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T5. Newtonian Universe

After Isaac Newton's discovery of the Law of Universal Gravitation, Pierre-Simón Laplace attempted to describe the behavior of the Universe as a cloud of moving matter. Laplace assumed that, in description of motion of this cloud, out of all interactions only gravity plays significant role at large distances between bodies. However, he lacked experimental data to accomplish construction of a realistic model.

Only in the 20th century, astronomical observations made it possible to establish that in the observed part of the Universe matter is distributed almost uniformly and isotropically (i.e. in the areas containing many galaxies the average density of matter is practically the same). In addition, Edwin Hubble discovered a law according to which distant objects move away from us along the line of sight with velocities V_R proportional to distance R: $V_R = H \cdot R$, where H is called the Hubble constant; it was found to be independent of distance and approximately equal to $7 \cdot 10^{-11}$ years^{-1}. George Gamow suggested that such velocity distribution is due to the Big Bang, an explosion that occurred in a small area of space, and then the matter flew in different directions at different speeds. Therefore, the particles, which flew faster, have by now gone farther away from the explosion area

A1	Does the Hubble law imply that the solar system is in the region of the Universe where the Big Bang occurred? (Because Hubble made his observations from the Earth!) Explain your answer with a drawing and formula.

In theoretical physics, Einstein's general relativity equations are used to describe the expansion of the Universe after the Big Bang. However, it's interesting to find the conclusions Laplace would have drawn if he used Hubble law and information about the homogeneity of the Universe in the model based on Newton's laws. To find such conclusions, let us consider the expansion of the so-called Newtonian Universe (NU). The NU is a homogeneous ball of total mass $M = 10^{55}$ kg, in which particles of matter interact due to the Newton's law of gravity with gravitational constant $G = 6.7 \cdot 10^{-11}$ m^3/(kg$\cdot$s^2), and velocities of matter are distributed according to Hubble law, in which the "constant" H is actually a function of time $H = H(t)$.

A2 Calculate the gravitational potential energy E_G of the NU at a time t when its radius is equal to R. Give your answer as a formula expressing E_G in terms of M and R.
Remark: if the mass of the system increases from m to $m + \Delta m$ without any changing the relative mass distribution, the gravitational potential energy increases by

$$\Delta E_G = \int_0^{\Delta m} \varphi \, dm,$$

where φ is a gravitational potential at the point of mass dm. Also φ is chosen so that $\varphi = 0$ on infinity.

A3	Calculate the kinetic energy E_K of the NU at the same time t. Give your answer as a formula expressing E_K in terms of M, H and R.

It is clear that in the process of further expansion (over time t) the matter in the NU will be decelerated by gravity. Let's assume that at time t, which is counted from the Big Bang, some "residents" of the NU measured the average density $\rho(t)$ of matter in it and the Hubble constant $H(t)$.

A4	At what relation between $\rho(t)$ and $H(t)$ will the expansion of the NU stop at a finit size and be replaced by compression? Give the answer in the form of an inequality.

A5	Let the total energy of matter in the NU, i.e. the sum of kinetic energy and potential energy of gravitational interaction, be equal to $$E = -\frac{2}{15}Mc^2,$$ where c is the speed of light in vacuum. Find the maximum radius of the NU in the process of expansion. Write down the formula and get a numerical answer in parsecs (1 parsec is approximately equal to 3.2 light years, or $3 \cdot 10^{16}$ m).

For the conditions described in question A5, find the total lifetime of the NU from the Big Bang to the Big Implosion. Write down the formula and get a numerical answer in years.

Remark:

$$\int\limits_0^1 \frac{\sqrt{x}}{\sqrt{1-x}}dx = (y = \sqrt{1-x}) = 2\int\limits_0^1 \sqrt{1-y^2}dy$$

A7	What are the options for further motion of matter in the Universe at different ratios between the density and the Hubble constant? Describe the qualitative behavior of the NU radius for each of the possible cases. Draw graphs: three different pairs of graphs showing the NU radius R and the rate of its expansion $\dot{R}$ versus time t (keep in mind that at the time of the Big Bang $t = 0$ and the radius of the NU is considered to be almost zero). Draw the graphs qualitatively, showing all the important details and features.

T6. Quintessence

According to data, collected by astronomers at the end of XXs century and the beginning of the XXIs centery, shining objects in the observable universe move away from each other with increasing velocity during last $2.5-4$ Myr (Myr $= 10^6$ yr). This observation haven't being explained by cosmological models, which assume that the universe consists only of ordinary matter. At the same time, it doesn't matter whether this matter visible (it radiates or scatters EM waves) or dark(it doesn't interact with EM waves) - the only important thing is that it normally gravitationally interacts with other objects. Therefore in the modern cosmology we add to the model various vacuum-like forms of matter with unusual properties. These forms of matter are called "dark matter". Certainly, scientific models of "quintessence" or "dark matter" are being developed with Metric-affine gravitation theory, Quantum field theory and cutting-edge math. However, several interesting phenomena could be studied with quite simple models. We will work with a similar one, which we will call the Toy Cosmological Model or TCM.

In the TCM not only an ordinary matter (which behaves according to Newton's law of gravity) is represented, but also the quintessence. When quintessence is mixed with an ordinary matter, it makes a negative contribution to the gravitational mass of that matter. Also, the quintessence creates a pressure p_q that acts on the ordinary matter and this pressure is determined by the mass density $\rho_q = M_q/V_q < 0$ of the quintessence:

$$p_q = A(-\rho_q)^{5/3},$$

where A is a constant. Note, that p_q is also an internal energy density and in order to move or deform a quintessence you must do work on it.

The total mass of the ordinary matter is $M_s > 0$ and the total mass of the quintessence is $M_q < 0$. Also $M_s + M_q > 0$. Moreover, we assume that the universe at any given time t is a ball with radius $R(t)$, homogeneously filled with a mixture of ordinary matter and quintessence. The motion of ordinary matter obeys Newton's laws and quintessence has no inertial mass, so its kinetic energy is equal to zero.

Remark: if the mass of system increases from m to $m + \Delta m$ without any changing the relative mass distribution, the gravitational potential energy increases by

$$\Delta E_G = \int_0^{\Delta m} \varphi \, dm,$$

where φ is a gravitational potential at the point of mass dm. Also φ is chosen so that $\varphi = 0$ on infinity.

A1	Find the total energy E of the universe as a function of its radius R in the form $$E = -\frac{\alpha}{R} + \frac{\beta}{R^2} + E_K,$$ where E_K is the kinetic energy of the universe.

A2	For given M_s, M_q, A and E of the universe find a range of radii which it can have. Write the expression in terms of α, β and E.

A3	Could the universe described by TCM be stable, i.e., have a constant radius? If so, describe it quantitatively.

Let's describe a particular realization of the universe in TCM. Let the total energy E of the universe be equal to zero and its history start from zero expansion velocity with the minimum radius. We also assume that in this universe the Hubble's law holds: at any given time, the distribution of velocities $v(r, t)$ of the ordinary matter as a function of distance r from the center is given by

$$v(r, t) = \frac{r}{R} v(R, t),$$

where R is the radius of the universe.

A4	How long will the universe continue to expand at a positive acceleration?