 + 3)(x - 5) + (x^3 + 3x)(1)$

At x = -1:

$(3(-1)^2 + 3)(-6) + (-1^3 + 3(-1)) = (3 + 3)(-6) + (-1 - 3) = -36 - 4 = -40$

$\tan(\theta) = -40 \Rightarrow \theta = \tan{-1}\,(-40) \approx -88.6°$

Angle with positive x-axis = 180 - 88.6 = 91.4°

Exercise 3

1

Find the first derivative of the function $y = \frac{2x+3}{x-2}$ when x = 3

$$\frac{dy}{dx} = \frac{(x-2)(2) - (2x+3)(1)}{(x-2)^2}$$
$$= \frac{(x-2)(2) - (2x+3)}{(x-2)^2}$$
$$= \frac{2x - 4 - 2x - 3}{(x-2)^2}$$
$$= \frac{-7}{(x-2)^2}$$

At x = 3:

$$= \frac{-7}{(3-2)^2} = -7$$

2

Find the slope of the tangent to the curve $y = x - \frac{2}{X}$ at x = 1

$$\frac{dy}{dx} = 1 + \frac{2}{X^2}$$

At x = 1:

$$= 1 + \frac{2}{1} = 3$$

3

Find the measure of the angle which the tangent to the curve $y = \frac{1+x}{1-x}$ makes with the positive direction of X-axis at the point (-1, 0)

$$y = \frac{1+x}{1-x}$$

$$\frac{dy}{dx} = \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}$$

$$= \frac{(1-x)+(1+x)}{(1-x)^2}$$

$$= \frac{2}{(1-x)^2}$$

At x = -1:

$$= \frac{2}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$$

$\tan(\theta) = \frac{1}{2} \rightarrow \theta = \tan\text{-}1\ (\frac{1}{2}) \approx 26.57°$

4

Find the points on the curve $y = \frac{x-2}{x-1}$ at which the slope of the tangent is equal to 1

$$\frac{dy}{dx} = \frac{(x-1)(1)-(x-2)(1)}{(x-1)^2}$$

$$= \frac{(x-1)-(x-2)}{(x-1)^2}$$

$$= \frac{x-1-x+2}{(x-1)^2} = \frac{1}{(x-1)^2}$$

$\frac{1}{(x-1)^2} = 1 \rightarrow (x - 1)^2 = 1 \rightarrow x - 1 = \pm 1$

x = 2 or x = 0

At x = 2: y = (2 - 2)/(2 - 1) = 0

At x = 0: y = (0 - 2)/(0 - 1) = 2

Points: (2, 0) and (0, 2)

Exercise 4

1

$Find \frac{dy}{dx}$ if $= (2x^2 - 3x - 5)^5$ at $x = 2$

$\frac{dy}{dx} = 5(2x^2 - 3x - 5)^4 * (4x-3)$

$=5(2(2)^2 - 3(2) - 5)^4 * (4(2) - 3) = 2025$

2

If $(x) = \frac{3x^2+1}{x^2+2}$, $fin\ '(x)\ at\ x=1$

$\frac{dy}{dx} = \frac{(x^2+2)(6x)-(3x^2+1)(2x)}{(x^2+2)^2} = \frac{(1^2+2)(6(1))-(3(1)^2+1)(2(1))}{(1^2+2)^2} = 1.111$

3

If $y=z^2$, $z= x^2+1$, $then\ find\ \frac{dy}{dx}\ at\ x = -1$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$\frac{dy}{dz} = 2z = 2(x^2 + 1)$

$\frac{dz}{dx} = 2x$

$\frac{dy}{dx} = 2(x^2 + 1) \,.\, 2x = 2((-1)^2 + 1) \,.\, 2(-1) = -8$

4

If $y = z^3$, $z = \frac{x}{x+1}$ $then, find \frac{dy}{dx}$ $at\ x$=1

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$\frac{dy}{dz} = 3z^2 = 3(\frac{x}{x+1})^{\ 2}$

$\frac{dz}{dx} = \frac{(x+1)-x}{(x+1)^2}$

$\frac{dy}{dx} = 3(\frac{x}{x+1})^{\ 2} .(\frac{(x+1)-x}{(x+1)^2}) = 3(\frac{1}{1+1})^{\ 2} .(\frac{(1+1)-1}{(1+1)^2}) = 0.1875$

5

If $y = z + 2$, $z =\ x^2 + 2x$, $then\ find \frac{dy}{dx}\ at\ x = 2$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$\frac{dy}{dz} = 1$

$\frac{dz}{dx} = 2x + 2$

$\frac{dy}{dx} = 1 \,.\, (2x + 2) = 1\ (2(2)+2) = 6$

6

If $y{=}z^3{+}, z = x^2{-}x\ then, find \frac{dy}{dx}\ at\ x{=}2$

$\frac{dy}{dx}{=}\frac{dy}{dz} \cdot \frac{dz}{dx}$

$\frac{dy}{dz}{=}3z^2 + 1$

$\frac{dz}{dx}{=}2x - 1$

$\frac{dy}{dx} = (3z^2 + 1) \,.\, (2x - 1) =(3((2)^2{-}(2)\ )^2 + 1) \,.\, (2(2) - 1)\ = 39$

Exercise 5

Find the first derivative of each of the following functions:

1 $y{=}tan2x$

$\frac{dy}{dx} = 2.\sec^2 2x$

2

$y = sin2x + sin3x$

$\frac{dy}{dx}$ =2.cos2x+3.cos3x

3

$y = xsinx - x^2$

$\frac{dy}{dx}$ = (x(cosx) + (1)(sinx)) - 2x = x. cosx + sinx -2x

4

$y = sin2x + cos3x$

$\frac{dy}{dx}$ = 2cos2x – 3 sin3x

5

$y = 2x + sin2x \quad at\ x = 0$

$\frac{dy}{dx}$ = 2+2cos2x = 2+2cos(2).(0) = 4

6

$y = \cos\ 2x + \sin 5$x

$\frac{dy}{dx}$ = -2sinsx + 5cos5x

7

$y = \tan(3x + 2)$

$\frac{dy}{dx}$ = 3.sec^2 (3x +2)

8

$y = cosx\ .\ sinx \quad at\ x = \pi$

$\frac{dy}{dx}$ = (-sinx)(sinx)+(cosx)(cosx)

= $\cos^2 x - \sin^2 x$

= cos2x

=$\cos(2(\pi))$ =1

9

$y = (3x - \cos\ 2x)^3$

$\frac{dy}{dx} = 3(3x - \cos\ 2x)^2.(3+2\sin 2x)$

10

$y = x^3\ \cos 2x$

$\frac{dy}{dx}$ = $(3x^2)(\cos 2x) + (x^3)(-2\sin 2x) = 3x^2 \cos 2x - 2x^3 \sin 2x$

Exercise 6

1

If $= Cosx\ .sinx$, find $\frac{dy}{dx}$ $at\ x = \frac{\pi}{4}$

$\frac{dy}{dx} = \sin x\ .\ (-\ \sin x) + \cos^2 x$

= $\cos^2 x - \sin^2 x$

= cos2x

= $\cos 2(\frac{\pi}{4}) = 0$

2 If $= (sinx+cosx)^{2}$, prove that: $y' = 2cos2x$

Note : cos2x = $\cos^2x - \sin^2x$

$\frac{dy}{dx} = 2(\sin x + \cos x) . (\cos x - \sin x)$

$= (2\sin x + 2\cos x) . (\cos x - \sin x)$

$= 2\sin x.\cos x - 2\sin^2 x + 2\cos^2 x - 2\sin x.\cos x$

$= 2\cos^2 x - 2\sin^2 x$

$= 2(\cos^2 x - \sin^2 x)$

$= 2(\cos 2x)$

3 Find $\frac{d}{dx}(2x^2 \, sinx \, cosx) \quad at \quad x = \pi$

Solution 1 :

Note : $\sin x \cos x = \frac{1}{2}\sin 2x$

$y = 2x^2 \, sinx \, cosx = (2x^2) . (\frac{1}{2}\sin 2x) = x^2 \sin 2x$

$\frac{dy}{dx} = 2x . (\sin 2x) + (2\cos 2x)(x^2) = 2(\pi) . (\sin 2\pi) + (2\cos 2\pi)(\pi^2) = 19.739$

Solution 2 :

If y = u(x) . v(x) . w(x)

$\mathbf{y' = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)}$

$y = 2x^2 \, sinx \, cosx$

$u = 2x^2$, $v = \sin x$, $w = \cos x$

$y' = (4x)(\sin x)(\cos x) + (2x^2)(\cos x)(\cos x) + (2x^2)(\sin x)(-\sin x)$

$= 4x \sin x \cos x + 2x^2 \cos^2 x - 2x^2 \sin^2 x$

$= 4\pi \sin\pi \cos\pi + 2\pi^2 \cos^2\pi - 2\pi^2 \sin^2\pi$

$= 19.739$

4 If $y = \sin x - \frac{1}{3}\sin^3 x$, prove that: $y \backslash = \cos^3 x$

Note : $\cos^2 x = 1 - \sin^2 x$

$\frac{dy}{dx} = \cos x - \left(\frac{1}{3}.3(\sin^2).(\cos x)\right)$

$= \cos x - \left((\sin^2 x).(\cos x)\right)$

$= \cos x - \sin^2 x \cos x$

$= \cos x(1 - \sin^2 x)$

$= \cos x(\cos^2 x)$

$= \cos^3 x$

5 If $= \frac{x}{\sin x}$, find $\frac{dy}{dx}$ $at\ x = \frac{\pi}{2}$

$\frac{dy}{dx} = \frac{(\sin x)(1) - (x)(\cos x)}{(\sin x)^2}$

$= \frac{\left(\sin\left(\frac{\pi}{2}\right)\right)(1) - \left(\frac{\pi}{2}\right)\left(\cos\left(\frac{\pi}{2}\right)\right)}{\left(\sin\left(\frac{\pi}{2}\right)\right)^2} = 1$

6 If $= x\cos 2x$, find $\frac{dy}{dx}$ $at\ x = \frac{\pi}{2}$

$\frac{dy}{dx} = 1(\cos 2x) + x(-2\sin 2x)$

$= 1(\ os2(\frac{\pi}{2})\ ) + \frac{\pi}{2}\ (\ -2\sin2(\frac{\pi}{2})\ ) = -1$

7 Find $\frac{d}{dx}\ (\frac{x}{\cos3x})$ $\quad a = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{(\cos3x)(1) - (x)(-3\sin3x)}{(\cos3x)^2}$$

$$= \frac{\left(\cos3(\frac{\pi}{3})\right)(1) - (\frac{\pi}{3})(-3\sin3(\frac{\pi}{3}))}{(\cos3(\frac{\pi}{3}))^2} = -1$$

8 Find the slope of the tangent to the curve $y = sin2x.cosx$ $\quad at\ x = \pi$

$\frac{dy}{dx} = 2\cos2x\ .\ \cos x + (-\ \sin x\ )\ .\ (sin2)$

$= 2\cos2\ \ .\ \cos\pi + (-\ \sin\pi\ )\ .\ (sin2\ \ ) = -2$