$$f(x,y) = \begin{cases} x^2+y^2-8x-4y+10 & \alpha(x,y) \text{ IF } x=2y \\ x^2+y^2-4x-2y & \beta(x,y) \text{ OTHERWISE} \end{cases}$$

EXAMINE POINT $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\frac{\partial f}{\partial x}(2,1) = \lim_{h \to 0} \frac{f(2+h,1) - f(2,1)}{h}$$

$$f(2+h,1) \overset{\text{because } x \neq 2y}{=} \beta(2+h,1)$$

$$f(2,1) \overset{\text{because } x = 2y}{=} \alpha(2,1) = -5 = \beta(2,1)$$

HENCE $\frac{\partial f}{\partial x}(2,1) = \frac{\partial \beta}{\partial x}(2,1) = 2x - 4 \overset{\text{substitute } x=2}{=} 0$

SIMILARLY, $\frac{\partial f}{\partial y}(2,1) = \frac{\partial \beta}{\partial y}(2,1) = 2y - 2 \overset{\text{substitute } y=1}{=} 0$

HENCE $\nabla_f(2,1) = \vec{0}$ 1

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(2,1) = \lim_{h \to \emptyset} \frac{\frac{\partial f}{\partial x}(2+h,1) - \frac{\partial f}{\partial x}(2,1)}{h}$$

這兩項不一樣
PARTIAL DERIVATIVE 是否本身
就有問題?

$$\frac{\partial f}{\partial x}(2+h,1) = \lim_{k \to \emptyset} \frac{f(2+h+k,1) - f(2+h,1)}{k}$$

$$f(2+h+k,1) \quad \text{IF} \quad \begin{aligned} & h \to \emptyset^-, k \to \emptyset^- \overset{\text{because } x \neq 2y}{=\!=\!=} \beta(2+h+k,1) \\ & h \to \emptyset^-, k \to \emptyset^+ \overset{\text{because } x = 2y}{=\!=\!=} \alpha(2,1) = -5 = \beta(2,1) \\ & h \to \emptyset^+, k \to \emptyset^- \overset{\text{because } x = 2y}{=\!=\!=} \alpha(2,1) = -5 = \beta(2,1) \\ & h \to \emptyset^+, k \to \emptyset^+ \overset{\text{because } x \neq 2y}{=\!=\!=} \beta(2+h+k,1) \end{aligned}$$

GENERALIZE AS $\beta(2+h+k,1)$

$$f(2+h,1) \overset{\text{because } x \neq 2y}{=\!=\!=} \beta(2+h,1)$$

HENCE $\frac{\partial f}{\partial x}(2+h,1) = \frac{\partial \beta}{\partial x}(2+h,1)$

ALSO, WE'VE PROVED THAT $\frac{\partial f}{\partial x}(2,1) = \frac{\partial \beta}{\partial x}(2,1)$

HENCE $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(2,1) = \frac{\partial}{\partial x}\left(\frac{\partial \beta}{\partial x}\right)(2,1) \overset{\text{because } \frac{\partial \beta}{\partial x}(2,1) = 2x-4}{=\!=\!=\!=\!=} 2$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(2,1) = \lim_{h\to\emptyset} \frac{\frac{\partial f}{\partial x}(2,1+h) - \frac{\partial f}{\partial x}(2,1)}{h}$$

$$\frac{\partial f}{\partial x}(2,1+h) = \lim_{k\to\emptyset} \frac{f(2+k,1+h) - f(2,1+h)}{k}$$

$f(2+k, 1+h)$ IF

$$h\to\emptyset^-,\ k\to\emptyset^- \ \overset{\text{because } x\neq 2y}{=\!=\!=} \ \beta(2+k, 1+h)$$

$$h\to\emptyset^-,\ k\to\emptyset^+ \ \overset{\text{because } x\neq 2y}{=\!=\!=} \ \beta(2+k, 1+h)$$

$$h\to\emptyset^+,\ k\to\emptyset^- \ \overset{\text{because } x\neq 2y}{=\!=\!=} \ \beta(2+k, 1+h)$$

$$h\to\emptyset^+,\ k\to\emptyset^+ \ \overset{\text{because } x\neq 2y}{=\!=\!=} \ \beta(2+k, 1+h)$$

$$f(2,1+h) \ \overset{\text{because } x\neq 2y}{=\!=\!=} \ \beta(2,1+h)$$

HENCE $\frac{\partial f}{\partial x}(2,1+h) = \frac{\partial \beta}{\partial x}(2,1+h)$

AGAIN, WE KNOW THAT $\frac{\partial f}{\partial x}(2,1) = \frac{\partial \beta}{\partial x}(2,1)$

HENCE $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(2,1) = \frac{\partial}{\partial y}\left(\frac{\partial \beta}{\partial x}\right)(2,1) \ \overset{\text{because } \frac{\partial \beta}{\partial x}(2,1) = 2x-4}{=\!=\!=} \ \emptyset$

DOING SIMILAR DEDUCTIONS FOR $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ AND $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$, WE HAVE

AT $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\frac{\partial^2 f}{(\partial x)^2} = 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{(\partial y)^2} = 2$$

$$H = \frac{\partial^2 f}{(\partial x)^2} \cdot \frac{\partial^2 f}{(\partial y)^2} - \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial^2 f}{\partial x \partial y} = 4 > 0 \quad (2)$$

BY (1) AND (2), $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ IS A LOCAL MINIMUM POINT

BUT THIS IS NOT TRUE, AS FOR $h \in \mathbb{R}$, $h \to 0^+$:

$$f(2+2h, 1+h) = \alpha(2+2h, 1+h) < \alpha(2,1) = f(2,1)$$

WHY THAT IS TRUE?

LET $x = 2y$, $\alpha(y) = 2y^2 - 12y + 10$

EXAMINE $\frac{d\alpha}{dy}(1) = 4y - 12 = -8 < 0$

OR $\lim_{h \to 0} \frac{\alpha(1+h) - \alpha(1)}{h} < 0$

$\Rightarrow \lim_{h \to 0^+} \frac{\alpha(1+h) - \alpha(1)}{h} < 0$

$\Leftrightarrow \alpha(1+h) - \alpha(1) < 0 \quad (h \to 0^+)$

OR $\alpha(1+h) < \alpha(1) \quad (h \to 0^+)$

I.E. $\alpha(2+2h, 1+h) < \alpha(2,1) \quad (h \to 0^+$

PROVED!