Energy Detection of Unknown Deterministic Signals

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Abstract—By using Shannon's sampling formula, the problem of the detection of a deterministic signal in white Gaussian noise, by means of an energy-measuring device, reduces to the consideration of the sum of the squares of statistically independent Gaussian variates. When the signal is absent, the decision statistic has a central chi-square distribution with the number of degrees of freedom equal to twice the time-bandwidth product of the input. When the signal is present, the decision statistic has a noncentral chi-square distribution with the same number of degrees of freedom and a noncentrality parameter λ equal to the ratio of signal energy to two-sided noise spectral density. Since the noncentral chi-square distribution has not been tabulated extensively enough for our purpose, an approximate form was used. This form replaces the noncentral chi-square with a modified chi-square whose degrees of freedom and threshold are determined by the noncentrality parameter and the previous degrees of freedom.

Sets of receiver operating characteristic (ROC) curves are drawn for several time-bandwidth products, as well as an extended nomogram of the chi-square cumulative probability which can be used for rapid calculation of false alarm and detection probabilities.

Related work in energy detection by J. I. Marcum and E. L. Kaplan is discussed.

Glossary of Principal Symbols

$s(t)$ = signal waveform
$n(t)$ = noise waveform
N_{02} = two-sided noise power density spectrum
E_s = signal energy
V' = normalized test (decision) statistic
V_T' = threshold value of V'
T = observation time interval, seconds
W = bandwidth, cycles per second
sinc $x = \sin \pi x/\pi x$
$a_i = n(i/2W)$ = noise sample value
$b_i = a_i/\sqrt{2WN_{02}}$ = normalized noise sample value
$\alpha_i = s(i/2W)$ = signal sample value
$\beta_i = \alpha_i/\sqrt{2WN_{02}}$ = normalized signal sample value
$\lambda = E_s/N_{02}$ = noncentrality parameter
D = modified number of degrees of freedom
G = divisor for modified threshold value
$T_B(m, n, r)$ = incomplete Toronto function
χ^2 = central chi-square variate
χ'^2 = noncentral chi-square variate
$n_c(t)$, $n_s(t)$ = in-phase and quadrature modulation components of $n(t)$
$s_c(t)$, $s_s(t)$ = in-phase and quadrature modulation components of $s(t)$
$a_{ci} = n_c(i/W)$
$a_{si} = n_s(i/W)$
$b_{ci} = a_{ci}/\sqrt{2WN_{02}}$
$b_{si} = a_{si}/\sqrt{2WN_{02}}$
$\alpha_{ci} = s_c(i/W)$
$\alpha_{si} = s_s(i/W)$
$\beta_{ci} = \alpha_{ci}/\sqrt{2WN_{02}}$
$\beta_{si} = \alpha_{si}/\sqrt{2WN_{02}}$
$y(t)$ = input to the energy detector
$N(m, \sigma^2)$ = a Gaussian variate with mean m and variance σ^2.

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I. Introduction

THE DETECTION OF a signal in the presence of noise requires processing which depends upon what is known of the noise characteristics and of the signal. When the noise is Gaussian and the signal has a known form, even with unknown parameters, the appropriate processing includes a matched filter or its correlator equivalent. When the signal has an unknown form, it is sometimes appropriate to consider the signal as a sample function of a random process. When the signal statistics are known, this knowledge can often be used to design suitable detectors.

In the situation treated here, so little is known of the signal form that one is reluctant to make unwarranted assumptions about it. However, the signal is considered to be deterministic, although unknown in detail. The spectral region to which it is approximately confined is, however, known. The noise is assumed to be Gaussian and additive with zero mean; the assumption of a deterministic signal means that the input with signal present is Gaussian but not zero mean.

In the absence of much knowledge concerning the signal, it seems appropriate to use an energy detector to determine the presence of a signal. The energy detector measures the energy in the input wave over a specific time interval. Since only the signal energy matters, and not its form, the results given in this paper apply to any deterministic signal.

It is assumed here that the noise has a flat band-limited power density spectrum. By means of a sampling plan, the energy in a finite time sample of the noise can be approximated by the sum of squares of statistically independent random variables having zero means and equal variances. This sum has a chi-square distribution for which extensive tables and a convenient nomogram (to be described later) exist. In the presence of a deterministic signal, the sampling plan yields an approximation to the energy consisting of the sum of squares of random variables, where the sum has a noncentral chi-square distribution. The tabulations of noncentral chi-square are not as extensive nor as convenient as those for central chi-square, making it desirable to use approximations.

Grenander, Pollak, and Slepian [1] have developed a method for obtaining the exact distribution of the energy in a finite time sample of Gaussian noise with a flat, band-limited power density spectrum. They compare the exact results with the chi-square approximation (in their Section 4.4) for several values of time-bandwidth product and conclude that the approximation is quite good even for moderate values of time-bandwidth product.

Jacobs [2] has obtained expressions for the distribution of energy by using the Karhunen-Loeve expansion. Again, he concludes that the chi-square approximation is a good one for large values of time-bandwidth product.

There appears to be little relevant work in finding exact distributions of the energy in a finite time sample of noise plus unknown deterministic signal. An appropriate procedure, then, is to use the sampling plan mentioned above and consider ways to evaluate the noncentral chi-square distribution which results. The closest previous work of which I am aware is that of Marcum [3] and Kaplan [4]; the connection with these works will be reviewed later.

II. Detection in White Noise: Lowpass Processes

The situation of interest is shown in Fig. 1. The energy detector consists of a square law device followed by a finite time integrator. The output of the integrator at any time is the energy of the input to the squaring device over the interval T in the past. The noise prefilter serves to limit the noise bandwidth; the noise at the input to the squaring device has a band-limited, flat spectral density.

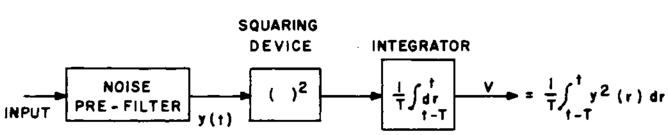

Fig. 1. Energy detection.

The detection is a test of the following two hypotheses.

1) H_0: The input $y(t)$ is noise alone:
   a) $y(t)=n(t)$
   b) $E[n(t)]=0$
   c) noise spectal density $=N_{02}$ (two-sided)
   d) noise bandwidth $= W$ cycles per second.
2) H_1: The input $y(t)$ is signal plus noise:
   a) $y(t)=n(t)+s(t)$
   b) $E[n(t)+s(t)]=s(t)$.

The output of the integrator is denoted by V and we concentrate on a particular interval, say, $(0, T)$, and take the test statistic as V or any quantity monotonic with V. We shall find it convenient to compute the false alarm and detection probabilities using the related quantity

$$V' = (1/N_{02})\int_0^T y^2(t)\,dt. \tag{1}$$

The choice of T as the sampling instant is a matter of convenience; any interval of duration T will serve. Another quantity of interest is the false alarm rate Q_0/T, where Q_0 is the false alarm probability based upon the energy in a sample of duration T.

It is known that a sample function, of duration T, of a process which has a bandwidth W (negligible energy outside this band) is described approximately by a set of sample values $2TW$ in number. In the case of a low-pass process, the values are obtained by sampling the process at times $1/2W$ apart. In the case of a relatively narrowband bandpass process, the values are obtained from the in-phase and quadrature modulation components sampled at times $1/W$ apart.

The expansion of a finite portion of a process in terms of sampling functions leaves something to be desired from the point of view of precision as well as rigor, although the approximations obtained are often considered satisfactory for engineering purposes. More satisfactory statements concerning the approximation to the energy by a finite sum of squares can be made. A discussion of this point is left to the Appendix. Here we merely state that very good approximations are obtained by $2TW$ terms except when $TW=1$. This special case can be treated separately, as we shall show later.

Let us start with a low-pass process. With an appropriate choice of time origin, we may express each sample of noise in the form [5]

$$n(t) = \sum_{i=-\infty}^{\infty} a_i \operatorname{sinc}(2wt - i), \tag{2}$$

where sinc $x = \sin \pi x/\pi x$, and

$$a_i = n\left(\frac{i}{2W}\right). \tag{3}$$

Clearly, each a_i is a Gaussian random variable with zero mean and with the same variance σ_i^2, which is the variance of $n(t)$; i.e.,

$$\sigma_i^2 = 2N_{02}W, \quad \text{all } i. \tag{4}$$

Using the fact that

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sinc}(2Wt - i)\operatorname{sinc}(2Wt - k)\,dt &= 1/2W, \quad i = k\\ &= 0, \qquad\quad i \neq k\end{aligned} \tag{5}$$

we may write

$$\int_{-\infty}^{\infty} n^2(t)\,dt = (1/2W)\sum_{i=-\infty}^{\infty} a_i^2. \tag{6}$$

Over the interval $(0, T)$, $n(t)$ may be approximated by a finite sum of $2TW$ terms, as follows:

$$n(t) = \sum_{i=1}^{2TW} a_i \operatorname{sinc}(2Wt - i), \qquad 0 < t < T. \tag{7}$$

Similarly, the energy in a sample of duration T is approximated by $2TW$ terms of the right-hand side of (6):

$$\int_0^T n^2(t)\,dt = (1/2W)\sum_{i=1}^{2TW} a_i^2. \tag{8}$$

This statement is supported by the Appendix and (8) may be considered to be obtained as an approximation (valid for large T) after substituting (7) into the left-hand side of (8), or by using (57) and the statements below it to justify taking $2TW$ terms of (6).

We can see that (8) is $N_{02}V'$, with V' here being the test statistic under hypothesis H_0.

Let us write

$$a_i/\sqrt{2WN_{02}} = b_i, \tag{9}$$

$$V' = \sum_{i=1}^{2TW} b_i^2. \tag{10}$$

Thus, V' is the sum of the squares of $2TW$ Gaussian random variables, each with zero mean and unity variance. V' is said to have a chi-square distribution with $2TW$ degrees of freedom, for which extensive tables exist [6]–[8].

Now consider the input $y(t)$ when the signal $s(t)$ is present. The segment of signal duration T may be represented by a finite sum of $2TW$ terms,

$$s(t) = \sum_{i=1}^{2TW} \alpha_i \operatorname{sinc}(2Wt - i), \tag{11}$$

where

$$\alpha_i = s(i/2W). \tag{12}$$

By following the line of reasoning above, we can approximate the signal energy in the interval $(0, T)$ by

$$\int_0^T s^2(t)\,dt = (1/2W)\sum_{i=1}^{2TW} \alpha_i^2. \tag{13}$$

Define the coefficient β_i by

$$\beta_i = \alpha_i/\sqrt{2WN_{02}}. \tag{14}$$

Then

$$(1/N_{02})\int_0^T s^2(t)\,dt = \sum_{i=1}^{2TW} \beta_i^2. \tag{15}$$

Using (11) and (2), the total input $y(t)$ with the signal present can be written as:

$$y(t) = \sum_{i=1}^{2TW} (a_i + \alpha_i) \operatorname{sinc}(2Wt - i). \tag{16}$$

The energy of $y(t)$ in the interval $(0, T)$ is approximated by

$$\int_0^T y^2(t)\,dt = (1/2W)\sum_{i=1}^{2TW} (a_i + \alpha_i)^2. \tag{17}$$

Under the hypothesis H_1, the test statistic V' is

$$V' = (1/N_{02})\int_0^T y^2(t)\,dt = \sum_{i=1}^{2TW} (b_i + \beta_i)^2. \tag{18}$$

The sum in (18) is said to have a noncentral chi-square distribution [9], [10] with $2TW$ degrees of freedom and a noncentrality parameter λ given by

$$\lambda = \sum_{i=1}^{2TW} \beta_i^2 = (1/N_{02})\int_0^T s^2(t)\,dt \equiv E_s/N_{02}. \tag{19}$$

λ, the ratio of signal energy to noise spectral density, provides a convenient definition of *signal-to-noise ratio.*

III. Detection in White Noise: Bandpass Processes

If the noise is a bandpass random process, each sample function may be expressed in the form [13]

$$n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t, \tag{20}$$

where ω_c is the reference angular frequency, and $n_c(t)$ and $n_s(t)$ are, respectively, the in-phase and quadrature modulation components. If $n(t)$ is confined to a frequency band W, $n_c(t)$ and $n_s(t)$ are low-pass functions whose spectral densities are confined to the region $|f| < W/2$. If $n(t)$ has a flat power density spectrum N_{02}, $n_c(t)$ and $n_s(t)$ will also have flat power density spectra, each equal to $2N_{02}$ over $|f| < W/2$. Thus, $n_c(t)$ and $n_s(t)$ each have TW degrees of freedom and variance $2N_{02}W$. Also, to a good approximation, which gets better as T increases,

$$\int_0^T n^2(t)\,dt = \frac{1}{2}\int_0^T [n_c^2(t) + n_s^2(t)]\,dt. \tag{21}$$

By following the reasoning in the previous section, we get similar series expansions for the energy in $n_c(t)$ and $n_s(t)$, as follows:

$$\int_0^T n_c^2(t)\,dt = (1/W)\sum_{i=1}^{TW} a_{ci}^2$$

$$\int_0^T n_s^2(t)\,dt = (1/W)\sum_{i=1}^{TW} a_{si}^2, \tag{22}$$

where

$$a_{ci} = n_c(i/W)$$

$$a_{si} = n_s(i/W). \tag{23}$$

Define b_{ci} and b_{si} as follows:

$$b_{ci} = a_{ci}/\sqrt{2WN_{02}}$$

$$b_{si} = a_{si}/\sqrt{2WN_{02}}. \tag{24}$$

Then, under H_0,

$$V' = (1/N_{02})\int_0^T n^2(t)\,dt = \sum_{i=1}^{TW} (b_{ci}^2 + b_{si}^2). \tag{25}$$

Since the variance of any b_{ci} or b_{si} is unity, the sum on the right-hand side of (25), which is the test statistic V' under

H_0, has a chi-square distribution with $2TW$ degrees of freedom.

A bandpass signal may be expressed in the form

$$s(t) = s_c(t)\cos\omega_c t - s_s(t)\sin\omega_c t. \quad (26)$$

$s_c(t)$ and $s_s(t)$ have frequency components confined to the band $|f| < W/2$. Using the sampling formula as above,

$$\begin{aligned} s_c(t) &= \sum_{i=1}^{TW} \alpha_{ci}\,\mathrm{sinc}(Wt - i) \\ s_s(t) &= \sum_{i=1}^{TW} \alpha_{si}\,\mathrm{sinc}(Wt - i), \end{aligned} \quad (27)$$

where

$$\begin{aligned} \alpha_{ci} &= s_c(i/W) \\ \alpha_{si} &= s_s(i/W). \end{aligned} \quad (28)$$

The total input $y(t)$ now may be written as

$$\begin{aligned} y(t) &= [s_c(t) + n_c(t)]\cos\omega_c t - [s_s(t) + n_s(t)]\sin\omega_c t \\ &= y_c(t)\cos\omega_c t - y_s(t)\sin\omega_c t. \end{aligned} \quad (29)$$

Again, using the sampling formula,

$$\begin{aligned} y_c(t) &= s_c(t) + n_c(t) = (1/W)\sum_{i=1}^{TW} (\alpha_{ci} + a_{ci})\,\mathrm{sinc}(Wt - i) \\ y_s(t) &= s_s(t) + n_s(t) = (1/W)\sum_{i=1}^{TW} (\alpha_{si} + a_{si})\,\mathrm{sinc}(Wt - i). \end{aligned} \quad (30)$$

Define the coefficients β_{ci} and β_{si} as follows:

$$\begin{aligned} \beta_{ci} &= s_c(i/W)/\sqrt{2WN_{02}} \\ \beta_{si} &= s_s(i/W)/\sqrt{2WN_{02}}. \end{aligned} \quad (31)$$

By following the reasoning of the previous section, we can write

$$(1/N_{02})\int_0^T s^2(t)\,dt = \sum_{i=1}^{TW} (\beta_{ci}^2 + \beta_{si}^2) = E_s/N_{02}. \quad (32)$$

Under hypothesis H_1, the total bandpass input $y(t)$ with the signal present has its energy given by

$$\begin{aligned} \int_0^T y^2(t)\,dt &= \frac{1}{2}\int_0^T [y_c^2(t) + y_s^2(t)]\,dt \\ &= \frac{1}{2}\int_0^T [s_c(t) + n_c(t)]^2\,dt \\ &\quad + \frac{1}{2}\int_0^T [s_s(t) + n_s(t)]^2\,dt. \end{aligned} \quad (32a)$$

Using (15), (18), (32), and (32a), we follow the line of reasoning of the previous section to show that under hypothesis H_1, the test statistic V' is

$$V' = \sum_{i=1}^{TW} \{(b_{ci} + \beta_{ci})^2 + (b_{si} + \beta_{ci})^2\}. \quad (33)$$

It is seen that V' has a noncentral chi-square distribution with $2TW$ degrees of freedom and a noncentrality parameter λ given by E_s/N_{02}, E_s being the signal energy. Thus, the results are the same for bandpass processes and for low-pass processes, provided that W is interpreted as the *positive* frequency bandwidth. That is, in the case of low-pass inputs, W refers to the interval between zero frequency and the upper end of the band.

IV. Computation of Detection and False Alarm Probabilities

The probability of false alarm Q_0 for a given threshold V_T' is given by

$$Q_0 = \mathrm{Prob}\{V' > V_T' | H_0\} = \mathrm{Prob}\{\chi_{2TW}^2 > V_T'\}. \quad (34)$$

The far right-hand side of (34) indicates a chi-square variable with $2TW$ degrees of freedom. For the same threshold level V_T', the probability of detection Q_d is given by

$$Q_d = \mathrm{Prob}\{V' > V_T' | H_1\} = \mathrm{Prob}\{\chi_{2TW}'^2(\lambda) > V_T'\}. \quad (35)$$

The symbol $\chi_{2TW}'^2(\lambda)$ indicates a noncentral chi-square variable with $2TW$ degrees of freedom and noncentrality parameter λ; in our case $\lambda = E_s/N_{02}$, and is defined as the signal-to-noise ratio. As mentioned above, extensive tables exist for the chi-square distribution, but the noncentral chi-square has not been as extensively tabulated. Approximations were sought which could be used to cover the ranges of interest.

Many approximations have been devised for the noncentral chi-square. For descriptions of some of these, the interested reader is referred to Owen [11] and to Greenwood and Hartley [12]. The approximation used in this paper is the one used by Patnaik [9], [10]. It was checked against the exact values used by Fix [10] for 0.01 and 0.05 false alarm probabilities; close agreement was obtained. The approximation replaces the noncentral chi-square with a central chi-square having a different number of degrees of freedom and a modified threshold level. If the noncentral chi-square variable has $2TW$ degrees of freedom and noncentrality parameter λ, define a modified number of degrees of freedom D and a threshold divisor G given by

$$\begin{aligned} D &= (2TW + \lambda)^2/(2TW + 2\lambda) \\ G &= (2TW + 2\lambda)/(2TW + \lambda). \end{aligned} \quad (36)$$

Then

$$\mathrm{Prob}\{\chi_{2TW}'^2(\lambda) > V_T'\} = \mathrm{Prob}\{\chi_D^2 > V_T'/G\}. \quad (37)$$

The receiver operating characteristic (ROC) curves of Figs. 3 through 6 were drawn for $TW > 1$, using (34) and (35) approximated by (37).

Figure 2, for $TW = 1$, was obtained in a different way. Referring to (20), the squared envelope of $n(t)$ is given by

$$\mathrm{Env}^2[n(t)] = n_c^2(t) + n_s^2(t). \quad (38)$$

Similarly, from (26), the squared envelope of the signal $s(t)$ is given by

$$\mathrm{Env}^2[s(t)] = s_c^2(t) + s_s^2(t). \quad (39)$$

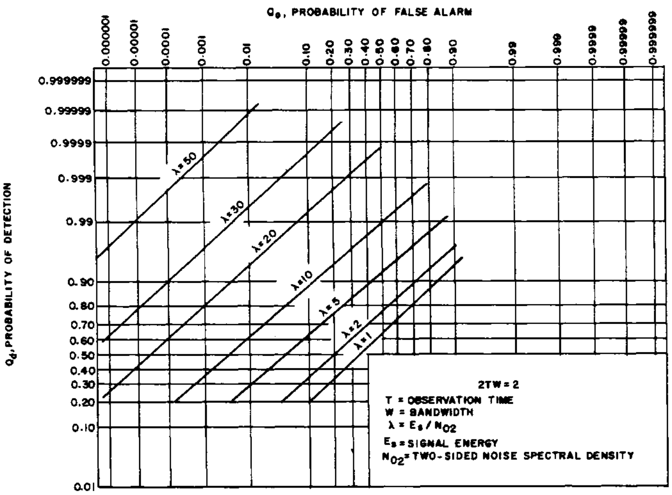

Fig. 2. Receiver operating characteristic (ROC) curves.

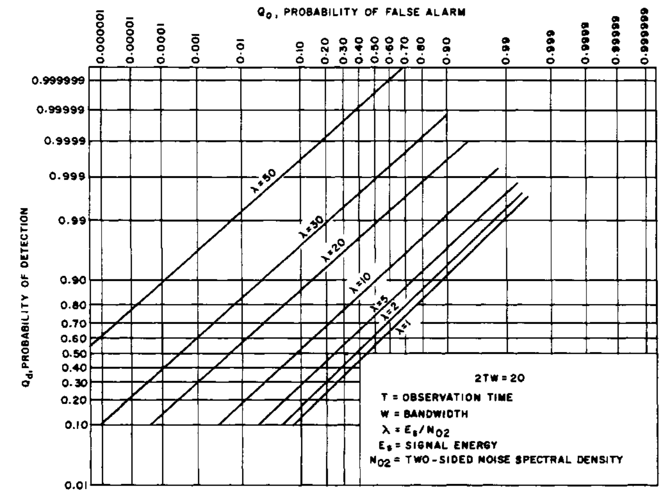

Fig. 4. Receiver operating characteristic (ROC) curves.

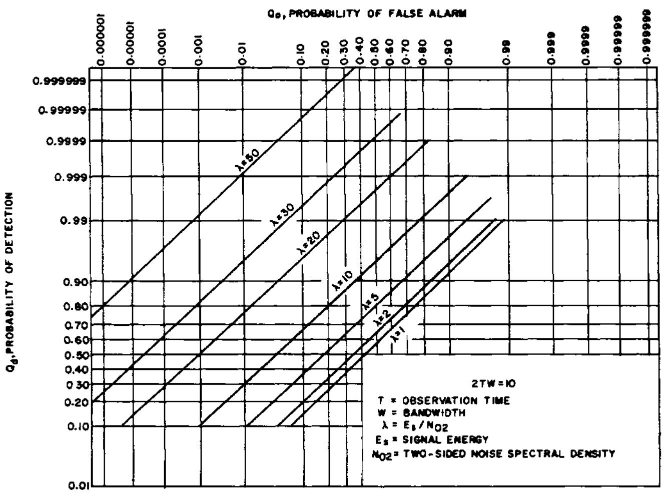

Fig. 3. Receiver operating characteristic (ROC) curves.

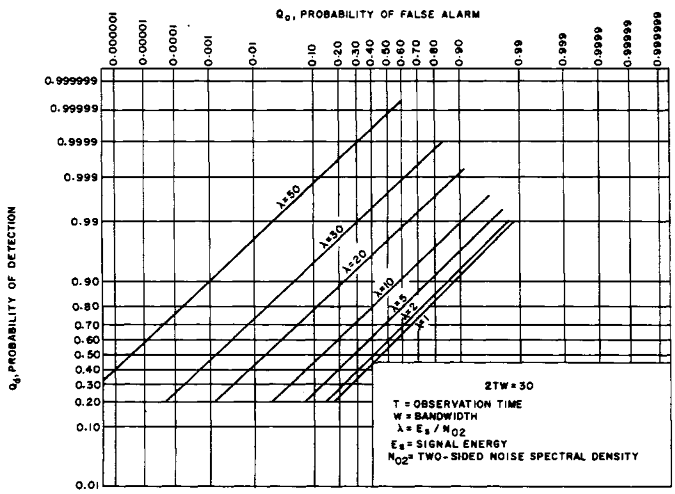

Fig. 5. Receiver operating characteristic (ROC) curves.

Thus, from (17),

$$a_{ci}^2 + a_{si}^2 = \mathrm{Env}^2\left[n(i/W)\right] \tag{40}$$

so that

$$b_{ci}^2 + b_{si}^2 = (1/2WN_{02})\,\mathrm{Env}^2\left[n(i/W)\right]. \tag{41}$$

By a similar process of reasoning, we show that

$$(b_{ci} + \beta_{ci})^2 + (b_{si} + \beta_{si})^2 = (1/2WN_{02})\,\mathrm{Env}^2\left[s(i/W) + n(i/W)\right]. \tag{42}$$

Therefore, the test statistic V' for $TW = 1$ can be considered to be obtained by sampling at $t = T$ the squared envelope of the input to the energy detector. In this instance, the conditional probability density functions of V' can be written in closed form, although direct computation of Q_d still presents some difficulty. However, we note that not only can V' be used as a test statistic, but so can any strictly monotonic function of V'; in particular, $\sqrt{V'}$, the *envelope* of the input to the energy detector, can be used. It is well known that the envelope of the Gaussian noise has a Rayleigh distribution and that adding a mean value function, $s(t)$, makes the resulting envelope of signal plus Gaussian noise have a Rice distribution [13]. Computation of false alarm and detection probabilities may be based upon the assumption of the envelope as the test statistic. Extensive computations for this situation have already been made, but the most convenient source is the nomogram of Bailey and

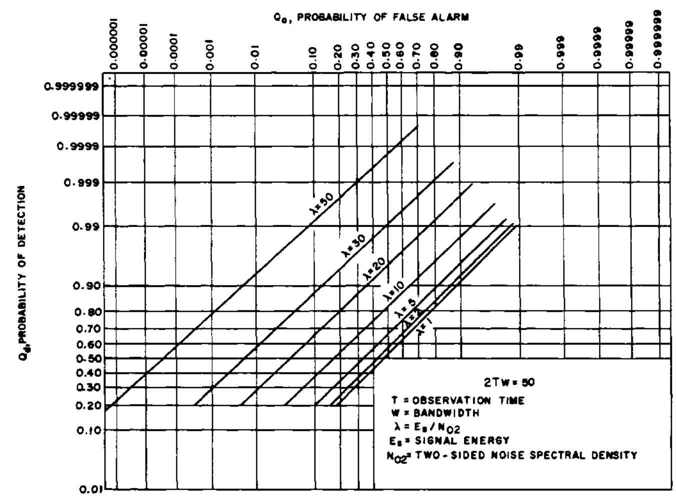

Fig. 6. Receiver operating characteristic (ROC) curves.

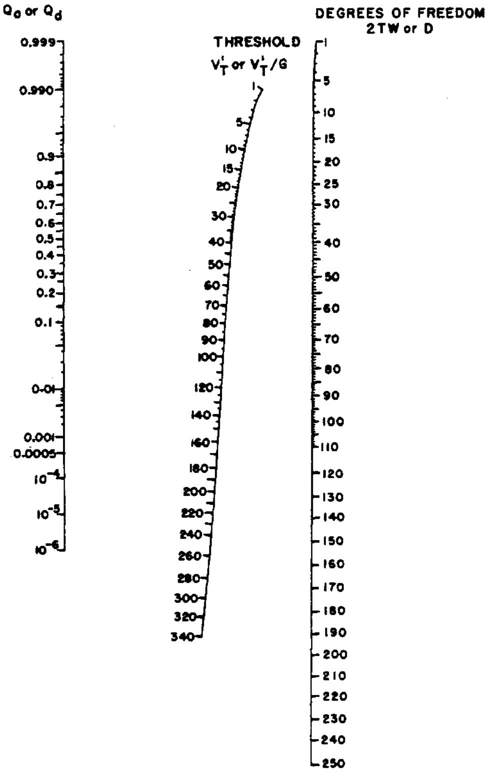

Fig. 7. Nomogram of the chi-square probability function.

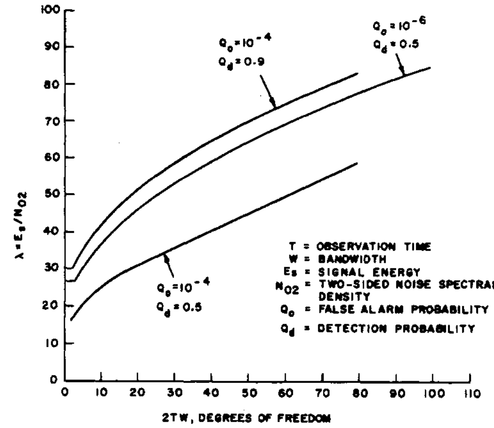

Fig. 8. Required signal energy.

Randall [14]; Fig. 2 was drawn using values obtained from [14].

Of course, many more sets of receiver operating characteristics (ROC's) could be computed, but it was felt better to present a few representative sets and then provide the means for rapidly computing false alarm and detection probabilities for given TW and λ. This means is provided by the nomogram of Fig. 7, which is an extension of the nomogram of Smirnov and Potapov [15]. The nomogram is used first to find the false alarm probability Q_0. Equation (36) is used to find D and G, the modified number of degrees of freedom, from the given values of TW and λ, as well as the dividing factor G to find the new threshold value of χ^2. Then the nomogram is used again to find the detection probability Q_d.

Example:

Given $2TW=40$, $\lambda=30$, $Q_0=10^{-4}$:

threshold value $V_T'=81$
new number of degrees of freedom D [from (36)]$=49$
threshold divisor G [from (36)]$=1.4$
new threshold value $V_T''=V_T'/G=58$
probability of detection $Q_d=0.18$.

In the general case, it is of interest to see how $2TW$ and the signal-to-noise ratio λ vary for given false alarm and detection probabilities. Figure 8 shows such a relationship for various Q_0 and Q_d. This figure was obtained by cross-plots from Figs. 4–6. It is clearly seen that increasing $2TW$, the number of degrees of freedom, causes an increase in the required signal-to-noise ratio. A natural question is: why does increasing the time-bandwidth product increase the required signal-to-noise ratio? It has been suggested[1] that the answer lies in the increased incoherence of the noise which tends to "dilute" the signal energy, somewhat analogously to the suppression encountered in incoherent detection.

V. Expressions for Large Time-Bandwidth Product

The nomogram of Fig. 7 is suitable for values of $2TW \leq 250$. For larger values, we may use Gaussian approximations to the probability density functions of the test statistic V' under either condition: noise alone, or signal plus noise. The appropriate expressions are found by using a normal variate, with the proper mean and variance, for finding the probability of exceeding the threshold. Let $N(m, \sigma^2)$ indicate a Gaussian variate with mean m and variance σ^2.

Reference to (10) or (25) shows that V', under the no-signal condition, is the sum of $2TW$ statistically independent random variables. The mean value of each variable is simply the variance of the noise variate and is unity. Thus, the mean value is $2TW$. Since each b_i is a normal variate with mean zero and unit variance, the variance of each b_i^2 is given by

$$\begin{aligned} \operatorname{var} b_i^2 &= \overline{b_i^4} - (\overline{b_i^2})^2 \\ &= 2. \end{aligned} \tag{43}$$

Thus, the variance of the sum is $4TW$. Therefore, under the no-signal conditions, V' is distributed as a Gaussian variate $N(2TW, 4TW)$. Then the false alarm probability Q_0 is given by

$$\begin{aligned} Q_0 &= \frac{1}{\sqrt{8\pi TW}} \int_{V_T'}^{\infty} \exp\left[-\frac{(x-2TW)^2}{8TW}\right] dx \\ &= \frac{1}{2} \operatorname{erfc}\left[\frac{V_T' - 2TW}{2\sqrt{2}\sqrt{TW}}\right], \end{aligned} \tag{44}$$

where

$$\operatorname{erfc}(z) = (2/\sqrt{\pi}) \int_z^{\infty} \exp(-x^2)\, dx. \tag{45}$$

[1] By one of the reviewers of this paper.

Turning to V' when signal is present, (18) or (33) leads to a mean value of $2TW+\lambda$ [6, p. 943]. Also, it can be shown [6, p. 943] that the variance of V' is $4(TW+\lambda)$. Thus, we say that V' with signal present is a Gaussian variate $N(2TW+\lambda, 4(TW+\lambda))$. The probability of detection Q_d is given by

$$Q_d = \frac{1}{2} \operatorname{erfc} \left[\frac{V_T' - 2TW - \lambda}{2\sqrt{2}\sqrt{TW + \lambda}} \right]. \tag{46}$$

Extensive tables of the normal probability integral are available [6].

VI. Discussion of Related Work

The earliest related work of which the author is aware is that of Marcum [3]. By using a sampling theorem, the problem we treat reduces to that of the sum of squares of statistically independent Gaussian variates. This is what Marcum treats, although his curves are not suitable for our purpose. Furthermore, although Marcum gives a series expansion for the distribution of the sum of the squared Gaussian variates, we felt that our approximation was far more convenient, because of the existence of extensive tables and the nomogram (Fig. 7) we constructed. We have confidence in our approximation because we could check it at several points using Fix's exact values [10] and Marcum's graphs of the incomplete Toronto function.

An interesting thing came to light during a rereading of Marcum's paper. He points out that the noncentral chi-square cumulative distribution is given by the incomplete Toronto function; yet, this is not even mentioned by Patnaik [9], Fix [10], or Zelen and Severo [6]. The incomplete Toronto function has been described by Fisher [16] and Heatley [17], as well as by Marcum [3, especially p. 182], who provides several graphs. It is useful to show the connection between noncentral chi-square and the Toronto function. The incomplete Toronto function is defined by

$$T_B(m, n, r) = 2r^{n-m+1} \exp(-r^2) \int_0^B t^{m-n} \cdot \exp(-t^2) I_n(2rt)\, dt. \tag{47}$$

The probability density of noncentral chi-square with f degree of freedom and noncentrality parameter λ is given by [18], [19]

$$\begin{aligned} p(\chi^2) d(\chi^2) &= \tfrac{1}{2}(\chi^2/\lambda)^{(f-2)/4} \\ &\quad \cdot \exp[-(\chi^2 + \lambda)/2] I_{(f-2)/2}(\sqrt{\lambda\chi^2})\, d\chi^2, \quad \chi^2 \geq 0 \\ &= 0, \qquad \chi^2 < 0. \end{aligned} \tag{48}$$

By appropriate comparison of variables, one may show that

$$\begin{aligned} V_T' &= 2B^2, \text{ the threshold level} \\ m &= f - 1 \\ n &= f/2 - 1 \\ r^2 &= \lambda/2. \end{aligned}$$

Then, the probability Q_d of exceeding the threshold is

$$Q_d = 1 - T_{\sqrt{V_T'/2}}(2TW - 1, TW - 1, \sqrt{\lambda/2}). \tag{49}$$

Some curves of the incomplete Toronto function given by Marcum were used to check a few of the points in Figs. 3 through 6. Agreement was very close.

Marcum's curves can be used for a number of values of $2TW$, but the values lying between those given by Marcum give some difficulty. Our feeling is that the nomogram of Fig. 7 represents a particularly convenient method for the engineer to compute quickly the false alarm and detection probabilities for any time-bandwidth product and signal-to-noise ratio. In this connection, it has been pointed out to the author that there exist computer programs for computing cumulative probabilities of the noncentral chi-square variable [20]. It is also interesting to note that the Patnaik approximation is equivalent to taking the first term in the Laguerre series expansion for the probability density function of a noncentral chi-square variable, as shown by Marcum [3, equation (118) *et seq.* of the Mathematical Appendix].

Another related work in energy detection is that of Kaplan [4] who treats the case, among others, of a steady sine wave in noise. Because of the infinite observation times assumed by Kaplan, his detector is a power detector rather than an energy detector. Nevertheless, the decision statistic has the distributions given above, if properly normalized. We can see this as follows: the properties of the detection test are described by $\lambda = E_s/N_{02}$, the ratio of signal energy to two-sided noise spectral density, and by $2TW$, twice the time-bandwidth product. Let

$$\begin{aligned} P_N &= \text{average noise power} = 2WN_{02} \\ P_s &= \text{average signal power} = E_s/T. \end{aligned} \tag{50}$$

Then

$$E_s/N_{02} = 2TWP_s/P_N. \tag{51}$$

Therefore, Kaplan's results can be compared with ours by using his ratio P_s/P_N multiplied by $2TW$, the number of degrees of freedom.

Kaplan assumes that a normal approximation to the distributions is good enough, although Marcum [3, p. 165] states that $2TW$ must be quite large before the normal approximation is adequate. We therefore checked a few of Kaplan's points with ours and we found that the discrepancies were not large.

VII. Discussion and Conclusions

If the form of a signal to be detected is unknown, it appears appropriate to consider an energy detector as a device for deciding whether or not the signal is present. Since an energy detector does not care about anything but the amount of energy in the given observation time, the form of the signal does not affect the conditional probability that a threshold will be exceeded when the signal is present. Of course, it is assumed that the noise is zero mean Gaussian. By using Shannon's sampling theorem, one can show that the energy in a finite time interval can be described as a sum of the square of a number of statistically independent Gaussian variates if the noise input is Gaussian and has a flat spectral density over a limited bandwidth. With noise alone, the output of the energy detector has a central chi-square distribution with a number of degrees of freedom equal to twice the time-bandwidth product at the input to the energy detector. In the presence of a deterministic signal, although the form of the signal may be unknown, the distribution of the decision statistic at the output of the energy detector is that of a noncentral chi-square with the number of degrees of freedom equal to twice the time-bandwidth product and a noncentrality parameter given by the ratio of signal energy to two-sided noise spectral density.

Although there are some tabulations of the noncentral chi-square distribution, many of the parameter values of interest to this paper are not included. However, sufficiently good approximations to the noncentral distribution can be obtained, using a modified central chi-square distribution. This modified distribution, with the aid of a nomogram, was used to compute several receiver operating characteristic curves for the energy detector. Figure 8 shows curves of energy required vs. time-bandwidth product, or degrees of freedom, for fixed false alarm and detection probabilities. The curves of Fig. 8 show that, as the number of degrees of freedom increases, a larger signal energy is required for a given detection capability. It has been suggested that the reason lies in the increased incoherence of the noise, producing an effect analogous to the modulation suppression encountered in incoherent detection. However, the curves of Fig. 8 do not show a proportionality between the number of degrees of freedom and the required signal energy, indicating that the trade-off is distinctly in favor of increasing observation time. This is another way of stating the obvious conclusion that the observation time should be as long as possible, provided the signal energy keeps arriving at a good rate as the interval is extended.

Although this paper has taken the point of view that the unknown signal is of deterministic form, there is nothing in it which changes results for any signal, known or unknown, deterministic or random, provided the probability of detection is considered a conditional probability of detection where the condition is a given amount of signal energy; i.e., if the signal present has a certain amount of energy, then its detection probability is given by the results of this paper, regardless of where the signal comes from. It may come from a random process, or may be a one-shot affair, or may come from a process which repeats signals of the same form at regular or irregular intervals.

Finally, it is of interest to compare an energy detector with a matched filter detector. This comparison is shown in Fig. 9, which shows the approximate increase in input signal-to-noise ratio necessary for an energy detector to achieve the same performance as a matched filter detector in which the decision statistic is the envelope of the matched filter output. One curve suffices for Fig. 9 because the curves of Fig. 8 are approximately parallel. The curve of Fig. 9 starts at $TW=1$, because the energy detector is equivalent to the matched filter detector (with noncoherent demodulation) when the time-bandwidth product is unity. The loss represented by Fig. 9 represents the price paid for ignorance of the signal structure.

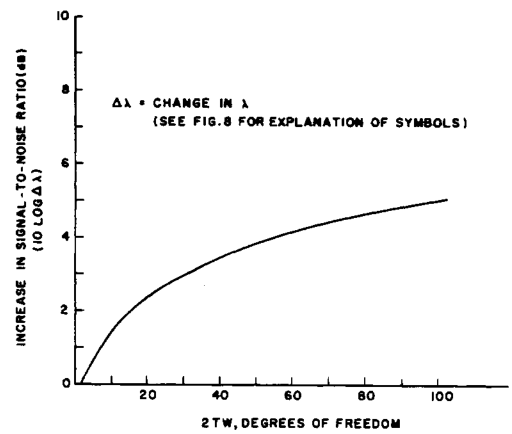

Fig. 9. Approximate increase in signal-to-noise ratio necessary for the energy detection to achieve the same performance as the matched filter detector.

Appendix

In this appendix, the Karhunen-Loeve expansion [13, p. 96] is used to show that $2TW$ terms suffice to approximate the energy in a finite duration sample of a band-limited process with a flat power density spectrum. This demonstration has somewhat more rigor than that using the sampling expansion.

Consider a zero mean, wide-sense stationary, Gaussian random process $n(t)$ with a flat power density spectrum extending over the frequency interval $(-W, W)$. Let its autocorrelation function $R(\tau)$ be given by

$$R(\tau) = \operatorname{sinc} 2W\tau, \tag{52}$$

where $\operatorname{sinc} x = \sin \pi x/\pi x$. The process $n(t)$ may be represented in the interval $(0, T)$ by the expansion of orthonormal functions $\phi_i(t)$, as follows:

$$n(t) = \sum_{i=1}^{\infty} \eta_i \phi_i(t) \tag{53}$$

where η_i is given by

$$\eta_i = \int_0^T n(t)\phi_i(t)\,dt, \tag{54}$$

and the $\phi_i(t)$ are eigenfunctions of the integral equation

$$\int_0^T R(t-\tau)\phi_i(\tau)\,d\tau = g_i\phi_i(t). \tag{55}$$

The g_i are the eigenvalues of the equation.

The number of terms in (53) which constitute a sufficiently good approximation with a finite number of terms depends upon how rapidly the eigenvalues decrease in value after a certain index. The eigenfunctions of (55) are the prolate spheroidal wave functions considered in [21] and

[22]. The cited sources show that the eigenvalues drop off rapidly after $2TW$ terms (except for $TW=1$, which is treated separately in the body of our paper). Therefore, we approximate (53) by

$$n(t) \simeq \sum_{i=1}^{2TW} \eta_i \phi_i(t). \tag{56}$$

The approximation (56) may be considered by some to be more satisfactory than the approximation (7) based upon sampling functions, because the rapidity of drop-off of the terms can be judged by how rapidly the eigenvalues g_i drop off after $2TW$ terms.

Since the $\phi_i(t)$ are orthonormal, the energy of $n(t)$ in the interval $(0, T)$ is, using (56),

$$\int_0^T n^2(t)\,dt \simeq \sum_{i=1}^{2TW} \eta_i^2. \tag{57}$$

Since the process is Gaussian, the η_i are Gaussian. The variance of η_i is g_i and these are nearly the same for $i \leq 2TW$. Thus, the energy in the finite duration sample of $n(t)$ is the sum of $2TW$ squares of zero mean Gaussian variates all having the same variance. With appropriate normalization, we are led to the chi-square distribution.

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