Discrete Mathematics 100 (1992) 303–311
North-Holland

The ubiquitous Petersen graph

Gary Chartrand*

Western Michigan University, Kalamazoo, MI 49008, USA

Héctor Hevia

Western Michigan University, Kalamazoo, MI 49008, USA
Universidad Católica de Valparaíso

Robin J. Wilson

The Open University, Milton Keynes, UK, MK7 6AA
The Colorado College, Colorado Springs, CO 80903, USA

Received 15 April 1991
Revised 25 September 1991

Abstract

Chartrand, G., H. Hevia and R.J. Wilson, The ubiquitous Petersen graph, Discrete Mathematics 100 (1992) 303–311.

1. Introduction

The Petersen graph is named after the Danish mathematician Julius Petersen (1839–1910). During the 1890s, Petersen was studying factorizations of regular graphs and, in 1891, published an important paper [33] that is commemorated in this volume. In his paper, Petersen proved that every 3-regular graph with at most two bridges contains a 1-factor. A few years earlier, Tait [35] had written that he had shown every 3-regular graph to be 1-factorable, but that this result was 'not true without limitation'. What Tait meant by this is unclear, but in 1898 Petersen interpreted Tait's remark to mean that every 3-regular bridgeless graph is 1-factorable. If this result had been true, then it would have been stronger than Petersen's theorem. However, Petersen showed it to be false by producing a 3-regular bridgeless graph that is not 1-factorable (see [34])—the graph which we now call the *Petersen graph* (see Fig. 1).

The main properties of the Petersen graph were discussed at length by two of us in 1985 [15]. However, the Petersen graph continues to appear throughout the literature of graph theory. In the present paper we update our earlier review by presenting additional recent results relating to the Petersen graph.

* Work supported in part by Office of Naval Research Contract N00014-91-J-1060.

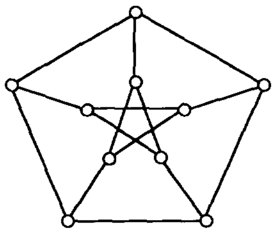

Fig. 1.

2. Some characterizations

In [15] several characterizations of the Petersen graph were presented. An *r-cage* is a 3-regular graph of smallest order with girth r. It is easy to see that the complete graph K_4 is the only 3-cage, and the complete bipartite graph $K_{3,3}$ is the only 4-cage. For $r = 5$, we have the following result.

Theorem 1. *The Petersen graph is the only* 5*-cage.*

More generally, a *(k, r)-cage* is a k-regular graph of smallest order with girth r. Thus the Petersen graph is the only $(3, 5)$-cage. The next characterization uses the fact that any two non-adjacent vertices are mutually joined to exactly one other vertex.

Theorem 2. *Apart from the complete graph K_4, the Petersen graph is the only* 3*-regular graph in which any two non-adjacent vertices are mutually adjacent to just one other vertex.*

In [22] Exoo and Harary defined a graph G to have the property $P_{1,n}$ if, for each sequence $v, v_1, v_2, \ldots, v_n$ of $n+1$ vertices, there is another vertex of G, adjacent to v but not to $v_1, v_2, \ldots, v_n$. For $1 \leq n \leq 6$, the $(n+1, 5)$-cage is a graph of smallest order with property $P_{1,n}$. In particular, they obtained the following result for $n = 2$, also obtained by Murty [31].

Theorem 3. *The Petersen graph is the smallest graph with the property that, given any three distinct vertices v, v_1, v_2, there is a fourth vertex adjacent to v but not to v_1 or v_2. In fact, the Petersen graph is the only graph with fewer than twelve vertices having property $P_{1,2}$.*

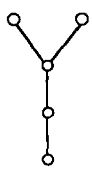

Fig. 2.

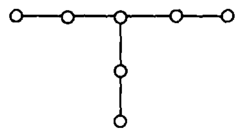

Fig. 3.

If G is a given graph, a graph F is a *G-frame* if it is a graph of smallest order with the property that, for each vertex v of G and each vertex w of F, there is an embedding of G into F as an induced subgraph with v appearing at w. For example, the product graph $K_2 \times K_3$ is the unique Y-frame for the tree Y of Fig. 2. The next result was established by Chartrand, Henning, Hevia and Jarrett [14].

Theorem 4. *The Petersen graph is the only T-frame for the tree T of Fig.* 3.

3. Cycles and cycle covers

It is well known that the Petersen graph has a cycle passing through any nine vertices, but no (Hamiltonian) cycle passing through all ten vertices. Holton, McKay, Plummer and Thomassen [24] have extended the first of these results by proving that, if G is any 3-connected 3-regular graph, then G has a cycle passing through any nine vertices. For cycles through ten vertices, Kelmans and Lomonov [27] and Ellingham, Holton and Little [19] have proved the following result.

Theorem 5. *If G is a 3-connected 3-regular graph, then G has a cycle passing through any ten vertices if and only if G is not of 'Petersen form'—that is, G cannot be contracted to the Petersen graph with each of the ten vertices mapped to a distinct vertex of the Petersen graph* (*see Fig.* 4).

An extension of Theorem 5 to eleven vertices was obtained by Aldred [1] (see also Aldred, Bau and Holton [2]). Together with G. Royle, these authors have obtained the following result [3].

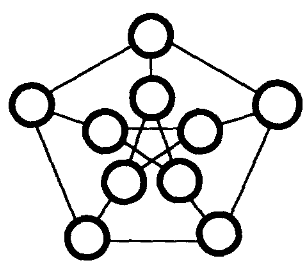

Fig. 4.

Theorem 6. *If G is a 3-connected 3-regular graph, then any set A of eleven vertices lies on at least three different cycles of G unless there is a contraction ϕ from G to the Petersen graph, where*

either $\phi(A) = V(P)$, in which case no cycle of G contains A.

or $|\phi(A)| = 9$, in which case at least two cycles of G contain A.

In the case of 2-connected graphs, the following result was obtained by Dean, Kaneko, Ota and Toft [17].

Theorem 7. *Except for the Petersen graph, every 2-connected graph with minimum degree at least* 3 *contains a cycle whose length is congruent to* 1 *modulo* 3.

In [20] Erdős mentioned Theorem 7 and attributed it to H. Jochens.

The following result, related to Theorem 5 but for graphs that are not necessarily 3-regular, has recently been announced by P.A. Catlin and Hong-Jian Lai, along with several other results of a similar nature.

Theorem 8. *If G is a 3-edge-connected graph with at most ten edge-cuts of size* 3, *then*

either G has a spanning closed trail,
or G is contractible to the Petersen graph.

A *double cycle cover* of a graph G is a collection C of cycles in G such that each edge of G lies in exactly two cycles in C. Catlin [12–13] and Lai [28] have proved a number of results involving double cycle covers and the Petersen graph. A typical one is as follows.

Theorem 9. *If G a bridgeless graph with at most* 13 *edge-cuts of size* 3, *then*

either G has a 3-colorable double cycle cover,
or G is contractible to the Petersen graph.

Catlin has conjectured that 'at most 13' in Theorem 9 can be replaced by 'at most 17'; this would be best possible, because of the existence of snarks of order 18.

4. Hamiltonian properties

It is well known (see [15], for example) that the Petersen graph is not Hamiltonian. It is *hypohamiltonian,* however, in that the removal of any vertex leaves a Hamiltonian graph. Thus, the Petersen graph is, in some sense, 'nearly Hamiltonian'. It is close to being Hamiltonian in another sense. In [26], Jackson proved the following result.

Theorem 10. *Every 2-connected k-regular graph of order at most 3k is Hamiltonian.*

Zhu, Liu and Yu [36] proved the following conjecture of Jackson (see also Bondy and Kouider [9]).

Theorem 11. *The only 2-connected k-regular non-Hamiltonian graph of order* $3k+1$ *is the Petersen graph.*

Jackson also conjectured that, for $k \geqslant 4$, all 2-connected k-regular graphs of order at most $3k+3$ are Hamiltonian. This conjecture has been proved for $k \geqslant 6$ by Zhu, Liu and Yu [37]. Regular Hamiltonian graphs have also been studied by Erdős and Hobbs [21], who proved the following result.

Theorem 12. *Let G be a 2-connected k-regular graph of order $2k+3$ or $2k+4$ $(k \geqslant 2)$. Then G is Hamiltonian if and only if G is not the Petersen graph.*

Lovász [29] has conjectured that every connected vertex-transitive graph contains a Hamiltonian path, and Thomassen (see [7]) has conjectured that there are only finitely many vertex-transitive non-Hamiltonian graphs. The known exceptions are the Petersen graph, the Coxeter graph, and two graphs obtained from these by replacing their vertices with triangles. It is also known (see, for example, Marusic and Parsons [30]) that, apart from the Petersen graph, every connected vertex-transitive graph of order p (prime), p^2, p^3, $2p$, $3p$, or $5p$ is Hamiltonian.

The Petersen graph also appears as the exceptional graph in the following result of Broersma, Van den Heuvel and Veldman [11], generalizing Ore's theorem on neighbourhood unions.

Theorem 13. *If G is a 3-connected graph of order n such that $|N(v) \cup N(w)| \geqslant \frac{1}{2}n$ for every pair v, w of non-adjacent vertices, then either G is Hamiltonian or G is the Petersen graph.*

One of the best known necessary conditions for a graph G to be Hamiltonian is that $k(G-S) \leqslant |S|$, for every proper subset S of $V(G)$, where $k(G-S)$ denotes the number of components of $G-S$. Let $\kappa(G)$ and $\beta(G)$ denote the connectivity and independence number, respectively, of a graph G. Chvátal and Erdős [16] have also proved that if G is a graph of order at least 3, with $\kappa(G) \geqslant \beta(G)$, then G is Hamiltonian. These two results have been combined by Bigalke and Jung [8], who proved the following result.

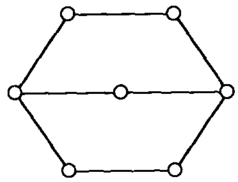

Fig. 5.

Theorem 14. *Let G be a 3-connected graph that is not the Petersen graph. If $k(G-S) \leqslant |S|$ and $\kappa(G) \geqslant \beta(G)-1$, then G is Hamiltonian.*

Another result along these lines concerns *pancyclic graphs*—graphs of order $n\,(\geqslant 3)$ containing cycles of all possible lengths $3, 4, \ldots, n$. Benhocine and Fouquet [6] have proved the following result.

Theorem 15. *If G is a 2-connected graph with $\kappa(G) \geqslant \beta(G)-1$, then its line graph $L(G)$ is pancyclic unless G is a cycle of length n $(4 \leqslant n \leqslant 7)$, the graph in Fig. 5, or the Petersen graph.*

A graph G is said to be *traceable* if it has a Hamiltonian path. Galvin, Rival and Sands [23] have proved that, for each triple r, s, n of positive integers, there exists a least positive integer $T(r, s, n)$ such that every traceable graph of order exceeding $T(r, s, n)$ either contains $K_{r,s}$ as a subgraph or has a chordless n-path. Among the results they have obtained are that $T(2, 2, 6) = 10$, and that the unique traceable graph of order 10 containing no $K_{2,2}\,(= C_4)$ and no chordless 6-path is the Petersen graph.

There has also been much interest in the generalized Petersen graph $P(n, k)$, consisting of an outer n-cycle, n spokes, and an inner n-cycle (or set of disjoint cylces of total length n) joined to the free ends of every kth spoke. (Thus $P(5, 2)$ is the Petersen graph.) Much attention has been paid to when these graphs are Hamiltonian, and the question finally settled by Alspach [4] in the following.

Theorem 16. *$P(n, k)$ is Hamiltonian unless it lies in one of the following sequences:*

(i) $P(5, 2)$, $P(11, 2)$, $P(17, 2)$, $P(23, 2)$, $P(29, 2)$, $\ldots$;

(ii) $P(8, 4)$, $P(12, 6)$, $P(16, 8)$, $P(20, 10)$, $P(24, 12)$, $\ldots$.

5. Extremal graph theory

We conclude this survey with a number of results from extremal graph theory. A graph G is 3-*saturated* if G contains no triangles, but $G + e$ contains a triangle for each edge e. Duffus and Hanson [18] have proved the following result.

Theorem 17. *Let G be a 3-saturated graph of minimum degree 3 and with at least 10 vertices. If* $|E(G)| \leq 3n - 15$, *then G has a subgraph isomorphic to the Petersen graph.*

Another result on triangle-free graphs was proved by Nenov and Khadzhiivanov [32] in the following.

Theorem 18. *If G has no triangles and has order 10, then G has at least five independent sets of 4 vertices; there are exactly five such sets if G is one of three graphs, one of which is the Petersen graph.*

Hopkins and Staton [25] have considered extremal bipartite subgraphs in 3-regular triangle-free graphs as follows.

Theorem 19. *Every 3-regular triangle-free graph has a bipartite subgraph with at least* $\frac{4}{5}$ *of the original edges.*

This result is best possible in the sense that neither the Petersen graph nor the graph of the dodecahedron contains a bipartite subgraph with more than $\frac{4}{5}$ of its edges.

The previous result was made a bit more specific by Bondy and Locke [10].

Theorem 20. *If G is a triangle-free 3-regular graph whose largest bipartite subgraph contains exactly* $\frac{4}{5}$ *of the edges of G, then G is either the Petersen graph or the graph of the dodecahedron.*

Finally, Beenker and Van Lint [5] have determined the maximum order of a generalized Petersen graph of given diameter in the following.

Theorem 21. *Let* $f(d)$ *be the maximum order of a generalized Petersen graph of diameter d. Then*

$$f(2) = 10 \qquad (\textit{the Petersen graph}),$$

$$f(3) = 14 \qquad (\textit{the graph } P(7, 2)), \textit{ and}$$

$$f(d) = 4d^2 - 12d + 10, \quad \textit{if } d \geq 4.$$

References

[1] R. Aldred, Cycles through seven vertices avoiding one edge in 3-connected cubic graphs, Ars Combin. 23B (1986) 79–86.

[2] R. Aldred, S. Bau and D.A. Holton, Primitive graphs, Ars Combin. 23B (1987) 183–193.
[3] R. Aldred, S. Bau, D. Holton and G. Royle, An 11-vertex theorem for 3-connected cubic graphs, J. Graph Theory 12 (1988) 561–570.
[4] B. Alspach, The classification of Hamiltonian generalized Petersen graphs, J. Combin. Theory Ser. B 34 (1983) 293–312.
[5] G.F.M. Beenker and J.H. Van Lint, Optimal generalized Petersen graphs, Philips J. Res. 43 (1988) 129–136.
[6] A. Benhocine and J.-L. Fouquet, The Chvátal–Erdős condition and pancyclic line graphs, Discrete Math. 66 (1987) 21–26.
[7] J.-C. Bermond, Hamiltonian graphs, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory (Academic Press, London, 1978) 163.
[8] A. Bigalke and H.A. Jung, Über Hamiltonsche Kreise und unabhängige Ecken in Graphen, Monatsh. Math. 88 (1979) 195–210.
[9] J.A. Bondy and M. Kouider, Hamilton cycles in regular 2-connected graphs, J. Combin. Theory Ser. B 44 (1988) 177–186.
[10] J.A. Bondy and S.C. Locke, Largest bipartite subgraphs in triangle-free graphs with maximum degree three, J. Graph Theory 10 (1986) 477–504.
[11] H.J. Broersma, J. van den Heuvel and H.J. Veldman, A generalization of Ore's theorem involving neighborhood unions, University of Twente Memorandum 825, 1989.
[12] P.A. Catlin, Double cycle covers and the Petersen graph, J. Graph Theory 13 (1989) 465–483.
[13] P.A. Catlin, Double cycle covers and the Petersen graph, II, to appear.
[14] G. Chartrand, M.A. Henning, H. Hevia and E.B. Jarrett, A new characterization of the Petersen graph, submitted.
[15] G. Chartrand and R.J. Wilson, The Petersen graph, in: F. Harary and J. Maybee, eds., Proc. 1st Colorado Conf. on Graph Theory (Wiley, New York, 1985) 69–100.
[16] V. Chvátal and P. Erdős, A note on hamiltonian circuits, Discrete Math. 2 (1972) 111–113.
[17] N. Dean, A. Kaneko, K. Ota, and B. Toft, Cycles modulo 3, submitted.
[18] D.A. Duffus and D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree, J. Graph Theory 10 (1986) 55–67.
[19] M.N. Ellingham, D.A. Holton and C.H.C. Little, Cycles through ten vertices in 3-connected cubic graphs, Combinatorica 4 (1984) 265–273.
[20] P. Erdős, Some recent problems and results in graph theory, combinatorics, and number theory, Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing. Congressus Num. XVII, Utilitas Math. (1976) 3–14.
[21] P. Erdős and A.M. Hobbs, A class of Hamiltonian regular graphs. J. Graph Theory 2 (1978) 129–135.
[22] G. Exoo and F. Harary, The smallest graphs with certain adjacency properties, Discrete Math. 29 (1980) 25–32.
[23] F. Galvin, I. Rival and B. Sands, A Ramsey-type theorem for traceable graphs, J. Combin. Theory Ser. B 33 (1982) 7–16.
[24] D.A. Holton, B. McKay, M.D. Plummer and C. Thomassen, A nine point theorem for 3-connected graphs, Combinatorica 2 (1982) 53–62.
[25] G. Hopkins and W. Staton, Extremal bipartite subgraphs of cubic triangle-free graphs, J. Graph Theory 6 (1982) 115–121.
[26] B. Jackson, Hamilton cycles in regular 2-connected graphs, J. Combin. Theory Ser. B 29 (1980) 27–46.
[27] A.K. Kelmans and M.V. Lomonov, A cubic 3-connected graph having no cycle through a given ten vertices has the 'Petersen form', J. Graph Theory 6 (1982) 495–496.
[28] H.-J. Lai, Reduced graphs of diameter two, J. Graph Theory 14 (1990) 77–87.
[29] L. Lovász, Combinatorial Structures and their Applications (Gordon and Breach, New York, 1970) 243–246.
[30] D. Marusic and T.D. Parsons, Hamiltonian paths in vertex-symmetric graphs of order $5p$, Discrete Math. 42 (1982) 227–242.
[31] U.S.R. Murty, On critical graphs of diameter 2, Math. Magazine 41 (1968) 138–140.
[32] N.D. Nenov and N. Khadzhiivanov, An extremal property of the Petersen graph, Annuaire University, Sofia Fac. Math. Mec. 75 (1984) 115–123.

[33] J. Petersen, Die Theorie der regulären graphs, Acta Math. 15 (1891) 193–200.
[34] J. Petersen [Sur Le théorème de Tait], Intermed. Math. 5 (1898) 225–227.
[35] P.G. Tait, Listing's Topologie, Phil. Mag. 17 (1884) 30–46.
[36] Y.J. Zhu, Z.H. Liu and Z.G. Yu, An improvement on Jackson's result on Hamiltonian cycles in 2-connected regular graphs, in: Cycles in Graphs (North-Holland, Amsterdam, 1982) 115.
[37] Y.J. Zhu, Z.H. Liu and Z.G. Yu, 2-connected k-regular graphs, $k \geqslant 6$, on at most $3k+3$ vertices are hamiltonian, J. Systems Sci. Math. Sci. 6 (1986) 36–49, 136–145.