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T15. Plasma oscillations

Plasma - is a state of matter, when it consists of mobile charged particles: ions and electrons. Its unique properties are realted with that fact that it's a gas of mobile chrages.

The mass of the electron $m = 9.10 \cdot 10^{-31}$ kg, the elementary charge $e = 1.60 \cdot 10^{-19}$ C.

Electrons are are much lighter than ions, so the electronic contribution is the main one in the most of processes. Let n_0 be the concentration of the ions (m^{-3} units) and n be the concentration of the electrons. Let's consider that all ions are similar and that their charge is $+e$.

A1	If we assume that the velocity of the electrons at a certain point is $\vec{v}$, what is the current density $\vec{j}_t$ at that point?

The macroscopic electrodynamic properties of the matter are described by its relative permittivity ε and conductivity σ. More precisely, if there is an electric field $\vec{E}$ in this matter there is also the polarization density $\vec{P}$ and the conduction current $\vec{j}$:

$$\vec{P} = \varepsilon_0(\varepsilon - 1)\vec{E}, \quad \vec{j} = \sigma\vec{E}.$$

A2	How to express the total current $\vec{j}_t$ from the conduction current $\vec{j}$ and the time derivative of the polarization density $\partial\vec{P}/\partial t$.

We can redefine the relative permittivity ε in such a way that the polarization density also compounds the conduction current and $\vec{j}_t = \partial\vec{P}/\partial t$. Note, that the relative permittivity becomes a complex number.

A3	Show that for the harmonic external field $\tilde{E}e^{-i\omega t}$ with the angular frequency ω, the redefinition of the ε is $$\tilde{\varepsilon} = \varepsilon + \frac{i\sigma}{\omega\varepsilon_0}.$$

We can find the relative permittivity of a plasma if we can describe the motion of the electrons in it. The electrons interact with the external field $\tilde{E}e^{-i\omega t}$ and the electric field of the other particles. Also, a plasma is electroneutral at fairly large distances and fields of individual particles is shielded.

Particulary, when the plasma is in equilibrium without the external field the average charge density is zero. According to Maxwell's equation $\operatorname{div}\vec{E} = \rho/\varepsilon_0$ the electric field is zero everywhere. However, every single electron has it's own electric field $\vec{E} = \pm e\vec{r}/\left(4\pi\varepsilon_0 r^3\right)$, so only the total electric field is zero by averaging over finite volume with a characteristic radius λ_D, called the Debye length. In the other words, at the distances of the order of λ_D, the electric field of the particular charge is shielded by the others.

Since the magnetic field generated by the system is always negligible in the presence of the electric interaction we can choose the potential $\Phi(\vec{r})$ of the electric field. Thus, the substitution $\vec{E} = -\operatorname{grad}\Phi$ in Maxwell's equation gives us

$$-\Delta\Phi = \frac{e}{\varepsilon_0}\left(n_0 - n\right).$$

Also, the value of the $\Phi(\vec{r})$ is proportional to the potential energy at that particular point and a thermodynamic equilibrium of particles is described by the Boltzmann distribution

$$n(\vec{r}) = \langle n\rangle \exp\left(\frac{e\Phi}{k_B T}\right), \quad n_0(\vec{r}) = \langle n_0\rangle \exp\left(-\frac{e\Phi}{k_B T}\right),$$

where $\langle n\rangle = \langle n_0\rangle$ are the average concentrations of electrons and ions in the plasma, so the constant in the potential is chosen so that if $V\langle n\rangle = \int n(\vec{r})\, dV$.

Finally, we have a nonlinear differential equation for Φ:

$$\Delta\Phi = \frac{e\langle n\rangle}{\varepsilon_0}\left[\exp\left(\frac{e\Phi}{k_\mathrm{B}T}\right) - \exp\left(-\frac{e\Phi}{k_\mathrm{B}T}\right)\right] \tag{1}$$

Let's consider how the electric field of the charge q behaves on the distances where the thermal energy is much larger than the potential electric energy, i.e. $e\Phi \ll k_\mathrm{B}T$. In this approach the equation (1) becomes linear and has a spherically symmetric solution corresponding to the shielded electric field of charge q:

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0 r}e^{-r/\lambda_D}.$$

A4	Using that for a spherically symmetric function, the Lapplacian could be written as $$\Delta\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right),$$ calculate the λ_D. Express your answer in terms of $\langle n\rangle$ and T. Estimate the value of λ_D in the gas discharge, where$\langle n\rangle = 10^{16}\ \mathrm{m}^{-3}$, $T = 10^4$ K.

A5	Estimate the average number of particles N_loc whose field is not shielded at the given point in the plasma.

If $N_\mathrm{loc} \gg 1$, the electric field at each point is the sum of numerous fields that cancel each other out, so the only field that interacting with the particular electron is the external field.

A6	Analyze the motion of the electron in the plasma in the external field $\tilde{E}e^{-i\omega t}$ and find the relative permittivity $\tilde{\varepsilon}$ of the plasma.

The plasma has the frequency of self-oscillations: the oscillations that could exist without a periodic external field. Such a oscillations are called the Langmuir waves.

A7	Find the circular frequency ω_p of the plasma self-oscillations.

T16. Collisions in plasma

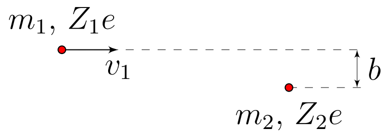
In the simplest model of the plasma we take into account only the interaction of the electrons with the external field and the shielding effect, so we write their equation of motion as

$$m\ddot{\vec{r}} = \tilde{E}e^{-i\omega t}.$$

In this model the conductivity of the plasma σ is zero. Let's add the interaction between charged particles, in particular their scattering on each other. We assume that the plasma consists of electrons with the charge $-e = -1.60 \cdot 10^{-19}$ C and the mass $m = 9.10 \cdot 10^{-31}$ kg and ions with the charge Ze and the mass $M \gg m$. The averaged over the Debye length concentration of ions in the plasma is n_0 and the concentration of electrons is n. Also, the plasma is "hot" i.e. $\lambda_D^3 n \gg 1$.

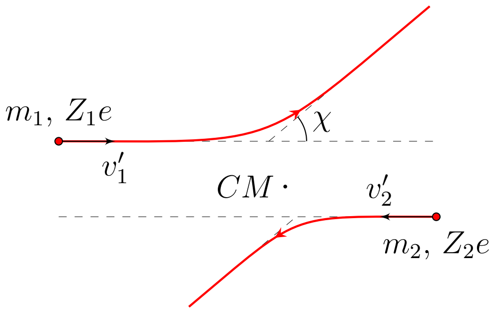
Part A. Rutherford scattering

Let's consider the particle with the charge $Z_1 e$ and the mass m_1 which is the projectile, and the particle with the charge $Z_2 e$ and the mass m_2 which is the target. The velocity of the first particle is v_1 when it's far from the target and the impact parameter is b.



A1	Write the motion equations for both particles and show that this system is equivalent to the interaction of both particles with the effective Columb's potential with a center similar to the center of mass CM.

Let's consider the dynamics in the center of mass frame: in this case the dynamic equations of the particles are separated. For further discussion we will use the polar coordinate system with the reference point CM.



A2	How does the angular momentum of the first particle behave?

A3	Write the second Newton's law in terms of r, θ and their time derivatives $\ddot{r}$ and $\dot{\theta}$.

A4	Introduce the Binet substitution $u = 1/r$ and obtain the differential equation for u as a function of θ in the form of a harmonic oscillator: $$u'' + Au + B = 0,$$ where u'' is the second derivative of u with respect to the angle θ.

A5	Express the deflection angle χ in the frame of the center of mass through b, v_1, $m_{1,2}$ and $Z_{1,2}$.

Let b_{90} the impact parameter b for which the defelction angle χ is $90°$.

A6	Find the b_{90}. Express you answer by Z_1, Z_2, m_1 and v_1.

A7	In this question we work in the lab frame where the target was originally motionless. What is the difference $\Delta p(b)$ of the momuntum of the first particle projection onto the direction of the original motion? Work with the small angle approximation $b \gg b_{90}$. Express your answer by the original momentum of the particle p, $m_{1,2}$, b and b_{90}.

Part B. Scattering in matter

Let's look at how scattering changes the dynamics of electrons in plasma. First of all, we have to take into account that in matter we have to average of $\Delta p(v)$ over several scattering events. The formula for the average Δp when the projectile has traveled the distance l in the matter consisting of motionless ions with the concentration n_0 is

$$\Delta p = \int_0^\infty \Delta p(b) \cdot n_0 2\pi b l \, db.$$

This intergral diverges at $b = 0$ and also at $b = \infty$, so we have to cut-off it. Firstly, we use simplified formula for $p(b)$ in approximation $b \gg b_{90}$, so let's use the b_{90} as the bottom limit. Secondly, according the shielding effect we have to use λ_D as the top limit. Let Λ be λ_D/b_{90}.

B1	Find the equation for Δp in approximation $M \gg m$. Express the answer by p, b_{90}, n_0, l and Λ.

Thermal motion of electrons is much faster than drifting under the external field. Let's consider the small volume ΔV of the electron gas with the size which approximately equal to λ_D. Let this volume drift with the velocity $\vec{v}_d$ in the matter consisting of the motionless ions. In this volume there is an electron velocity distribution which we will consider to be Mawellian:

$$f(\vec{v}) = \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} \exp\left(-\frac{m(\vec{v} - \vec{v}_d)^2}{2k_{\mathrm{B}} T}\right),$$

where $f(\vec{v})$ is the probability density function, i.e. the probability dP that the velocity is in the region $v_x \in (v_x, v_x + dv_x)$, $v_y \in (v_y, v_y + dv_y)$, $(v_z, v_z + dv_z)$ is

$$dP = f(\vec{v}) \, dv_x \, dv_y \, dv_z.$$

The Gauss integral can be useful for further calculations:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

B2	Show, that the probability P_{any} that the electron's velocity is any is unity, i.e. $$P_{\text{any}} = \int_{-\infty}^{+\infty} dv_x \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z f(v_x, v_y, v_z) = 1.$$

B3	Find the mean value of the velocity $\langle v \rangle$. Ignore any corrections associated with the presence of v_d.

B4	Estimate the mean force $\langle\vec{F}\rangle$ that acting on the volume ΔV from the ions. For simplicity, assume that Λ is independent of the electron velocity v and use the value of Λ for $v = \langle v\rangle$. Also, ignore that $\lvert\vec{v}\rvert$ (not a vector, just it's absolute value) depends on the $\vec{v}_d$. Finally, you understand approximations properly if $$\langle\vec{F}\rangle \propto \vec{v}_d \int u^3 e^{-u^2} du.$$ Note, that all of these approximations don't change the crux of the phenomena, the only thing that might be affected is a different numerical coefficient before.

B5	Write the second Newton's law for ΔV. Consider, that the interaction with neighboring volumes of the electron gas is canceled out.

B6	Find the current density $\vec{j}_t$ in the model under the external field $\tilde{E}e^{-i\omega t}$.

B7	With $\sigma = \mathfrak{Re}\left\{\dfrac{j}{\tilde{E}e^{-i\omega t}}\right\}$ find the conductivity σ of the plasma.