Paul Holsen IMSE 884 HW #4

① Knapsack constraint:

$$(K) \quad 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39$$

ⓐ Find an extended cover inequality that is facet-defining; prove that it is a facet.

Consider $C = \{2, 7, 10\}$

This is a cover: $27 + 12 + 3 > 39$

Then $E(C) = \{1, 2, 7, 10\}$

and the extended cover inequality is

$$x_1 + x_2 + x_7 + x_{10} \leq 2 \qquad (*)$$

We know the knapsack polytope for this problem has $\dim(P_{ck}) = 10$; to see the dimension of the face F induced by $(*)$:

- the point $(0,0,0,0,0,0,0,0,0,0)$ is in P_{ck} and gives $\emptyset < 39$, so $\dim(F) \leq 9$
- these ten affinely independent points on F mean that $\dim(F) \geq 9$ (max # AI points -1)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

so $\dim(F) = 9 = \dim(P_{ck}) - 1$

→ Hence F is a facet of P_{ck}, and the extended cover inequality defines it.

4 point cut off by $(*)$: $\left(\frac{12}{31}, \frac{12}{31}, 0, 0, 0, 0, 1, 0, 0, 1\right)$ ☒

(1)

(b) Take the cover {6, 7, 8, 9} and sequentially lift it to make it facet-defining; prove that it is facet-defining.

This cover yields the inequality $x_6 + x_7 + x_8 + x_9 \leq 3$. The face F induced by this inequality is not a facet — $\dim(F) \geq 4$, with affinely independent points:

(0, 0, 0, 0, 0, 1, 1, 1, 0, 1)
(0, 0, 0, 0, 0, 1, 1, 0, 1, 1)
(0, 0, 0, 0, 0, 1, 0, 1, 1, 1)
(0, 0, 0, 0, 0, 0, 1, 1, 1, 1)
(0, 0, 0, 0, 0, 0, 1, 1, 1, 0)

Let (K) be the original knapsack constraint.

Sequentially uplift x_1:

solve: $\max\ x_6 + x_7 + x_8 + x_9$

s.t. (K)

$x_1 = 1$

$x_i \in \{0, 1\}$

$z^* = \emptyset \rightarrow \alpha_1 = 3 - \emptyset = 3$

$\rightarrow$ new inequality: $3x_1 + x_6 + x_7 + x_8 + x_9 \leq 3$

LR point cut off by this inequality

$\left(\frac{4}{35}, 0, 0, 0, 0, 1, 1, 1, 0, 0\right)$

dim(its face) ≥ 5; add affinely indep. point

(1, 0, 0, 0, 0, 0, 0, 0, 0, 1)

(cont'd ⟹)

1f) (cont'd)

Sequentially uplift x_2:

solve: $\max\ 3x_1 + x_6 + x_7 + x_8 + x_9$

s.t. (K)

$x_2 = 1$

$x_i \in \{0,1\}$

$z^* = 1 \rightarrow \alpha_2 = 3 - 1 = 2$

$\rightarrow$ new inequality: $3x_1 + 2x_2 + x_6 + x_7 + x_8 + x_9 \leq 3$

LR point cut off by this inequality:

$(\frac{3}{31}, \frac{3}{31}, 0, 0, 0, 1, 1, 0, 0, 0)$

dim (its face) ≥ 6; add affinely indep. point

$(0, 1, 0, 0, 0, 0, 1, 0, 0, 0)$

Sequentially uplift x_3:

solve: $\max\ 3x_1 + 2x_2 + x_6 + x_7 + x_8 + x_9$

s.t. (K)

$x_3 = 1$

$x_i \in \{0,1\}$

$z^* = 2 \rightarrow \alpha_3 = 3 - 2 = 1$

$\rightarrow$ new inequality: $3x_1 + 2x_2 + x_3 + x_6 + x_7 + x_8 + x_9 \leq 3$

LR point cut off by this inequality:

$(\frac{1}{31}, \frac{1}{31}, 1, 0, 0, 0, 0, 1, 1, 0)$

dim (its face) ≥ 7; add affinely indep. point

$(0, 0, 1, 0, 0, 0, 0, 1, 1, 0)$

$\Longrightarrow$ (cont'd)

(13) (cont'd)

Sequentially uplift x_4:

Solve: $\max\ 3x_1 + 2x_2 + x_3 + x_6 + x_7 + x_8 + x_9$

s.t. (K)

$x_4 = 1$

$x_i \in \{0,1\}$

$z^* = 2 \rightarrow \alpha_4 = 3 - 2 = 1$

$\rightarrow$ new inequality: $3x_1 + 2x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9 \leq 3$

LR point cut off by this inequality:

$\left(\frac{1}{26}, \frac{1}{26}, \frac{1}{26}, \frac{1}{26}, 0, 1, 1, 1, 0, 0\right)$

dim(its face) ≥ 8; add affinely indep. point

$(0,0,0,1,0,0,1,1,0,0)$

Sequentially lift x_5:

solve: $\max\ 3x_1 + 2x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9$

s.t. (K)

$x_5 = 1 \quad x_i \in \{0,1\}$

$z^* = 2 \rightarrow \alpha_5 = 3 - 2 = 1$

$\rightarrow$ new inequality: $3x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leq 3$

LR point cut off by this inequality:

$\left(\frac{4}{119}, \frac{4}{119}, \frac{4}{119}, \frac{4}{119}, \frac{4}{119}, 1, 1, 1, 0, 0\right)$

dim(its face) ≥ 9; add affinely indep. point

$(0,0,0,0,1,1,0,1,0,0)$

Origin doesn't meet induced face at equality, so induced face isn't whole space, so $\dim(F) < \dim(P_{ch})$

$\rightarrow \dim(F) = 9 = \dim(P_{ch}) - 1 \rightarrow$ facet. $\boxtimes$

1c) cover $\{6,7,8,9\}$

$\rightarrow x_6 + x_7 + x_8 + x_9 \le 3$

Simultaneously lift $\{3,4,5\}$

$\rightarrow$ solve: $\max M(x_3+x_4+x_5) + x_6 + x_7 + x_8 + x_9$

s.t. (K)

$x_i \in \{0,1\}$

$\rightarrow z^* = 2M, \quad \vec{x}^* = (0,0,0,1,1,0,0,0,0,0)$

$z^* > 3$, so inequality $M(x_3+x_4+x_5)+x_6+x_7+x_8+x_9 \le 3$ is invalid.

$\alpha(0+1+1)+0+0+0+0 = 3$

$2\alpha = 3$

$\alpha = 3/2$

$\rightarrow$ solve: $\max \frac{3}{2}(x_3+x_4+x_5) + x_6 + x_7 + x_8 + x_9$

s.t. (K)

$x_i \in \{0,1\}$

$\rightarrow z^* = 7/2, \quad \vec{x}^* = (0,0,0,0,1,0,0,1,1,0)$

$z^* > 3$, so inequality $\frac{3}{2}(x_3+x_4+x_5)+x_6+x_7+x_8+x_9 \le 3$ is invalid

$\alpha(0+0+1)+0+0+1+1 = 3$

$\alpha + 2 = 3$

$\alpha = 1$

$\rightarrow$ solve: $\max x_3+x_4+x_5+x_6+x_7+x_8+x_9$

s.t. (K)

$x_i \in \{0,1\}$

$\rightarrow z^* = 3$, so $x_3+x_4+x_5+x_6+x_7+x_8+x_9 \le 3$ is valid

$\vec{x}^* = (0,0,0,0,0,0,1,1,1,0)$

LP point cut off:

$\left(0,0,\frac{4}{57},\frac{4}{57},\frac{4}{57},1,1,1,0,0\right)$ $\Longrightarrow$ (cont'd)

1c (cont'd)

Next, simultaneously lift $\{1,2\}$

$\to$ solve: $\max\ M(x_1+x_2)+x_3+\ldots+x_9$
s.t. (K)
$x_i \in \{0,1\}$

$\to z^* = M+1$, $\vec{x}^* = (0,1,0,0,0,0,0,1,0,0)$

$z^* > 3$, so $M(x_1+x_2)+x_3+\ldots x_9 \le 3$ is invalid

$\alpha(0+1)+0+\ldots+0+1+0+0 = 3$

$\alpha+1 = 3$

$\alpha = 2$

$\to$ solve: $\max\ 2(x_1+x_2)+x_3+\ldots+x_9$
s.t. (K)
$x_i \in \{0,1\}$

LR point cut off: $\left(\frac{4}{119},\frac{4}{119},\frac{4}{119},\frac{4}{119},\frac{4}{119},1,1,1,0,0\right)$

$\to z^* = 3$, $\vec{x}^* = (0,0,0,0,0,0,1,1,1,0)$

$\hookrightarrow$ so $2x_1+2x_2+x_3+\ldots+x_9 \le 3$ is valid. (*)

The origin of B^{10} does not meet the face induced by F, so $\dim(F) < \dim(P_{ch})$.
The greatest number of affinely independent points here is 9:

$$\begin{array}{c}\begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0&1\\ 1&0&0&0&0&1&0&0&1\\ 0&1&1&0&1&1&1&1&1\\ 0&0&0&1&1&0&0&1&0\\ 0&1&0&1&0&0&0&0&1 \end{bmatrix}\\ \begin{matrix} 1&1&1&1&1&1&1&1&1 \end{matrix}\end{array}$$

So $\dim(F) = 8$, which is $< \dim(P_{ch})-1$

$\to$ F is not a facet

$\to$ (*) is not facet-defining

(Skipping 1d re: Balas' theorem as indicated in lecture.)

2) Find and prove a facet-defining inequality to:

$$36x_1 + 27x_2 + 12x_3 + 6x_4 + 4x_5 + 2x_6 = 39$$
$$\vec{x} \in \{0,1\}^6$$

our polyhedron $P = \{\vec{x} \in \mathbb{B}^6 : 36x_1 + 27x_2 + 12x_3 + 6x_4 + 4x_5 + 2x_6 = 39\}$

$= \{(0,1,1,0,0,0), (0,1,0,1,1,1)\}$

Thus $P_{ch} = \{\lambda(0,1,1,0,0,0) + (1-\lambda)(0,1,0,1,1,1) : \forall \lambda \in [0,1]\}$,

the convex combinations of the two points in P.

The two points in P are affinely independent
$\rightarrow$ max # of affinely indep points in $P_{ch} = 2$
$\rightarrow \dim(P_{ch}) = 1$.

Consider the inequality $x_2 + x_3 \leq 2$
Any point in P_{ch} must satisfy this inequality, so it is valid, and the induced face $F = \{\vec{x} \in P_{ch} : x_2 + x_3 = 2\}$.

Feasible point $(0,1,0,1,1,1) \notin F$, so $\dim(F) < \dim(P_{ch})$; feasible point $(0,1,1,0,0,0) \in F$, and there are no other feasible points, so
$\dim(F) = 0 = \dim(P_{ch}) - 1$, and so
$x_2 + x_3 \leq 2$ is a facet.

I take that back - I'm taking a swipe at:

1d Given minimal cover $\{6, 7, 8, 9\}$,
and its cover inequality $x_6 + x_7 + x_8 + x_9 \leq 3$,

For all h in $\{1, \ldots, |C|\} = \{1, 2, 3, 4\}$:

construct: $\mu_0 = 0$

$\mu_1 = 15$ (a_6)

$\mu_2 = 15 + 12 = 27$ $(a_6 + a_7)$

$\mu_3 = 27 + 8 = 35$ $(a_6 + a_7 + a_8)$

$\mu_4 = 35 + 6 = 41$ $(a_6 + a_7 + a_8 + a_9)$

$\lambda = 41 - 39 = 2$

and the intervals:

$(h=0)$:	$0 \leq a_\ell \leq 15 - 2 = 13$	$\emptyset$	(A)
	$13 < a_\ell < 15$	$[\emptyset, 1]$	(B)
$(h=1)$	$15 \leq a_\ell \leq 27 - 2 = 25$	1	(C)
	$25 < a_\ell < 27$	$[1, 2]$	(D)
$(h=2)$	$27 \leq a_\ell \leq 35 - 2 = 33$	2	(E)
	$33 < a_\ell < 35$	$[2, 3]$	(F)
$(h=3)$	$35 \leq a_\ell \leq 41 - 2 = 39$	3	(G)
	$39 < a_\ell < 41$	$[3, 4]$	(H)

$a_1 = 35$, range (G) $\to 3$

$a_2 = 27$, range (E) $\to 2$

$a_3 = 23$, range (C) $\to 1$

$a_4 = 19$, range (C) $\to 1$

$a_5 = 15$, range (C) $\to 1$

$a_{10} = 3$, range (A) $\to \emptyset$

so $3x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leq 3$
is an approximate-lifting valid inequality

Interesting that sequential lifting @ 1b arrived at same result.