Домашняя работа

(4.) $f(z) = u + iv$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ - усл. Коши-Римана

1. $w(z) = x^2 + y^2$, $u = x^2 + y^2$, $v = 0$

$\frac{\partial u}{\partial x} = 2x \neq 0$

2. $w(z) = x^2 - y^2 + 2ixy$ $u = x^2 - y^2$ $v = 2xy$

$2x = 2x$ $-2y = -2y$

3. $w(z) = \frac{1}{(x+iy)} = \frac{x-iy}{x^2+y^2}$, $u = \frac{x}{x^2+y^2}$, $v = -\frac{y}{x^2+y^2}$

$$\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = -\frac{(x^2+y^2-2y^2)}{(x^2+y^2)^2}$$

$$+\frac{2xy}{(x^2+y^2)^2} = +\frac{2xy}{(x^2+y^2)^2}$$

5 $|f| = e^{r^2\cos 2\varphi}$

$z = re^{i\varphi}$ $\quad f(z) - ?$

$|f| = e^{r^2(\cos^2\varphi - \sin^2\varphi)} = e^{x^2-y^2}$

$|f|^2 = ff^* = e^{2(x^2-y^2)}$.

$z = x+iy$

$z^2 = (x^2-y^2) + 2ixy$

$z^{*2} = (x^2-y^2) - 2ixy$

$2(x^2-y^2) = z^2 + z^{*2}$

$ff^* = e^{z^2+z^{*2}}$

$f = e^{z^2}$

~~Arg f = xy~~

7 $\int_c z\,dz = \left|\begin{matrix} z = e^{i\varphi} \\ dz = ie^{i\varphi}d\varphi \end{matrix}\right| = \int_0^{2\pi} ie^{2i\varphi}\,d\varphi = i\int_0^{2\pi}(\cos 2\varphi +$

$+ i\sin 2\varphi)\,d\varphi = 0$

$\int_c z^*\,dz = \left|\begin{matrix} z = e^{i\varphi} \\ z^* = e^{-i\varphi} \\ dz = ie^{i\varphi}d\varphi \end{matrix}\right| = i\int_0^{2\pi} d\varphi = 2\pi i$

8) $\int_c \frac{y\,dx - x\,dy}{x^2+y^2}$

1. $x=0,\ y=0.$

$$\int_c \frac{y\,dx - x\,dy}{x^2+y^2} = \int_c -\mathrm{Im}\,\frac{dz}{z} = \left| \begin{matrix} z = e^{i\varphi} \\ dz = ie^{i\varphi}d\varphi \end{matrix} \right| =$$

$$= -\int_c \mathrm{Im}\,\frac{ie^{i\varphi}d\varphi}{e^{i\varphi}} = -2\pi$$

2. $x=2,\ y=0.$ по т. Коши:

$$\int_c \frac{y\,dx - x\,dy}{x^2+y^2} = 0.$$

9) $p(n) = \frac{1}{2\pi i}\int_c dz\, z^{-1-n} \prod_{k=1}^{\infty} \frac{1}{1-z^k}$

$$\frac{1}{1-z^k} = \cancel{1 + \ldots} = 1 + z^k + z^{2k} + z^{3k} \ldots$$

1. $p(1) = \frac{1}{2\pi i}\int_c \frac{dz}{z^2} \prod_{k=1}^{\infty} \frac{1}{1-z^k} =$

$$= \frac{1}{2\pi i}\int_c \frac{dz}{z^2}(1+z+\ldots)(1+z^2+\ldots)(1+z^3+\ldots) =$$

$$= \frac{1}{2\pi i}\int_c \frac{dz}{z^2}(1+z+\ldots) = \frac{1}{2\pi i}\int_c \frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \cdot 2\pi i = 1$$

2. $p(4) = \frac{1}{2\pi i}\int \frac{dz}{z^5} \prod_{k=1}^{\infty} \frac{1}{1-z^k} = \frac{1}{2\pi i}\int_c \frac{dz}{z^5}(1+z+z^2+z^3+z^4+z^5+\ldots)\ \cancel{\ldots}\ (1+z^2+z^4+\ldots)(1+z^3+\ldots)(1+z^4+\ldots)(1+z^5+\ldots) = \frac{1}{2\pi i} 5\int \frac{dz}{z} = \frac{5}{2\pi i} \cdot 2\pi i = 5$

⑥ Для аналитической ф-ции:

$$\partial_{xy}^2 \begin{matrix} \nearrow \partial_x\partial_x u = \partial_x^2 u \\ \searrow \partial_y\partial_x v = -\partial_y^2 u \end{matrix} \Bigg\} \quad (\partial_x^2 + \partial_y^2)\, u = 0$$

$u = \operatorname{Re} f(z)$, $\quad u = \varphi(x^2 - y^2)$

$\partial_x^2 \varphi(x^2 - y^2) = \partial_x\left(2x\varphi'(x^2-y^2)\right) = [2\varphi'(x^2-y^2) +$

$+ 4x^2 \varphi''(x^2 - y^2)]$.

$\partial_y^2 \varphi(x^2 - y^2) = \partial_y\left(-2y\varphi'(x^2-y^2)\right) = (-2\varphi'(x^2-y^2) +$

$+ 4y^2 \varphi''(x^2 - y^2))$

$(\partial_x^2 + \partial_y^2) u = (x^2+y^2)\varphi''(x^2-y^2) = 0 \Rightarrow$

$\varphi(t) = at + b$

$u(x, y) = b + a(x^2 - y^2)$

Из усл. Коши-Римана:

$\frac{\partial v}{\partial y} = 2ax$, $\quad v = 2axy + c$

$f(x, y) = b + a(x^2 - y^2) + i\, 2axy =$

$= b + a(x^2 - y^2 + 2ixy) = b + az^2 = f(z)$

$$u = \varphi\left(\frac{y}{x}\right)$$

$$\partial_x^2 \varphi\left(\frac{y}{x}\right) = \partial_x\left(\varphi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\right) = \varphi''\cdot\left(-\frac{y}{x^2}\right)^2 +$$

$$+ \varphi'\cdot\left(\frac{2y}{x^3}\right)$$

$$\partial_y^2 \varphi\left(\frac{y}{x}\right) = \partial_y\left(\varphi'\cdot\frac{1}{x}\right) = \varphi''\cdot\frac{1}{x^2}$$

$$\varphi''\frac{y^2}{x^4} + 2\varphi'\frac{y}{x^3} + \varphi''\frac{1}{x^2} = 0$$

$$\varphi''\frac{y^4}{x^4} + 2\varphi'\frac{y^3}{x^3} + \varphi''\frac{y^2}{x^2} = 0$$

$$\varphi''(t^4+t^2) + 2\varphi' t^3 = 0$$

$$\varphi' = \xi$$

$$\frac{d\xi}{dt}(t^4+t^2) = 2\xi t^3$$

$$\frac{d\xi}{\xi} = 2\frac{t^3}{t^4+t^2}dt$$

$$\frac{d\xi}{\xi} = 2\frac{t}{t^2+1}dt$$

$$\frac{d\xi}{\xi} = \frac{d(t^2+1)}{t^2+1}$$

$$\ln\xi = \ln(t^2+1) + \ln C$$

$$\xi = (t^2+1)C_0$$

$$\frac{d\varphi}{dt} = (t^2+1)C_0$$

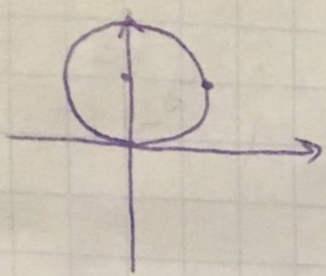
$$\varphi = \frac{t^3}{3}C_0 + tC_0 + C_1$$

$$u = \left(\frac{y}{x}\right)^3\frac{C_0}{3} + \frac{y}{x}C_0 + C_1$$

$$-\frac{\partial v}{\partial x} = \frac{\cancel{3}y^2}{x^3}\cdot\frac{C_0}{\cancel{3}} + \frac{C_0}{x}$$

$$v = +\frac{y^2}{2x^2}C_0 - C_0\ln x + C_1$$

③ (ii) $|z-i| = 1$ $z \to w(z) = \frac{1}{z-2i}$



Прямая.

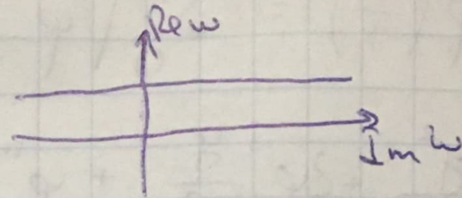
$w(0) = -\frac{1}{2i} = \frac{i}{2}$

$w(1+i) = \frac{1}{1+i-2i} = \frac{1}{1-i} =$

$= \frac{1+i}{1+1} = \frac{1}{2} + \frac{i}{2}$.

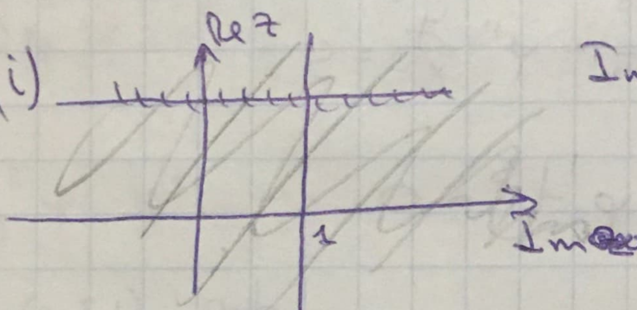
$A = (0; \frac{1}{2})$

$B = (\frac{1}{2}; \frac{1}{2})$.

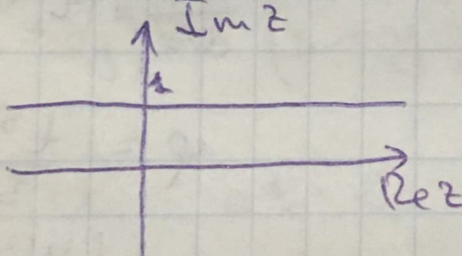


$\operatorname{Im} w = \frac{1}{2} + 0 \times \operatorname{Re} w$

(i)



$\operatorname{Im} z = 1$



$\operatorname{Im} z = 1$

$z = i + \operatorname{Re} z$

$w(z) = (i + \operatorname{Re} z)^3 + 3(i + \operatorname{Re} z) - i = -i - 3\operatorname{Re} z +$

$+ 3i \operatorname{Re} z^2 + \operatorname{Re} z^3 + 3i + 3\operatorname{Re} z - i = i + 3i \operatorname{Re} z^2 + \operatorname{Re} z^3$

$\operatorname{Re} w = \operatorname{Re} z^3$

$\operatorname{Im} w = 1 + 3\operatorname{Re} z^2 = 1 + 3 |\operatorname{Re} w|^3$

$$\oint = 2\pi i \sum_{z_i \in in} \operatorname{res} f(z)$$

Продолжение ДЗ.

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$$f = |f| e^{i \arg f}$$

$$w = \ln f = \overbrace{\ln |f|}^{u} + i \overbrace{\arg f}^{v}$$

Из усл. Коши-Римана

$$\frac{\partial v}{\partial y} = x = \frac{\partial u}{\partial x} \qquad u = \frac{x^2}{2} + \varphi(y)$$

$$\frac{\partial v}{\partial x} = y = -\frac{\partial u}{\partial y} = -\varphi'(y)$$

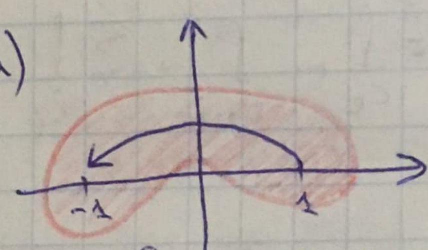
$$\varphi(y) = -\frac{y^2}{2} + C$$

$$u = \frac{x^2 - y^2}{2} + C$$

$$|f| = e^{\frac{x^2-y^2}{2} + C}$$

$$f = e^{\frac{x^2-y^2}{2} + ixy + C} = e^{\frac{1}{2}(x^2 - y^2 + 2ixy) + C} = e^{\frac{z^2}{2} + C}$$

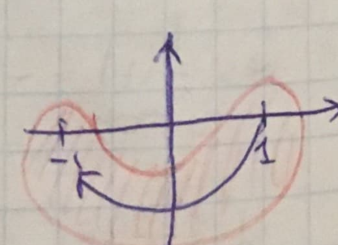
(10) а)



Т.к. против часовой

$$y(-1) = y(1) + \int_1^{-1} y'(z)\,dz = \int_{1}^{-1} \frac{dz}{2z} = \left| \begin{matrix} z = Re^{i\varphi} \\ dz = iRe^{i\varphi}d\varphi \end{matrix} \right| =$$

$$= \int_0^{\pi} \frac{i\,d\varphi}{2} = \frac{i\pi}{2}.$$



б) Т.к. по часовой.

$$y(-1) = y(1) + \int_1^{-1} y'(z)\,dz = +\int_0^{-\pi} \frac{i\,d\varphi}{2} = -\frac{i\pi}{2}.$$

Домашняя работа №1 (Продолжение)

(11) $\dfrac{1+2z^2}{z^3+z^5} = \dfrac{1+2z^2}{z^3(1+z^2)} = \dfrac{(1+2z^2)(1-z^2+\dots)}{z^3} =$

$$= \frac{1-z^2+2z^2+\dots}{z^3} = \frac{1+z^2}{z^3}$$

(12) $f(z) = \dfrac{1}{z(e^z-1)} = \dfrac{1}{z(1-1+z+\dots)} \sim \dfrac{1}{z^2}$, т.е

порядок полюса – 2.

$$f(z) = \frac{1}{z\left(1-1+z+\frac{z^2}{2}-\dots\right)} = \frac{1}{z\left(z+\frac{z^2}{2}\right)} =$$

$$= \frac{1}{z^2\left(1+\frac{z}{2}\right)} = \frac{1}{z^2}\left(1-\frac{z}{2}\right) = \underline{-\frac{1}{2z}} + \frac{1}{z^2} + \dots$$

(13) $f(z) = \dfrac{1}{z(z-1)}$

1. $|z| \in (0;1)$

$$\frac{1}{z(z-1)} = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$$

2. $|z| \in (1;+\infty)$

$$\frac{1}{z(z-1)} = \frac{1}{z^2\left(1-\frac{1}{z}\right)} = -\frac{1}{z} + \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z} + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n =$$

$$= -\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n = -\frac{1}{z} + \frac{1}{z} + \sum_{n=2}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=2}^{\infty}\left(\frac{1}{z}\right)^n$$

(14) $f(z) = \dfrac{z}{z^2+1}$ $\qquad z = i$

$z = \varepsilon + i$

$$\frac{z}{(z-i)(z+i)} = \frac{\varepsilon+i}{\varepsilon(\varepsilon+2i)} = \frac{\varepsilon+i}{2\varepsilon i\left(1+\frac{\varepsilon}{2i}\right)} =$$

$$= \frac{\varepsilon+i}{2\varepsilon i\left(1-\frac{i\varepsilon}{2}\right)} = \frac{i}{2i\varepsilon} - \frac{i/4}{1-i\frac{\varepsilon}{2}} =$$

$$= \frac{1}{2\varepsilon} - \frac{i}{4}\sum_{n=0}^{\infty}\left(\frac{i}{2}\varepsilon\right)^n =$$

$\to \left|\frac{\varepsilon}{2}\right| < 1$

$|\varepsilon| < 2$

$|z-i| < 2$

$$= \frac{1}{2(z-i)} - \frac{i}{4}\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n (z-i)^n$$

(1) $2 \le |z-i| \le 4$

$2 \le |x+i(y-1)| \le 4$

$2 \le \sqrt{x^2+(y-1)^2} \le 4$

Центр: $i = z_0$

Площадь: $S = \pi(R^2-r^2) = \pi(16-4) = 12\pi$

$|z-4i| + |z+4i| = 10$

$|x+i(y-4)| + |x+i(y+4)| = 10$

$f(x,y) = \sqrt{x^2+(y-4)^2} + \sqrt{x^2+(y+4)^2} = 10$

$f(-x,-y) = f(x,-y) = f(-x,y) = f(x,y)$ –

картина сим-тна отн-но $(0;0)$ – центр!

Получим:

$$\sqrt{x^2+(y-4)^2}+\sqrt{x^2+(y+4)^2}=10$$

$x=0:$

$$|y-4|+|y+4|=10$$

$$y-4+y+4=10$$

$\underline{a=5}$

$y=0:$

$$2\sqrt{x^2+16}=10$$

$$x^2=\cancel{8}\,9$$

$\underline{b=3}$

$$\operatorname{Im}\frac{1}{z}=1$$

$$\operatorname{Im}\frac{z^*}{|z|^2}=1$$

$$-y=y^2+x^2$$

$$x^2+y^2+y+\frac{1}{4}=\frac{1}{4}$$

$$x^2+\left(y+\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$$

② $$1+\varepsilon+\varepsilon^2+\dots+\varepsilon^n = \frac{\varepsilon^n-1}{-\varepsilon+1}$$

$$1+2\varepsilon+3\varepsilon^2+\dots+n\varepsilon^{n-1} = \frac{n\varepsilon^n(1-\varepsilon)+(\varepsilon^n-1)}{(1-\varepsilon)^2} =$$

$$= \frac{n\left(\varepsilon^n-\varepsilon^{n+1}\right)-\varepsilon^n+1}{(\varepsilon-1)^2}$$

$=$