On the Error Probability of Linearly Modulated Signals on Frequency-Flat Ricean, Rayleigh, and AWGN Channels

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***Abstract*—The method used in [1] for finding the error probability of linearly modulated signals on Rayleigh frequency-flat fading channels has been applied to the more general case of Ricean fading. A signal received on a fading channel is subject to a multiplicative distortion (MD) and to the usual additive noise. Following a compensation of the MD, the signal provided to the detector may be thought to include only a single additive distortion term ("final noise"), which comprises the effects of the original additive noise, the MD, and the error in MD compensation. In this paper, an exact expression for the probability density function of the final noise is derived. This allows calculation of error probability for arbitrary types of linear modulations. Results for many cases of interest are presented here. Furthermore, as special cases of Ricean fading, error probability for Rayleigh fading and non-fading channels are obtained which either match the results or complete the approximate derivations formerly known from the literature.**

I. Introduction

The communication link of many satellite mobile systems may be characterized as a Ricean frequency-flat fading (F^3) channel. In a Ricean channel, the transmitted signal reaches the receiver through fixed scatterers or direct paths in addition to random scatterers. The effect of the channel on the transmitted signal is in the form of a random multiplicative distortion (MD) with a Gaussian distribution (according to the central limit theorem), having nonzero mean. Furthermore, the usual additive white Gaussian noise (AWGN) is also present at the receiver.

Rayleigh fading which occurs, e.g., in land-mobile communications is a special case of Ricean fading where no fixed scatterer or direct path is available between the transmitter and the receiver. On the other hand, in the absence of random scatterers, the availability of only fixed scatterers or direct paths yields the special case of a nonfading AWGN channel.

Coherent detection on F^3 channels (where an unambiguous estimate of the phase of the MD is formed at the receiver) has many advantages over noncoherent detection when issues such as power efficiency and robustness against random FM or co-channel interference are considered [2]. In this paper we examine coherent detection of linearly modulated signals such as M-PSK and M-QAM which are known to be power efficient with small M or bandwidth efficient with large M. Exact results for bit and symbol error probabilities over a Ricean F^3 channel are obtained. The results found are also applied to the special cases of Rayleigh fading and nonfading AWGN channels.

This paper is organized as follows: Section II introduces the system model. In this model the multiplicative distortion and the original additive noise are combined and form a single additive noise ("final noise"). In section III the probability density function (pdf) of the final noise is calculated. In sections IV and V bit and symbol error probabilities are obtained in two cases of perfect (error-free) and non-perfect MD estimation. Section VI presents the conclusions.

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II. System Model

Ref. [1] provides a detailed description of the baseband equivalent model of the system. It will, therefore, suffice to explain it briefly. A linearly modulated signal is transmitted on a Ricean F^3 channel. The fading is assumed slow in the sense that, at the receiver, the maximum frequency deviation (due to Doppler spread and Doppler shift) of a transmitted pure test tone is small compared to the channel symbol rate[1]. After down conversion of the received signal, matched filtering and sampling with perfect symbol timing at the rate of $1/T$, a baseband, T-spaced discrete-time, complex-valued signal is obtained as

$$z_k = y_k a_k + n_k. \tag{1}$$

The zero mean sequence a_k represents the M-ary channel symbols. The frequency-flat assumption implies the absence of intersymbol interference. The sequence y_k is the MD which is complex, nonzero mean, Gaussian, and independent of a_k and n_k. Mainly, y_k represents the fading and a possible Doppler frequency shift. Any remaining error in the phase and amplitude references at the receiver may also be thought to be included in y_k. Without loss of generality, both a_k and y_k are normalized to have unity power. n_k is a complex white Gaussian process with zero mean and average power P_n. Thus, the signal to noise ratio (SNR) equals to $1/P_n$.

In coherent detection we need to have an unambiguous estimate $\hat{y}_k$ of the MD. The estimation error is then

[1] This definition of slow fading actually covers most cases of interest in aeronautical and mobile communications.

$$\tilde{y}_k := y_k - \hat{y}_k . \quad (2)$$

It is assumed that $\tilde{y}_k$ and $\hat{y}_k$ are uncorrelated (and since Gaussian, are independent). This assumption is valid, for example, when the minimum mean square error (MMSE) method is used for estimation [3]. Furthermore, with no loss of practicality, both $\tilde{y}_k$ and $\hat{y}_k$ are assumed independent of n_k [1].

Assuming that the channel symbols form an independent sequence, it is reasonable to aim for symbol-by-symbol detection and work on the decision variable

$$D := z / \hat{y} = (y/\hat{y})a + n/\hat{y} \quad (3)$$

$$= a + \frac{n + \tilde{y}a}{y - \tilde{y}} =: a + m. \quad (4)$$

Observe that the original additive noise n, the channel symbol a, the MD y, and the MD estimation error $\tilde{y}$ are combined to form an additive noise (the "final noise") m. Now, the symbol error rate (SER) may be written as

$$\mathrm{SER} = 1 - \sum_{i=1}^{M} p(a^i) \iint_{R^i} p_{m^i}\left(m - a^i\right) dm \quad (5)$$

where R^i, $i = 1...M$, is the decision region for the ith symbol a^i of the constellation and m^i is the final noise corresponding to a^i. It is usually the case that $p(a^i)=1/M$. Then, in order to find the SER, it is sufficient to obtain $p(m^i) := p(\mathrm{Re}[m^i], \mathrm{Im}[m^i])$, the joint pdf of the real and imaginary parts of m^i.

III. PDF OF THE FINAL NOISE

Recall that for the general case of a Rice channel the MD y is a complex Gaussian variable with nonzero mean ($\bar{y} := E[y]$; $s := |\bar{y}| \neq 0$) and unity power ($P_y = E[|y|^2] = 1$). We also assume that $\hat{y}$ is a complex Gaussian process with the same mean value of y (i.e. it is an unbiased estimate), and has power $P_{\hat{y}}$. Similar to $|y|$, the magnitude $|\hat{y}|$ will have a Rice pdf which we will shortly need:

$$P_{|\hat{y}|}(\alpha) = \frac{2\alpha}{P_{\hat{y}} - s^2} \exp\left(-\frac{s^2 + \alpha^2}{P_{\hat{y}} - s^2}\right) I_0\left(\frac{2\alpha s}{P_{\hat{y}} - s^2}\right); \quad \alpha > 0 \quad (6)$$

Here, $I_0(\cdot)$ is the modified Bessel function of order zero. It is possible to write s as a function of the Rice factor K_R which is defined as the ratio of fixed path power to random scatterers power :

$$K_R := \frac{|\bar{y}|^2}{E\left[|y - \bar{y}|^2\right]} \quad (7)$$

$$s^2 = \frac{K_R}{K_R + 1}. \quad (8)$$

Similar to y and $\hat{y}$, the estimation error $\tilde{y} = y - \hat{y}$ is a complex Gaussian process but with zero mean. Its average power $P_{\tilde{y}}$ depends on system parameters such as K_R, SNR, and bandwidth of the random process y_k (Doppler spread). Since $\tilde{y}$ and $\hat{y}$ are uncorrelated, it is easily shown that

$$P_{\tilde{y}} = P_y - P_{\hat{y}} = 1 - P_{\hat{y}}. \quad (9)$$

From (4), the final noise for a specific channel symbol a^i may be written as

$$m^i = \frac{n + \tilde{y}a^i}{y - \tilde{y}} = \frac{n + \tilde{y}a^i}{\hat{y}}, \quad (10)$$

where the random variables n, $\tilde{y}$, and $\hat{y}$ have been discussed earlier to be mutually independent. Now, the circular symmetry [4] of both n and $\tilde{y}a^i$ (which results from independence of their Gaussian real and imaginary parts) yields a circularly symmetric numerator so that we may ignore the arg[$\hat{y}$] from the denominator and work on a new random variable

$$m'^i := m^i \exp\{j \arg[\hat{y}]\} = \left(n + \tilde{y}a^i\right) / |\hat{y}| =: An'^i ; \quad (11)$$

$$A := 1/|\hat{y}| \quad ; \quad n'^i := n + \tilde{y}a^i . \quad (12)$$

n'^i has the same pdf as found in [1]:

$$p\left(n'^i\right) = p\left(\mathrm{Re}\left[n'^i\right]\right) \cdot p\left(\mathrm{Im}\left[n'^i\right]\right) \quad (13)$$

$$= \frac{1}{\pi\left(P_n + |a^i|^2 P_{\tilde{y}}\right)} \exp\left(-\frac{|n'^i|^2}{P_n + |a^i|^2 P_{\tilde{y}}}\right). \quad (14)$$

The pdf of A of (12) is found by using (6) in relation (5.7) of [4]:

$$P_A(\alpha) = p_{1/|\hat{y}|}(\alpha) = \frac{1}{\alpha^2} p_{|\hat{y}|}(1/\alpha) \quad ; \quad \alpha > 0 \quad (15)$$

$$= \frac{2}{\alpha^3 \left(P_{\hat{y}} - s^2\right)} \exp\left\{-\frac{s^2 + 1/\alpha^2}{P_{\hat{y}} - s^2}\right\} I_0\left[\frac{2s}{\alpha\left(P_{\hat{y}} - s^2\right)}\right]. \quad (16)$$

Now, we are in the stage to point out from (11) and (12)

$$p_{\text{Rice}}(m^i) = p_{\text{Rice}}(m'^i) = \int_0^\infty p(m'^i, A)\, dA \qquad (17)$$

$$= \int_0^\infty p_{m'^i|A}(m'^i|\alpha)\, p_A(\alpha)\, d\alpha \qquad (18)$$

$$= \int_0^\infty \frac{1}{\pi\alpha^2\left(P_n + |a^i|^2 P_{\tilde{y}}\right)} \exp\left(-\frac{|m'^i|^2}{\alpha^2\left(P_n + |a^i|^2 P_{\tilde{y}}\right)}\right)$$

$$\cdot \frac{2}{\alpha^3\left(P_{\hat{y}} - s^2\right)} \exp\left\{-\frac{s^2 + 1/\alpha^2}{P_{\hat{y}} - s^2}\right\} I_0\left[\frac{2s}{\alpha\left(P_{\hat{y}} - s^2\right)}\right] d\alpha. \qquad (19)$$

Fortunately the pdf of final noise m^i is obtainable in closed form (see Appendix):

$$p_{\text{Rice}}(m^i) = \frac{1}{\pi} \exp\left[\frac{-K_R}{1 - P_{\tilde{y}}(K_R + 1)}\right]$$

$$\cdot \left\{ \frac{\bar{\gamma}^i \left[\frac{1 - P_{\tilde{y}}(K_R+1)}{(1-P_{\tilde{y}})(K_R+1)}\right]}{\left(1 + |m^i|^2 \bar{\gamma}^i \left[\frac{1 - P_{\tilde{y}}(K_R+1)}{(1-P_{\tilde{y}})(K_R+1)}\right]\right)^2} + \frac{\bar{\gamma}^i \left[\frac{K_R}{(1-P_{\tilde{y}})(K_R+1)}\right]}{\left(1 + |m^i|^2 \bar{\gamma}^i \left[\frac{1 - P_{\tilde{y}}(K_R+1)}{(1-P_{\tilde{y}})(K_R+1)}\right]\right)^3} \right\}$$

$$\cdot \exp\left\{ \frac{\frac{K_R}{1 - P_{\tilde{y}}(K_R+1)}}{1 + |m^i|^2 \bar{\gamma}^i \left[\frac{1 - P_{\tilde{y}}(K_R+1)}{(1-P_{\tilde{y}})(K_R+1)}\right]} \right\}; \qquad (20)$$

$$\bar{\gamma}^i := \frac{\bar{\gamma}(1 - P_{\tilde{y}})}{1 + \bar{\gamma}|a^i|^2 P_{\tilde{y}}} \qquad (21)$$

=: average SNR for each constellation member a^i;

$$\bar{\gamma} := 1/P_n = \text{average SNR}. \qquad (22)$$

Note that $p(m^i)$ depends only on $|m^i|$, i.e. circularly symmetric. Also it is a function of the Rice factor, power of the MD estimation error, and, through (21) & (22) of the SNR and $|a^i|$.

For M-PSK modulations, $|a^i| = 1\ \forall i$ so that the following simpler form for (21)

$$\bar{\gamma}^i = \frac{\bar{\gamma}(1 - P_{\tilde{y}})}{1 + \bar{\gamma} P_{\tilde{y}}} =: \bar{\gamma}' \qquad (23)$$

makes $p(m^i) = p(m)$ identical for all i.

The expression in (20) becomes much simpler for the special case of Rayleigh fading with $K_R = 0$ (cf. (22) of [1]). However, the special case of an AWGN channel does not meet some of the independence assumptions which led to (9) and (11) and, therefore, will not be considered at this stage of the paper. ($K_R = \infty$ in (7) would mean $y = \bar{y} =$ non random and it can be shown that $\hat{y}$ and $\tilde{y}$ will not be uncorrelated if $\tilde{y} \neq 0$.)

A. Perfect MD Estimation

When there is a perfect (error-free) estimate of the MD, the pdf of the final noise m^i is independent of a^i. To see this, we set $P_{\tilde{y}} = 0$ so that (21) yields $\gamma^i = \gamma$ and we get from (20)

$$p_{\text{Rice,PE}}(m) = \frac{e^{-K_R}}{\pi} \left(\frac{\frac{\bar{\gamma}}{K_R+1}}{\left(1 + |m|^2 \frac{\bar{\gamma}}{K_R+1}\right)^2} + \frac{\frac{\bar{\gamma} K_R}{K_R+1}}{\left(1 + |m|^2 \frac{\bar{\gamma}}{K_R+1}\right)^3} \right)$$

$$\cdot \exp\left\{ \frac{K_R}{1 + |m|^2 \frac{\bar{\gamma}}{K_R+1}} \right\} \qquad (24)$$

where PE stands for perfect estimation (of the MD). For the Rayleigh channel, $K_R = 0$ and (24) simplifies to

$$p_{\text{Ray.,PE}}(m) = \frac{\bar{\gamma}}{\pi\left[1 + \bar{\gamma}|m|^2\right]^2} \qquad (25)$$

which is the same as (35) in [1]. For the AWGN channel $y_k = \bar{y}$ is a non-random complex constant so that from (7) $K_R = \infty$ and we get from (24) and (22)

$$p_{\mathrm{AWGN,PE}}(m) = \frac{1}{\pi P_n} \exp\left(-\frac{|m|^2}{P_n} \right) \tag{26}$$

which is indeed the complex Gaussian pdf.

In order to simplify the forthcoming calculations, we will use polar coordinates in the following section and take advantage of the circular symmetry of $p(m)$ in (24). In so doing, the following relation is used [4]

$$p_{|m|,\arg[m]}(r,\theta) = r\, p_{\mathrm{Re}[m],\mathrm{Im}[m]}(r\cos\theta, r\sin\theta) \quad ; \quad r \geq 0 \tag{27}$$

Circular symmetry requires that $p(r,\theta) = p(r)/2\pi$. Thus, from (24),

$$p_{|m|;\mathrm{Rice,PE}}(r) = 2re^{-K_R} \left(\frac{\frac{\bar{\gamma}}{K_R+1}}{\left(1+r^2\frac{\bar{\gamma}}{K_R+1}\right)^2} + \frac{\frac{\bar{\gamma} K_R}{K_R+1}}{\left(1+r^2\frac{\bar{\gamma}}{K_R+1}\right)^3} \right)$$

$$\cdot \exp\left\{ \frac{K_R}{1+r^2\frac{\bar{\gamma}}{K_R+1}} \right\}. \tag{28}$$

IV. Error Probability with Perfect MD Estimation

In this section, the error probability will be obtained asssuming $P_{\tilde{y}} = 0$, which was shown to yield a final noise with identical statistics for all a^i. In other words, we assume the ideal case of perfect phase synchronization and AGC at the receiver. In the next section we show how the results may be extended to the non-perfect MD estimation case.

A. Symbol Error Rate

We start from (5), transfer it into polar coordinates, and without loss of generality, assume that the center of coordinates is moved to each a^i when the corresponding double integration is being performed

$$\mathrm{SER} = 1 - \frac{1}{M} \sum_{i=1}^{M} \iint_{R^i} \frac{1}{2\pi} p_{|m|;\mathrm{Rice,PE}}(r)\, dr\, d\theta. \tag{29}$$

The first integral has an analytical solution. The result of its indefinite integration is given by the function $S(\cdot)$:

$$S(r) := \int \frac{1}{2\pi} p_{|m|;\mathrm{Rice,PE}}(r)\, dr$$

$$= -\frac{e^{-K_R}}{2\pi} \frac{1}{1+r^2\frac{\bar{\gamma}}{K_R+1}} \exp\left\{ \frac{K_R}{1+r^2\frac{\bar{\gamma}}{K_R+1}} \right\}. \tag{30}$$

Therefore, with any decision region R^i definable in terms of r_1^i, r_2^i, θ_1^i, and θ_2^i, we have

$$\mathrm{SER} = 1 - \frac{1}{M} \sum_{i=1}^{M} \int_{\theta_1^i}^{\theta_2^i} S(r) \Big|_{r_1^i(\theta)}^{r_2^i(\theta)} d\theta =: 1 - \frac{1}{M} \sum_{i=1}^{M} I^i. \tag{31}$$

Here, optimum decision regions are the usual ones, based on the distance between symbols. The reason is that $p(m)$ is the same for all channel symbols when perfect estimation of MD is available.

As the first example, the symbol set for M-PSK has the decision region shown in Fig. 1 which turns out identical for all symbols. We have divided it into three subregions such that

$$\mathrm{SER}_{MPSK} = 1 - I = 1 - 2(I_1 + I_2 + I_3). \tag{32}$$

I_1, I_2, and I_3 are the integrals of the final noise pdf over the subregions R_1, R_2, and R_3, respectively.

$$I_1 = \int_0^{\pi/M} S(r) \Big|_0^{\infty} d\theta \tag{33-a}$$

In obtaining I_2, we may use the circular symmetry of the final noise pdf and rotate R_2 clockwise by the angle π/M

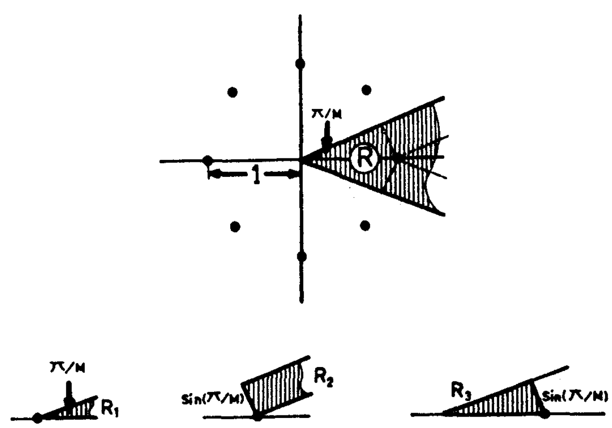

Fig. 1. Decision region R for an M-PSK symbol and its division into three subregions: $R = 2(R_1 + R_2 + R_3)$.

$$I_2 = \int_0^{\pi/2} S(r)\Big|_0^{\frac{\sin \pi/M}{\sin\theta}} d\theta$$

A change of variable from θ to $\pi/2-\theta$ gives

$$I_2 = \int_0^{\pi/2} S(r)\Big|_0^{\frac{\sin \pi/M}{\cos\theta}} d\theta \tag{33-b}$$

Finally, a clockwise rotation of R_3 by the angle $(\pi/2+\pi/M)$ yields

$$I_3 = \int_0^{\frac{\pi}{2}-\frac{\pi}{M}} S(r)\Big|_0^{\frac{\sin \pi/M}{\cos\theta}} d\theta. \tag{33-c}$$

In all the cases considered in this paper, the values of r where $S(r)$ is needed in equations such as (33) turn out to have the form $d/\cos\theta$. For compactness, we use this fact to define a function F to represent a general definite integral of $-2S(r) = -2S(d/\cos\theta)$ of (30):

$$F(\theta_1,\theta_2,\bar{\gamma},K_R,d) := \int_{\theta_1}^{\theta_2} \frac{e^{-K_R}}{\pi} \frac{1}{1+\frac{\bar{\gamma}}{K_R+1}\cdot\frac{d^2}{\cos^2\theta}} \cdot \exp\left(K_R \Big/ \left(1+\frac{\bar{\gamma}}{K_R+1}\cdot\frac{d^2}{\cos^2\theta}\right)\right) d\theta \tag{34}$$

which, in general, will have to be solved numerically. Then, it is possible to write SER in terms of F.

Continuing our M-PSK example, (32) and (33) can be shown to turn into

$$\mathrm{SER}_{M\mathrm{PSK-Rice}} = F\left(0,\frac{\pi}{2},\bar{\gamma},K_R,\sin\frac{\pi}{M}\right) + F\left(0,\frac{\pi}{2}-\frac{\pi}{M},\bar{\gamma},K_R,\sin\frac{\pi}{M}\right) \tag{35}$$

The result for Rayleigh fading may be obtained by setting $K_R = 0$ in the above equation. This allows (34) to be solved analytically and we get

$$\mathrm{SER}_{M\mathrm{PSK-Ray.}} = \frac{M-1}{M} - \sqrt{\frac{\bar{\gamma}\sin^2\frac{\pi}{M}}{\bar{\gamma}\sin^2\frac{\pi}{M}+1}} \cdot \left\{\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{\bar{\gamma}\sin^2\frac{\pi}{M}}{\bar{\gamma}\sin^2\frac{\pi}{M}+1}}\cot\frac{\pi}{M}\right)\right\} \tag{36}$$

which reconciles with the results from the literature obtained through other methods (e.g. [6], eq. (7.6.1) with diversity 1). Also by setting $K_R = \infty$ in (35), we obtain the error probability for AWGN channel as follows

$$\mathrm{SER}_{M\mathrm{PSK-AWGN}} = \mathrm{erfc}\left(\sqrt{\bar{\gamma}}\sin\pi/M\right) - \frac{1}{\pi}\int_{\frac{\pi}{2}-\frac{\pi}{M}}^{\pi/2} \exp\left(-\gamma\frac{\sin^2\pi/M}{\cos^2\theta}\right) d\theta \tag{37}$$

where

$$\mathrm{erfc}(x) := \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2} dt. \tag{38}$$

Note that most of the time, an approximate relation for M-PSK on AWGN channel, e.g.

$$\mathrm{SER} \approx \mathrm{erfc}(\sqrt{\gamma}\sin\pi/M)$$

is used, while (37) is an exact expression.

For 8-AMPM and 16-QAM, shown in Fig. 2, precise SER is derived similarly and the results are listed here:

$$\mathrm{SER}_{8\mathrm{AMPM-Rice}} = \frac{1}{2}F\left(0,\frac{\pi}{2},\bar{\gamma},K_R,\sqrt{0.4}\right) + \frac{1}{4}F\left(0,\frac{\pi}{2},\bar{\gamma},K_R,\sqrt{0.2}\right) + 2F\left(0,\frac{\pi}{4},\bar{\gamma},K_R,\sqrt{0.2}\right) \tag{39}$$

$$\mathrm{SER}_{8\mathrm{AMPM-Ray.}} = \frac{7}{8} - \frac{1}{4}\sqrt{\frac{\bar{\gamma}}{\bar{\gamma}+2.5}} - \sqrt{\frac{\bar{\gamma}}{\bar{\gamma}+5}}\left\{\frac{1}{8}+\frac{2}{\pi}\tan^{-1}\left(\sqrt{\frac{\bar{\gamma}}{\bar{\gamma}+5}}\right)\right\} \tag{40}$$

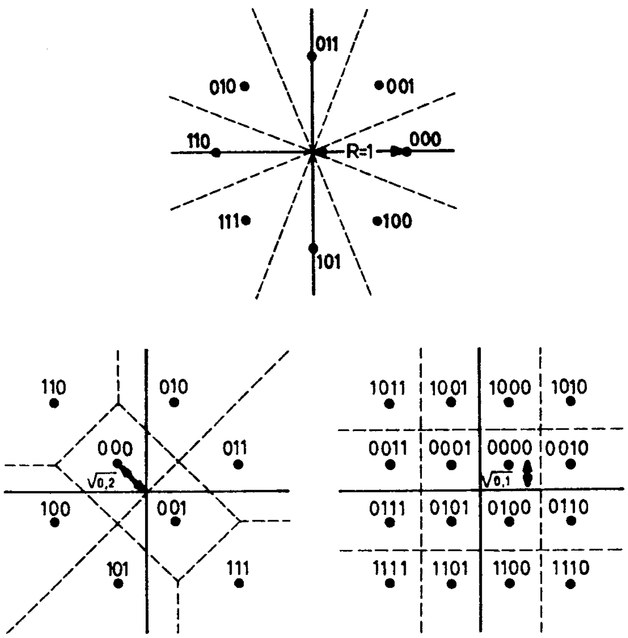

Fig. 2. 8-PSK, 8-AMPM and 16-QAM with Gray coding.

$$\mathrm{SER}_{\mathrm{8AMPM-AWGN}} = \frac{1}{4}\operatorname{erfc}\left(\sqrt{\gamma/2.5}\right) + \frac{9}{8}\operatorname{erfc}\left(\sqrt{\gamma/5}\right) - \frac{2}{\pi}\int_{\pi/4}^{\pi/2} \exp\left(\frac{-\gamma}{5\cos^2\theta}\right) d\theta \tag{41}$$

$$\mathrm{SER}_{\mathrm{16QAM-Rice}} = \frac{3}{4}F\left(0,\frac{\pi}{2},\bar{\gamma},K_R,\sqrt{0.1}\right) + \frac{9}{4}F\left(0,\frac{\pi}{4},\bar{\gamma},K_R,\sqrt{0.1}\right) \tag{42}$$

$$\mathrm{SER}_{\mathrm{16QAM-Ray.}} = \frac{15}{16} - \sqrt{\frac{\bar{\gamma}}{\bar{\gamma}+10}}\left\{\frac{3}{8} + \frac{9}{4\pi}\tan^{-1}\left(\sqrt{\frac{\bar{\gamma}}{\bar{\gamma}+10}}\right)\right\} \tag{43}$$

$$\mathrm{SER}_{\mathrm{16QAM-AWGN}} = \frac{3}{2}\operatorname{erfc}\left(\sqrt{\frac{\gamma}{10}}\right)\left[1 - \frac{3}{8}\operatorname{erfc}\left(\sqrt{\frac{\gamma}{10}}\right)\right]. \tag{44}$$

B. Bit Error Rate (BER)

A single symbol error may cause from 1 up to $\log_2 M$ bits of error. By correct combination of all different types of error, we can write the average BER in the form

$$\mathrm{BER} = \frac{1}{\log_2 M}\sum_{b=1}^{\log_2 M} b\,p(b), \tag{45}$$

where b is the number of bits detected erroneously and $p(b)$ is the probability of b-bit errors happening. We assume that Gray coding, as in Fig. 2, is used in forming the symbols. In this way, the BERs for M-PSK and 16-QAM have been derived [5] and the results are listed here.[2]

1) 4-PSK:

$$\mathrm{BER}_{\mathrm{4PSK-Rice}} = F\left(0,\frac{\pi}{2},\bar{\gamma}_b,K_R,1\right) \tag{46}$$

$$\mathrm{BER}_{\mathrm{4PSK-Ray.}} = \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_b}{\bar{\gamma}_b+1}}\right] \tag{47}$$

$$\mathrm{BER}_{\mathrm{4PSK-AWGN}} = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\gamma_b}\right) \tag{48}$$

2) 8-PSK:

$$\mathrm{BER}_{\mathrm{8PSK-Rice}} = \frac{1}{3}F\left(0,\frac{\pi}{2},3\bar{\gamma}_b,K_R,\cos\frac{\pi}{8}\right) + \frac{1}{3}F\left(0,\frac{\pi}{8},3\bar{\gamma}_b,K_R,\cos\frac{\pi}{8}\right) + \frac{1}{3}F\left(0,\frac{\pi}{2},3\bar{\gamma}_b,K_R,\cos\frac{3\pi}{8}\right) + \frac{1}{3}F\left(0,\frac{3\pi}{8},3\bar{\gamma}_b,K_R,\cos\frac{3\pi}{8}\right) \tag{49}$$

$$\mathrm{BER}_{\mathrm{8PSK-Ray.}} = (1/2) - \frac{1}{3}\sqrt{\frac{3\bar{\gamma}_b\cos^2\frac{\pi}{8}}{3\bar{\gamma}_b\cos^2\frac{\pi}{8}+1}}\left\{\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{3\bar{\gamma}_b\cos^2\frac{\pi}{8}}{3\bar{\gamma}_b\cos^2\frac{\pi}{8}+1}}\tan\frac{\pi}{8}\right)\right\} - \frac{1}{3}\sqrt{\frac{3\bar{\gamma}_b\cos^2\frac{3\pi}{8}}{3\bar{\gamma}_b\cos^2\frac{3\pi}{8}+1}}\left\{\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{3\bar{\gamma}_b\cos^2\frac{3\pi}{8}}{3\bar{\gamma}_b\cos^2\frac{3\pi}{8}+1}}\tan\frac{3\pi}{8}\right)\right\} \tag{50}$$

[2]Note the use of the usual notation $\bar{\gamma}_b := \bar{\gamma}/(\log_2 M)$ = average SNR per bit.

$$\mathrm{BER}_{\mathrm{8PSK-AWGN}} = \frac{1}{3}\mathrm{erfc}\left(\sqrt{3\gamma_b}\cos\frac{\pi}{8}\right)$$
$$+\frac{1}{3}\mathrm{erfc}\left(\sqrt{3\gamma_b}\cos\frac{3\pi}{8}\right)\left[1-\frac{1}{2}\mathrm{erfc}\left(\sqrt{3\gamma_b}\cos\frac{\pi}{8}\right)\right] \quad (51)$$

3) 16-PSK:

$$\mathrm{BER}_{\mathrm{16PSK-Rice}} = \frac{1}{4}F\left(0,\frac{\pi}{2},4\bar{\gamma}_b,K_R,\cos\frac{7\pi}{16}\right)$$
$$+\frac{1}{4}F\left(0,\frac{7\pi}{16},4\bar{\gamma}_b,K_R,\cos\frac{7\pi}{16}\right)+\frac{1}{2}F\left(0,\frac{5\pi}{16},4\bar{\gamma}_b,K_R,\cos\frac{5\pi}{16}\right)$$
$$+\frac{1}{4}F\left(\frac{\pi}{16},\frac{\pi}{2},4\bar{\gamma}_b,K_R,\cos\frac{\pi}{16}\right) \quad (52)$$

$$\mathrm{BER}_{\mathrm{16PSK-Ray.}} = \frac{1}{2}-\frac{1}{4}\sqrt{\frac{4\bar{\gamma}_b\cos^2\frac{7\pi}{16}}{4\bar{\gamma}_b\cos^2\frac{7\pi}{16}+1}}$$
$$\cdot\left\{\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{4\bar{\gamma}_b\cos^2\frac{7\pi}{16}}{4\bar{\gamma}_b\cos^2\frac{7\pi}{16}+1}}\tan\frac{7\pi}{16}\right)\right\}-$$
$$\frac{1}{4}\sqrt{\frac{4\bar{\gamma}_b\cos^2\frac{\pi}{16}}{4\bar{\gamma}_b\cos^2\frac{\pi}{16}+1}}\left\{\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{4\bar{\gamma}_b\cos^2\frac{\pi}{16}}{4\bar{\gamma}_b\cos^2\frac{\pi}{16}+1}}\tan\frac{\pi}{16}\right)\right\}$$
$$-\frac{1}{2\pi}\sqrt{\frac{4\bar{\gamma}_b\cos^2\frac{5\pi}{16}}{4\bar{\gamma}_b\cos^2\frac{5\pi}{16}+1}}\cdot\tan^{-1}\left(\sqrt{\frac{4\bar{\gamma}_b\cos^2\frac{5\pi}{16}}{4\bar{\gamma}_b\cos^2\frac{5\pi}{16}+1}}\tan\frac{5\pi}{16}\right) \quad (53)$$

$$\mathrm{BER}_{\mathrm{16PSK-AWGN}} =$$
$$\frac{1}{4}\mathrm{erfc}\left(\sqrt{4\gamma_b}\cos\frac{7\pi}{16}\right)-\frac{1}{4\pi}\int_{7\pi/16}^{\pi/2}\exp\left(-4\gamma_b\frac{\cos^2\frac{7\pi}{16}}{\cos^2\theta}\right)d\theta$$

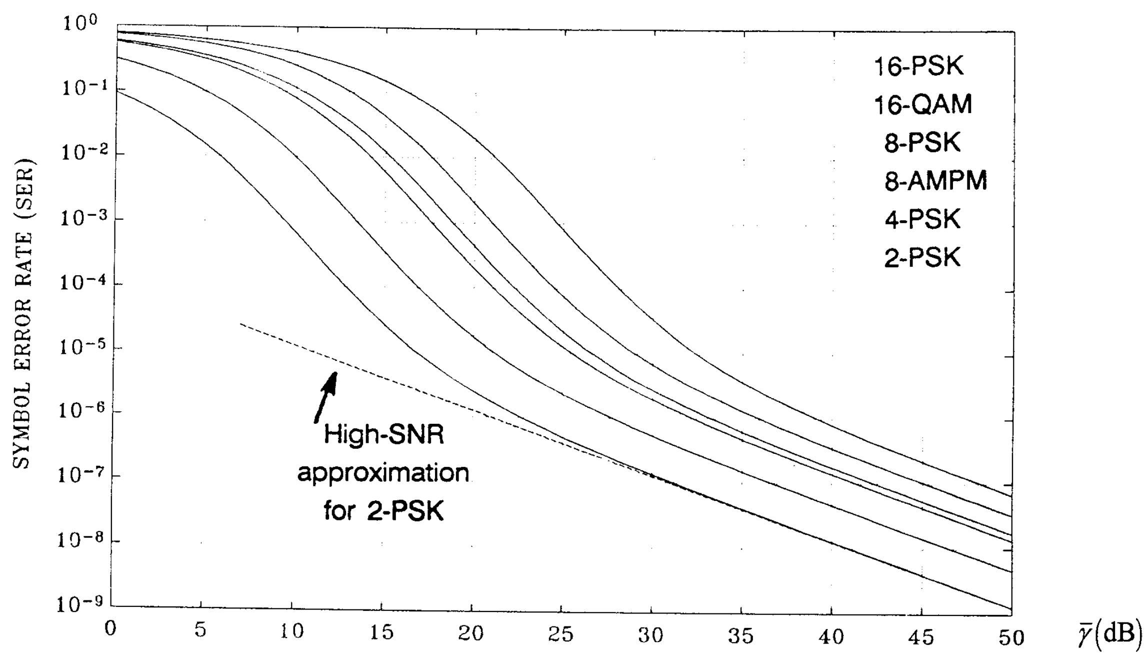

Fig. 3. Symbol error rates for a few types of linear modulations with perfect (error-free) MD estimation on a frequency-flat Ricean channel (K_R=10 dB). The legends are in the same order as the curves.

$$+\frac{1}{2\pi}\int_0^{5\pi/16}\exp\left(-4\gamma_b\frac{\cos^2\frac{5\pi}{16}}{\cos^2\theta}\right)d\theta$$

$$+\frac{1}{4\pi}\int_{\pi/16}^{\pi/2}\exp\left(-4\gamma_b\frac{\cos^2\frac{\pi}{16}}{\cos^2\theta}\right)d\theta \quad (54)$$

4) 16-QAM:

$$\mathrm{BER}_{16\mathrm{QAM-Rice}}=\frac{1}{2}F\left(0,\frac{\pi}{2},4\bar{\gamma}_b,K_R,\sqrt{0.1}\right)$$

$$+\frac{1}{4}F\left(0,\tan^{-1}3,4\bar{\gamma}_b,K_R,\sqrt{0.1}\right)+\frac{1}{2}F\left(0,\frac{\pi}{2},4\bar{\gamma}_b,K_R,3\sqrt{0.1}\right)$$

$$+\frac{1}{4}F\left(0,\tan^{-1}\frac{1}{3},4\bar{\gamma}_b,K_R,3\sqrt{0.1}\right)-\frac{1}{4}F\left(0,\frac{\pi}{2},4\bar{\gamma}_b,K_R,5\sqrt{0.1}\right) \quad (55)$$

$$\mathrm{BER}_{16\mathrm{QAM-Ray.}}=\frac{1}{2}+\frac{1}{8}\sqrt{\frac{10\bar{\gamma}_b}{10\bar{\gamma}_b+1}}$$

$$-\frac{1}{4}\sqrt{\frac{0.4\bar{\gamma}_b}{0.4\bar{\gamma}_b+1}}\left\{1+\frac{1}{\pi}\tan^{-1}\left(3\sqrt{\frac{0.4\bar{\gamma}_b}{0.4\bar{\gamma}_b+1}}\right)\right\}$$

$$-\frac{1}{4}\sqrt{\frac{3.6\bar{\gamma}_b}{3.6\bar{\gamma}_b+1}}\left\{1+\frac{1}{\pi}\tan^{-1}\left(\frac{1}{3}\sqrt{\frac{3.6\bar{\gamma}_b}{3.6\bar{\gamma}_b+1}}\right)\right\} \quad (56)$$

$$\mathrm{BER}_{16\mathrm{QAM-AWGN}}=\frac{3}{8}\mathrm{erfc}\left(\sqrt{0.4\gamma_b}\right)+\frac{3}{8}\mathrm{erfc}\left(\sqrt{3.6\gamma_b}\right)$$

$$-\frac{1}{8}\mathrm{erfc}\left(\sqrt{10\gamma_b}\right)-\frac{1}{4\pi}\int_{\tan^{-1}3}^{\pi/2}\exp\left(-\frac{0.4\gamma_b}{\cos^2\theta}\right)d\theta$$

$$-\frac{1}{4\pi}\int_{\tan^{-1}(1/3)}^{\pi/2}\exp\left(-\frac{3.6\gamma_b}{\cos^2\theta}\right)d\theta \quad (57)$$

We emphasize that all of the above results are exact, whereas an approximation such as BER $\approx (1/\log_2 M)\cdot$SER would be used in the absence of such results. Korn [7] analyzes a specific practical receiver of M-PSK and uses this approximation to find BER curves. The special cases of M-PSK on AWGN and Rayleigh channels have been treated in [8] and [9], respectively, where exact results are given for M up to 8. Fitz [10] concentrates on the error probability analysis with "a noisy phase reference" in a non-fading channel, a case which could not be handled by our modeling in this paper.

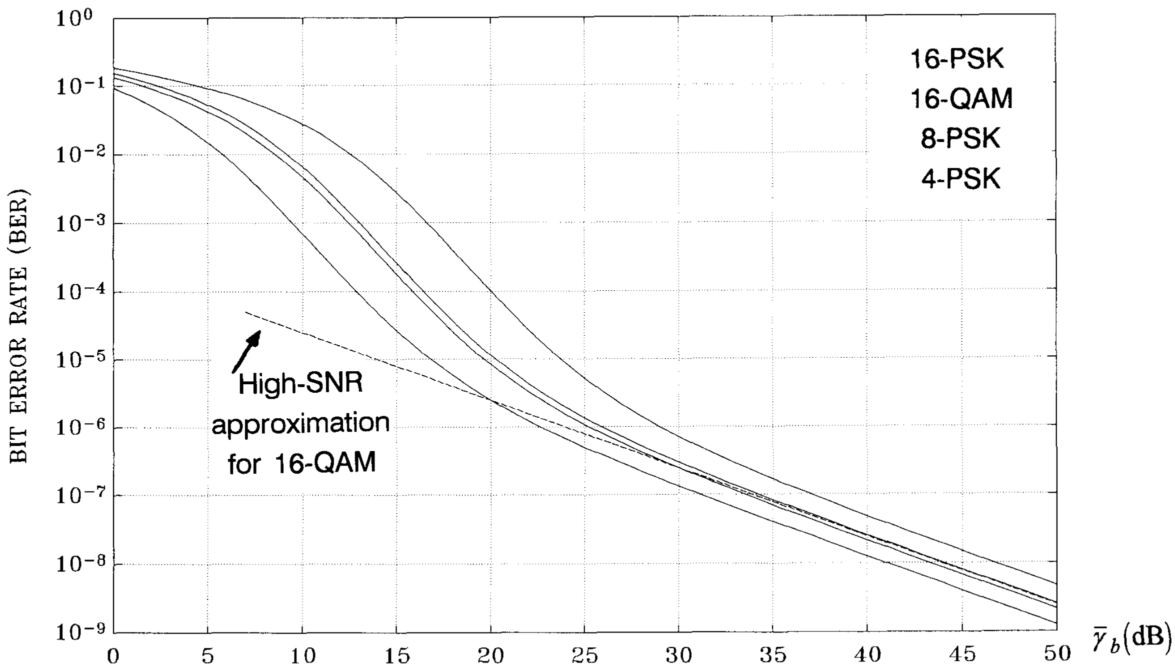

Fig. 4. Bit error rates for a few types of linear modulations with perfect (error-free) MD estimation on a frequency-flat Ricean channel (K_R =10 dB). The legends are in the same order as the curves.

Basically there is no problem for finding SER or BER of other types of linearly modulated signals. Of course, in some cases such as BER of 8-AMPM, one may end up with calculations too long for one's patience.

The general case of Ricean fading does not have any analytical solution and the single integral in (34) must be solved by numerical methods. Figs. 3 and 4 show SER and BER of some modulations on Ricean channel with K_R=10 and error-free MD estimation. The well-known Romberg numerical integration with precision of 10^{-12} was applied for calculating (34). Implemented on a 286-personal computer clocked at 12 MHZ, the curve for BER of 16QAM-Rice needed about 2 seconds run-time, where (55) was calculated at 50 different values of $\bar{\gamma}_b$. In Fig. 5, BER of 4-PSK is plotted for different values of K_R in the range 0 to ∞.

C. Error Probability for Large SNR

It would be interesting to search for simpler expressions for error probability when the SNR is large. For this purpose we allow some approximations. It has been shown above that error probabilities on Ricean channel comprise of several similar single integrals which are not solvable analytically. Considering (34), we first ignore the exp{·} term from the integrand when $\bar{\gamma}$ is large. Then, the result of the integration is

$$F(\theta_1,\theta_2,\bar{\gamma},K_R,d) \approx$$

$$\frac{e^{-K_R}}{\pi}\left\{\theta - \sqrt{\frac{d^2\bar{\gamma}}{d^2\bar{\gamma}+K_R+1}}\tan^{-1}\left(\sqrt{\frac{d^2\bar{\gamma}}{d^2\bar{\gamma}+K_R+1}}\tan\theta\right)\right\}_{\theta=\theta_1}^{\theta=\theta_2} \tag{58}$$

Next, the following approximations are applied

$$\sqrt{\frac{d^2\bar{\gamma}}{d^2\bar{\gamma}+K_R+1}} \approx 1-\frac{1}{2}\frac{K_R+1}{d^2\bar{\gamma}} \quad ; \quad \bar{\gamma} >> \frac{K_R+1}{d^2} \tag{59}$$

$$\tan^{-1}\left(\sqrt{\frac{d^2\bar{\gamma}}{d^2\bar{\gamma}+K_R+1}}\tan\theta\right) \approx$$

$$\theta - \frac{K_R+1}{4d^2\bar{\gamma}}\sin 2\theta \quad ; \quad \bar{\gamma} >> \frac{K_R+1}{d^2} \tag{60}$$

where we have used Taylor series expansion, retaining only the first two terms. Now using (59) and (60) in (58), and ignoring the resulting $1/\bar{\gamma}^2$ term, we get

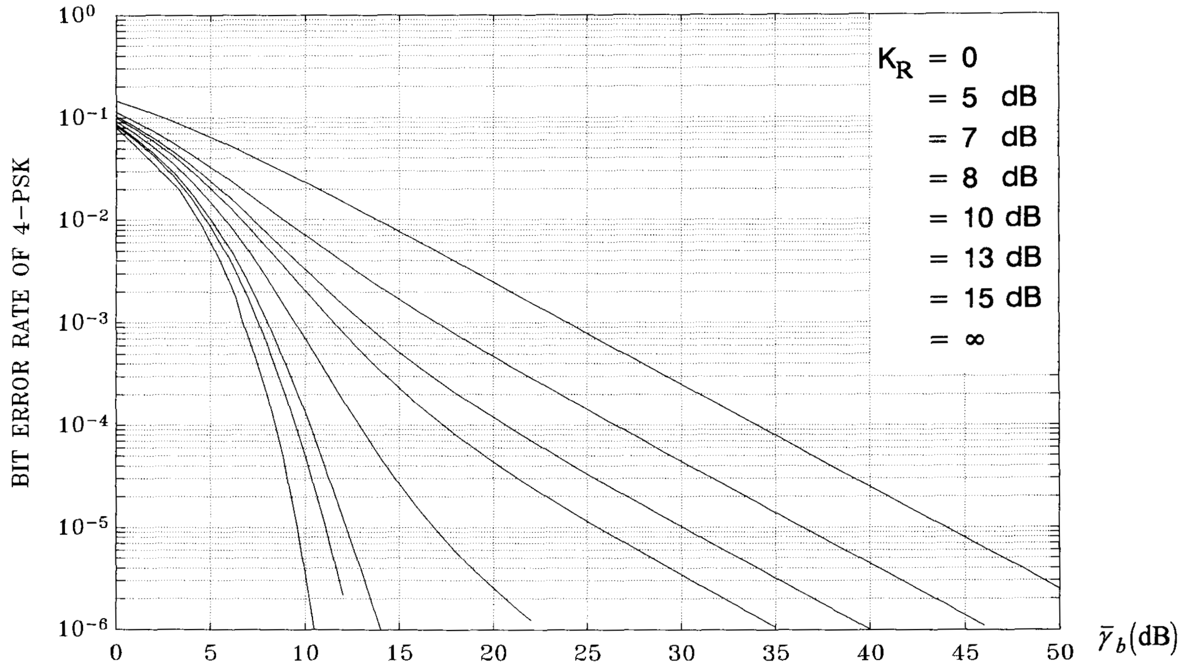

Fig. 5. Bit error rate of 4-PSK with perfect (error-free) MD estimation for different values of K_R (Rayleigh, Ricean, and AWGN). The legends are in the same order as the curves.

$$F(\theta_1,\theta_2,\bar{\gamma},K_R,d) \approx$$

$$(K_R+1)e^{-K_R}\frac{1}{4\pi d^2\bar{\gamma}}\left[2\theta+\sin 2\theta\right]_{\theta=\theta_1}^{\theta=\theta_2} \;;\quad \bar{\gamma} >> \frac{K_R+1}{d^2} \tag{61}$$

This implies that the error probability is proportional to $1/\bar{\gamma}$ (linear, when the axes have logarithmic scale). For example, for M-PSK we get

$$\mathrm{SER}_{M\mathrm{PSK-Rice}} \approx (K_R+1)e^{-K_R}$$

$$\cdot\frac{\frac{2(M-1)}{M}+\frac{1}{\pi}\sin\frac{2\pi}{M}}{4\sin^2\pi/M}\cdot\frac{1}{\bar{\gamma}} \;;\quad \bar{\gamma} >> \frac{K_R+1}{\sin^2\pi/M} \tag{62}$$

Approximate results based on (61) are obtainable [5] for all other cases and are in a very good agreement with the more exact ones obtained by numerical methods (cf. Figs. 3 and 4 for two examples).

The case of Rayleigh fading with large SNR may also be handled by approximations similar to (59) and (60), applied to all the corresponding expressions given in sections IV-A and IV-B. The result is that, again, for large $\bar{\gamma}$ the error probabilities are proportional to $1/\bar{\gamma}$. Also it is easily shown that for a given modulation type and a fixed SNR, the following interesting relation holds

$$\frac{\mathrm{SER}_{\mathrm{Rice}}}{\mathrm{SER}_{\mathrm{Ray.}}} = \frac{\mathrm{BER}_{\mathrm{Rice}}}{\mathrm{BER}_{\mathrm{Ray.}}} \approx (K_R+1)e^{-K_R} \;;\quad \bar{\gamma} >> \frac{K_R+1}{d^2} \tag{63}$$

Note that for AWGN channel with $K_R=\infty$, the condition in (61) for its validity is never met and the large SNR approximation does not apply.

V. Error Probability with Non-Perfect MD Estimation

In the preceding section, we considered perfect MD estimation, i.e. when error-free phase and amplitude references are available at the receiver. Now we go on to a more realistic situation with non-perfect MD estimation and present two methods for handling the case. Note that the material in this section does not apply to the special case of AWGN channel (cf. paragraph following (23)).

For M-PSK modulations, by comparing $p(m)$ in the cases of perfect (eq. (24)) and non-perfect MD estimation ((23) inserted in (20)), we see that the same result can be obtained if we replace $\bar{\gamma}$ by $\bar{\gamma}'$ and K_R by $K_R/(1-P_{\bar{\gamma}}(K_R+1))$. Defining

$$L := \frac{\bar{\gamma}}{\bar{\gamma}'} = \frac{1+\bar{\gamma}P_{\bar{\gamma}}}{1-P_{\bar{\gamma}}} \tag{64}$$

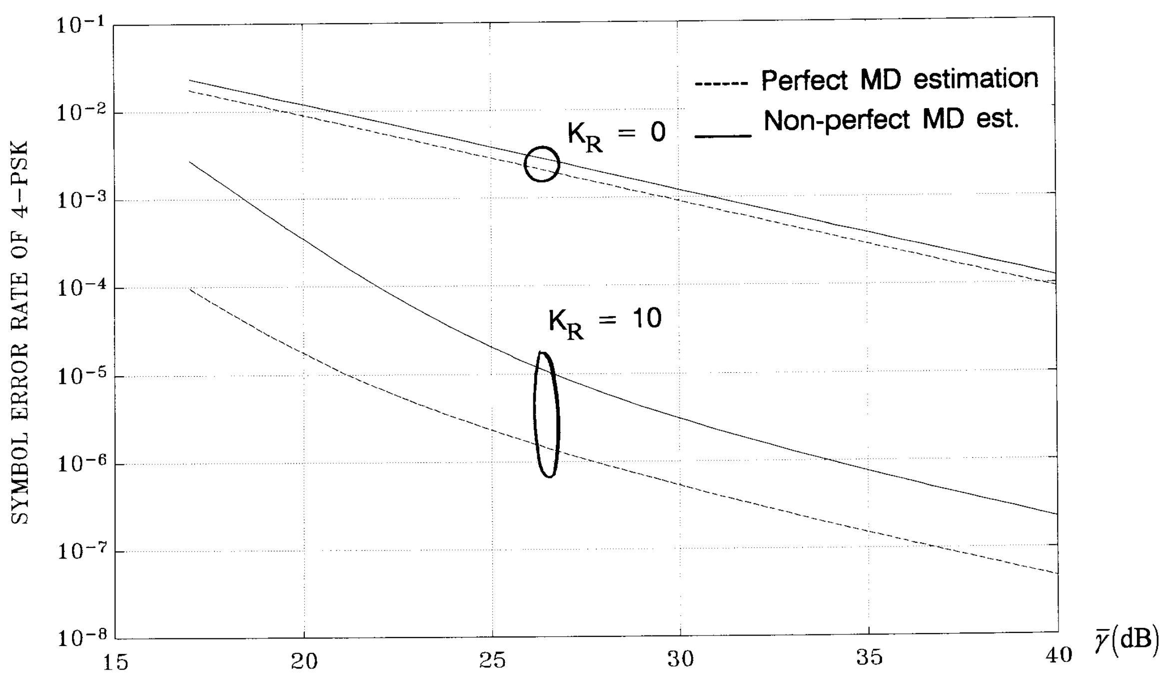

Fig. 6. Performance degradation due to error in phase and amplitude references, with the error power assumed as $P_{\bar{\gamma}} = (K_R+1)/(3\bar{\gamma})$.

$$K_R' := \frac{K_R}{1 - P_{\tilde{y}}(K_R + 1)} \quad ; \quad L_R := \frac{K_R}{K_R'} = 1 - P_{\tilde{y}}(K_R + 1), \quad (65)$$

we obtain the SER for the non-perfect case in terms of SER for the perfect case.

$$\mathrm{SER}(\bar{\gamma}, K_R, P_{\tilde{y}}) = \mathrm{SER}_{\mathrm{PE}}\left(\frac{\bar{\gamma}}{L}, \frac{K_R}{L_R}\right) \quad (\text{finite } K_R) \quad (66)$$

PE: Perfect Estimation (of the MD).

As an example, we used (64)-(66) to find the SER of 4-PSK for two values of K_R=0 and K_R=10. The MD estimation error power $P_{\tilde{y}}$ was assumed as $(K_R + 1)/(3\bar{\gamma})$. The results are displayed in Fig. 6, where the case of perfect estimation is also shown for comparison.

For BER, it is important to know if a fraction (say, $1/E$) of the transmitter power is spent (for the sake of receiver synchronization and AGC) on sending non-information-carrying components such as pilot tones or time multiplexed pilot symbols [2]. If so, there will be an effective loss of SNR for the information bits. We may find the BER of M-PSK modulations with non-perfect MD estimation as follows

$$L_b := \frac{E}{E-1} L \quad (67)$$

$$\mathrm{BER}(\bar{\gamma}_b, K_R, P_{\tilde{y}}) = \mathrm{BER}_{\mathrm{PE}}\left(\frac{\bar{\gamma}_b}{L_b}, \frac{K_R}{L_R}\right). \quad (\text{finite } K_R) \quad (68)$$

For non-PSK modulations, $p_{Rice}(m^i)$ of (20) depends on $|a^i|$ through (21). The individual average SNR of each symbol depends inversely on its distance from the origin, i.e. when $P_{\tilde{y}} \neq 0$, (21) shows that symbols with larger magnitude face a stronger "final noise". The method used in [1], (40)-(42), defines an L which may be used in (66)-(68) to yield actual SER and BER with the help of the results from the perfect estimation cases. The approximation involved in that method is satisfactory when $\{\bar{\gamma}\, P_{\tilde{y}} \max[|a^i|^2]\} << 1$.

Alternatively, for all linear modulation types it is possible to find the error probability with non-perfect MD estimation directly and exactly. It turns out that K_R' of (65) and $\bar{\gamma}^i$ of (21) will have to be substituted respectively, for K_R and $\bar{\gamma}$ in (30) in order to give $S(\cdot)$ a new form $S^i(\cdot)$. Then, for an arbitrary set of decision regions R^i, the function $S^i(\cdot)$ will be used instead of $S(\cdot)$ in (31) to give the actual SER. A similar procedure will yield the BER.

VI. Conclusions

We considered linearly modulated signals on Ricean frequency-flat fading channels with coherent detection at the receiver. We provided a general method and showed specific examples of deriving exact expressions for bit and symbol error probabilities. The end results involved several versions of a single integral which was easily solved by numerical methods. We showed that, with good approximation, the error probability for large SNR is proportional to its inverse. We also showed how to find error probabilities with non-perfect phase and amplitude references either directly or from the results of the perfect case.

As a special case of Ricean fading, Rayleigh fading was formed when the Rice factor K_R was set equal to zero. Then, all integrations involved in error probability calculations were solvable analytically and exact results were obtained with no need for numerical integration. The approximation with large SNR and the case of non-perfect references were handled similarly as for Ricean fading.

The special case of a nonfading AWGN channel was obtained by letting $K_R = \infty$. Then, the assumptions used in our "final noise" method applied only to the perfect-reference case. The corresponding exact error rates, involving the usual erfc(.) function and other similar integrations, were presented in sections IV-A and IV-B.

The general framework of this paper yields a unified approach for exact assessment of error performance of linearly modulated signals on three practical classes of channel models. On two of these (i.e. Ricean and Rayleigh), this approach allows also for non-perfect phase and amplitude references.

Appendix
Derivation of the PDF of the Final Noise

In this appendix we solve the integral in (19) and find the result in closed form. Taking the constants out of the integral yields

$$p_{\mathrm{Rice}}(m'^i) = p_{\mathrm{Rice}}(m^i) = \frac{2}{\pi\left(P_n + |a^i|^2 P_{\tilde{y}}\right)\left(P_{\hat{y}} - s^2\right)}$$

$$\cdot \exp\left(-\frac{s^2}{P_{\hat{y}} - s^2}\right)\int_0^\infty \frac{1}{\alpha^5} \exp\left(-\frac{|m^i|^2\left(P_{\hat{y}} - s^2\right) + \left(P_n + |a^i|^2 P_{\tilde{y}}\right)}{\alpha^2\left(P_n + |a^i|^2 P_{\tilde{y}}\right)\left(P_{\hat{y}} - s^2\right)}\right)$$

$$\cdot I_0\left(\frac{2s}{\alpha\left(P_{\hat{y}} - s^2\right)}\right) d\alpha. \quad (\mathrm{A.1})$$

For simplification, B and C are defined as:

$$B := \frac{2}{\pi\left(P_n + |a^i|^2 P_{\tilde{y}}\right)\left(P_{\hat{y}} - s^2\right)} \exp\left(-\frac{s^2}{P_{\hat{y}} - s^2}\right) \quad (\mathrm{A.2})$$

$$C := \frac{\left|m^i\right|^2\left(P_{\hat{y}} - s^2\right) + \left(P_n + \left|a^i\right|^2 P_{\tilde{y}}\right)}{\left(P_n + \left|a^i\right|^2 P_{\tilde{y}}\right)\left(P_{\hat{y}} - s^2\right)}. \qquad \text{(A.3)}$$

Also we use the following series form

$$I_0(x) = \sum_{k=0}^{\infty} (x/2)^{2k} / (k!)^2$$

and (A.1) turns into

$$p_{\text{Rice}}\left(m^i\right) =$$

$$B\int_0^{\infty} \frac{1}{\alpha^5} \exp\left(-\frac{C}{\alpha^2}\right) \cdot \left\{\sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{s}{\alpha\left(P_{\hat{y}} - s^2\right)}\right)^{2k}\right\} d\alpha. \quad \text{(A.4)}$$

Now, we change the order of summation and integration.

$$p_{\text{Rice}}\left(m^i\right) =$$

$$B\sum_{k=0}^{\infty}\left\{\int_0^{\infty} \frac{1}{\alpha^5}\exp\left(-C/\alpha^2\right) \cdot \left[\frac{1}{(k!)^2}\left(\frac{s}{\alpha\left(P_{\hat{y}} - s^2\right)}\right)^{2k}\right] d\alpha\right\}$$

$$= B\sum_{k=0}^{\infty}\left\{\left(\frac{s}{\left(P_{\hat{y}} - s^2\right)}\right)^{2k} \frac{1}{(k!)^2}\int_0^{\infty} \frac{\exp\left(-C/\alpha^2\right)}{\alpha^{2k+5}} d\alpha\right\} \quad \text{(A.5)}$$

The above integral is easily solved analytically by a change of variable ($x := 1/\alpha^2$) and using a table of integrals

$$p_{\text{Rice}}\left(m^i\right) = B\sum_{k=0}^{\infty}\left\{\left(\frac{s}{\left(P_{\hat{y}} - s^2\right)}\right)^{2k} \frac{1}{(k!)^2}\frac{(k+1)!}{2C^{k+2}}\right\}$$

$$= \frac{B}{2C^2}\sum_{k=0}^{\infty}\frac{k}{k!}\left(\frac{s^2}{C\left(P_{\hat{y}} - s^2\right)^2}\right)^k + \frac{B}{2C^2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{s^2}{C\left(P_{\hat{y}} - s^2\right)^2}\right)^k. \quad \text{(A.6)}$$

Using

$$\sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x \quad ; \quad \sum_{k=0}^{\infty}\frac{k}{k!}x^k = xe^x,$$

we get

$$p_{\text{Rice}}\left(m^i\right) = \frac{B}{2C^2}\exp\left(\frac{s^2}{C\left(P_{\hat{y}} - s^2\right)^2}\right)\left[\frac{s^2}{C\left(P_{\hat{y}} - s^2\right)^2} + 1\right]. \quad \text{(A.7)}$$

Next a new variable is defined as

$$\bar{\gamma}^i := \frac{P_{\hat{y}}}{P_n + \left|a^i\right|^2 P_{\tilde{y}}} = \frac{\bar{\gamma}\left(1 - P_{\tilde{y}}\right)}{1 + \bar{\gamma}\left|a^i\right|^2 P_{\tilde{y}}} \qquad \text{(A.8)}$$

=:average SNR for each constellation member a^i

$$\bar{\gamma} := 1/P_n = \text{average SNR}. \qquad \text{(A.9)}$$

Then by using (A.2), (A.3), (A.8), and (A.9) in (A.7) we obtain

$$p_{\text{Rice}}\left(m^i\right) = \frac{1}{\pi}\exp\left(-\frac{s^2}{P_{\hat{y}} - s^2}\right)$$

$$\cdot\left\{\frac{\bar{\gamma}^i\left(1 - s^2/P_{\hat{y}}\right)}{\left[1 + \left|m^i\right|^2 \bar{\gamma}^i\left(1 - s^2/P_{\hat{y}}\right)\right]^2} + \frac{\bar{\gamma}^i s^2/P_{\hat{y}}}{\left[1 + \left|m^i\right|^2 \bar{\gamma}^i\left(1 - s^2/P_{\hat{y}}\right)\right]^3}\right\}$$

$$\cdot\exp\left(\frac{s^2/\left(P_{\hat{y}} - s^2\right)}{1 + \left|m^i\right|^2 \bar{\gamma}^i\left(1 - s^2/P_{\hat{y}}\right)}\right). \qquad \text{(A.10)}$$

Finally using (8) and (9) in (A.10) yields (20).

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