Home work Problems Taken From Nemhauser's Wolsey

Theorem 7.8, for bounded mixed-integer constraint sets, appears in Jeroslow (1979b). Its generalization to unbounded sets is given by Bachem and Schrader (1980) and by Bachem, Johnson, and Schrader (1982). Also see Blair (1978) and Jeroslow (1985).

9. EXERCISES

1. Let $S = \{x \in Z^2_+: 4x_1 + x_2 \leq 28,\ x_1 + 4x_2 \leq 27,\ x_1 - x_2 \leq 1\}$. Determine the facets of conv(S) graphically (see Exercise 10 of Section I.4.8). Then derive each of the facets of conv(S) as a C–G inequality.
2. Let $S = \{x \in Z^3_+: 19x_1 + 28x_2 - 184x_3 = 8\}$. Derive the valid inequality $x_1 + x_2 + 5x_3 \geq 8$ using modular arithmetic.
3. For $S = \{x \in B^4: 9x_1 + 7x_2 - 2x_3 - 3x_4 \leq 12,\ 2x_1 + 5x_2 + 1x_3 - 4x_4 \leq 10\}$ show that $4x_1 + 5x_2 - 2x_3 - 4x_4 \leq 12$ is a valid inequality by disjunctive arguments.
4. Consider the node-packing problem on the graph of Figure 9.1. Show that $\Sigma_{i=1}^7 x_i \leq 2$ is a valid inequality, both combinatorially and algebraically.
5. Prove the following:
   i) Let $P = \{x \in R^n: Ax \leq b\} \neq \emptyset$. $\pi x \leq \pi_0$ is a valid inequality for P if and only if there exists $u \in R^m_+$ such that $uA = \pi$ and $ub \leq \pi_0$.
   ii) Let $P = \{x \in R^m_+: Ax \leq b, x \leq d\}$. $\pi x \leq \pi_0$ is a valid inequality for P if and only if there exist $u \in R^m_+$ and $w \in R^n_+$ such that $uA + w \geq \pi$ and $ub + wd \leq \pi_0$.
6. Let $P_i = \{x \in R^n_+: A^i x \leq b_i\}$ for $i = 1, 2$. Show that $\pi x \leq \pi_0$ is a valid inequality for $P_1 \cup P_2$ if there exists $u^i \in R^m_+$ such that $u^i A^i \geq \pi$ and $u^i b_i \leq \pi_0$ for $i = 1, 2$. Under what restrictions on P_1 and P_2 does the converse hold?
7. (The Davis–Putnam Procedure). Consider the satisfiability problem for $S \subseteq B^n$ defined by

$$\sum_{j \in C_k} x_j + \sum_{j \in \tilde{C}_k} (1 - x_j) \geq 1 \quad \text{for } k = 1, \ldots, K, x \in B^n$$

where $C_k \cap \tilde{C}_k = \emptyset$ and $C_k, \tilde{C}_k \subseteq N$ for $k = 1, \ldots, K$.

i) Given $q \in N$ and a pair of constraints k, l such that $q \in C_k \cap \tilde{C}_l$, show that

$$\sum_{j \in (C_k \cup C_l) \setminus \{q\}} x_j + \sum_{j \in (\tilde{C}_k \cup \tilde{C}_l) \setminus \{q\}} (1 - x_j) \geq 1$$

is a valid inequality for S.

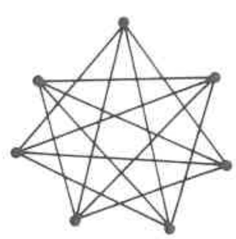

Figure 9.1

ii) Show that the inequality is a D-inequality.

iii) Show that if $S = \emptyset$, it is possible to generate the valid inequality $0x \leq -1$ by a finite number of replications of the procedure i.

iv) Show that the resulting algorithm is polynomial if $|C_k \cup \tilde{C}_k| \leq 2$ for all k.

8. What is the rank of conv(S) in Exercise 1?

9. Prove Propositions 2.9 and 2.10.

10. Show that the rank of conv(S^t) is $t - 1$, where $S^t = P^t \cap Z^2$ and

$$P^t = \{x \in R^2_+ \colon tx_1 + x_2 \leq 1 + t,\ -tx_1 + x_2 \leq 1,\ x_1 \leq 1\}.$$

11. Show that if $P = \{x \in R^n_+ \colon x_i + x_j \leq 1 \text{ for } 1 \leq i < j \leq n\}$ and $S = P \cap B^n$, the rank of $\sum_{j=1}^n x_j \leq 1$ is $O(\log n)$.

12. Use Theorem 2.5 to show that every valid inequality is a D-inequality for mixed 0-1 programs.

13. Consider the integer program $\max\{2x_1 + 5x_2 \colon x \in S\}$, where S is given in Exercise 1. Using the optimal basis of the corresponding linear program, the problem can be rewritten as

$$\begin{array}{lrrrrrl} \max z & & & & & & \\ z & & + \frac{1}{5}x_3 & + \frac{6}{5}x_4 & & = & 38 \\ & x_1 & + \frac{4}{15}x_3 & - \frac{1}{15}x_4 & & = & \frac{17}{3} \\ & x_2 & - \frac{1}{15}x_3 & + \frac{4}{15}x_4 & & = & \frac{16}{3} \\ & & - \frac{1}{3}x_3 & + \frac{1}{3}x_4 & + x_5 & = & \frac{2}{3} \\ & & x \in Z^5_+. & & & & \end{array}$$

Derive a Gomory fractional cut from each equation. Express each cut in terms of the original variables (x_1, x_2). Derive each cut as a rank 1 C–G inequality.

14. For $S = P \cap Z^2$ as given in Exercise 1 show that

i) $x_1 \leq 5$,

ii) $x_1 + 2x_2 \leq 15$, and

iii) $2x_1 + 5x_2 \leq 36$

are superadditive valid inequalities.

15. What conditions must be imposed on F so that $\sum_{j=1}^n F(a_j)x_j \leq F(b)$, is a valid inequality for $S = \{x \in Z^n \colon Ax \leq b\}$?

16. Show that the following functions are superadditive:

i) $G(d) = \max\{\alpha, F(d)\}$, where $\alpha < 0$ and F is superadditive.

ii) $G(d) = \max_{h \in Z^m} \{F_1(h) + F_2(d - h)\}$, where F_1 and F_2 are superadditive on Z^m.

iii) $G_\alpha(d) = \max\{\alpha, \min(0, d)\}$ for $d \in R^1$ and $\alpha < 0$.

Grötschel, 1980b). Another such class of graphs has been studied by Papadimitriou and Yannakakis (1984).

Other polyhedral results for the symmetric traveling salesman problem have been obtained by Cornuejols and Pulleyblank (1982), Cornuejols, Naddef and Pulleyblank (1983), and Cornuejols, Fonlupt and Naddef (1985).

Facets for the convex hull of tours on a directed graph have been studied by Grötschel and Padberg (1975) and Grötschel and Wakabayashi (1981a,b). Grötschel and Padberg (1985) surveyed these results.

Section II.2.4

The basic results for the variable upper-bound flow model are from Padberg, Van Roy and Wolsey (1985). Martin and Schrage (1985) obtained similar inequalities using different arguments. Van Roy and Wolsey (1986) have generalized these results to handle variable lower bounds.

The facet-defining inequalities for the lot-size model (4.8) were developed in Barany et al. (1984). Extensions to handle capacities are given in Leung and Magnanti (1986) and Pochet (1988), and those to treat backlogging are given in Pochet and Wolsey (1988). Valid inequalities for more general fixed-cost network problems are given in Van Roy and Wolsey (1985).

6. EXERCISES

1. Use clique inequalities, odd hole inequalities, and lifting to derive facets for the convex hull of node packings for the graph in Figure 6.1.

2. Prove Proposition 1.2.

3. Consider the uncapacitated facility location problem (UFL) introduced in Section I.1.3, with

$$T = \left\{x \in B^n, y \in R_+^{mn}: \sum_{j \in N} y_{ij} = 1 \text{ for } i \in M,\ y_{ij} \leqslant x_j \text{ for all } i \in M, j \in N\right\}.$$

   i) Show that $\dim(\text{conv}(T)) = mn - m + n$.

   ii) Show that $y_{ij} \leqslant x_j$ define facets of $\text{conv}(T)$.

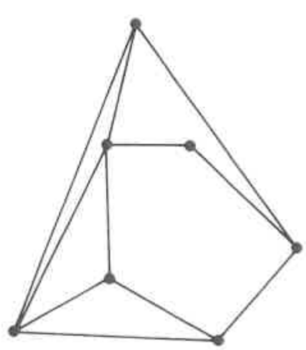

Figure 6.1

4. Let $G = (V, E)$ be a graph where each node has degree at least 3. Consider the set

$$S = \left\{ x \in B^{|E|}: x_j - \sum_{e \in \delta(v) \setminus \{j\}} x_e \leqslant 0 \text{ for all } j \in \delta(v) \text{ and } v \in V \right\}$$

where $\delta(v)$ denotes the set of edges incident to node v.

i) Show that the inequalities $x_e \leqslant 1$ define facets of conv(S).

ii) Show that the inequalities $x_j - \Sigma_{e \in \delta(v) \setminus \{j\}} x_e \leqslant 0$ define facets of conv(S).

5. Consider the linear ordering problem of determining a permutation π: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ formulated as

$$\max \sum_{ij} c_{ij} \delta_{ij}$$

$$\delta_{ij} + \delta_{ji} = 1 \quad \text{for all } i < j$$

$$\delta_{j_1 j_2} + \cdots + \delta_{j_r j_1} \leqslant |C| - 1 \quad \text{for all cycles } C = \{j_1, \ldots, j_r\}$$

$$\delta \in B^{n(n-1)},$$

where $\delta_{ij} = 1$ if i precedes j.

i) Show that the inequalities with $|C| \geqslant 4$ are unnecessary in the description of the problem.

ii) Show that for $|C| = 3$, the inequalities define facets.

6. For $S = \{x \in B^n: \Sigma_{j \in N} a_j x_j \leqslant b\}$, show that $x_j \geqslant 0$ and $x_j \leqslant 1$ define facets of conv(S) when $a \in Z_+^n$ and $a_j + a_k \leqslant b$ for all $j, k \in N$ with $j \neq k$.

7. For Example 2.3 use Propositions 2.3 and 2.6 to find as many facets as you can. Use these results to solve

$$\max 12x_1 + 5x_2 + 8x_3 + 7x_4 + 5x_5 + 5x_6 + 4x_7 + 3x_8 + 2x_9 + x_{10}$$

$$35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leqslant 39$$

$$x \in B^{10}$$

as a linear programming problem.

8. Let $S = \{x \in B^6: 27x_1 + 23x_2 + 17x_3 + 12x_4 + 8x_5 + 2x_6 \leqslant 40\}$.

i) Describe as many facet-defining inequalities as possible for S based on Proposition 2.3 and Corollary 2.4.

ii) What inequalities are obtained for S from Proposition 2.6?

9. Let $S = \{x \in B^n: \Sigma_{i \in I} \Sigma_{j \in Q_i} a_j x_j \leqslant b, \Sigma_{j \in Q_i} x_j \leqslant 1 \text{ for } i \in I\}$ with $N = \cup_{i \in I} Q_i$.

i) Show that if C is a minimal dependent set with $|C \cap Q_i| \leqslant 1$, $C \cap Q_i = \{j(i)\}$ when $C \cap Q_i \neq \emptyset$, and

$$\tilde{E}(C) = E(C) \cup \bigcup_{\{i: C \cap Q_i \neq \emptyset\}} \{j \in Q_i: a_j \geqslant a_{j(i)}\},$$

then $\Sigma_{j \in \tilde{E}(C)} x_j \leqslant |C| - 1$ is a valid inequality for S.

ii) Specify conditions under which this valid inequality defines a facet of conv(S).

10. Let

$$\begin{aligned} S = \{x \in B^7:\ & 5x_1 + 7x_2 + 11x_3 - 8x_4 - 10x_5 - 15x_6 \leqslant -2 \\ & x_1 + x_2 + x_3 \leqslant 1 \\ & x_4 + x_5 + x_6 \leqslant 1\} \end{aligned}$$

(see Exercise 14 of Section I.1.8).

i) Derive facets for conv(S).

ii) Can you show that these facet-defining inequalities give conv(S)?

11. Consider the symmetric traveling salesman polytope for the complete graphs on 5 and 7 nodes, respectively. Try to write down all of the facet-defining inequalities and see if you can give a proof that you have them all.

12. Give a nontrivial lower bound on the number of facets of the symmetric traveling salesman polytope for complete graphs with $n = 5, 7, 10, 100$, and 1000 nodes.

13. Prove the validity of the generalized comb inequalities (3.14).

14. Prove that $\Sigma_{e \in E'} x_e \leqslant 9$ is valid for the complete graph on 10 nodes, where $G = (V, E')$ is the Petersen graph, by showing it to be a C–G inequality.

15. Prove Proposition 4.1.

16. i) Use Proposition 4.3 to derive valid inequalities for

$$\begin{aligned} T = \{x \in B^6, y \in R^6_+:\ & y_1 + y_2 + y_3 + y_4 - y_5 - y_6 \leqslant 12, \\ & y_1 \leqslant 8x_1, y_2 \leqslant 7x_2, y_3 \leqslant 4x_3, y_4 \leqslant 2x_4, y_5 \leqslant 3x_5, y_6 \leqslant x_6\}. \end{aligned}$$

ii) Which of these inequalities define facets?

17. Under what conditions does (4.4) define a facet of

$$T' = \left\{x \in B^n, y \in R^n_+: \sum_{j \in N^+} y_j - \sum_{j \in N^-} y_j = b,\ y_j \leqslant a_j x_j \text{ for } j \in N = N^+ \cup N^-\right\}?$$

18. Consider the capacitated facility problem with feasible region

$$T = \left\{x \in B^n, y \in R^{mn}_+: \sum_j y_{ij} = a_i \text{ for } i \in M,\ \sum_i y_{ij} \leqslant b_j x_j \text{ for } j \in N\right\}.$$

Let $I \subseteq M$ and $z_j = \Sigma_{i \in I}\, y_{ij}$ so that the z_j satisfy

$$\sum_{j \in N} z_j = \sum_{i \in I} a_i \text{ and } \qquad z_j \leqslant b_j x_j.$$

i) Derive valid inequalities for T.

ii) Can you show that the inequalities define facets?