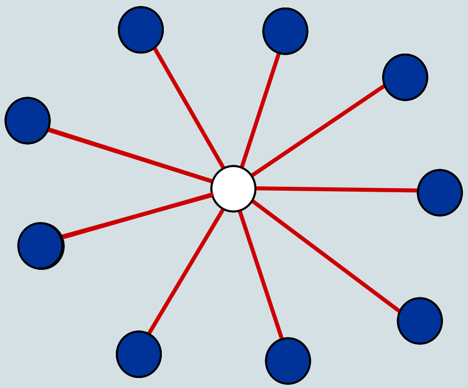
Algorithms (2IL15) – Lecture 11

Approximation Algorithms



NP-complete problems: problems A in NP such that B $\leq_P$ A for any B in NP

Examples: Circuit-Sat, SATISFIABILITY, 3-SAT, CLIQUE, VertexCover, Subset Sum, Hamiltonian Cycle, …

NP-complete problems cannot be solved in polynomial time, unless P = NP

What to do when you want to solve an NP-complete (optimization) problem ?

- still use exact algorithm: algorithm guaranteed to compute optimal solution
  ➔ will not be polynomial-time, but perhaps fast enough for some instances
- heuristic: algorithm without guarantees on the quality of the solution
- approximation algorithm: algorithm that computes solution whose quality is guaranteed to be within a certain factor from OPT

Approximation algorithms: terminology

Consider minimization problem

OPT(I) = value of optimal solution for problem instance I

ALG(I) = value of solution computed by algorithm ALG for problem instance I

ALG is ρ–approximation algorithm if ALG(I) ≤ ρ · OPT(I) for all inputs I

approximation factor

Note: ρ can be a function of input size *n* (e.g.: *O*(log *n*)-approximation)

Example: Vertex Cover

if ALG computes, for any graph G, a cover with at most twice the minimum number of nodes in any vertex cover for G, then ALG is 2-approximation

Approximation algorithms: terminology (cont'd)

Consider ~~minimization~~ problem
maximization

OPT(I) = value of optimal solution for problem instance I

ALG(I) = value of solution computed by algorithm ALG for problem instance I

ALG(I) ≥ (1/ρ) · OPT(I)

ALG is ρ–approximation algorithm if ~~ALG(I) ≤ ρ · OPT(I)~~ for all inputs I

Example: CLIQUE

if ALG computes, for any graph G, a clique with at least half the maximum number of nodes in any clique in G, then ALG is 2-approximation.

NB In the literature, one often talks about (1/ρ)-approximation instead of ρ-approximation for maximization problems. In example, ALG would be (1/2)-approximation algorithm.

For ρ-approximation algorithm ALG (for minimization problem) we must

- describe algorithm …
- … and analyze running time …
- … and prove that $ALG(I) \leq \rho \cdot OPT(I)$ for all inputs I

How can we prove that $ALG(I) \leq \rho \cdot OPT(I)$ if we don't know OPT ?!

we need to prove a lower bound on OPT(I)

for maximization problems we need upper bound

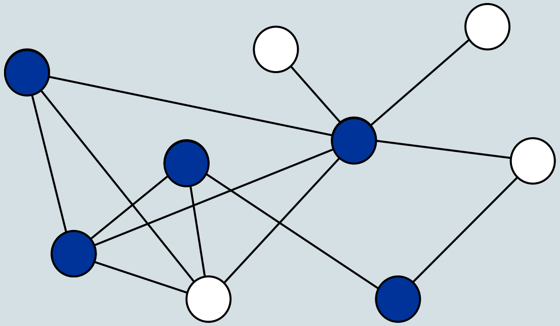
Example:

$$\left.\begin{array}{r} OPT(I) \geq 100 \\ ALG(I) \leq 200 \end{array}\right\} \Longrightarrow ALG(I) \leq 2\, OPT(I)$$

Example 1: Vertex Cover

G = (V,E) is undirected graph

vertex cover in G: subset $C \subset V$ such that for each edge (u,v) in E
we have u in C or v in C (or both)



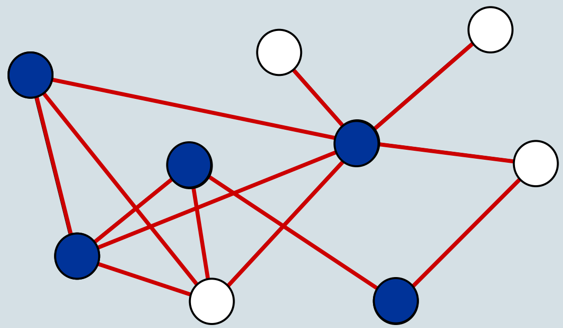
Vertex Cover (optimization version)

Input: undirected graph G = (V,E)

Problem: compute vertex cover for G with minimum number of vertices

Vertex Cover is NP-hard.

An approximation algorithm for Vertex Cover: first attempt

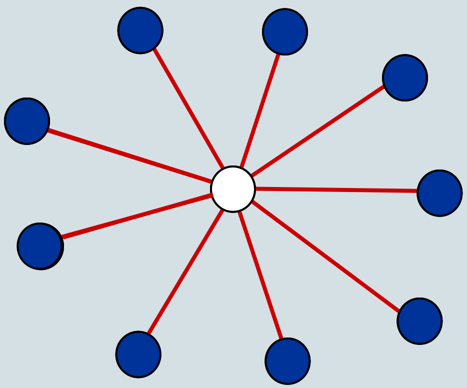


Approx(?)-Vertex-Cover (*G*)

1. $C \leftarrow$ empty set; $E^* \leftarrow E$ // E^* = set of edges not covered by C
2. **while** E^* not empty
3. **do** take edge (u,v) in E^*
4. $C \leftarrow C \cup \{u\}$
5. remove from E^* all edges that have u as an endpoint
6. **return** C

Does Approx(?)_Vertex-Cover have good approximation ratio?

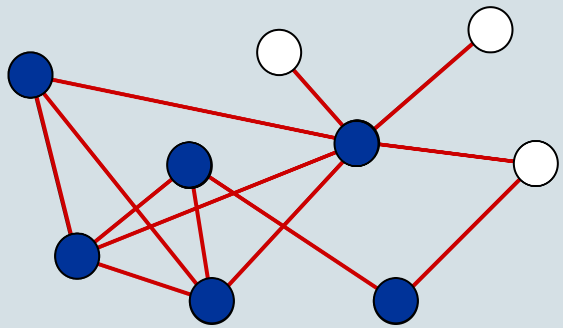
No: if we pick the wrong vertices, approximation ratio can be $|V|-1$



Second attempt: greedy

- always pick vertex that covers largest number of uncovered edges
- gives O(log $|V|$)-approximation algorithm: better, but still not so good

An approximation algorithm for Vertex Cover: third attempt



Approx~~(2)~~-Vertex-Cover (G)

$C \leftarrow$ empty set; $E^* \leftarrow E$ // E^* = set of edges not covered by C

while E^* not empty

  do take edge (u,v) in E^*

    $C \leftarrow C \cup$ ~~$\{u\}$~~ $\{u,v\}$

    remove from E^* all edges that have u as an endpoint

and/or v

return C

Approx-Vertex-Cover (G)

1. $C \leftarrow$ empty set; $E^* \leftarrow E$ // E^* = set of edges not covered by C
2. **while** E^* not empty
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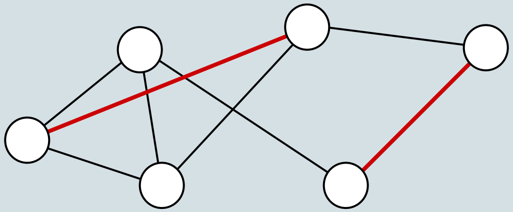
Proving approximation ratio, step 1: try to get lower bound on OPT



two edges are non-adjacent if they do not have an endpoint in common

Lemma. Let $N \subseteq E$ be any set of non-adjacent edges. Then OPT $\geq |N|$.

Approx-Vertex-Cover (G) // $N \leftarrow$ empty set

1. $C \leftarrow$ empty set; $E^* \leftarrow E$ // E^* = set of edges not covered by C
2. **while** E^* not empty
3. **do** take edge (u,v) in E^*
4. $C \leftarrow C \cup \{u,v\}$ // $N \leftarrow N \cup \{ (u,v) \}$
5. remove from E^* all edges that have u and/or v as an endpoint
6. **return** C

Theorem. Approx-Vertex-Cover is 2-approximation algorithm and can be implemented to run in $O(|V| + |E|)$ time.

Proof (of approximation ratio). Consider any input graph G.

Let N be set of edges selected in step 3 during the algorithm.

- N is set of non-adjacent edges, so $\mathrm{OPT}(G) \geq |N|$
- $|C| = 2|N|$

Hence, $\mathrm{ALG}(G) = |C| = 2|N| \leq 2\,\mathrm{OPT}(G)$. ■

Theorem. Approx-Vertex-Cover is 2-approximation algorithm and can be implemented to run in $O(|V| + |E|)$ time.

Can we do better? (1.5)-approximation, or (1.1)-approximation, or … ?)

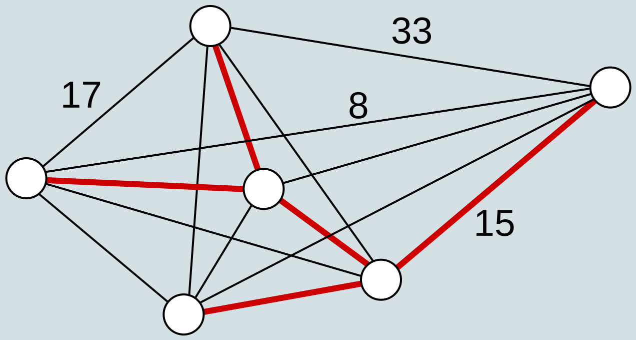
Slightly better algorithm is known: approximation ratio $2 - \frac{1}{\Theta(\log |V|)}$

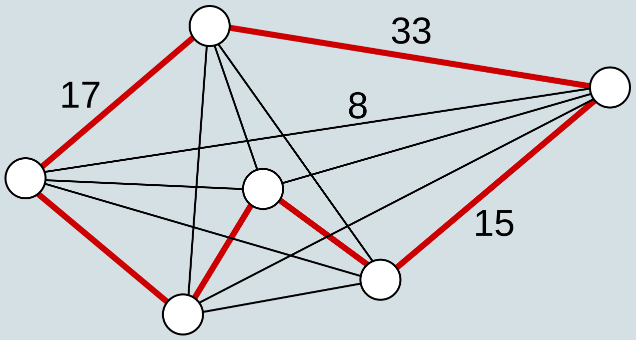
Theorem. No polynomial-time (1.3606)-approximation algorithm exists for Vertex Cover unless P = NP.

but some other problems can be approximated much better …

Example 2: TSP

MST versus TSP





Min Spanning Tree

Input: weighted graph

Output: minimum-weight tree connecting all nodes

greedy: O($|E| + |V| \log |V|$)

Traveling Salesman (TSP)

Input: complete weighted graph (non-negative edge weights)

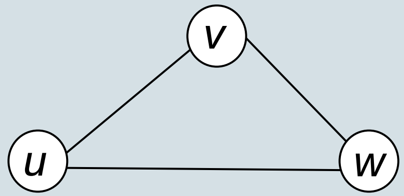
Output: minimum-weight tour connecting all nodes

NP-complete

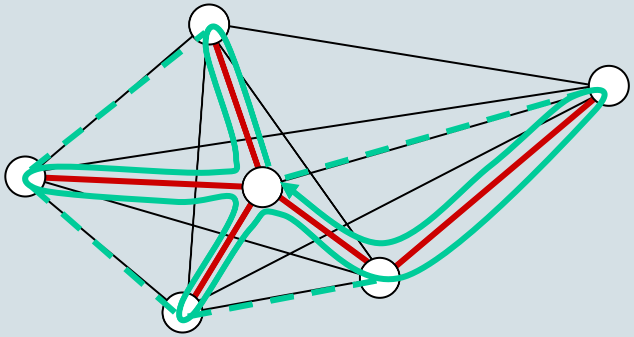
what about approximation algorithm ?

Lemma. MST is lower bound: for any graph G, we have TSP(G) ≥ MST(G).

Triangle inequality: weight(u,w) ≤ weight(u,v) + weight(v,w)
for any three vertices u,v,w



A simple 2-approximation for TSP with triangle inequality



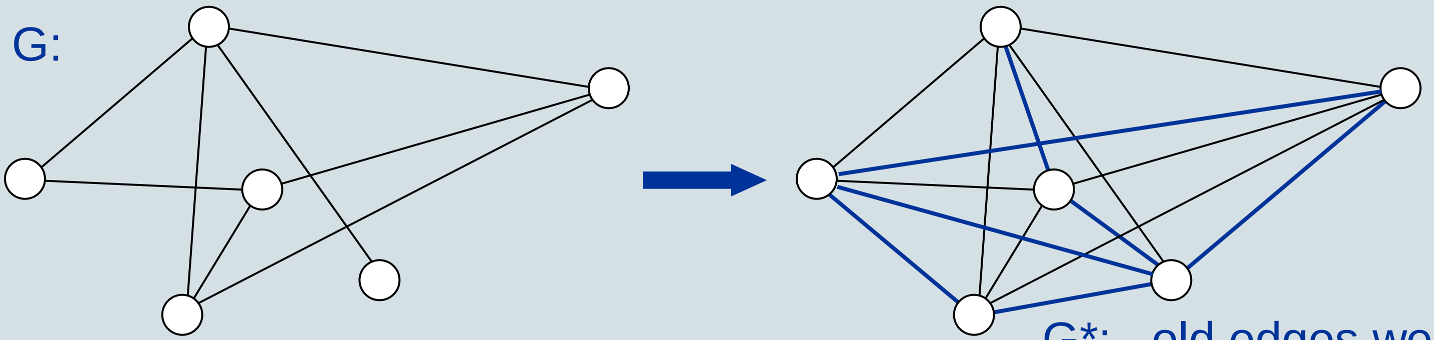
Approx-TSP

1. Compute MST on G.
2. Double each edge of MST to get a cycle.
3. Take shortcuts to get a tour visiting every vertex exactly once.

Hardness of approximation of TSP without triangle inequality

Theorem. Let ρ ≥ 1 be any constant. Then there cannot be a polynomial-time ρ-approximation algorithm for general TSP, unless P = NP.

even for positive edge weights



G*: old edges weight 1

new edges: weight $\rho \cdot |V| + 1$

G has Hamiltonian cycle iff G* has tour of length $|V|$

➔ ρ-approx algorithm computes tour T with length(T) ≤ ρ· OPT = $\rho \cdot |V|$

and such a tour can only use edges of length 1

Polynomial-time approximation schemes

Consider minimization problem.

A polynomial-time approximation scheme (PTAS) is an algorithm ALG with two parameters, an input instance I and a number $\varepsilon > 0$, such that:

i. $ALG(I, \varepsilon) \leq (1+\varepsilon) \cdot OPT(I)$ (so it's $(1+\varepsilon)$-approximation algorithm)
ii. running time of ALG is polynomial in *n* for any constant $\varepsilon>0$.

Example of possible running times: $O(n^2 / \varepsilon^5)$, $O(2^{1/\varepsilon}\, n \log n)$, $O(n^{3/\varepsilon})$, etc.

FPTAS — $O(n^2 / \varepsilon^5)$

not FPTAS — $O(2^{1/\varepsilon}\, n \log n)$, $O(n^{3/\varepsilon})$

FPTAS = fully polynomial-time approximation scheme
= PTAS with running time not only polynomial in n, but also in $1/\varepsilon$

Polynomial-time approximation schemes

Consider ~~minimization~~ maximization problem.

A polynomial-time approximation scheme (PTAS) is an algorithm ALG with two parameters, an input instance I and a number $\varepsilon > 0$, such that:

i. $\text{ALG}(I, \varepsilon) \; \cancel{\leq (1+\varepsilon)} \; \geq 1/(1+\varepsilon) \cdot \text{OPT}(I)$ (so it's $(1+\varepsilon)$-approximation algorithm)

ii. running time of ALG is polynomial in *n* for any constant $\varepsilon>0$.

Example of possible running times: $O(n^2 / \varepsilon^5)$, $O(2^{1/\varepsilon}\, n \log n)$, $O(n^{3/\varepsilon})$, etc.

FPTAS

not FPTAS

FPTAS = fully polynomial-time approximation scheme

= PTAS with running time not only polynomial in n, but also in $1/\varepsilon$

Subset Sum (optimization version)

"Find maximum load for a truck, under the condition that the load does not exceed a given weight limit for the truck."

Input: multi-set X of non-negative integers, non-negative integer t

Problem: compute subset S that maximizes $\sum_{x \text{ in } S} x$

under the condition that $\sum_{x \text{ in } S} x \leq t$?

Example: $X = \{ 3, 5, 7, 10, 16, 23, 41 \}$ $t = 50$

$S = \{ 3, 5, 41\}$ gives sum 49, which is optimal

For simplicity we will only compute value of optimal solution, not solution itself.

An exponential-time exact algorithm for Subset Sum

Use backtracking: generate all subset sums that are at most t

Exact-Subset-Sum (X, t)
1. $L \leftarrow$ Generate-Subset-Sums (X, t)
2. **return** the maximum sum in L

Generating subset sums

- $X = \{ x_1, x_2, \ldots, x_n \}$
- choices: do we take x_1 into subset ? do we take x_2 into subset ? ETC

Define L_i = { all possible sums you can make with $x_1,\ldots,x_i$ }

Example: X = 2, 7, 5

$L_0 = \{ 0 \}$

$L_1 = \{ 0 \} \cup \{0+2\} = \{0, 2\}$ // for each sum in L_0, we can add x_1 to it or not

$L_2 = \{0, 2\} \cup \{0+7, 2+7\} = \{0, 2, 7, 9\}$

$L_3 = \{0,2,7,9\} \cup \{0+5, 2+5, 7+5, 9+5\} = \{0, 2, 5, 7, 9, 12, 14\}$

Generating subset sums

- $X = \{ x_1, x_2, \ldots, x_n \}$
- choices: do we take x_1 into subset ? do we take x_2 into subset ? ETC

Generate-Subsets (X,t)
1. $L_0 \leftarrow \{ 0 \}$
2. **for** $i \leftarrow 1$ **to** n
3. **do** Copy L_{i-1} to list M_{i-1}
4. Add x_i to each number in M_{i-1} and remove numbers larger than t
5. $L_i \leftarrow L_{i-1} \cup M_{i-1}$
6. Remove duplicates from L_i
7. **return** L_n

running time: Ω (size of list L_n) = $\Omega(2^n)$

Turning the exponential-time exact algorithm into an FPTAS

Speeding up the algorithm: don't keep all numbers in L_i (trim L_i)

L_i = 0, 104, 134, 135, 145, 149, 204, 207, 224, 300

if we keep only one of them,
we make only a small mistake

if numbers y and z in L_i are almost the same, then throw away one of them

when exactly can we afford to do this ?

throw away y when $1 \leq y / z \leq 1 + \delta$
for a suitable parameter δ

which one ?

safer to keep the smaller number, let's say z

Trim (L, δ)

// L is a sorted list with elements $z_0, z_1, z_2, \dots, z_m$ (in order)   NB $z_0 = 0$

- last ← z_1
- **for** $j \leftarrow 2$ **to** m
- **do if** z_j / last ≤ 1+δ
- **then** remove z_j from L // trim
- **else** last ← z_j // don't trim

Generate-Trimmed-Subsets (X,t)

1. L_0 ← empty set
2. **for** $i \leftarrow 1$ **to** n
3. **do** Copy L_{i-1} to list M_{i-1}
4. Add x_i to each number in M_{i-1} and remove numbers larger than t
5. $L_i \leftarrow L_{i-1} \cup M_{i-1}$
6. ~~Remove duplicates from L_i~~ Trim (L_i, δ)
7. **return** L_n

Lemma. The size of any list L_i is $O((1/\delta) \cdot \log t)$

Proof. Let $L = z_0, z_1, z_2, \dots , z_m$ (in order). Then $z_j / z_{j-1} \geq 1+\delta$,

so
$$\begin{aligned} z_m &\geq (1+\delta) \cdot z_{m-1} \\ &\geq (1+\delta)^2 \cdot z_{m-2} \\ &\geq \dots \\ &\geq (1+\delta)^{m-1} \cdot z_1 \\ &\geq (1+\delta)^{m-1} \end{aligned}$$

Hence, $(1+\delta)^{m-1} \leq z_m \leq t$, so $(m-1) \cdot \log (1+\delta) \leq \log t$

Conclusion: $m = O((1/\delta) \cdot \log t)$ since $\log(1+\delta) \approx \delta$ for small δ

■

Lemma. If we choose $\delta = \varepsilon/(2n)$, then approximation ratio is $1+\varepsilon$.

Proof. L_i = list we maintain

U_i = list we would maintain if we would never do any trimming

Claim: For any y in U_i there is z in L_i such that $z \leq y \leq (1+\delta)^i \cdot z$

Proof of Claim. By induction. ■

OPT = value of optimal solution.

Then OPT is max number in U_n, so there is z in L_i with

$$z \geq (1 / (1+\delta)^n) \cdot \text{OPT}$$

$$= (1 / (1+ \varepsilon/(2n))^n) \cdot \text{OPT}$$

$$\geq (1 / (1+ \varepsilon)) \cdot \text{OPT}$$

■

Lemma. If we choose $\delta = \varepsilon/(2n)$, then approximation ratio is $1+\varepsilon$.

Lemma. The size of any list L_i is $O((1/\delta) \cdot \log t))$.

Plug in $\delta = \varepsilon/(2n)$: size of L_i is $O((n/\varepsilon) \cdot \log t))$

Need $\log t$ bits to represent t

➔ size of L_i polynomial in input size and $1/\varepsilon$

Running time is $O(\Sigma_i |L_i|)$

➔ running time polynomial in input size and $1/\varepsilon$

Theorem. There is an FPTAS for the Subset Sum optimization problem.

Summary

- ρ-approximation algorithm:
  algorithm for which computed solution is within factor ρ from OPT

- to prove approximation ratio we usually need lower bound on OPT
  (or, for maximization, upper bound)

- PTAS = polynomial-time approximation scheme
  = algorithm with two parameters, input instance and ε > 0, such that
  - approximation ratio is1+ ε
  - running time is polynomial in *n* for constant ε

- FPTAS = PTAS whose running time is polynomial in 1/ ε

- some problems are even hard to approximate