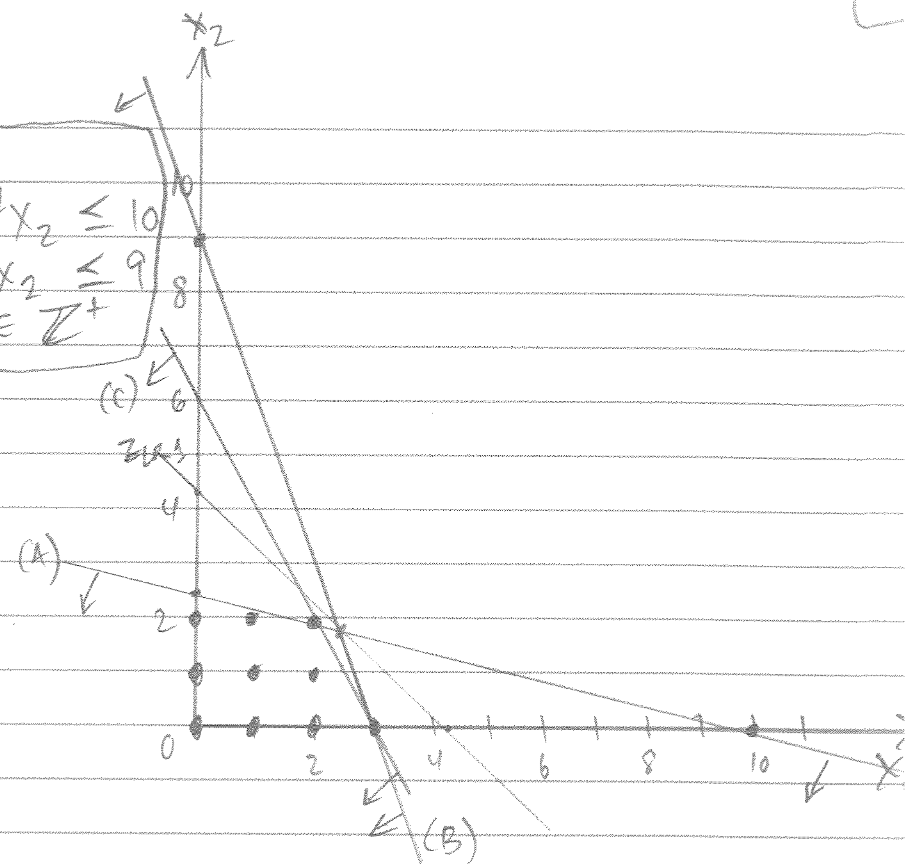
Paul Holser IMSE 884 Hw #3

1) $\max\ x_1 + x_2$

s.t. $x_1 + 4x_2 \le 10$ (A)

$3x_1 + x_2 \le 9$ (B)

$x_1, x_2 \in \mathbb{Z}^+$



$4x_1 + 5x_2 = 19$

$5x_2 = -4x_1 + 19$

$x_2 = \frac{19}{5} - \frac{4}{5}x_1$

$3x_1 + \left(\frac{19}{5} - \frac{4}{5}x_1\right) = 9$

$\frac{11}{5}x_1 + \frac{19}{5} = 9$

$11/5\, x_1 = \frac{26}{5}$

$x_1 = 26/11$

$x_2 = \frac{19}{5} - \frac{4}{5}\left(\frac{26}{11}\right)$

$= \frac{\frac{209}{11} - \frac{104}{11}}{5}$

$= \frac{105}{55}$

$= \frac{21}{11}$

$x^*_{LR} = \left(\frac{26}{11}, \frac{21}{11}\right)$

$z^*_{LR} = 47/11$

$\rightarrow$ Choose cutting plane through $(2,2)$ and $(3,0)$

$m = \frac{0-2}{3-2} = -2$

$x_2 = -2x_1 + b$

$\cancel{b = x_2 + 2x_1 = 0 + 2(3) = 6}$

$\rightarrow 2x_1 + x_2 \le 6$ (C)

new LR: $x^* = (2,2)$, $z^* = 4$

This is an integer soln to an LR optimal

$\rightarrow$ it's optimal for the IP. Done.

(cont'd) $\Rightarrow$

1) (cont'd)

Proving that $2x_1 + x_2 \leq 6$ is facet-defining:

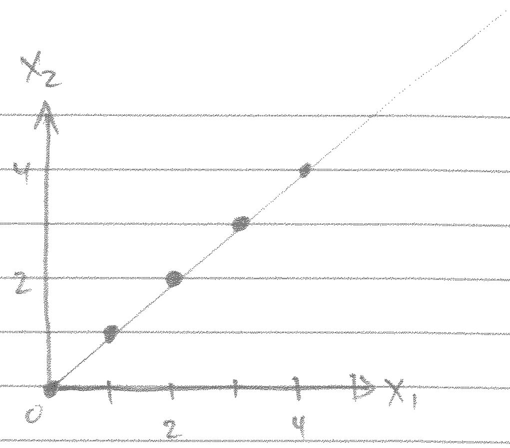
★ Finding $\dim(P_{ch})$:
- $(0,0)$, $(0,1)$, and $(1,0)$ are three affinely independent points in P_{ch}
  $\rightarrow \dim(P_{ch}) \geq 2$
- $P_{ch} \subset \mathbb{R}^2 \rightarrow \dim(P_{ch}) \leq 2$
- Taking these two inequalities together, $\underline{\dim(P_{ch}) = 2}$.

★ From the graph, we see that our cutting plane does not eliminate any points from P from consideration
$\rightarrow$ it is a valid inequality.
Also can test each member of P against the inequality to see that $2x_1 + x_2 \leq 6 \; \forall (x_1, x_2) \in P$

★ The induced face F is $\{(x_1, x_2) \in P_{ch} : 2x_1 + x_2 \leq 6\}$.
Consider points $(2,2)$ and $(3,0)$ in F.
They are affinely independent, so $\dim(F) \geq 1$.
Further, consider point $(0,0) \in P_{ch}$ — this is $\notin F$, so F is not the entire space
$\rightarrow \dim(F) < \dim(P_{ch})$
$\rightarrow \dim(F) = 1$
$\rightarrow$ F is a facet, and $2x_1 + x_2 \leq 6$ defines it

$\blacksquare$

(2) Consider the IP:



$$\begin{aligned} \max\ & -x_1 \\ \text{s.t. } & x_1 - x_2 \le \emptyset \\ & -x_1 + x_2 \le \emptyset \\ & x_1, x_2 \in \mathbb{Z}^+ \end{aligned}$$

Then $P = \{(0,0) + (i,i)\ \forall i \in \mathbb{Z}^+\}$
$P_{ch} = \{(0,0) + (c,c)\ \forall c \in \mathbb{R}^+\}$

$\dim(P_{ch})$ = # of linearly independent vectors
= $\underline{1}$, since you can always express one member as a multiple of another.

The two inequalities induce faces
$F_1 = \{(x_1, x_2) \in P_{ch} : x_1 - x_2 = \emptyset\}$ and
$F_2 = \{(x_1, x_2) \in P_{ch} : x_2 - x_1 = \emptyset\}$.
But $F_1 = F_2 \rightarrow$ call the single face F.
For F to be a facet, we would need for $\dim(F) = \dim(P_{ch}) - 1 = \emptyset$.
But notice that the set of points in F must be exactly the points of $P_{ch} \rightarrow$ only those non-negative reals (x_1, x_2) where $x_1 = x_2$.
So $F = P_{ch}$, thus $\dim(F) = \dim(P_{ch}) = 1$ and F is not a facet.

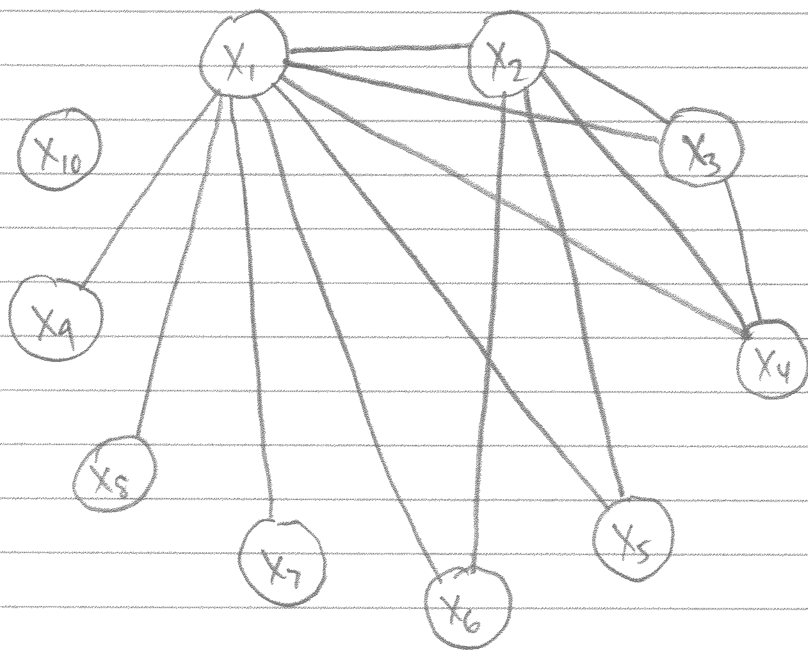
No other inequalities describe P / P_{ch}, hence the IP has no facets.

∎

(3) Knapsack constraint:

$$35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39$$

(a) Conflict graph



There is a clique $\{x_1, x_2, x_3, x_4\}$, hence the clique inequality $x_1 + x_2 + x_3 + x_4 \leq 1$.

Is it facet-defining?

★ Finding $\dim(P_{ch})$:

Take e_i $\forall i \in \{x_1, x_2, \ldots, x_{10}\}$, plus the origin.
These are 11 affinely independent points in P_{ch}
$\rightarrow \dim(P_{ch}) \geq 10$
There are 10 variables $\rightarrow \dim(P_{ch}) \leq 10$
$\rightarrow \dim(P_{ch}) = 10$.

(cont'd) $\Longrightarrow$

3a (cont'd)

★ The inequality $x_1 + x_2 + x_3 + x_4 \le 1$ is satisfied by all points in P since setting any two of the involved variables to 1 would violate the knapsack constraint; so, the inequality is valid.

★ The induced face F is $\{\vec{x} \in P_{ch} : x_1 + x_2 + x_3 + x_4 = 1\}$. Consider point $(1, 0, \ldots, 0)$; it is in P_{ch} but not in F, so F doesn't define the whole space
$\rightarrow \dim(F) < \dim(P_{ch})$

Consider these ten points in P_{ch} that meet F, where the specified variables are 1 and the rest $\emptyset$:
$\{x_1\}$ $\{x_2\}$ $\{x_3\}$ $\{x_4\}$
$\{x_2, x_7\}$, $\{x_2, x_8\}$, $\{x_2, x_9\}$, $\{x_2, x_{10}\}$,
$\{x_3, x_5\}$, $\{x_3, x_6\}$.

$$\begin{array}{c}
\begin{bmatrix}
1 & & & & & & & & & \\
 & 1 & & & 1 & 1 & 1 & 1 & & \\
 & & 1 & & & & & & 1 & 1 \\
 & & & 1 & & & & & & \\
 & & & & & & & & 1 & \\
 & & & & & & & & & 1 \\
 & & & & 1 & & & & & \\
 & & & & & 1 & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 1 & &
\end{bmatrix} \\
\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}
\end{array}$$

(0's implied in other positions)

These points are affinely independent, so
$\dim(F) \ge 9$
$\rightarrow \dim(F) = 9$ (since also $< \dim(P_{ch})$
$= 10$)
$= \dim(P_{ch}) - 1$.

Thus $x_1 + x_2 + x_3 + x_4 \le 1$ is facet-defining. ∎

3b) If we set x_7, x_8, x_9, and x_{10} (the variables with the smallest coefficients) to 1, and the rest to $\emptyset$, we're still feasible. Then setting the next-highest coefficient's variable (x_6) also to 1 makes us infeasible. Since setting the five variables with the least coefficients to 1 makes us infeasible, no other set of five variables set to 1 can be a feasible solution.

→ We have a conflict hyperclique, with edge size 5, so that we have the hyperclique inequality

$$x_1 + x_2 + \dots + x_{10} \leq 4. \quad (*)$$

This can cut off sections of the feasible region of the linear relaxation, e.g.:

$(0, 0, 0, 0, \underbrace{\ ,\ }, \underbrace{1, 1, 1, 1})$

either one of these $= 2/3$, the other $\emptyset$.

29

★ $\dim(P_{ch}) = 10$, as shown in part (a)

★ All feasible points in P must satisfy the hyperclique inequality $(*)$

★ There are nine points in P_{ch} that meet the face F induced by $(*)$ at equality:

$\{x_4, x_8, x_9, x_{10}\}$ $\{x_6, x_7, x_8, x_{10}\}$
$\{x_5, x_6, x_9, x_{10}\}$ $\{x_6, x_7, x_9, x_{10}\}$
$\{x_5, x_7, x_8, x_{10}\}$ $\{x_6, x_8, x_9, x_{10}\}$
$\{x_5, x_7, x_9, x_{10}\}$ $\{x_7, x_8, x_9, x_{10}\}$
$\{x_5, x_8, x_9, x_{10}\}$

(cont'd) ⟹

3b (cont'd)

$$
\begin{array}{c}
\left[\begin{array}{ccccccccc}
0 & \cdots & & & & & & & 0 \\
0 & \cdots & & & & & & & 0 \\
0 & \cdots & & & & & & & 0 \\
1 & 0 & \cdots & & & & & & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & \cdots & & 0 \\
0 & 1 & & & & 1 & 1 & 1 & \\
0 & & 1 & 1 & & 1 & 1 & & 1 \\
1 & & 1 & & 1 & 1 & & 1 & 1 \\
1 & 1 & & 1 & 1 & & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] \\
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\end{array}
$$

There are six affinely independent points here, so $\dim(F) \geq 6$, which ~~is~~ is not $\dim(P_{cl}) - 1$.
$\rightarrow$ the hyperclique inequality does not define a facet.

$\blacksquare$

(4) The unit 4-D hypercube is defined by the inequalities:

$$x_1 \geq \emptyset \;;\; x_1 \leq 1$$
$$x_2 \geq \emptyset ; \;\; x_2 \leq 1$$
$$x_3 \geq \emptyset ; \;\; x_3 \leq 1$$
$$x_4 \geq \emptyset ; \;\; x_4 \leq 1$$

Clearly $\dim(P_{ch})$ for the hypercube $= 4$.

* Consider the inequality $x_4 \leq 1$.
  Then the induced face $F = \{\vec{x} \in P_{ch} : x_4 = 1\}$.

  = a unit cube, "advanced forward" one "unit" in time.
  This is certainly a facet — $\dim(\text{that cube}) = 3$.
  $= \dim(P_{ch}) - 1$.

* Consider the inequality $x_1 \leq 1$.
  Then the induced face $F = \{\vec{x} \in P_{ch} : x_1 = 1\}$

  This is like a cube induced by moving the width and depth of a unit cube through one "unit" of time.

  Similar to the first cube, $\dim(\text{this cube}) = 3$
  $= \dim(P_{ch}) - 1$,
  hence a facet.

∎