Домашно №1 по Дискр. Стр-ри, спец. Информатика, летен семестър 2018/2019 г.

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Задача 1

S = $\emptyset$
P - едноместен предикат над S
P - силен $\leftrightarrow \exists x\ (P(x)) \rightarrow \forall y\ (P(y))$
Да се докаже, че $\forall P : P$ - силен $\forall x \forall y\ (P(x) \leftrightarrow P(y))$

$\forall P\ (\ \exists x(P(x))\ \forall y(P(y))\ )\overset{?}{\rightarrow} \forall x \forall y\ (P(x) \iff P(x))$

$(a \rightarrow b) \leftrightarrow (\neg a \vee b)$

$\forall P\ A(B) \rightarrow B(P)$
$A(P) \rightarrow B(P)$

$\Rightarrow\ A(P) \leftrightarrow\ \neg(\exists x P(x)) \vee (\forall y P(y))$
$\iff\ [\forall x(\neg P(x))] \vee [\forall y P(y)]$
$\iff\ \forall x\ P(x) \vee \neg P(y)$

$B(P) \leftrightarrow \forall x \forall y\ [P(x) \leftrightarrow P(y)]$
$\iff\ \forall x \forall y\ [P(x) \leftrightarrow P(y)]$
$\iff\ \forall x \forall y\ [(P(x) \wedge P(y)) \vee (\neg P(x) \wedge \neg P(y))]$
$\iff\ \forall x\ (P(x) \vee \neg P(x))$

$A(P) \rightarrow B(P)$
$\Longrightarrow\ \forall P : P$ - силен $\rightarrow \forall x \forall y\ (P(x) \leftrightarrow P(y))$

Задача 2

A,B,C - множества
$X = (A \cup A) \times (C \cap D)$
$Y = [(A \times C) \cap (A \times D)] \cup [(B \times C) \cap (B \times D)]$

Да се докаже, че $X = Y$

I: $X \subseteq Y$

$(a_1, a_2) = a \in X \rightarrow a \in [(A \times B) \times (C \cap D)]$
$\iff\ (a_1 \in (A \cup B)) \wedge (a_2 \in (C \cap D))$

$\implies (a_1 \in A \vee a_1 \in B) \wedge (a_2 \in C \wedge a_2 \in D)$
$\implies (a_1 \in A \wedge a_2 \in C \wedge a_2 \in D) \vee (a_1 \in B \wedge a_2 \in C \wedge a_2 \in D)$
$\implies [a = (a_1, a_2) \in (A \times C) \wedge a \in (A \times D)] \vee [a \in (B \times C) \wedge a \in (B \times D)]$
$\implies a \in [(A \times C) \cap (A \times D)] \vee a \in [(B \times C) \cap (B \times D)]$
$\implies a \in [(A \times C) \cap (A \times D)] \cup [(B \times C) \cap (B \times D)]$
$\implies X \subset Y$

II: $X \supseteq Y$

$(a_1, a_2) = a \in X \to a \in [(A \times C) \cap (A \times D)] \cup [(B \times C) \cap (B \times D)]$
$\implies a \in [(A \times C)\ cap(A \times D)]] \ \vee\ a \in [(B \times C) \cap (B \times D)]$
$\implies a \in [(A \times C) \cap (A \times D)] \ \vee\ a \in [(B \times C) \cap (B \times D)]$
$\implies [a \in (A \times C) \ \wedge\ a \in (A \times D)] \vee [a \in [(B \times C) \wedge a \in (B \times D)]]$

$\implies [a_1 \in A \ \wedge\ a_2 \in C \ \wedge\ a_1 \in A \ \wedge\ a_2 \in D] \vee$
$[a_1 \in B \ \wedge\ a_2 \in C \ \wedge\ a_1 \in B \ \wedge\ a_2 \in D]$
$\implies [a_1 \in A \ \wedge\ a_2 \in C \ \wedge\ a_2 \in D] \ \vee\ [a_1 \in B \ \wedge\ a_2 \in C \ \wedge\ a_2 \in D]$
$\implies [a_1 \in A \ \wedge\ a_2 \in (C \cap D)] \ \vee\ [a_1 \in B \ \wedge\ a_2 \in C \ \wedge\ a_2) \in D]$
$\implies [a_1 \in A \ \vee\ a_2 \in B] \ \wedge\ a_2 \in (C \cap D)$
$\implies a_1 \in (A \cup B) \ \wedge\ a_2 \in (C \cap D)$
$\implies (a_1, a_2) = a \in [(A \cup B) \times (c \cap D)]$

$(X \supseteq Y) \wedge (X \subseteq Y) \to X = Y$

Задача 3

$\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$
$R \subseteq \mathbb{N}^+ \times \mathbb{N}^+$
$x\ R\ y \leftrightarrow \exists k \in \mathbb{Z}, n \in N : \frac{x}{y} = 2^k \ \wedge\ xy = n^2$

I. Рефлексивност

$x \in \mathbb{N}^+ \overset{?}{\to} xRx$

$$(1) \quad \frac{x}{x} = 2^k = 1 \to k = 0$$
$\implies k = 0 \in \mathbb{Z} \to xRx$

$$(2) \quad x^2 = n^2 \to x = n$$
$\implies (n = x \in \mathbb{N}^+ \subset \mathbb{N}) \to n \in \mathbb{N}$

$(1) \wedge (2) \implies$ Рефлексивност

II. Симетричност

$x, y \in \mathbb{N}^+$
$xRy \overset{?}{\to} yRx$

$$(1)\quad \frac{x}{y} = 2^k \ \wedge \ xy = n^2 \ \wedge \ k \in \mathbb{Z} \ \wedge \ n \in \mathbb{N}$$
$$\implies \frac{y}{x} = \frac{1}{2^k} = 2^{-k} \to -k \in \mathbb{Z}$$
$$\implies k \in \mathbb{Z}$$

$$(2)\quad xy = n^2 \to yx = n^2$$
$$\implies n \in \mathbb{N}$$

$(1) \wedge (2) \implies$ Симетричност

III. Транзитивност

$\frac{a}{b} = 2^k \qquad \frac{b}{c} = 2^l$
$ab = n^2 \qquad bc = m^2$

$\frac{a}{b}b^2 = n^2$
$2^k b^2 = n^2$
$n \in \mathbb{N} \to 2^{\frac{k}{2}} b = n \in \mathbb{N}$
$\implies k \equiv 0 \ (mod\ 2)$

$\frac{b}{c}c^2 = m^2$
$2^l c^2 = m^2$
$m \in \mathbb{N} \to 2^{\frac{l}{2}} c = m \in \mathbb{N}$
$\implies l \equiv 0 \ (mod\ 2)$

$abbc = n^2 m^2$
$ac = \frac{n^2 m^2}{b^2} = \frac{abbc}{b^2} = \frac{a}{b} \cdot \frac{c}{b} . b^2 = 2^{k-l} b^2 = (2^{\frac{k-l}{2}} b)^2$
$\implies (k \equiv 0 \ (mod\ 2)) \wedge (l \equiv 0 \ (mod\ 2)) \to (k - l \equiv 0 \ (mod\ 2))$
$\implies k - l \in \mathbb{Z}$
$\implies k > l (R)$
$\implies \frac{k-l}{2} \in \mathbb{N} -> 2^{\frac{k-l}{2}} \in \mathbb{N}$
$\implies b \in \mathbb{N}^+ \to 2^{\frac{k-l}{2}} . b \in \mathbb{N}$
$\implies$ Транзитивност

От (I),(II) и (III) следва, че R е релация на еквивалентност

Задача 4

I: $f \cap g$ - частична

Допускаме, че $f \cap g$ не е частична
$\implies \exists a \in A \ \exists b, c \in B, b \neq c : (a, b) \in f \cap g \ \wedge \ (a, c) \in f \cap g$
$\implies (a, b) \in f \cap g \ \wedge \ (a, c) \in f \cap g$
$\implies (a, b) \in f \ \wedge \ (a, b) \in g \ \wedge \ (a, c) \in f \ \wedge \ (a, c) \in g \ \wedge \ b \neq c$
$\implies$ f и g не са функции
$\implies$ противоречие с условието

$\implies f \cap g$ е частична

II: $f \cap g$ - тотална

Ако f=g
$\implies f \cap g = f$ - функция
$\implies f \cap g$ е функция

Нека $f \neq g$
Допускаме, че $f \cap g$ е тотална
$\implies \forall a \in A\ \exists b \in B : (a, b) \in f \cap g$
$\implies (a, b) \in f\ \land\ (a, b) \in g$
$\implies \forall a \in A f(a) = g(a) \leftrightarrow f = g$
$\implies$ противоречие с $f \neq g$
$\implies f \cap g$ не е тотална

III: $f \subseteq g \overset{?}{\to} f \cup g$ - функция

$f \subseteq g \to f \cup g = f$
f е тотална $\implies f \cup g$ е тотална

IV: $f \cup g$ - функция

Допускаме, че $f \cup g$ е функция
$\implies \forall a \in A(\exists b \in B\ (a, b) \in f) \to (\forall c \in B\ (a, c) \notin g\ \lor\ b = c)$
$\implies \forall a \in A(\forall b \in B\ (a, b) \notin f)\ \lor\ (\forall c \in B\ (a, c) \notin g\ \lor\ b = c)$
$\implies \forall a \in A\ \forall b, c \in B\ (a, b) \notin f\ \lor\ (a, c) \notin g\ \lor\ b = c$
$\implies$ f не е дефинирана $\forall a \in A$, или g не е дефинирана $\forall a \in A$, или f=g
$\implies$ f не е функция, или g не е функция, или f=g
$\implies (f \cup g)$ е функция $\leftrightarrow (f = g)$

V: h е функция

$\forall n \in \mathbb{N} f_n : \mathbb{N} \to \mathbb{N}$
$$f_n(x) = \begin{cases} x, & x \leq n \\ \text{недефинирана}, & \text{в противен случай} \end{cases}$$
$$h = \bigcup_{i=1}^{\infty} f_i$$
Да се докаже, че h е функция
$$f_0(x) = \begin{cases} 0, & x = 0 \\ \text{недефинирана}, & \text{в противен случай} \end{cases}$$
$\forall f_i,\ i \in I_n\ f_i = f_{i-1} \cup \{(i, i)\}$
$\implies \forall f_i\ f_i \subseteq f_{i+1}\ , i \in \mathbb{N}$
$\implies \forall f_i\ f_i \cup f_{i+1} = f_{i+1}$

$\implies k \in \mathbb{N} \cup \{\infty\},\ \bigcup_{i=0}^{k} f_i = f_k$
$\implies \forall x \in \mathbb{N},\ x \leq k f_k(x) = x$
$\implies f_k = \bigcup_{i=1}^{\infty} f_i = h$
Нека k=∞. Тогава $\forall x \in \mathbb{N},\ f_k(x) = x$
$\implies$ h е дефинирана $\forall x < \infty,\ x \in \mathbb{N}$ и h(x)=x
$\implies \forall a \in \mathbb{N}\ (\exists b \in \mathbb{N}\ (a,b) \in h)\ \wedge\ (\exists c \in \mathbb{N}\ (a,c) \in h \rightarrow b = c)$
$\implies$ h е функция