1 Polyhedral Theory Primer

An excellent reference is Integer and Combinatorial Optimization by Nemhauser and Wolsey, Wiley 1988.

Define an integer program (IP) as

Maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$ and $x \in \mathbb{Z}^n$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

A key idea of integer programming is to use the linear relaxations of an integer program (LR), which is defined as

Maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

Solve the following IP using cutting planes and discuss the separation problem.

Maximize $5x_1 + 2x_2$

subject to $10x_1 + 2x_2 <= 23$

$4x_1 + 2x_2 \leq 13$

$x_1, x_2 \geq 0$ and $x_1, x_2 \in \mathbb{Z}^n$

Define the feasible region of an integer program to be $P = \{x \in \mathbb{Z}^n : Ax \leq b\}$. Find P for the above problem.

Define the feasible region of the linear relaxation to be $P^{LR} = \{x \in \mathbb{R}^n : Ax \leq b\}$. Find P^{LR} for the above problem.

A set is convex if and only if every point on the line segment between any two points in the set is also in the set.

Definition Let x^1 and x^2 be any two points in some set S, then S is convex if and only if $x' \in S$ for every $x' = \lambda x^1 + (1-\lambda)x^2$ where $\lambda \in [0,1]$.

Definition The convex hull of any set S is defined as the intersection of all convex sets that contain S. Examples:

Define P^{ch} as the convex hull of P. Thus, one could write an integer program as Max $\{c^T x : x \in P\}$, which is equivalent to Max $\{c^T x : x \in P^{ch}\}$ as long as this last version is solved with the simplex method (guarantees a corner, extreme point or extreme ray). If we can find the convex hull of P, then there is no need to perform any branching. This is the goal of polyhedral theory in integer programming, completely describe P^{ch}. Find P^{ch} for the above example.

1.1 Preliminary Theory from Matrix Theory

Dimension plays a critical role in determining how to define P^{ch}. From linear algebra you should recall that the dimension of a space is the maximum number of linearly independent vectors that define the space.

Definition The vectors $v^1, ..., v^q \in \mathbb{R}^n$ are linearly independent if and only if the unique solution to $\sum_{i=1}^{q} \lambda_i x_i = 0$ is $\lambda_i = 0$ for all $i = 1, ..., q$. Furthermore, the dimension of a set of q vectors is the number of linearly independent vectors.

Typically the dimension is found through Gaussian elimination. Are the following set of vectors linearly independent (1,3,7), (6, 3, -1) and (13, 9, 5)? What about (1,2,1), (1,5,4) and (3,6,4)?

Definition The number of linear independent columns is equal to the number of linear independent rows, which is called the rank, rank(A) where A is a matrix.

One of the biggest problems with integer programs is that no vector ever stays in the feasible region, P. However, we can get a dimension on P^{ch}. The dimension of the convex hull of a set of points is determined through affine independence.

Definition The points $x^1, ..., x^q$ in $\mathbb{R}^n$ are affinely independent if and only if the unique solution to $\sum_{i=1}^{q} \lambda_i x^i = 0$ and $\sum_{i=1}^{q} \lambda_i = 0$ is $\lambda_i = 0$ for all $i = 1, ..., q$.

Are these point affinely independent (1,0), (0,1) and (1,1)? What about (1,3), (2,6), (4,12)? Remember to write the points in a column and with an extra row of ones.

Comments: Linear independence implies affine independence, but not the converse.

Theorem 1.1 *The following statements are equivalent (if and only ifs)*
(i) The points $x^1, ..., x^k$ *are affinely independent*
(ii) The vectors $x^1 - x^k, x^2 - x^k ..., x^{k-1} - x^k$ *are linearly independent*
(iii) $(x_1, 1),\ \ldots,\ (x_k, 1)$ *are linearly independent.*

Discussion

Definition The dimension of the convex hull of a set of points is the maximum number of affinely independent points minus 1.

1.2 Polyhedrons

Definition Let $a, x \in \mathbf{R}^n,\ b \in \mathbf{R}$, then $a^T x = b$ is a hyperplane in n dimensional space.

Definition Every hyperplane can form 2 halfspaces that take the form $a^T x \leq b$ and $a^T x \geq b$. The last one can be flipped by multiplying by a negative one and so it is now assumed that all halfspaces take the form $a^T x \leq b$.

Definition A set $T \subseteq \mathbb{R}^n$ is a polyhedron if and only if T can be expressed as a finite number of halfspaces; that is, $T = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Find the polyhedron of IP^{LR} in the previous example. Discuss why a circle is not a polyhedron. Why will P never be a polyhedron?

Integer programming is confusing because there are two polyhedrons. They are IP^{LR} and P^{ch}. Polyhedrons are totally independent from integer programming so you need to have care that you are always referring to the correct polyhedron. In these notes, Q will be used for a generic polyhedron.

Definition A polyhedron $Q \subset \mathbf{R}^n$ is a bounded if an only if there exist an M such that $Q \subset \{x \in \mathbf{R}^n : -M \leq x \leq M\}$.

Definition A polytope is a bounded polyhedron.

Definition A polyhedron is said to be rational if A and b are rational.

Remark Observe that the feasible region of every LP is a polyhedron.

Remarks By definition, a polyhedron is convex. Notice that a polyhedron is exactly the feasible region of an LP. Observe that every bounded P^{ch} polytope would be rational. Examples

Definition $r \in \mathbf{R}^n$ is a ray of feasibility if for every $\lambda \geq 0$ and every feasible x, the point $x + \lambda r$ is feasible.

Definition A set $C \subset \mathbf{R}^n$ is a cone if any $c \in C$ implies that $\lambda c \in C$ for all $\lambda \geq 0$.

The polyhedron $\{x \in \mathbf{R}^n : Ax \leq 0\}$ is a cone.

The extreme rays of a polyhedron are a cone.

Every point in a polyhedron can be expressed as a convex combinations of its extreme points plus a nonnegative linear combination of its extreme rays.

Thus $Q = \{x \in \mathbf{R}^n : Ax \leq b, x \geq 0\}$ can be expressed as $\sum_{i=1}^{r} \lambda_i x_i + \sum_{i=1}^{s} \alpha_i x_i$, $\sum_{i=1}^{r} \lambda_i = 0$ and $\lambda_i \geq 0$ for $i = 1, ..., r$ and $\alpha_i \geq 0$ for $i = 1, ..., s$ where the extreme points are $x^1, ..., x^r$. This is the main idea of Dantzig-Wolfe decomposition.

1.3 Dimension

The dimension plays a critical role in finding P^{ch}.

Definition: A polyhedron T has dimension k, $dim(T) = k$, if and only if the maximum number of affinely independent points in T is $k+1$.

For convenience, if $q = \emptyset$, then $dim(Q) = -1$.

This definition can cause enormous problems. First how does one know if he or she has found the "maximum" number of affinely independent points? Typically one needs to prove both a lower and an upper bound on the dimension of a polyhedron. Many times the upper bound is the number of variables. Theorems 1.2, 1.3 and 1.6 help provide an upper bound on the dimension of a polyhedron.

Theorem 1.2 *The maximum number of linear independent vectors in $\mathbf{R}^n$ is n.*

Therefore, the maximum number of affinely independent points in $\mathbb{R}^n$ is $n+1$.

Definition: Given some set of points $P \in \mathbb{R}^n$. Define $(A^=, b^=)$ to be the set of equations such that $A^=x = b^=$ for every $x \in P$.

At the very least, $(A^=, b^=)$ contains the equality constraints of an integer program, but this is not always the case.

Theorem 1.3 *Let T be a polyhedron in $\mathbb{R}^n$, then $dim(T) + rank(A^=, b^=) = n$.*

Example find the dimension of P^{ch} where P is defined by the following conditions:

$x_1 + x_2 + x_3 \leq 5$

$-x_1 - x_2 - x_3 \leq -5$

$-x_1 \leq 0$

$-x_2 \leq 0$

$-x_3 \leq 0$

$x_3 \leq 3$

x_1, x_2, x_3 integers.

Definition: A polyhedron with dimension n is called full dimensional.

Definition: The constraints of a polyhedron Q can be partitioned into $A^{\leq}b^{\leq}$ and $A^{=}b^{=}$ where every $x \in Q$ satisfies $A^{=}x = b^{=}$ and there exists at least one $x \in Q$ such that $A^{\leq}x < b^{\leq}$. For convenience let $M^{=}$ be the set of row indices that fall into $A^{=}b^{=}$ and $M^{\leq}$ be the set of row indices that fall into $A^{\leq}b^{\leq}$.

Definition: A point $x \in Q$ is an called an inner point of Q if $A_j x_i < b_j$ for all $A_j b_j \in A^{\leq}b^{\leq}$.

Definition: A point $x \in Q$ is an called an interior point of Q if $Ax < b$.

Theorem 1.4 *Every full dimensional has an interior point.*

We shall use this example throughout these sections. Throughout the rest of this section, let P be defined by the following restrictions.

$10x_1 + 2x_2 \leq 23$

$4x_1 + 2x_2 \leq 13$

$-x_1 \leq 0$, $-x_2 \leq 0$ and integer.

Graph the feasible points and draw the convex hull. Find the dimension of P^{ch}. Also discuss other polyhedra that can be graphed and their dimension.

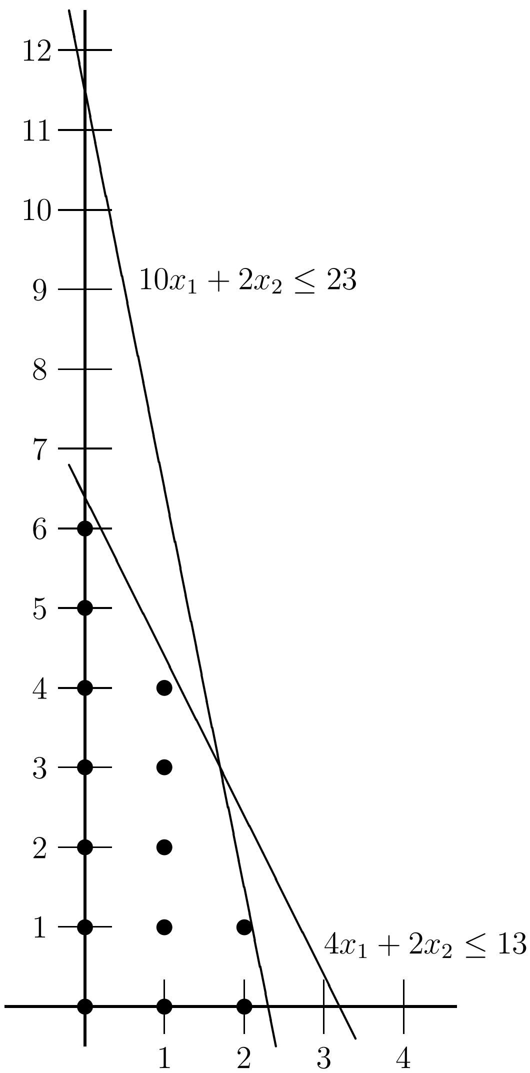

Figure 1: Graph for the IP

1.4 Valid Inequalities, Faces and Facets

Definition: The inequality $\alpha^T x \leq \beta$ or (α, β) or $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ is called a valid inequality of P^{ch} if and only if every $x \in P$ satisfies $\alpha^T x \leq \beta$.

Definition: A valid inequality is trivial if it is implied by the bounds of the problem ($x \geq 0$ or $2x_1 + 4x_2 + x_3 \leq 7$ where x_1, x_2, x_3 are binary.)

Valid inequalities are also called cutting planes or cuts.

Classify the following inequalities as valid and comment about the usefulness of any inequalities.

$x_1 \leq 200$

$x_1 \leq 2$

$2x_1 \leq 4$

$x_1 \leq 1$

$x_1 + 2x_2 \leq 9$

$5x_1 + 2x_2 \leq 12$

$x_1 - x_2 \leq 2$

$x_3 \geq 0$

Let $\alpha^T x \leq \beta$ and $\gamma^T x \leq \omega$ be two valid inequalities.

Then $\alpha^T x \leq \beta$ and $\gamma^T x \leq \omega$ are equivalent if and only if there exists some $\lambda \geq 0$ such that $\lambda\alpha = \gamma$ and $\lambda\beta = \omega$.

If there exists some λ such that $\lambda\alpha \geq \gamma$ and $\lambda\beta \leq \omega$, then $\alpha^T x \leq \beta$ dominates $\gamma^T x \leq \omega$.

Discuss implied dominance.

Definition If $\alpha^T x \leq \beta$ is a valid inequality for P^{ch}, then define the induced face of (α, β) to be

$F = \{x \in P^{ch} : \alpha^T x = \beta\}$. If $F \neq \emptyset$, then F is called a face of P^{ch} and we say that $\alpha^T x \leq \beta$ supports P^{ch}. A face is proper if $F \neq P^{ch}$.

In some weird sense, the dimension of the face determines the strength of the inequality. Thus, larger dimensional faces are theoretically stronger, unless the face is the whole space, in which case the inequality is useless. With each of the previous inequalities determine the dimension of the induced face.

A maximal inequality is not dominated by any other inequality.

Every maximal inequality induces a nonempty face of P^{ch}.

A maximal inequality is said to support Q.

Proposition 1.5 *If $Q = \{x \in \mathbf{R}^N : Ax \leq b\}$ with equality set $M^=$ and F is a nonempty face of Q, then F is a polyhedron and $F = \{x \in \mathbf{R}^n : A^i x = b_i$ for $i \in M^=$, $A^i x \leq b_i$ for $i \in M^{\leq}\}$ and the number of distinct faces is finite.*

Furthermore, every face is a polyhedron.

Definition A face F of P^{ch} is a facet of P if and only if $dim(F) = dim(P^{ch}) - 1$.

Thus a facet defining inequality must be valid and induce a face of the proper dimension. Discuss the idea behind my recent research. Any non facet defining inequality is

The purposes of facets and faces is to decide, which inequalities are necessary for the description of a polyhedron. Observe that facets and faces are independent of the objective function.

Facet-defining inequalities are extremely important. Some of the following are straightforward results that help indicate the power and importance of facet-defining inequalities.

If F is a facet of P^{ch}, then there exists at least one linear inequality $\alpha^T x \leq \beta$ (there may be infinite) that represents F.

For each facet $F \subset P^{ch}$, one of the inequalities representing F is necessary in the description of P^{ch}. Therefore, any non facet-defining inequality is redundant and can be removed from the problem.

A full-dimensional polyhedron P^{ch} has a unique (to within scalar multiplication) minimal representation by a finite set of linear inequalities (each facet-defining inequality is unique up to scaling). If the dimension of P^{ch} is not full, then there exists an infinite number of facet-defining inequalities for each facet. These infinite number of inequalities are based upon the linearly independent rows of $(A^=, b^=)$.

If the polyhedron is not full dimensional, then there exist an infinite number of inequalities that could represent each facet.

The following theorem will be used repeatedly in this class.

Theorem 1.6 *Let $\alpha^T x \leq \beta$ be a valid inequality, then if there exists a point $x \in P$ such that $\alpha^T x < \beta$, then the $dim(\{x \in P^{ch} : \alpha^T x = \beta\}) \leq dim(P^{ch}) - 1$..*

Notice that this theorem provides an upper bound on the dimension of a face.

There exist 3 steps in proving something is a facet in P^{ch}.

(Although there are some alternate ways to prove something is a facet.)

1. Find the dimension of P^{ch}.

a. Bound $dim(P^{ch})$ from above.

Typically using the number of variables and Theorem 1.3.

b. Bound $dim(P^{ch})$ from below.

Typically by finding a number of affinely independent points.

2. Show the inequality $\alpha^T x \leq \beta$ is a valid inequality for P^{ch}.
Show that no $x \in P$ has $\alpha^T x > \beta$

3. Find the dimension of induced face F in P^{ch}.

a. Bound the dimension of the face from above.

Typically using Theorem 1.6.

b. Bound the dimension of the face from below.

Typically by finding a number of affinely independent points.

Find and prove a facet or two in our example.

1.5 Separation

Given a point x^* and a polyhedron P^{ch}, the separation problem seeks a valid inequality $\alpha^T x^* \leq \beta$ such that $\alpha^T x > \beta$ and $\alpha^T x \leq \beta$ for all $x \in P$.

Valid inequalities are also known as cuts or cutting planes. The basic purpose of cuts is to find the x^{*LP} from the linear relaxation and then solve the separation problem, add the cut and resolve the LP. For the previous example, solve the IP by successively solving the separation problem. (This is asked for on the homework).

So let's solve the main problem by cutting planes. Assume that the objective function is to maximize $x_1 + x_2$.

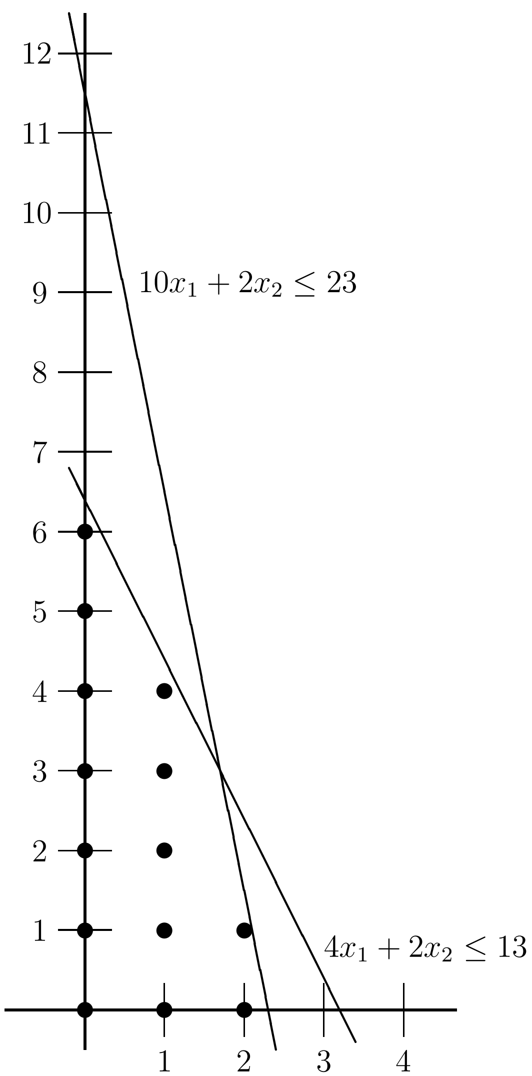

Figure 2: Graph for the IP

One of the most common techniques to solve integer programs is called branch and cut. It is branch and bound, but also has the option to include cuts. There can be local (all descendants) and global cuts (every relative) (CPLEX assumes global cuts).

2 Facet Defining Inequalities

Draw a graph on the board and write out the node packing formulation. Find and prove that a maximal clique is always a facet defining inequality.

Some of the most frequently used cuts are called clique cuts. These are typically derived from a conflict graph.

Create the conflict graph for the following IP, find a maximal clique and its cut. Show that this cut is not a trivial cut.

$5x_1 + 7x_2 + 8x_3 + 9x + 4 \leq 14$

$8x_1 + 6x_2 + 3x_3 + 5x + 4 \leq 10$

$x_1, x_2, x_3, x_4 \in \{0, 1\}$

Create the conflict hypergraph for the following IP, find a maximal hyperclique and its cut. Show that this cut is not a trivial cut.

$5x_1 + 7x_2 + 8x_3 + 9x + 4 \leq 21$

$8x_1 + 6x_2 + 3x_3 + 5x + 4 \leq 18$

$x_1, x_2, x_3, x_4 \in \{0, 1\}$

Clearly finding affinely independent points are critical. Let $C_{m,k}$ be an $m \times m$ matrix consisting of 0s and 1s. Let each column consist of a cyclic permutation of k 1s. A $C_{7,3}$ and a $C_{8,6}$ are pictured below.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

Theorem 2.1 *The rank of $C_{m,k}$ equals $m - GCD(m, k) + 1$ where GCD is the greatest common divisor.*

Draw a graph on the board and write out a matching formulation. Find and prove that an odd hole can be facet defining inequality and determine what conditions must be met for this inequality to be facet defining.

Draw a K_4 and find as many facet defining inequalities as you can for the TSP problem.

2.1 Knapsack Problem and Cover Cuts

This section is taken almost directly from Talia Gutierrez's M.S. Thesis with her permission.

The knapsack problem (KP) is a special case of integer programming problems, where the objective is to maximize the benefit obtained from packing n items, $N = \{1, ..., n\}$, in a knapsack of capacity b. Due to its immediate applications, many computational and theoretical studies have been performed on this problem. The standard knapsack problem can be defined as $Max \sum_{i=1}^{n} c^T x : a^T x \leq b$, where c and $a \in \mathbb{R}^n_+$, $b \in \mathbb{R}_+$, and $x \in \{0, 1\}^n$. Let P_K be the set of feasible points of a KP problem, $P_K = \{x \in \mathbb{B}^n : \sum_{i=1}^{n} a_i x_i \leq b\}$. Define P^{KP} to be the convex hull of P_K. Without loss of generality assume that the a_i's are sorted in descending order (i.e. if $i, j \in N$ and $i \leq j$, then $a_i \geq a_j$).

Assume $a_i \geq 0$ for all $i \in N$. If this is not the case, then either $a_i = 0$ and x_i can be removed from the problem or $a_i \leq 0$ and x_i can be replaced by $1 - x_{i'}$ to create an equivalent problem with $a'_i \geq 0$. Furthermore, assume that $a_1 \leq b$. If not, $x_1 = 0$ for every $x \in P_K$ and x_1 can be removed from the problem.

With the above assumptions, prove that P^{KP} is full dimensional (i.e. $dim(P^{PK}) = n$).

In numerous cases, covers are used to find KP valid and facet-defining inequalities. A cover is defined as any set $C \subset N$ such that $\sum_{j \in C} a_j > b$. A cover is minimal if $\sum_{j \in C \setminus \{l\}} a_j \leq b$ for each $l \in C$. Every cover induces a valid inequality, known as a cover inequality, of the form $\sum_{j \in C} x_j \leq |C| - 1$. Minimal covers are known to produce inequalities of dimension at least $|C| - 1$. The following example helps to explain cover inequalities.

Example 2.2 Consider the feasible region defined by the following knapsack constraint.

$$19x_1 + 18x_2 + 17x_3 + 12x_4 + 12x_5 + 11x_6 + 10x_7 + 9x_8 \leq 46$$
$$x_i = \{0, 1\} \; \forall \; i = 1, ..., n.$$

Find 3 minimal covers and at least one cover that isn't minimal. Then find their corresponding inequalities and the dimension of the induced faces.

Clearly, a cover is $\{4,5,6,7,8\}$ because $a_4+a_5+a_6+a_7+a_8 = 12+12+11+10+9 = 54 > 46$. Therefore, at most 4 of the corresponding variables can be selected in any feasible solution. Hence, the cover inequality becomes $x_4+x_5+x_6+x_7+x_8 \leq 4$ and it is clearly valid. Furthermore, this is a minimal cover because if any of the elements in the cover is removed, then the sum of coefficients would be less than or equal to the right hand side of the KP constraint.

Prove that $x_4+x_5+x_6+x_7+x_8 \leq 4$ defines a face of at least 4.

Observe that the cover inequality cuts off the linear relaxation point $(0,0,0,\frac{5}{6},\frac{5}{6}, \frac{5}{6}, \frac{5}{6},\frac{5}{6})$, which is the main goal of the cutting planes technique. However, this inequality is not facet-defining because, if $x_1 = 1$, then at most 2 or fewer variables from the cover inequality can be set to 1 and then $x_4+x_5+x_6+x_7+x_8 \leq 2 < 4$. This cover inequality can be strengthened to eliminate a larger portion of the fractional space through lifting. The next section provides a more detail explanation of the lifting technique.

2.2 Extended Covers

The extended cover $E(C) = C \cup \{i \in N : a_i \geq a_j \text{ for all } j \in C\}$.

The extended cover inequality is $\sum_{i \in E(C)} x_i \leq |C| - 1$

Derive conditions for an extended cover to be facet-defining?

2.3 Lifting

Gomory introduced lifting as a way to obtain strong coefficients for valid inequalities of bounded integer programs. Since that time numerous individuals have used lifting to create useful cutting planes for various problems.

Definition: Given an integer program, define the restricted space of P when $x_i = 0$ to be $P_{x_i=0} = \{x \in P : x_i = 0\}$ and $P^{ch}_{x_i=0} = conv(P^{ch}_{x_i=0})$.

For convenience, this restricted space definition can extend to any number of equalities or inequalities in the obvious manner.

Briefly, let $E \subseteq N = \{1, 2, \dots, n\}$ and $F \subseteq N \setminus E$ be any nonempty set. Now let $\sum_{i \in E} \alpha_i x_i \leq \beta$ be a valid inequality of $P^{ch}_{x_i=y_i\ \forall\ i \in F}$ where $0 \leq y_i \leq u_i$ for all $i \in F$. Lifting seeks to create a valid inequality of P^{ch}, which takes the form $\sum_{i \in F} \alpha_i x_i + \sum_{i \in E} \alpha_i x_i \leq \gamma$. There are various types of lifting, such as sequential, simultaneous, approximate, up and down lifting techniques. In this class we will only focus on sequential uplifting and if time permits, we will talk briefly about simultaneous lifting.

Typically, sequential uplifting is an exact lifting technique, which seeks to find the strongest α_1 possible. Thus, exact sequential uplifting finds the maximum value of α_1 that still maintains the validity of the inequality. If such a value of α_1 is obtained, then the dimension of the face induced by the sequentially lifted inequality typically increases by at least 1.

Sequential uplifting assumes that $\sum_{i=2}^{n} \alpha_i x_i \leq \beta$ is valid for $P^{ch}_{x_i=0}$ and seeks to create an inequality of the form $\alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i \leq \beta$.

Here, we present the commonly used technique to sequentially uplift binary variables. Suppose $\sum_{j=2}^{n} \alpha_j x_j \leq \beta$ is a valid inequality for $P^{ch}_{x_1=0}$. If $P_{x_1=0} = \emptyset$, then $x_1 \leq 0$ is valid for P^{ch}. If $P_{x_1=0} \neq \emptyset$, then $\alpha_1 x_1 + \sum_{j=2}^{n} \alpha_j x_j \leq \beta$ is a valid inequality for P^{ch} for any $\alpha_1 \leq \beta - Z^*$, where

$$
\begin{aligned}
Z^* = \text{Maximize } & \textstyle\sum_{j=2}^{n} \alpha_j x_j \\
\text{subject to } & Ax \leq b \\
& x_1 = 1 \\
& x \in \{0,1\}^n.
\end{aligned}
$$

If $\sum_{j=2}^{n} \alpha_j x_j \leq \beta$ defines a r dimensional face of $P^{BIP}_{x_1=0}$, then there exists $x^q \in P_{x_1=0}$ for $q = 1, \dots, r+1$ that are affinely independent and satisfy $\sum_{j=2}^{n} \alpha_j x_j \leq \beta$ at equality. Since $x_1^q = 0$, it follows that x^q also satisfies $\alpha_1 x_1 + \sum_{j=2}^{n} \alpha_j x_j \leq \beta$ at equality for $i = 1, \dots, r+1$. Setting $\alpha_1 = \beta - Z^*$ implies that the x^* from the above integer program satisfies $\alpha_1 x_1 + \sum_{j=2}^{n} \alpha_j x_j \leq \beta$ at equality. Therefore, $\alpha_1 x_1 + \sum_{j=2}^{n} \alpha_j x_j \leq \beta$ defines a face of dimension at least $r+1$ in P^{ch}.

The following example helps to clarify these notions of sequential uplifting to create facet-defining inequalities.

$$19x_1 + 18x_2 + 17x_3 + 12x_4 + 12x_5 + 11x_6 + 10x_7 + 9x_8 + 4x_9 \leq 46$$
$$x_i = \{0, 1\} \; \forall \; i = 1, ..., n.$$

Sequentially uplift x_1 into the cover inequality generated from the cover $\{4, 5, 6, 7, 8\}$. Then sequentially uplift x_2 followed by x_3. Once you have obtained the inequality, find a LR point that this cut cuts off, which the cover inequality doesn't.

The order in which variables are lifted has a direct impact on the lifting coefficient value. Variables that are lifted earlier achieve larger lifting coefficients. When the order in which variables are lifted in this example is changed from x_1, x_2, x_3 to x_3, x_2, x_1, the resulting sequentially uplifted inequality is $x_1 + x_2 + 2x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 4$. The third lifting order (x_2, x_1, x_3) results in the uplifted inequality $x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 4$.

2.3.1 Simultaneous Lifting

The attention now shifts to simultaneous uplifting sets of integer variables. Observe that the average of the three sequentially uplifted inequalities is $\frac{4}{3}x_1+\frac{4}{3}x_2+\frac{4}{3}x_3+x_4+x_5+x_6+x_7+x_8 \leq 4$. Simultaneous lifting is a way to create an inequality that is stronger than this average inequality. The simultaneously lifted inequality is $\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{3}{2}x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 4$.

3 Simultaneously Uplifting Sets of Integer Variables

This section presents an algorithm to perform exact simultaneous uplifting for sets of general integer variables. Again, let $E \subset N = \{1, 2, \ldots, n\}$ and let $\emptyset \neq F \subseteq N \setminus E$. Assume that $\sum_{i\in E} \alpha_i x_i \leq \beta$ is a valid inequality of $P^{ch}_{x_i=0 \ \forall \ i\in F}$. Define $w_i > 0$ as the scaling coefficient for the lifting coefficient of x_i for each $i \in F$. Simultaneously uplifting the variables of F into the aforementioned valid inequality results in inequalities of the form

$$\alpha \sum_{i\in F} w_i x_i + \sum_{i\in E} \alpha_i x_i \leq \beta.$$

Clearly the goal is to seek the maximum α value for which this inequality is valid.

To aid with the clarity of the results, a method to simultaneously uplift sets of general integer variables by solving many integer programs (SLA) is presented. This result can be strengthened to only require solving a single integer program.

The input to SLA is the constraints of a bounded general integer program $Ax \leq b$, $0 \leq x \leq u$, $x \in \mathbb{Z}^n$, sets $E,\ F \subset N$ such that $F \subseteq N \setminus E$, an inequality $\sum_{i\in E} \alpha_i x_i \leq \beta$ that is vald for $P^{ch}_{x_i=0 \ \forall \ i\in F}$ and scaling coefficients w_i for all $i \in F$. SLA outputs either a simultaneously lifted inequality or the fact that none exist.

The basic idea of SLA is to guess a value for α and assumes that the resulting inequality is valid. An integer program is used to test this hypothesis and if the integer program has a value above β, then the inequality is not valid. The optimal solution to the integer program is used to change the α value so that this new point meets the next candidate inequality at equality. The process continues until the inequality is shown to be valid by having the solution to an integer program be less than or equal to β. Formally,

The Simultaneous Lifting Algorithm (SLA)

Initialization

Set $j := 0$.

Set $\alpha := M$ where $M > \beta + \frac{1}{\sum_{i\in F} w_i} + \sum i \in E|\alpha_i|u_i$.

Set $Z^j := \beta + 1$.

MainStep

While $Z^j > \beta$

Set $j := j + 1$.

Solve the following lifting integer program

$$(SLAIP^j) \quad Z^j := \begin{array}{ll} \text{Maximize} & \sum_{i \in F} \alpha^{j-1} w_i x_i + \sum_{i \in E} \alpha_i x_i \\ \text{Subject to} & \begin{array}{rl} Ax & \leq b \\ \sum_{i \in F} x_i & \geq 1 \\ 0 \leq x_i \leq u_i & \text{ and } x_i \in \mathbb{Z}_+ \ \forall\ i = 1, ..., n. \end{array} \end{array}$$

If $(SLAIP^j)$ is infeasible, *then*

Set $Z^j := \beta$.

Set $\alpha^j := \infty$.

else

if $Z^j > \beta$, *then*

Let x^{j*} be the x that provided the optimal solution to $(SLAIP^j)$,

Set $\alpha^j := \frac{\beta - \sum_{i \in E} x_i^*}{\sum_{i \in F}(w_i x_i^*)}$.

End while

Output

Set $\alpha := \alpha^{j-1}$ and report $\alpha \sum_{i \in F} w_i x_i + \sum_{i \in E} \alpha_i x_i \leq \beta$ as a valid inequality for P^{ch}.

The following example demonstrates this algorithm and provides some fundamental insights into the differences between sequential lifting and this new simultaneous lifting.

Example 3.1 Consider any integer program with feasible region defined by

$$\begin{array}{rl} 6x_1 + 5x_2 + 7x_3 & \leq 28 \\ 5x_1 + 6x_2 + 7x_3 & \leq 28 \\ x_1, x_2, x_3 & \in \mathbb{Z}_+. \end{array}$$

Observe that $x_3 \leq 4$ is clearly a valid inequality and now we will simultaneous lift x_1 and x_2. Let the scaling coefficients be $w_1 = 1$ and $w_2 = 1$. Set $\alpha^1 = M$. The optimal solution to the first IP is $Z = 4M > 4$ from the point $(4, 0, 0)$. Using the formula $\alpha^2 = \frac{4-0}{4} = 1$, which is equivalent to solving for α^1 in $0 + \alpha^1(4+0) = 4$. Now the objective function changes to $1(x_1 + x_2) + x_3$, which has an optimal solution of $Z = 5 > 4$ from the point $(2, 3, 0)$. Solving for α^2 results in $\alpha^2 = \frac{5}{4}$. Now the objective function changes to $\frac{4}{5}(x_1 + x_2) + x_3$, which has an optimal solution of $Z = 4$. Therefore, $\frac{4}{5}x_1 + \frac{4}{5}x_2 + x_3 \leq 4$ is a valid inequality. The points $(0, 0, 4)$, $(3, 2, 0)$ and $(2, 3, 0)$ all satisfy this inequality at equality and so this inequality is facet-defining.

If one performs sequential lifting on $x_3 \leq 4$, then the two resulting facet defining inequalities would be $x_1 + \frac{1}{2}x_2 + x_3 \leq 4$ and $\frac{1}{2}x_1 + x_2 + x_3 \leq 4$. Observe that the average of these two sequentially lifted inequalities is $\frac{3}{4}x_1 + \frac{3}{4}x_2 + x_3 \leq 4$, which is dominated by the simultaneously lifted inequality $\frac{4}{5}x_1 + \frac{4}{5}x_2 + x_3 \leq 4$. None of the other sequentially lifted inequalities can derive this inequality and so these simultaneously lifted inequalities can result in new classes of facet defining inequalities for bounded general integer programs.

Perform this simultaneous lifting on the aforementioned knapsack problem with $w = (1, 1, 1)$. Show that there exists a point that this inequality cuts off which is not cut off by any of the sequentially lifted inequalities or the original knapsack constraint. Show that this inequality is facet-defining. Switch w to $(2, 1, 1)$ and repeat.

Discuss approximate lifting and Balas' result

A well-known lifting result by Balas [**?**] provides some guidelines for selecting sets of variables that are likely to produce strong and interesting inequalities. Balas' result requires a minimal cover $C = \{i_1, \ldots, i_{|C|}\}$ and defines $\mu_h = \sum_{k=1}^{h} a_{i_k}$ for $h = 1, \ldots, |C|$ and $\lambda = \mu_{|C|} - b$. The result then bounds the lifting coefficients as follows:

Theorem 3.2 *If C is a minimal cover, then every facet-defining inequality of the form*

$$\sum_{l \in N \setminus C} \rho_l x_l + \sum_{i \in C} x_i \leq |C| - 1$$

satisfies the following conditions:

(i) *If $\mu_h \leq a_l \leq \mu_{h+1} - \lambda$, then $\rho_l = h$.*

(ii) *If $\mu_{h+1} - \lambda < a_l < \mu_{h+1}$, then $\rho_l \in [h, h+1]$.*

Balas' result partitions variables into two classes. These classes are variables that are lifted with an exact value (i) or variables that are lifted with a range of values (ii). Letting E contain variables that fall into (i) will likely not produce fractional values of ρ and such inequalities are typically weaker than sequentially lifted inequalities. In contrast, allowing E to contain variables that fall into (ii) will likely produce fractional values for ρ. Since each E is only allowed a single ρ, one expects that selecting E from variables in (ii) for a specific h would tend to provide strong and useful inequalities.

There also exists sequence independent lifting and other versions.