QCQI Exercises

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Chapter 2

Exercise 2.1

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0. \tag{1}$$

Exercise 2.2

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Exercise 2.3

For each v_i, we have $A\left|v_i\right\rangle = \sum_j A_{ji}\left|w_j\right\rangle$. For each w_j, we have $B\left|w_j\right\rangle = \sum_k B_{kj}\left|x_k\right\rangle$.
So for each v_i, we have:

$$(BA)\left|v_i\right\rangle = B(A\left|v_i\right\rangle = B(\sum_j A_{ji}\left|w_j\right\rangle) = \sum_j A_{ji}B\left|w_j\right\rangle \tag{3}$$

$$= \sum_j A_{ji}\sum_k B_{kj}\left|x_k\right\rangle = \sum_{jk} B_{kj}A_{ji}\left|x_k\right\rangle = \sum_k (BA)_{ki}\left|x_k\right\rangle. \tag{4}$$

Therefore the matrix presentation for linear transformation BA is the matrix product of the matrix representation for B and A.

Exercise 2.4

If I is the identity operator, then for each $\left|v_i\right\rangle$, there must be $I\left|v_i\right\rangle = \sum_j I_{ji}v_i = v_i$. So $I_{ji} = \delta_{ji}$, which means the matrix representation of I is the identity matrix.

Exercise 2.5

If $(|u\rangle, |v\rangle) = \sum_i u_i^* v_i$, then we can verify that:

1. For $|v\rangle$ and $\sum_i \lambda_i |w_i\rangle$, we have:

$$(|v\rangle, \sum_i \lambda_i |w_i\rangle) = \sum_j v_j^* \sum_i \lambda_i w_{ij} = \sum_i \lambda_i (\sum_j v_j^* w_{ij}) = \sum_i \lambda_i (|v\rangle, |w_i\rangle). \tag{5}$$

2. For $|v\rangle$ and $|w\rangle$, we have:

$$(|v\rangle, |w\rangle) = \sum_i v_i^* w_i = \sum_i w_i v_i^* = \sum_i (w_i^* v_i)^* = (\sum_i w_i^* v_i)^* = (|w\rangle, |v\rangle)^*. \tag{6}$$

3. For $|v\rangle$, we have:

$$(|v\rangle, |v\rangle) = \sum_i v_i^* v_i = \sum_i |v_i|^2 \geq 0. \tag{7}$$

The equivalence holds only when all $v_i = 0$, which means $|v\rangle = 0$.

Exercise 2.6

We can verify that:

$$\left(\sum_i \lambda_i |w_i\rangle, |v\rangle\right) = \left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right)^* = \left(\sum_i \lambda_i (|v\rangle, |w_i\rangle)\right)^* = \sum_i \lambda_i^* (|w_i\rangle, |v\rangle). \tag{8}$$

Exercise 2.8

Use the induction method.
First, v_1 is normal, and the set $\{v_1\}$ is orthonormal.
Now suppose the vectors $v_1, v_2, \cdots, v_{j-1}$ are orthonormal, which means $\langle v_m | v_n \rangle = \delta_{mn}$ for all $m, n \leq j - 1$.
For any $|v_i\rangle$ and $|v_j\rangle$ with $j > i$, we have:

$$\begin{aligned}
\langle v_i | v_j \rangle &= \frac{1}{|w_j'|} \left(\langle v_i | \left(|w_j\rangle - \sum_{t=1}^{j-1} \langle v_t | w_j \rangle |v_t\rangle \right) \right) &(9)\\
&= \frac{1}{|w_j'|} \left(\langle v_i | w_j \rangle - \sum_{i=1}^{j-1} \langle v_t | w_j \rangle \langle v_i | v_t \rangle \right) &(10)\\
&= \frac{1}{|w_j'|} \left(\langle v_i | w_j \rangle - \sum_{i=1}^{j-1} \langle v_t | w_j \rangle \delta_{it} \right) &(11)
\end{aligned}$$

$$= \frac{1}{|w_j'|} \left(\langle v_i | w_j \rangle - \langle v_i | w_j \rangle \right) = 0. \tag{12}$$

Therefore $|v_j\rangle$ is orthogonal to all $|v_i\rangle$ for $i < j$. Additionally $|v_j\rangle$ is normal.
So the Gram-Schmidt procedure produces an orthonormal basis.

Exercise 2.9

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|. \tag{13}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{14}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i|1\rangle\langle 0| + i|0\rangle\langle 1|. \tag{15}$$

Exercise 2.10

$M = |v_j\rangle \langle v_k|$, and its each element is: $M_{ab} = \langle v_a| M |v_b\rangle = \delta_{aj}\delta_{bk}$. So only $M_{jk} = 1$, and other elements are all 0.

Exercise 2.11

X has eigenvalue 1 and eigenvector $1/\sqrt{2}(|0\rangle + |1\rangle)$, eigenvalue -1 and eigenvector $1/\sqrt{2}(|0\rangle - |1\rangle)$.
Y has eigenvalue 1 and eigenvector $1/\sqrt{2}(|0\rangle + i\,|1\rangle)$, eigenvalue -1 and eigenvector $1/\sqrt{2}(|0\rangle - i\,|1\rangle)$.
Z has eigenvalue 1 and eigenvector $|0\rangle$, eigenvalue -1 and eigenvector $|1\rangle$.
So the diagonal representations are:

$$X = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) - \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|). \tag{16}$$

$$Y = \frac{1}{2}(|0\rangle + i\,|1\rangle)(\langle 0| - i\,\langle 1|) - \frac{1}{2}(|0\rangle - i\,|1\rangle)(\langle 0| + i\,\langle 1|). \tag{17}$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|\,. \tag{18}$$

Exercise 2.12

$\det(A-\lambda I) = (1-\lambda)^2 = 0$ has only one eigenvalue $\lambda = 1$, but $\mathrm{rank}(I-A) < 2$. So it is not diagonalizable.

Exercise 2.13

For any two vectors $|a\rangle, |b\rangle$, we have:

$$\begin{aligned}
(|a\rangle, (|w\rangle\langle v|)^\dagger |b\rangle) &= ((|w\rangle\langle v|)|a\rangle, |b\rangle) && (19)\\
&= (\langle v|a\rangle |w\rangle, |b\rangle) && (20)\\
&= \langle v|a\rangle^* \langle w|b\rangle && (21)\\
&= \langle a|v\rangle \langle w|b\rangle && (22)\\
&= (|a\rangle, (|v\rangle\langle w|)|b\rangle). && (23)
\end{aligned}$$

So $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Exercise 2.14

Because

$$\begin{aligned}
\left(\left(\sum_i a_i A_i\right)^\dagger |v\rangle, |w\rangle\right) &= \left(|v\rangle, \sum_i a_i A_i |w\rangle\right) && (24)\\
&= \sum_i a_i \left(A_i^\dagger |v\rangle, |w\rangle\right) && (25)\\
&= \left(\sum_i a_i^* A_i^\dagger |v\rangle, |w\rangle\right), && (26)
\end{aligned}$$

So the equation holds, which means the adjoint operation is anti-linear.

Exercise 2.15

Because

$$\begin{aligned}
\left((A^\dagger)^\dagger |v\rangle, |w\rangle\right) &= (|v\rangle, A^\dagger |w\rangle) && (27)\\
&= (A^\dagger |w\rangle, |v\rangle)^* && (28)
\end{aligned}$$

$$= (|w\rangle, A|v\rangle)^* \tag{29}$$

$$= (A|v\rangle, |w\rangle), \tag{30}$$

so $(A^\dagger)^\dagger = A$.

Exercise 2.16

$$P^2 = \left(\sum_{i=1}^{k} |i\rangle\langle i|\right)\left(\sum_{i=1}^{k} |i\rangle\langle i|\right) \tag{31}$$

$$= \sum_{i,j=1}^{k} |i\rangle\langle i|j\rangle\langle j| = \sum_{i,j=1}^{k} \langle i|j\rangle |i\rangle\langle j| \tag{32}$$

$$= \sum_{i,j=1}^{k} \delta_{ij} |i\rangle\langle j| \tag{33}$$

$$= \sum_{i=1}^{k} |i\rangle\langle i| = P. \tag{34}$$

Exercise 2.17

We know that any normal matrix A can be diagonalized by an unitary matrix U, which means $A = U^\dagger D U$, where D is a diagonal matrix.
If A is Hermitian, then for any eigenvalue i and the corresponding eigenvector $|i\rangle$ for A, we have $A|i\rangle = i|i\rangle$, and $A^\dagger|i\rangle = i^*|i\rangle$. Additionally, $A = A^\dagger$, so $A|i\rangle = A^\dagger|i\rangle$, which means $i|i\rangle = i^*|i\rangle$. So $i = i^*$, which means all eigenvalues of A are real.
If all eigenvalues of A are real, which means $D = D^\dagger$, then $U^\dagger D U = (U^\dagger D U)^\dagger = U^\dagger D^\dagger U$, so $A = A^\dagger$, which means A is Hermitian.

Exercise 2.18

Because for any unitary matrix U, we have $U^\dagger U = I$. And for any eigenvalue i and the corresponding eigenvector $|i\rangle$, we have $U|i\rangle = i|i\rangle$. Additionally, $\langle i| U^\dagger U |i\rangle = \langle i| i^* i |i\rangle = |i|^2 \langle i|i\rangle$, while we also have $\langle i| U^\dagger U |i\rangle = \langle i|i\rangle$. So $|i| = 1$.

Exercise 2.20

For A', we have:

$$A'_{ij} = \langle v_i| A |v_j\rangle \tag{35}$$

$$= \sum_{k,t} \langle v_i|w_k\rangle \langle w_k| A |w_t\rangle \langle w_t|v_j\rangle \tag{36}$$

$$= \sum_{k,t} \langle v_i|w_k\rangle A''_{kl} \langle w_t|v_j\rangle \tag{37}$$

Exercise 2.21

If M is Hermitian, then $M = M^\dagger$, with $M = (P+Q)M(P+Q) = PMP + PMQ + QMP + QMQ$, where P is the projector onto the λ eigenspace, and Q is the projector onto the orthogonal complement space. So PMQ and QMP are both 0, and $M = PMP + QMQ$. We now prove QMQ is normal. Because we have:

$$QMQQM^\dagger Q = QM^\dagger QQMQ, \tag{38}$$

so QMQ is normal. Here we use $M = M^\dagger$ to simplify the proof. By the induction, QMQ is diagonal with respect to some orthonormal basis for Q, and PMP is already diagonal with respect to some orthonormal basis for P.

Exercise 2.22

If $H|a\rangle = a|a\rangle$, $H|b\rangle = b|b\rangle$, where a, b are two differnet eigenvalues, then we have:

$$\langle b| H |a\rangle = \langle b| a |a\rangle = a \langle b|a\rangle \tag{39}$$

$$= \langle a| H |b\rangle = \langle a| b |b\rangle = b \langle b|a\rangle . \tag{40}$$

Because $a \neq b$, so we must have $\langle b|a\rangle = 0$, which means $|a\rangle$ and $|a\rangle$ are orthogonal.

Exercise 2.23

Because for any projector P, we have $P^2 = P$. Then for any eigenvalue i and the corresponding eigenvector $|i\rangle$, we have $P^2|i\rangle = P(P|i\rangle) = P(i|i\rangle) = i(P|i\rangle) = i^2|i\rangle$, so $i^2|i\rangle = i|i\rangle$. So $i^2 = i$, which means $i = 0$ or $i = 1$.

Exercise 2.24

For any positive operator, we can write it as:

$$A = \frac{1}{2}(A + A^\dagger) + i\frac{1}{2i}(A - A^\dagger) \tag{41}$$

$$= B + iC. \tag{42}$$

It's evident that $B = 1/2(A + A^\dagger)$ and $C = 1/2i(A - A^\dagger)$ are both Hermitian. Then for any vector $|v\rangle$, we have:

$$\langle v| A |v\rangle = \langle v| (B + iC) |v\rangle = \langle v| B |v\rangle + i \langle v| C |v\rangle . \tag{43}$$

Because B and C are both Hermitian, so $\langle v| B |v\rangle$ and $\langle v| C |v\rangle$ are both real number. And A is a positive operator, so we should have $\langle v| A |v\rangle$ be real, so $\langle v| C |v\rangle = 0$. Therefore we have $A = A^\dagger$, which means A is Hermitian.

Exercise 2.25

For any operator A and vector $|v\rangle$, we have:

$$\langle v| A^\dagger A |v\rangle = |A |v\rangle |^2 \geq 0. \tag{44}$$

So $A^\dagger A$ is positive.

Exercise 2.28

$$(A \otimes B)^* = \begin{pmatrix} A_{11}^* B^* & A_{12}^* B^* & \cdots & A_{1n}^* B^* \\ A_{21}^* B^* & A_{22}^* B^* & \cdots & A_{2n}^* B^* \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}^* B^* & A_{n2}^* B^* & \cdots & A_{nn}^* B^* \end{pmatrix} = A^* \otimes B^* \tag{45}$$

$$(A \otimes B)^T = \begin{pmatrix} A_{11} B^T & A_{21} B^T & \cdots & A_{n1} B^T \\ A_{12} B^T & A_{22} B^T & \cdots & A_{n2} B^T \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} B^T & A_{2n} B^T & \cdots & A_{nn} B^T \end{pmatrix} = A^* \otimes B^* \tag{46}$$

$$(A \otimes B)^\dagger = \begin{pmatrix} A_{11}^* B^\dagger & A_{21}^* B^\dagger & \cdots & A_{n1}^* B^\dagger \\ A_{12}^* B^\dagger & A_{22}^* B^\dagger & \cdots & A_{n2}^* B^\dagger \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n}^* B^\dagger & A_{2n}^* B^\dagger & \cdots & A_{nn}^* B^\dagger \end{pmatrix} = A^\dagger \otimes B^\dagger \tag{47}$$

Exercise 2.29

For any two unitary operator U_1 and U_2, we have:

$$\begin{aligned}(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) &= (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) && (48)\\ &= (U_1^\dagger U_1) \otimes (U_2^\dagger U_2) = I \otimes I = I. && (49)\end{aligned}$$

Exercise 2.30

For any two Hermitian operator H_1 and H_2, we have:

$$\begin{aligned}(H_1 \otimes H_2)^\dagger (H_1 \otimes H_2) &= (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) && (50)\\ &= (U_1^\dagger U_1) \otimes (U_2^\dagger U_2) = I \otimes I = I. && (51)\end{aligned}$$

Exercise 2.31

For any two positive operator A_1 and A_2, we have:

$$\langle u| \otimes \langle v| \left(A_1 \otimes A_2\right) |v\rangle \otimes |u\rangle = \langle u| A |u\rangle \langle v| B |v\rangle \geq 0. \tag{52}$$

Exercise 2.32

For any two projectors P_1 and P_2, we have:

$$(P_1 \otimes P_2)^2 = P_1^2 \otimes P_2^2 = P_1 \otimes P_2. \tag{53}$$

Exercise 2.33

Because we can write the Hadmard operator of one qubit as:

$$\begin{aligned}H &= \frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|] && (54)\\ &= \frac{1}{\sqrt{2}}[|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|] && (55)\\ &= \frac{1}{\sqrt{2}} \sum_{x,y\in\{0,1\}} (-1)^{x\cdot y}|x\rangle\langle y| && (56)\end{aligned}$$

So for n qubits, we have:

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x_1, y_1 \in \{0,1\}} (-1)^{x_1 \cdot y_1} |x_1\rangle \langle y_1| \otimes \sum_{x_2, y_2 \in \{0,1\}} (-1)^{x_2 \cdot y_2} |x_2\rangle \langle y_2| \otimes \cdots \tag{57}$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x, y \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle\langle y|. \tag{58}$$

And we can calculate $H^{\otimes 2}$:

$$H^{\otimes 2} = \frac{1}{2}[|00\rangle\langle 00| + |01\rangle\langle 00| + |00\rangle\langle 01| - |01\rangle\langle 01| \tag{59}$$

$$+ |10\rangle\langle 00| + |11\rangle\langle 00| + |10\rangle\langle 01| - |11\rangle\langle 01| \tag{60}$$

$$+ |00\rangle\langle 10| + |01\rangle\langle 10| + |00\rangle\langle 11| - |01\rangle\langle 11| \tag{61}$$

$$- |10\rangle\langle 10| - |11\rangle\langle 10| - |10\rangle\langle 11| + |11\rangle\langle 11|] \tag{62}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{63}$$

Exercise 2.34

Let

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}. \tag{64}$$

Then $\det(A - \lambda I) = (4 - \lambda)^2 - 9$. Therefore A has two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 7$. The eigenvectors are $|\alpha\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$ and $|\beta\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ respectively.
Therefore, we can rewrite A as:

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = |\alpha\rangle \langle\alpha| + 7 |\beta\rangle \langle\beta|. \tag{65}$$

So its root is:

$$\sqrt{A} = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = |\alpha\rangle \langle\alpha| + \sqrt{7} |\beta\rangle \langle\beta| \tag{66}$$

$$= \frac{1}{2} \begin{pmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{pmatrix}. \tag{67}$$

Its logarithm is:

$$\sqrt{A} = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = \log 1 \left|\alpha\right\rangle \left\langle\alpha\right| + \log 7 \left|\beta\right\rangle \left\langle\beta\right| \tag{68}$$

$$= \frac{\log 7}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{69}$$

Exercise 2.35

Let $\vec{v} = (v_1, v_2, v_3)$, then we have:

$$\vec{v} \cdot \vec{\sigma} = \sum_{i=1}^{3} v_i \cdot \sigma_i \tag{70}$$

$$= \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}. \tag{71}$$

Additionally,

$$\det(\vec{v} \cdot \vec{\sigma} - \lambda I) = \lambda^2 - (v_1^2 + v_2^2 + v_3^2) = \lambda^2 - 1. \tag{72}$$

Therefore, the eigenvalues are $\lambda_1 = 1$, and $\lambda_2 = -1$. Assume the eigenvectors are $\left|\lambda_1\right\rangle$ and $\left|\lambda_2\right\rangle$. So we can write $\vec{v} \cdot \vec{\sigma}$ as:

$$\vec{v} \cdot \vec{\sigma} = \left|\lambda_1\right\rangle \left\langle\lambda_1\right| - \left|\lambda_2\right\rangle \left\langle\lambda_2\right|. \tag{73}$$

Therefore,

$$\exp(i\theta \vec{v} \cdot \vec{\sigma}) = \exp(i\theta) \left|\lambda_1\right\rangle \left\langle\lambda_1\right| + \exp(-i\theta) \left|\lambda_2\right\rangle \left\langle\lambda_2\right|. \tag{74}$$

Exercise 2.37

$$\mathrm{tr}(AB) = \sum_i (AB)_{ii} = \sum_{i,j} A_{ij} B_{ji} = \sum_{ji} B_{ji} A_{ij} = \sum_j (AB)_{jj} = \mathrm{tr}(BA). \tag{75}$$

Exercise 2.38

$$\mathrm{tr}(A+B) = \sum_i (A_{ii} + B_{ii}) = \sum_i A_i i + \sum_i B_i i = \mathrm{tr}(A) + \mathrm{tr}(B). \tag{76}$$

$$\mathrm{tr}(zA) = \sum_i zA_{ii} = z\sum_i A_{ii} = z\mathrm{tr}(A). \tag{77}$$

Exercise 2.39

(1) We now prove this definition satisfies the 3 rules of inner product.

$$(A, A) = \mathrm{tr}(A^\dagger A) = \sum_{ij} |A_{ij}|^2 \geq 0. \tag{78}$$

The equation holds only when $A = 0$.

$$(A, B)^* = (\mathrm{tr}(A^\dagger B))^* = \mathrm{tr}((A^\dagger B)^\dagger) = \mathrm{tr}(B^\dagger A) = (B, A). \tag{79}$$

$$(A, \sum_i \lambda_i B_i) = \mathrm{tr}\left(A^\dagger \left(\sum_i \lambda_i B_i\right)\right) = \mathrm{tr}\left(\sum_i \lambda_i A^\dagger B_i\right) \tag{80}$$

$$= \sum_i \lambda_i \mathrm{tr}\left(A^\dagger B_i\right) = \sum_i \lambda_i (A, B_i). \tag{81}$$

(2) Any linear transformation from V to V can be represented as a $d \times d$ matrix, where V is a $d-$dimensional space.

Because the number of independent $d \times d$ matrix is d^2, so L_v has dimension d^2.

(3) Let the orthonormal basis of V be $|v_1\rangle, |v_2\rangle, \cdots, |v_d\rangle$. Then define 3 sets of Hermitian matrices: $A_{ij} = 1/2(|v_i\rangle\langle v_j| + |v_j\rangle\langle v_i|)$, where $1 \leq i < j \leq d$, and $B_{ij} = 1/2(|v_i\rangle\langle v_j| - |v_j\rangle\langle v_i|)$, where $1 \leq i < j \leq d$, and $C_i = |v_i\rangle\langle v_i|$, where $1 \leq i \leq d$. The set $\{A_{ij}, B_{ij}, C_i\}$ is an orthonormal basis for L_v.

Exercise 2.43

$$\sigma_j \sigma_k = \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2}. \tag{82}$$

When $j \neq k$, $\{\sigma_j, \sigma_k\} = 0$, and when $j = k$. $\sigma_j\sigma_k = I$. So $\{\sigma_j, \sigma_k\} = 2\delta_{jk}I$. Additionally, $[\sigma_j, \sigma_k] = 2i\sum_{l=1}^{3} \epsilon_{jkl}\sigma_l$.
Therefore, using equation 82, we have:

$$\sigma_j\sigma_k = \delta_{jk}I + i\sum_{l=1}^{3} \epsilon_{jkl}\sigma_l. \tag{83}$$

Exercise 2.44

If $[A, B] = 0$ and $\{A, B\} = 0$, then $AB + BA = AB - BA = 0$. So $BA = 0$.
Because A is invertible, then $BAA^{-1} = BI = 0$. So B must be 0.

Exercise 2.45

$$[A, B]^\dagger = (AB - BA)^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = \left[B^\dagger, A^\dagger\right] \tag{84}$$

Exercise 2.47

Because A, B are Hermitian, so $A = A^\dagger$ and $B = B^\dagger$. Use the conclusion of 2.45 and 2.46, we have:

$$(i[A, B])^\dagger = -i[B^\dagger, A^\dagger] = -i[B, A] = i[A, B]. \tag{85}$$

So $i[A, B]$ is Hermitian.

Exercise 2.48

(1) For a positive matrix P, we have $P = \sum_i \lambda_i |i\rangle\langle i|$, where $\lambda_i \geq 0$.

So $J = \sqrt{P^\dagger P} = \sum_i \sqrt{\lambda_i^2} |i\rangle\langle i| = P$. Therefore the polar decomposition is $P = IP$.

(2) For a unitary matrix U, we have $U^\dagger U = I$, so $J = \sqrt{U^\dagger U} = I$. Therefore the polar decomposition is $U = UI$.

(3) For a Hermitian matrix H, we have $H = \sum_i \lambda_i |i\rangle\langle i|$, where λ_i are all real.

So $J = \sqrt{H^\dagger H} = \sum_i \sqrt{\lambda_i^2} |i\rangle\langle i| = \sum_i |\lambda_i| |i\rangle\langle i|$. Therefore the polar decomposition is $H = U\sum_i |\lambda_i| |i\rangle\langle i|$, where $U = \sum_i |e_i\rangle\langle i|$.

Exercise 2.49

For a normal matrix A, we have $A = \sum_i \lambda_i |i\rangle \langle i|$. So $J = \sqrt{A^\dagger A} = \sum_i \sqrt{\lambda_i^* \lambda} |i\rangle \langle i| = \sum_i |\lambda_i| |i\rangle \langle i|$. Therefore the polar decomposition is $A = U \sum_i |\lambda_i| |i\rangle \langle i|$, where $U = \sum_i |e_i\rangle \langle i|$.

Exercise 2.50

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A^\dagger A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{86}$$

We have $\det(A^\dagger A - \lambda I) = (2 - \lambda)(1 - \lambda) - 1 = \lambda^2 - 3\lambda + 1$.
So the eigenvalues are $\lambda_1 = (3 + \sqrt{5})/2$ and $\lambda_2 = (3 - \sqrt{5})/2$, with the eigenvectors:

$$|v_1\rangle = \frac{1}{\sqrt{10 - 2\sqrt{5}}} \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix}, |v_2\rangle = \frac{1}{\sqrt{10 + 2\sqrt{5}}} \begin{pmatrix} 2 \\ -1 - \sqrt{5} \end{pmatrix}. \tag{87}$$

So $J = \sqrt{A^\dagger A} = \sqrt{\lambda_1} |v_1\rangle \langle v_1| + \sqrt{\lambda_2} |v_2\rangle \langle v_2|$, and $U = AJ^{-1}$.

Exercise 2.53

Because $\det(H - \lambda I) = (1/\sqrt{2} - \lambda)(-1/\sqrt{2} - \lambda) - 1/2 = \lambda^2 - 1$, so H has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, with the eigenvectors:

$$|v_1\rangle = \frac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}, |v_2\rangle = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \begin{pmatrix} 1 \\ -\sqrt{2} - 1 \end{pmatrix}. \tag{88}$$

Exercise 2.54

Because A and B are commuting, then A and B have the same eigenvectors. So A and B can be diagonalized as:

$$A = \sum_i a_i |i\rangle \langle i|, B = \sum_i b_i |i\rangle \langle i|. \tag{89}$$

Then we have:

$$\exp(A)\exp(B) = \left(\sum_i \exp(a_i) |i\rangle \langle i| \right) \left(\sum_j \exp(b_j) |j\rangle \langle j| \right) \tag{90}$$

$$= \sum_{ij} \exp(a_i)\exp(b_j) |i\rangle \langle i|j\rangle \langle j| \tag{91}$$

$$= \sum_{ij} \delta_{ij} \exp(a_i) \exp(b_j) |i\rangle \langle j| \tag{92}$$

$$= \sum_{i} \exp(a_i + b_i) |i\rangle \langle i| \tag{93}$$

$$= \exp(A + B). \tag{94}$$

Exercise 2.55

$$U(t_1, t_2) = \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right], U^\dagger(t_1, t_2) = \exp\left[\frac{iH(t_2 - t_1)}{\hbar}\right]. \tag{95}$$

Then we have:

$$U(t_1, t_2)U^\dagger(t_1, t_2) = \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right] \exp\left[\frac{iH(t_2 - t_1)}{\hbar}\right] \tag{96}$$

$$= \sum_E \exp\left[\frac{-iE(t_2 - t_1)}{\hbar}\right] |E\rangle \langle E| \sum_{E'} \exp\left[\frac{iE'(t_2 - t_1)}{\hbar}\right] |E'\rangle \langle E'| \tag{97}$$

$$= \sum_{E,E'} \exp\left[\frac{i(E' - E)(t_2 - t_1)}{\hbar}\right] |E\rangle \langle E|E'\rangle \langle E'| \tag{98}$$

$$= \sum_{E,E'} \delta_{E,E'} \exp\left[\frac{i(E' - E)(t_2 - t_1)}{\hbar}\right] |E\rangle \langle E'| \tag{99}$$

$$= \sum_{E'} \exp(0) |E\rangle \langle E| \tag{100}$$

$$= I. \tag{101}$$

Exercise 2.56

For a unitary operator U, we have $U = \sum_k \lambda_k |v_k\rangle \langle v_k|$, where each $|\lambda_k| = 1$.
So we can also rewrite as $U = \sum_i e^{i\theta_k} |v_k\rangle \langle v_k|$, where each θ_k is real.
Additionally, $K = -i \log(U)$, so we have:

$$K = -i \sum_k \log\left(e^{i\theta_k}\right) |v_k\rangle \langle v_k| \tag{102}$$

$$= -i \sum_k i\theta_k |v_k\rangle \langle v_k| \tag{103}$$

$$= \sum_k \theta_k |v_k\rangle \langle v_k| . \tag{104}$$

Because each θ_k is real, then K is Hermitian.

Exercise 2.57

We first use L_l to measure $|\psi\rangle$ and get the result $|\psi_1\rangle$:

$$|\psi_1\rangle = \frac{L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}}. \tag{105}$$

Then we use M_m to measure $|\psi_1\rangle$ and get the result $|\psi_2\rangle$:

$$\begin{aligned}
|\psi_2\rangle &= \frac{M_m |\psi_1\rangle}{\sqrt{\langle\psi_1| M_m^\dagger M_m |\psi_1\rangle}} && (106)\\
&= M_m \frac{L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} \frac{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}}{\sqrt{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}} && (107)\\
&= \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}}. && (108)
\end{aligned}$$

The result equals to using $(M_m L_l)$ to measure $|\psi\rangle$ directly.

Exercise 2.58

The average value is:

$$E(M) = \langle\psi| M |\psi\rangle = \langle\psi| m |\psi\rangle = m. \tag{109}$$

This is because $|\psi\rangle$ is the eigenvector of eigenvalue m of M.
So the standard deviation is:

$$\begin{aligned}
\sqrt{[\Delta(M)]^2} &= \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{\langle\psi| M^2 |\psi\rangle - m^2} && (110)\\
&= \sqrt{\langle\psi| M(m |\psi\rangle) - m^2} = \sqrt{m^2 \langle\psi|\psi\rangle - m^2} = 0. && (111)
\end{aligned}$$

Exercise 2.59

The average value is:

$$E(X) = \langle 0| X |0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \tag{112}$$

The standard deviation is:

$$\sqrt{[\Delta(X)]^2} = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\langle 0| X^2 |0\rangle} \tag{113}$$

$$= \sqrt{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = 1. \tag{114}$$

Exercise 2.60

$$\vec{v} \cdot \vec{\sigma} = \sum_{k=1}^{3} v_k \cdot \sigma_k = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \tag{115}$$

Additionally,

$$\det(\vec{v} \cdot \vec{\sigma} - \lambda I) = \lambda^2 - (v_1^2 + v_2^2 + v_3^2) = \lambda^2 - 1. \tag{116}$$

Therefore, the eigenvalues are $\lambda_1 = 1$, and $\lambda_2 = -1$.
For $\lambda_1 = 1$and $\lambda_2 = -1$, the eigenvectors satisfies

$$\begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} |\psi_1\rangle = |\psi_1\rangle , \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} |\psi_2\rangle = - |\psi_2\rangle . \tag{117}$$

So the eignvectors are:

$$|\psi_1\rangle = \frac{1}{\sqrt{2 - 2v_3}} \begin{pmatrix} v_1 - iv_2 \\ 1 - v_3 \end{pmatrix} , |\psi_2\rangle = \frac{1}{\sqrt{2 + 2v_3}} \begin{pmatrix} v_1 - iv_2 \\ -1 - v_3 \end{pmatrix} . \tag{118}$$

Then the projectors are:

$$P_1 = |\psi_1\rangle \langle\psi_2| = \frac{1}{2 - 2v_3} \begin{pmatrix} v_1^2 + v_2^2 & (v_1 - iv_2)(1 - v_3) \\ (v_1 + iv_2)(1 - v_3) & (1 - v_3)^2 \end{pmatrix} \tag{119}$$

$$= \frac{1}{2} \begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} = \frac{1}{2}(I + \vec{v} \cdot \vec{\sigma}), \tag{120}$$

$$P_2 = |\psi_2\rangle \langle\psi_2| = \frac{1}{2 + 2v_3} \begin{pmatrix} v_1^2 + v_2^2 & (iv_2 - v_1)(1 + v_3) \\ -(v_1 + iv_2)(1 + v_3) & (1 + v_3)^2 \end{pmatrix} \tag{121}$$

$$= \frac{1}{2} \begin{pmatrix} 1 - v_3 & -v_1 + iv_2 \\ -v_1 - iv_2 & 1 + v_3 \end{pmatrix} = \frac{1}{2}(I - \vec{v} \cdot \vec{\sigma}). \tag{122}$$

Exercise 2.61

The probability of getting +1 is:

$$p(+1) = |0\rangle P_1 \langle 0| = \frac{1+v_3}{2}. \tag{123}$$

The state after the measurement is:

$$|\phi\rangle = \frac{P_1 |0\rangle}{\sqrt{p(+1)}} = \frac{1}{2}\sqrt{\frac{2}{1+v_3}} \begin{pmatrix} 1+v_3 \\ v_1 + iv_2 \end{pmatrix} \tag{124}$$

$$= \frac{1}{\sqrt{2+2v_3}} \frac{1+v_3}{v_1 - iv_2} \begin{pmatrix} v_1 - iv_2 \\ \frac{v_1^2+v_2^2}{1+v_3} \end{pmatrix} \tag{125}$$

$$= \frac{1}{\sqrt{2+2v_3}} \frac{1+v_3}{v_1 - iv_2} \begin{pmatrix} v_1 - iv_2 \\ 1 - v_3 \end{pmatrix} = |\psi_1\rangle . \tag{126}$$

Exercise 2.62

If M_m is the measurement operator, then its POVM measurement operator is $E_m = M_m^\dagger M_m$. And if they coincide, then $M_m = M_m^\dagger M_m$. So for any state $|\psi\rangle$:

$$\langle\psi| M_m |\psi\rangle = \langle\psi| M_m^\dagger M_m |\psi\rangle \geq 0. \tag{127}$$

So M_m is positive, which means M_m is Hermitian. Then $M_m^2 = M_m^\dagger M_m = M_m$, so M_m is a projector.

Exercise 2.63

Because M_m has a polar decomposition $M_m = U_m J_m$, where U_m is unitary and J_m is Hermitian.
Then $M_m^\dagger M_m = J_m^\dagger U_m^\dagger U_m J_m = J_m^\dagger J_m = J_m^2$. So $J_m = \sqrt{E_m}$, where $E_m = M_m^\dagger M_m$ is the POVM associated to M_m.

Exercise 2.64

We first construct a set of orthonormal basis from $|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_m\rangle$. Define $|\phi_j\rangle$ as:

$$|\phi_j\rangle = \frac{|\psi_j\rangle - \sum_{i=1}^{j-1} \langle\phi_i|\psi_j\rangle |\phi_i\rangle}{|| \, |\psi_j\rangle - \sum_{i=1}^{j-1} \langle\phi_i|\psi_j\rangle |\phi_i\rangle \, ||}. \tag{128}$$

We know that each $|\phi_j\rangle$ is orthogonal to all $|\psi_i\rangle, i \neq j$. Then we define E_j as:

$$E_j = |\phi_j\rangle \langle\phi_j|, 1 \leq i \leq m, \tag{129}$$

and define $E_{m+1} = I - \sum_{i=1}^m E_i$.
Here it's evident that each E_j is positive. Additionally $\langle\psi_i| E_i |\psi_i\rangle = |\langle\psi_i|\phi_i\rangle|^2 > 0$ because $|\psi_i\rangle$ and $|\phi_i\rangle$ are not orthogonal.
And if outcome E_i occurs, then it means the state $|\psi_k\rangle$ given to Bob satisfies $\langle\psi_k| E_i |\psi_k\rangle > 0$, so $\langle\psi_k| E_i |\psi_k\rangle = |\langle\psi_k|\phi_i\rangle|^2 > 0$. So it must be $k = i$.

Exercise 2.65

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{130}$$

Exercise 2.66

$$\langle X_1 Z_2\rangle = \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) \tag{131}$$

$$= \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) \left(\frac{((X|0\rangle) \otimes (Z|0\rangle) + (X|1\rangle) \otimes (Z|1\rangle))}{\sqrt{2}}\right) \tag{132}$$

$$= \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) \left(\frac{|10\rangle - |01\rangle}{\sqrt{2}}\right) = 0. \tag{133}$$

Exercise 2.67

Let $\overline{W}$ be the orthogonal complement of W in V. Let $|w_1\rangle, |w_2\rangle, \cdots, |w_n\rangle$ be the orthonormal basis of W, and $|w_1'\rangle, |w_2'\rangle, \cdots, |w_m'\rangle$ be the orthonormal basis of $\overline{W}$. Let $I(U)$ be the set of images of operator U. Then let $|u_1\rangle, |u_2\rangle, \cdots, |u_m\rangle$ be the orthonormal basis of the orthogonal complement of $I(U)$. (Because U preserves inner product, so the dimension of $I(U)$ should equal to the dimension of W). Then we have $\langle u_k| U |w_t\rangle = 0$ for all k, t. We also have $\langle w_k'|w_t\rangle = 0$ for all k, t.
Therefore, $\{|w_i\rangle\} \cup \{|w_i'\rangle\}$ is a set of orthogonal basis of W, while $\{U|w_i\rangle\} \cup \{|u_i\rangle\}$ is another set of orthogonal basis of W.
Then define U' as:

$$U' = \sum_{i=1}^n U|w_i\rangle \langle w_i| + \sum_{j=1}^m |u_j\rangle \langle w_j'|. \tag{134}$$

Then for each $|w_k\rangle$, we have:

$$U' |w_k\rangle = \left(\sum_{i=1}^{n} U |w_i\rangle \langle w_i| + \sum_{j=1}^{m} |u_j\rangle \langle w'_j| \right) |w_k\rangle = U |w_k\rangle \langle w_k|w_k\rangle = U |w_k\rangle . \tag{135}$$

Additionally, because U preserves inner product, so for any k, t, we have $\langle w_k| U^\dagger U |w_t\rangle = \langle w_k|w_t\rangle$. So we have:

$$(U')^\dagger U' = \left(\sum_{i=1}^{n} |w_i\rangle \langle w_i| U^\dagger + \sum_{j=1}^{m} |w'_j\rangle \langle u_j| \right) \left(\sum_{i=1}^{n} U |w_i\rangle \langle w_i| + \sum_{j=1}^{m} |u_j\rangle \langle w'_j| \right) \tag{136}$$

$$= \sum_{i=1}^{n} |w_i\rangle \langle w_i| + \sum_{j=1}^{m} |w'_j\rangle \langle w'_j| = I. \tag{137}$$

$$U'(U')^\dagger = \left(\sum_{i=1}^{n} U |w_i\rangle \langle w_i| + \sum_{j=1}^{m} |u_j\rangle \langle w'_j| \right) \left(\sum_{i=1}^{n} |w_i\rangle \langle w_i| U^\dagger + \sum_{j=1}^{m} |w'_j\rangle \langle u_j| \right) \tag{138}$$

$$= \sum_{i=1}^{n} U |w_i\rangle \langle w_i| U^\dagger + \sum_{j=1}^{m} |u_j\rangle \langle u_j| = I. \tag{139}$$

Therefore, U' is a unitary operator which extends U.

Exercise 2.68

If $|\psi\rangle = |a\rangle |b\rangle$, suppose $|a\rangle = a_1 |0\rangle + a_2 |1\rangle$, and $|b\rangle = b_1 |0\rangle + b_2 |1\rangle$. Then we have:

$$|a\rangle |b\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{140}$$

$$\Rightarrow a_1 b_1 |00\rangle + a_1 b_2 |01\rangle + a_2 b_1 |10\rangle + a_2 b_2 |11\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{141}$$

Therefore, we have $a_1 b_1 = a_2 b_2 = 1/\sqrt{2}$, which means $a_1, a_2, b_1, b_2 \neq 0$. Then $a_1 b_2 \neq 0, a_2 b_1 \neq 0$. This leads to contradiction.

Exercise 2.69

Define the 4 bell states as:

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{142}$$

$$|\psi_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \tag{143}$$

$$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \tag{144}$$

$$|\psi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{145}$$

It's easy to verify that each $|\psi_i\rangle$ has module equals to 1, and $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. Additionally we need to prove they are linear independent. If there exists a_1, a_2, a_3, a_4 such that

$$a_1 |\psi_1\rangle + a_2 |\psi_2\rangle + a_3 |\psi_3\rangle + a_4 |\psi_4\rangle = 0, \tag{146}$$

then there must be

$$\begin{cases} a_1 + a_2 = 0 \\ a_3 + a_4 = 0 \\ a_1 - a_2 = 0 \\ a_3 - a_4 = 0. \end{cases} \tag{147}$$

Then $a_1 = a_2 = a_3 = a_4 = 0$. So the states are linear independent. Therefore they form a set of orthonormal basis.

Exercise 2.70

For any two qubits $|ab\rangle \in \{0,1\}^2$, we have:

$$\langle ab| E \otimes I |ab\rangle = \langle ab| (E|a\rangle \otimes I|b\rangle) = \langle a| E |a\rangle . \tag{148}$$

So for any $|\psi\rangle$ of the four bell state, we have:

$$\langle\psi| E \otimes I |\psi\rangle = \frac{\langle 0| E |0\rangle + \langle 1| E |1\rangle}{2}. \tag{149}$$

If Alice and Bob share a state $|\psi\rangle$, and Eve gets the Alice's qubit and measures it using M_m. Then Eve gets a result $\langle\psi| (M_m^\dagger M_m) \otimes I |\psi\rangle$. Because $M_m^\dagger M_m$ is positive, then the results equals on all $|\psi\rangle$. So Eve cannot distinguish the bit string that Alice wants to send.

Exercise 2.71

Because ρ is a density operator, then $\rho = \sum_i p_i |i\rangle \langle i|$. Then:

$$\begin{aligned}
\rho^2 &= (\sum_i p_i |i\rangle \langle i|)(\sum_i p_i |j\rangle \langle j|) &(150)\\
&= \sum_{ij} p_i p_j |i\rangle \langle i|j\rangle \langle j| &(151)\\
&= \sum_{ij} p_i p_j \delta_{ij} |i\rangle \langle j| &(152)\\
&= \sum_i p_i^2 |i\rangle \langle i|, &(153)
\end{aligned}$$

and the trace is:

$$\begin{aligned}
\mathrm{tr}(\rho^2) &= \mathrm{tr}(\sum_i p_i^2 |i\rangle \langle i|) &(154)\\
&= \sum_i p_i^2 \mathrm{tr}(|i\rangle \langle i|) &(155)\\
&= \sum_i p_i^2 \langle i|i\rangle &(156)\\
&= \sum_i p_i^2. &(157)
\end{aligned}$$

Because each p_i indicates the probability that $|i\rangle$ occurs, so $0 < p_i \leq 1$, and $\sum_i p_i = 1$. So $0 < p_i^2 \leq 1$ and $\sum_i p_i^2 \leq 1$, and the equation holds only when there exists only one p_i and $p_i = 1$, which means the state is a pure state.

Exercise 2.72

(1) Because the Puli matrices $I, \sigma_x, \sigma_y, \sigma_z$ form a basis for 2-dimensional Hilbert space, then any density operator can be represented using this basis. We know that any density operator is Hermitian with the form

$$\rho = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}, \tag{158}$$

where $a + d = 1$.

Then let $r_3 = 2a - 1, r_1 = 2\mathrm{Re}(b), r_2 = -2\mathrm{Im}(b)$, we have:

$$\frac{I + \vec{r} \cdot \vec{\sigma}}{2} = \mathrm{Re}(b) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathrm{Im}(b) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + (a - \frac{1}{2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{I}{2} \tag{159}$$

$$= \begin{pmatrix} a & \mathrm{Re}(b) + \mathrm{Im}(b)i \\ \mathrm{Re}(b) - \mathrm{Im}(b)i & 1-a \end{pmatrix} \tag{160}$$

$$= \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix} \tag{161}$$

$$= \rho. \tag{162}$$

Now we prove $\|\vec{r}\| \leq 1$. Because ρ is positive, then all its eigenvalues are no less than 0.

$$\det(\rho - \lambda I) = (a-\lambda)(d-\lambda) - \|b\|^2 = 0 \tag{163}$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + ad - \|b\|^2 = 0 \tag{164}$$

$$\Rightarrow \lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - \|b\|^2)}}{2} \tag{165}$$

$$= \frac{1 \pm \sqrt{1 - 4(\frac{1-r_3^2}{4} - \frac{r_1^2+r_2^2}{4})}}{2} \tag{166}$$

$$= \frac{1 \pm \sqrt{1 - 1 + r_1^2 + r_2^2 + r_3^2}}{2} \tag{167}$$

$$= \frac{1 \pm \|\vec{r}\|}{2}. \tag{168}$$

Because $\lambda_{1,2} \geq 0$, then $\|\vec{r}\| \leq 1$.

(2) $\vec{r} = 0$, ρ is at the origin of the Bloch sphere, representing the maximally mixed state.

(3) If ρ is pure, then $\mathrm{tr}(\rho^2) = \lambda_1^2 + \lambda_2^2 = 1$. Then

$$\frac{1 + \|\vec{r}\|^2 + 2\|\vec{r}\|}{4} + \frac{1 + \|\vec{r}\|^2 - 2\|\vec{r}\|}{4} = 1 \tag{169}$$

$$\Rightarrow \|\vec{r}\| = 1. \tag{170}$$

If $\|\vec{r}\| = 1$, then $\mathrm{tr}(\rho^2) = \lambda_1^2 + \lambda_2^2 = 1$, so ρ is pure.

(4) If ρ is pure, suppose $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then $\mathrm{tr}(\rho) = \alpha^2 + \beta^2 = 1$. So the state can be written as:

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right). \tag{171}$$

Exercise 2.73

By the spectral theorem, Let $\rho = \sum_{k=1}^{d} \lambda_k |k\rangle \langle k|$, where each $\lambda > 0$, so $d = \text{rank}(\rho)$. Then $|1\rangle, |2\rangle, \cdots, |d\rangle$ is a set of minimal ensemble of ρ.
If $|\psi_i\rangle$ is in the support of ρ, then $|\psi_i\rangle = \sum_{k=1}^{d} a_{ik} |k\rangle$, and $\sum_{k=1}^{d} \|a_{ik}\|^2 = 1$. The probability that $|\psi_i\rangle$ occurs is:

$$p_i = \frac{1}{\sum_k \frac{\|a_{ik}\|^2}{\lambda_k}}. \tag{172}$$

Define a unitary operator u:

$$u_{ik} = \sqrt{\frac{p_i}{\lambda_k}} a_{ik}. \tag{173}$$

It's evident that $\sum_k u_{ik}^2 = 1$ for each i. Now we define a new set of ensemble:

$$\sqrt{p_i} |\psi_i\rangle = \sum_{k=1} u_{ik} \sqrt{\lambda_k} |k\rangle . \tag{174}$$

Using theorem 2.6, we have:

$$\sum_k p_k |\psi_k\rangle \langle \psi_k| = \sum_k \sqrt{p_k} |\psi_k\rangle \langle \psi_k| \sqrt{p_k} = \sum_k \sqrt{\lambda_k} |k\rangle U^T U^* \langle k| \sqrt{\lambda_k} \tag{175}$$

$$= \sum_k \lambda_k |k\rangle \langle k| = \rho. \tag{176}$$

So we construct a new set of minimal ensemble of ρ including $|\psi\rangle$.
Besides $\rho^{-1} = \sum_k 1/\lambda_i |k\rangle \langle k|$, then:

$$\langle \psi_i| \rho^{-1} |\psi_k\rangle = \sum_k \frac{\|a_{ik}\|^2}{\lambda_k} = \frac{1}{p_i}. \tag{177}$$

Exercise 2.74

The density matrix of system AB is $\rho_{AB} = (|a\rangle |b\rangle)(\langle a| \langle b|) = |a\rangle \langle a| \otimes |b\rangle \langle b|$.
Then we have:

$$\rho_A = \text{Tr}_B \rho_{AB} = |a\rangle \langle a| \text{Tr}(|b\rangle \langle b|) = |a\rangle \langle a| . \tag{178}$$

It's evident ρ_A is pure.

Exercise 2.75

(1) For $(|00\rangle + |11\rangle)/\sqrt{2}$:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \rho_1 = \rho_2 = \frac{I}{2}. \tag{179}$$

(2) For $(|00\rangle - |11\rangle)/\sqrt{2}$:

$$|\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \rho_1 = \rho_2 = \frac{I}{2}. \tag{180}$$

(3) For $(|01\rangle + |10\rangle)/\sqrt{2}$:

$$|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \rho_1 = \rho_2 = \frac{I}{2}. \tag{181}$$

(4) For $(|01\rangle - |10\rangle)/\sqrt{2}$:

$$|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \rho_1 = \rho_2 = \frac{I}{2}. \tag{182}$$

Exercise 2.76

Let H_1 and H_2 be the two Hilbert spaces with dimension m and n. Without loss of generality let $m \geq n$. Then for any $|\psi\rangle \in H_1 \otimes H_2$, we have:

$$|\psi\rangle = \sum_{1 \leq j \leq m, 1 \leq k \leq n} a_{jk} |j\rangle |k\rangle, \tag{183}$$

where a is a $m \times n$ matrix.
Then by the singular value decomposition, we can find a $m \times m$ unitary matrix U, a $n \times n$

unitary matrix V and a $m \times n$ matrix D such that

$$a = UDV, \tag{184}$$

and D can be written as:

$$D = \begin{pmatrix} D' \\ 0 \end{pmatrix} \tag{185}$$

where D' is a $n \times n$ diagonal matrix. Then we can rewrite $|\psi\rangle$ as:

$$|\psi\rangle = \sum_{1\le j\le m, 1\le k, i\le n} U_{ji} D_{ii} V_{ik} |j\rangle |k\rangle . \tag{186}$$

Then let $|i_A\rangle = \sum_{1\le j\le m} U_{ji} |j\rangle$, $|i_B\rangle = \sum_{1\le k\le n} V_{ik} |k\rangle$, $\lambda_i = D_{ii}$, we have:

$$|\psi\rangle = \sum_{1\le i\le n} \lambda_i |i_A\rangle |i_B\rangle . \tag{187}$$

Exercise 2.77

$$|\psi\rangle = |0_A\rangle \otimes \left(\frac{|0_B 0_C\rangle + |1_B 1_C\rangle}{\sqrt{2}} \right) . \tag{188}$$

Then for any set of basis, we can write $|\psi\rangle$ as:

$$|\psi\rangle = (\alpha_A |i_A\rangle + \beta |j_A\rangle) \otimes (\alpha_{BC} |i_B i_C\rangle + \beta_{BC} |j_B j_C\rangle) . \tag{189}$$

There are always some cross terms.

Exercise 2.78

(1) If $|\psi\rangle$ is a product state, then it can be written as the $|\psi\rangle = |\psi_A\rangle |\psi_B\rangle$. So it's obvious it has Schmidt number 1.

Additionally, if it has Schmidt number 1, then the state can be written as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ directly.

(2) If $|\psi\rangle$ is a product state, which means $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, then the density operator of A is $|\psi_A\rangle \langle\psi_A|$, and the density operator of B is $|\psi_B\rangle \langle\psi_B|$, both of which are pure.

If ρ_A and ρ_B are pure, then they can be written as: $\rho_A = |\psi_A\rangle \langle\psi_A|$, and $\rho_B = |\psi_B\rangle \langle\psi_B|$. Then $\rho_{AB} = \rho_A \otimes \rho_B = (|\psi_A\rangle \otimes |\psi_B\rangle)(\langle\psi_A| \otimes \langle\psi_B|)$. Then $|\psi\rangle$ can be written as $|\psi_A\rangle \otimes |\psi_B\rangle$, which means $|\psi\rangle$ is the product state.

Exercise 2.79

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} |0\rangle |0\rangle + \frac{1}{\sqrt{2}} |1\rangle |1\rangle . \tag{190}$$

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right). \tag{191}$$

As for $|\psi\rangle = (|00\rangle + |01\rangle + |10\rangle)\sqrt{3}$, we have:

$$\rho = |\psi\rangle \langle\psi| = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \rho_1 = \rho_2 = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{192}$$

So $|\psi\rangle$ is actually a pure state. Then the eigenvalues are:

$$\det(\rho_1 - \lambda I) = (2/3 - \lambda)(1/3 - \lambda) - 1/9 = 0 \tag{193}$$

$$\Rightarrow 9\lambda^2 - 9\lambda + 1 = 0 \tag{194}$$

$$\Rightarrow \lambda_{1,2} = \frac{3 \pm \sqrt{5}}{6}. \tag{195}$$

And the corresponding eigenvectors are:

$$|\alpha_1\rangle = \sqrt{\frac{2}{5+\sqrt{5}}}\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, |\alpha_2\rangle = \sqrt{\frac{2}{5+\sqrt{5}}}\begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}. \tag{196}$$

Then $|\psi\rangle$ can be written as $\sqrt{\lambda_1}\,|\alpha_1\rangle\,|\alpha_1\rangle + \sqrt{\lambda_2}\,|\alpha_2\rangle\,|\alpha_2\rangle$.

Exercise 2.80

We can write $|\psi\rangle$ and $|\varphi\rangle$ as:

$$|\psi\rangle = \sum_i \lambda_i |\psi_{iA}\rangle |\psi_{iB}\rangle , |\varphi\rangle = \sum_i \lambda_i |\varphi_{iA}\rangle |\varphi_{iB}\rangle . \tag{197}$$

Let $U = \sum_i |\psi_{iA}\rangle \langle\varphi_{iA}| , V = \sum_i |\psi_{iB}\rangle \langle\varphi_{iB}|$. Then:

$$(U \otimes V) |\varphi\rangle = \sum_i \lambda_i (U |\varphi_{iA}\rangle) \otimes (V |\varphi_{iB}\rangle) \tag{198}$$

$$= \sum_i \lambda_i |\psi iA\rangle |\psi iB\rangle = |\psi\rangle . \tag{199}$$

Exercise 2.81

If $|AR_1\rangle$ and $|AR_2\rangle$ are two purifications, then $|AR_1\rangle = \sum_i \alpha_i |\alpha_{iA}\rangle |\alpha_{iR_1}\rangle$, and $|AR_2\rangle = \sum_i \beta_i |\beta_{iA}\rangle |\beta_{iR_2}\rangle$. Additionally, ρ_A equals to the partial trace of both ρ_{AR_1} and ρ_{AR_2}, which must be the same. Therefore, we have:

$$\mathrm{Tr}_{R_1}(|AR_1\rangle \langle AR_1|) = \mathrm{Tr}_{R_2}(|AR_2\rangle \langle AR_2|) \tag{200}$$

$$\Rightarrow \sum_i \alpha_i |\alpha_{iA}\rangle \langle \alpha_{iA}| = \sum_i \beta_i |\beta_{iA}\rangle \langle \beta_{iA}| \tag{201}$$

Then without loss of generality, we can just let $\alpha_i = \beta_i = \lambda_i$, where λ_i is the eigenvalue of ρ_A, and let $|\alpha_{iA}\rangle = |\beta_{iA}\rangle = |\lambda_{iA}\rangle$, where $|\lambda_{iA}\rangle$ is the eigenvector of ρ_A. Then we have:

$$|AR_1\rangle = \sum_i \lambda_i |\lambda_{iA}\rangle |\alpha_{iR_1}\rangle , |AR_2\rangle = \sum_i \lambda_i |\lambda_{iA}\rangle |\beta_{iR_2}\rangle \tag{202}$$

By the conclusion of Exercise 2.80, we can let $V = I_A, U_R = \sum_i |\alpha_{iR_1}\rangle \langle \beta_{iR_2}|$.
Therefore $|AR_1\rangle = (I_A \otimes U_R) |AR_2\rangle$.

Exercise 2.82

(1) If $|\psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$, then:

$$\mathrm{Tr}_R(|\psi\rangle\langle\psi|) = \sum_{i,j} \sqrt{p_i}\sqrt{p_j} |\psi_i\rangle \langle\psi_j| \mathrm{Tr}_R(|i\rangle\langle j|) \tag{203}$$

$$= \sum_{i,j} \sqrt{p_i}\sqrt{p_j} |\psi_i\rangle \langle\psi_j| \delta_{ij} \tag{204}$$

$$= \sum_i p_i |\psi_i\rangle \langle\psi_i| = \rho. \tag{205}$$

So $|\psi\rangle$ is a purification of ρ.

(2) The measurement can be defined as $M_i = I \otimes (|i\rangle \langle i|)$.

Then the probability of getting $|i\rangle$ is $\langle\psi| M_i |\psi\rangle = p_i \langle i| \langle\psi_i|\psi_i\rangle |i\rangle = p_i$, and the post-measurement state is:

$$|\psi'\rangle = \frac{M_i |\psi\rangle}{\sqrt{p_i}} = \frac{\sqrt{p_i} |\psi_i\rangle |i\rangle}{\sqrt{p_i}} = |\psi_i\rangle |i\rangle . \tag{206}$$

Then for the system A, is the corresponding state is $|\psi_i\rangle$.

(3) Suppose $|AR\rangle$ is a purification of ρ, with $|AR\rangle = \sum_i \sqrt{\lambda_i}\,|\phi_{iA}\rangle\,|\phi_{iR}\rangle$.

Then the partial trace of $|AR\rangle\langle AR|$ should equals to ρ, which means:

$$\mathrm{Tr}_R(|AR\rangle\langle AR|) = \sum_i \lambda_i\,|\phi_{iA}\rangle\langle\phi_{iA}| = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|\,. \tag{207}$$

Using theorem 2.6, there exists a unitary operator U such that:

$$\sqrt{\lambda_i}\,|\phi_{iA}\rangle = \sum_j u_{ij}\sqrt{p_j}\,|\psi_j\rangle\,. \tag{208}$$

Therefore, for $|AR\rangle$, we have:

$$|AR\rangle = \sum_i \sqrt{\lambda_i}\,|\phi_{iA}\rangle\,|\phi_{iR}\rangle \tag{209}$$

$$= \sum_i \left(\sum_j u_{ij}\sqrt{p_j}\,|\psi_j\rangle\right)|\phi_{iR}\rangle \tag{210}$$

$$= \sum_j \sqrt{p_j}\,|\psi_j\rangle \left(\sum_i u_{ij}\,|\phi_{iR}\rangle\right) = \sum_j \sqrt{p_j}\,|\psi_j\rangle\,|j\rangle \tag{211}$$

by defining $|j\rangle = \sum_i u_{ij}\,|\phi_{iR}\rangle$.

Because u is a unitary operator, and $\{|\phi_{iR}\rangle\}$ is an orthogonal basis for R, then $\{|j\rangle\}$ is also an orthogonal basis for R.

Now, using the defined basis, we can get the same result to (2), which means R be measured such that the corresponding post-measurement state for system A is $|\psi_i\rangle$ with probability p_i.

Chapter 4

Exercise 4.1

The point (θ, φ) on the bloch sphere represents the state

$$|v\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}(\cos\varphi + i\sin\varphi)\,|1\rangle\,. \tag{212}$$

For X, the eigenvectors and the points on the bloch sphere are:

$$|v_{+1}\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, (\theta, \varphi) = (\frac{\pi}{2}, 0), \tag{213}$$

$$|v_{-1}\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, (\theta, \varphi) = (\frac{\pi}{2}, \pi). \tag{214}$$

For Y, the eigenvectors and the points on the bloch sphere are:

$$|v_{+1}\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}, (\theta, \varphi) = (\frac{\pi}{2}, \frac{\pi}{2}), \tag{215}$$

$$|v_{-1}\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}, (\theta, \varphi) = (\frac{\pi}{2}, \frac{3\pi}{2}). \tag{216}$$

For Z, the eigenvectors and the points on the bloch sphere are:

$$|v_{+1}\rangle = |0\rangle\,, (\theta, \varphi) = (0, 0), \tag{217}$$

$$|v_{-1}\rangle = |1\rangle\,, (\theta, \varphi) = (\pi, 0). \tag{218}$$

Exercise 4.2

Because $A^2 = I$, then for any of A's eigenvalue v and eigenvector $|v\rangle$, we have:

$$A^2\,|v_i\rangle = A(v_i\,|v_i\rangle) = v_i^2\,|v_i\rangle = |v_i\rangle\,. \tag{219}$$

So $v_i = \pm 1$. Therefore, $\cos(v_i x) = \cos(x), \sin(v_i x) = v_i \sin(x)$
Then we have:

$$\exp(iAx) = \exp\left(i\sum_i v_i\,|v_i\rangle\,\langle v_i|\,x\right) \tag{220}$$

$$= \sum_i \exp(iv_i x)\,|v_i\rangle\,\langle v_i| \tag{221}$$

$$= \sum_i (\cos(v_i x) + i\sin(v_i x))\,|v_i\rangle\,\langle v_i| \tag{222}$$

$$= \sum_i (\cos(x) + i\sin(x)v_i)\,|v_i\rangle\,\langle v_i| \tag{223}$$

$$= \cos(x)I + i\sin(x)A. \tag{224}$$

Exercise 4.3

$$R_z(\pi/4) = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}, \tag{225}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8} R_z(\pi/4). \tag{226}$$

Exercise 4.4

$$
\begin{aligned}
R_z(\pi/2)R_x(\pi/2)R_z(\pi/2) &= \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} && (227) \\
&= \begin{pmatrix} e^{-i\pi/2}\cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ -i\sin\frac{\pi}{4} & e^{i\pi/2}\cos\frac{\pi}{4} \end{pmatrix} && (228) \\
&= \frac{-i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, && (229)
\end{aligned}
$$

$$
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{-i\pi/2} R_z(\pi/2)R_x(\pi/2)R_z(\pi/2). \tag{230}
$$

Exercise 4.5

We have proved the anti-communicator relationship in Chapter 2:

$$
\{\sigma_i, \sigma_j\} = 2\delta_{ij} I. \tag{231}
$$

Therefore, we have:

$$
\begin{aligned}
(\hat{n}\cdot\vec{\sigma})^2 &= (n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3)^2 && (232) \\
&= \sum_i n_i^2\sigma_i^2 + n_1n_2(\sigma_1\sigma_2 + \sigma_2\sigma_1) + n_2n_3(\sigma_2\sigma_3 + \sigma_3\sigma_2) + n_3n_1(\sigma_3\sigma_1 + \sigma_1\sigma_3) && (233) \\
&= (n_1^2 + n_2^2 + n_3^2) I && (234) \\
&= I. && (235)
\end{aligned}
$$

Using Taylor expansion and the equation, we have:

$$
\begin{aligned}
R_{\hat{n}}(\theta) &= \exp(-i\theta\hat{n}\cdot\vec{\sigma}/2) && (236) \\
&= 1 - i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma} - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 I + \frac{i}{3!}\left(\frac{\theta}{2}\right)^3 \hat{n}\cdot\vec{\sigma} + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 I - \cdots && (237) \\
&= \left(1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - \frac{1}{6!}\left(\frac{\theta}{2}\right)^6 + \cdots\right) I \\
&\quad - i\left(\frac{\theta}{2} - \frac{i}{3!}\left(\frac{\theta}{2}\right)^3 + \frac{i}{5!}\left(\frac{\theta}{2}\right)^5 - \frac{i}{7!}\left(\frac{\theta}{2}\right)^7 + \cdots\right) \hat{n}\cdot\vec{\sigma} && (238) \\
&= \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\hat{n}\cdot\vec{\sigma}. && (239)
\end{aligned}
$$

Exercise 4.6

We can first prove that the effect of the rotation $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$ on any state is to rotate it by α about the corresponding axis of the Bloch sphere.
For (θ, φ) on a Bloch sphere. The state it represents is:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}(\cos\varphi + i\sin\varphi)|1\rangle . \tag{240}$$

Applying rotation $R_z(\alpha)$ on $|\psi\rangle$, we have:

$$\begin{aligned}
|\psi'\rangle &= R_z(\alpha)|\psi\rangle && (241)\\
&= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}(\cos\varphi + i\sin\varphi) \end{pmatrix} && (242)\\
&= e^{-i\alpha/2}\cos\frac{\theta}{2}|0\rangle + e^{i\alpha/2}\sin\frac{\theta}{2}(\cos\varphi + i\sin\varphi)|1\rangle && (243)\\
&= e^{-i\alpha/2}\left(\cos\frac{\theta}{2}|0\rangle + e^{i(\alpha+\varphi)}\sin\frac{\theta}{2}|1\rangle\right) . && (244)
\end{aligned}$$

Here the parameter $e^{-i\alpha/2}$ can be ignored. If we rotate it by α about z axis, then the new point is $(\theta, \varphi + \alpha)$. Then the the state is indeed the point $(\theta, \varphi + \alpha)$ on the Bloch sphere.
By symmetric propriety of the x, y and z axis, the rotation operation about any axis has the same feature.
Additionally, we can represent the rotation about any axis $\hat{n}$ by the combination of rotations about the three axis:

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)(\hat{n} \cdot \vec{\sigma}), \tag{245}$$

So we can rotate a state by rotating its Bloch sphere representation.

Exercise 4.7

$$XYX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -Y. \tag{246}$$

$$\begin{aligned}
XR_y(\theta)X &= X\left(\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y\right)X && (247)\\
&= \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}Y && (248)
\end{aligned}$$

$$= \cos \frac{-\theta}{2} I - i \sin \frac{-\theta}{2} Y \tag{249}$$

$$= R_y(-\theta). \tag{250}$$

Exercise 4.8

1. An arbitrary single qubit unitary operator is a 2×2 unitary matrix.

   For $U = \exp(i\alpha) R_{\hat{n}}(\theta)$, we have:

$$UU^\dagger = \exp(i\alpha) R_{\hat{n}}(\theta)(\exp(i\alpha))^\dagger (R_{\hat{n}}(\theta))^\dagger \tag{251}$$

$$= \exp(i\alpha)\exp(-i\alpha)\exp(-i\theta\hat{n}\cdot\vec{\sigma}/2)\exp(i\theta\hat{n}\cdot\vec{\sigma}/2) \tag{252}$$

$$= I. \tag{253}$$

   Therefore, any $U = \exp(i\alpha) R_{\hat{n}}(\theta)$ is unitary.

   For any unitary operator, we can write it as $U = t_0 I + t_1 X + t_2 Y + t_3 Z$, with:

$$\sum_{i=0}^{3} t_i^2 = 1, t_0 t_i^* + t_0^* t_i = 0. \tag{254}$$

   Additionally, we have:

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) - i \sin\left(\frac{\theta}{2}\right)(n_x X + n_y Y + n_z Z). \tag{255}$$

   Then we can let:

$$\begin{cases} \exp(i\alpha)\cos(\theta/2) = t_0 \\ i\exp(i\alpha)\sin(\theta/2) n_x = -t_1 \\ i\exp(i\alpha)\sin(\theta/2) n_y = -t_2 \\ i\exp(i\alpha)\sin(\theta/2) n_z = -t_3. \end{cases} \tag{256}$$

   Because $\cos(\theta/2)$ is real, then we can use $\cos(\theta/2) = |t_0|$ to calculate θ and α.

   We can verify that equation 254 always holds on condition that $\hat{n}$ is a real vector. Then we can calculate $\hat{n}$.

2. The Hadmard gate satisfies that:

$$H = \frac{1}{\sqrt{2}} X + \frac{1}{\sqrt{2}} Z. \tag{257}$$

Then, let $\theta = \pi$, $\hat{n} = (1/\sqrt{2}, 0, 1/\sqrt{2})$, $\alpha = \pi/2$.

3. The phase gate satisfies that:

$$S = \frac{1+i}{2} I + \frac{1-i}{2} Z. \tag{258}$$

Then, $\cos(\theta/2) = |(1+i)/2| = 1/\sqrt{2}$, so $\theta = \pi/2$. So $\exp(i\alpha) = (1+i)\sqrt{2}$, so $\alpha = \pi/4$, and $\vec{n} = (0, 0, 1)$.

Exercise 4.9

It's evident that equation 4.12 is a unitary operator. We have proved that any unitary operator can be written as:

$$\begin{aligned}
U &= \exp(i\alpha')R_{\hat{n}}(\theta) && (259)\\
&= \exp(i\alpha') \begin{pmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(n_x + n_z) & -n_y \sin\frac{\theta}{2} \\ n_y \sin\frac{\theta}{2} & \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(n_x - n_z) \end{pmatrix} && (260)\\
&= \exp(i\alpha') \begin{pmatrix} \left(1 - i\tan\frac{\theta}{2}(n_x + n_z)\right)\cos\frac{\theta}{2} & -n_y \sin\frac{\theta}{2} \\ n_y \sin\frac{\theta}{2} & \left(1 - i\tan\frac{\theta}{2}(n_x - n_z)\right)\cos\frac{\theta}{2} \end{pmatrix}. && (261)
\end{aligned}$$

Therefore, let $\sin(\gamma/2) = n_y \sin(\theta/2)$, $\alpha = \alpha'$, and setting proper β and δ, the equation holds.

Exercise 4.11

We can just first rotate z axis $\hat{n}$, and rotate y axis to $\hat{m}$, and after rotating by $\hat{n}$ and $\hat{m}$, we can then rotate z and y back.

Exercise 4.17

Applying H on the target qubit, and then control-Z, and then H on the target qubit.

Exercise 4.22

To achieve this, we should first depart $C^2(U)$ into the combination of $C(V)$ according to figure 4.8, and then depart $C^2(V)$ into the combination of single qubit gates and $CNOT$ gates. After that, we should combine some single qubit gate, and swap some CNOT gates. The process looks like:

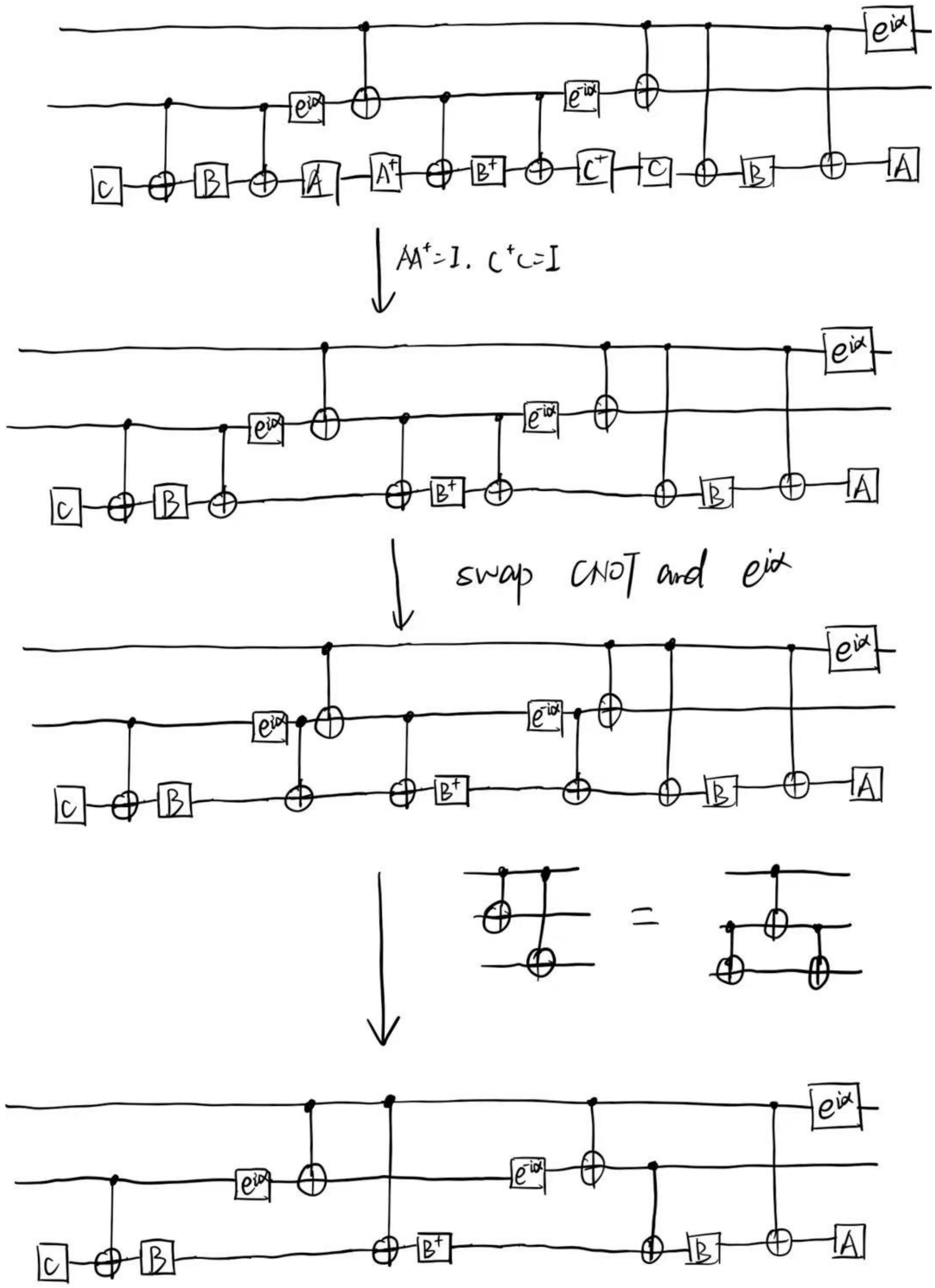

Figure 1:

Exercise 4.23

We should just find proper gates for this decomposition:

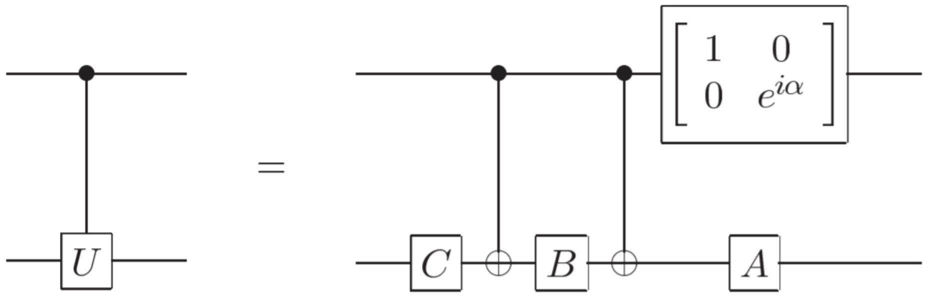

Figure 2:

If $U = R_y(\theta)$, then $e^{i\alpha} = \det(U)$ and $\alpha = 0$. Therefore, we can just let $\alpha = 0, A = I, B = R_y(-\theta/2)$ and $C = R_y(\theta/2)$.
As for $U = R_x(\theta)$, we can similarly rotate the system, and then apply $HZH = X$. Therefore, we can just let $\alpha = 0, A = H, B = R_z(-\theta/2)$ and $C = HR_y(\theta/2)$.

Exercise 4.30

To achieve $C^n(U)$, we can first achieve it using $C^n(X)$ and other single qubit gates, and then the problem is how to achieve $C^n(X)$ using Tollifo gates and single qubit gates without work qibits.
To do this, we can let $A^2 = X$, and apply $O(n^2)$ $C^2(A)$ gates and $C^2(X)$ gates.