Paul Holser IMSE 884 HW #5

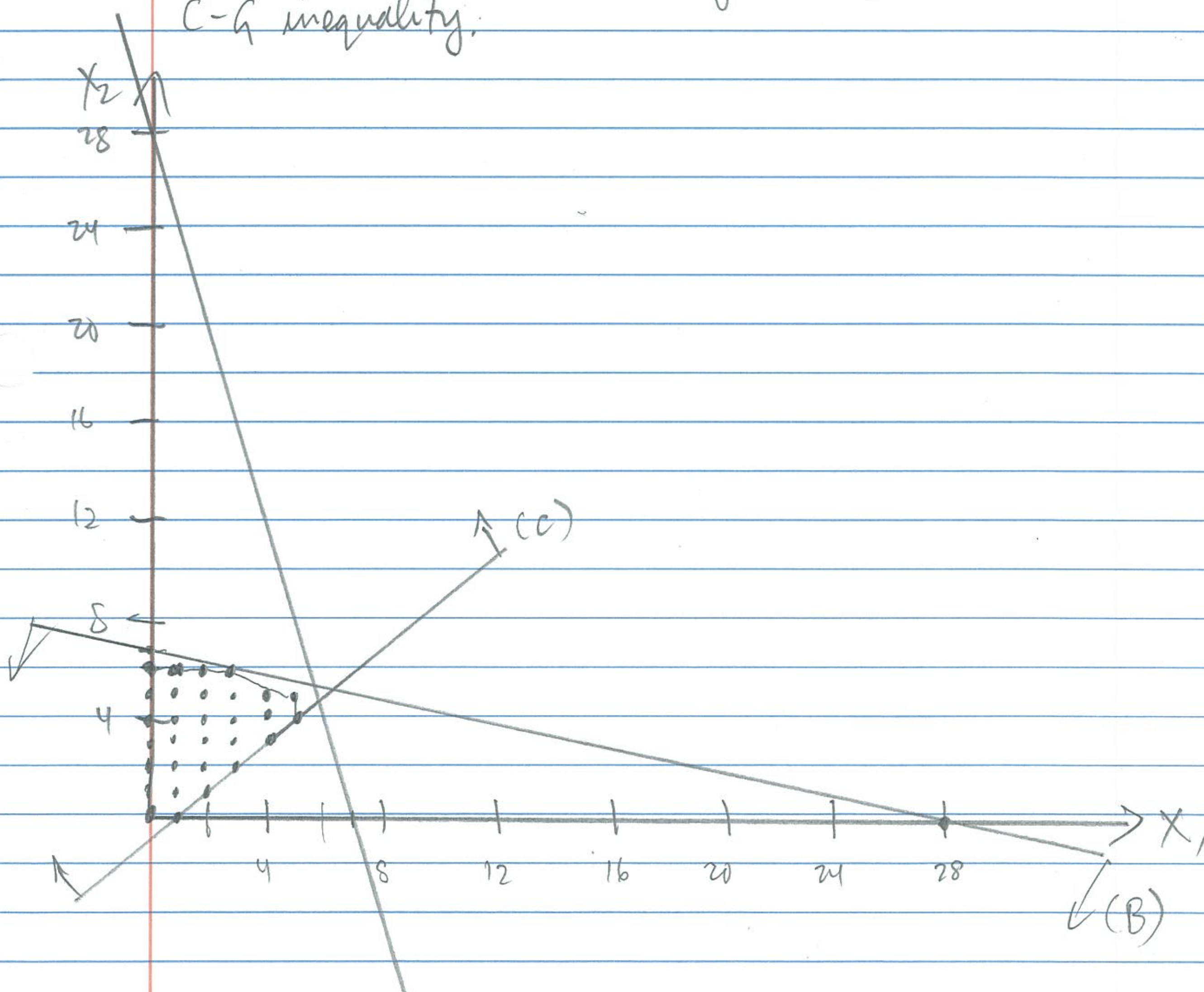
II.9.1

Let $S = \{\vec{x} \in \mathbb{Z}_+^2 : 4x_1 + x_2 \le 28 \quad (A)$

$x_1 + 4x_2 \le 27 \quad (B)$

$x_1 - x_2 \le 1 \} \quad (C)$

Determine the facets of conv(S) graphically. Then derive each of the facets of conv(S) as a C-G inequality.



Extreme points of conv(S):

(0,6), (3,6), (5,5), (5,4), (1,0), (0,0)

(⟹ cont'd)

II.9.1 (cont'd)

- thru (0,6) and (3,6) : $\boxed{x_2 \le 6}$ (i)

thru (3,6) and (5,5) :

$m = \frac{5-6}{5-3} = -1/2 \qquad x_2 = -1/2 x_1 + b \qquad 6 = -\frac{1}{2}(3) + b$

$b = 15/2$

$\rightarrow 1/2 x_1 + x_2 \le \frac{15}{2}$

$\rightarrow \boxed{x_1 + 2x_2 \le 15}$ (ii)

thru (5,5) and (5,4) : $\boxed{x_1 \le 5}$ (iii)

thru (5,4) and (0,1) (this is from the formulation):

$x_1 - x_2 \le 1$ (iv)

(Not including $x_1 \ge 0$ and $x_2 \ge 0$ here.)

☆ For (i) - take $\vec{u} = (0, 1/4, 0)$

$\rightarrow 1/4 x_1 + x_2 \le 27/4$

$\rightarrow \lfloor 1/4 \rfloor x_1 + x_2 \le \lfloor 27/4 \rfloor$

$\rightarrow x_2 \le 6.$ ■

$\Rightarrow$ (cont'd)

II, 9.1 (cont'd)

For (ii): combining (A) and (B) gives

$$5x_1 + 5x_2 \le 55$$
$$\rightarrow x_1 + x_2 \le 11$$

All feasible integer points obey this – it is a valid inequality – but by inspecting the set of feasible integer points we can see that none of them has $x_1 + x_2 > 10$. So in fact, we can write

$$x_1 + x_2 \le 10$$
$$\rightarrow 3x_1 + 3x_2 \le 30$$
$$x_1 + 4x_2 \le 27 \quad (B)$$
$$4x_1 + 7x_2 \le 57$$
$$x_2 \le 6 \quad (i)$$
$$4x_1 + 8x_2 \le 63$$
$$\rightarrow x_1 + 2x_2 \le 63/4$$
$$\rightarrow x_1 + 2x_2 \le \lfloor 63/4 \rfloor$$
$$\rightarrow x_1 + 2x_2 \le 15$$

$\blacksquare$

II.a.1 (cont'd)

For iii, take $\vec{u} = (1/5, 0, 1/5)$

$$\frac{4}{5}x_1 + \frac{1}{5}x_2 \le \frac{28}{5}$$

$$\frac{1}{5}x_1 - \frac{1}{5}x_2 \le \frac{1}{5}$$

$$\rightarrow \left\lfloor \frac{4}{5} + \frac{1}{5} \right\rfloor x_1 \le \left\lfloor \frac{29}{5} \right\rfloor$$

$$\rightarrow \boxed{x_1 \le 5}.$$

iv: take $\vec{u} = (0, 0, 1)$ – use (c) as-is:

$$\boxed{x_1 - x_2 \le 1}$$

II.9.2 Let $S = \{\vec{x} \in \mathbb{Z}_+^3 : 19x_1 + 28x_2 - 184x_3 = 8\}$.
Derive the valid inequality $x_1 + x_2 + 5x_3 \geq 8$
using modular arithmetic.

Choose $d = 9$. Note that:

$$19 \equiv 1 \pmod 9 \quad \text{since } 19 = 2(9) + 1$$
$$28 \equiv 1 \pmod 9 \quad \text{since } 28 = 3(9) + 1$$
$$-184 \equiv 5 \pmod 9 \quad \text{since } -184 = -21(9) + 5$$
$$8 \equiv 8 \pmod 9 \quad \text{since } 8 = 0(9) + 8$$

We can then construct a modular cut
using these remainders:

$$\rightarrow 1x_1 + 1x_2 + 5x_3 \geq 8$$

$$\rightarrow x_1 + x_2 + 5x_3 \geq 8. \quad \blacksquare$$

II.9.3

$$S = \{ \vec{x} \in \mathbb{B}^4 : 9x_1 + 7x_2 - 2x_3 - 3x_4 \le 12, \quad (A)$$
$$2x_1 + 5x_2 + x_3 - 4x_4 \le 10 \} \quad (B)$$

Show that $4x_1 + 5x_2 - 2x_3 - 4x_4 \le 12$
is a valid inequality by disjunctive arguments.

Transform (B) like so:

$$\to \quad 2x_1 + 5x_2 + x_3 - 4x_4 + 2 \le 12$$
$$\to \quad 4x_1 + 5x_2 + x_3 - 4x_4 + 2(1 - x_1) \le 12$$

so that, when $x_1 = \emptyset$,

$$4x_1 + 5x_2 + x_3 - 4x_4 + 2 \le 12$$
$$\to 4x_1 + 5x_2 + x_3 - 4x_4 \le 10 \quad (C)$$

Let $S_1 = S$, $S_2 = S \setminus \{ \vec{x} \in \mathbb{B}^4 : x_1 = 1 \}$

Then $S_1 \cup S_2 = S$,
(A) is valid for S_1,
(C) is valid for S_2,

so
$$\min(9,4)x_1 + \min(7,5)x_2 + \min(-2,1)x_3 + \min(-3,-4)x_4 \le \max(12,10)$$

$$\to 4x_1 + 5x_2 - 2x_3 - 4x_4 \le 12$$

is valid for S.

∎

II.9.13 $\max\ 2x_1 + 5x_2$

s.t. $4x_1 + x_2 \le 28$

$x_1 + 4x_2 \le 27$

$x_1 - x_2 \le 1 \qquad x_1, x_2 \in \mathbb{Z}_+$

Solving LR gives optimal tableau:

$$\begin{array}{llllll} z & & +\tfrac{1}{5}x_3 & +\tfrac{6}{5}x_4 & & = 38 \\ x_1 & & +\tfrac{4}{15}x_3 & -\tfrac{1}{15}x_4 & & = 17/3 \\ & x_2 & -\tfrac{1}{15}x_3 & +\tfrac{4}{15}x_4 & & = 16/3 \\ & & -\tfrac{1}{3}x_3 & +\tfrac{1}{3}x_4 & +x_5 & = 2/3 \end{array}$$

Gomory cut for row 3 of tableau:

$$\left(-\tfrac{1}{3} - \lfloor -\tfrac{1}{3} \rfloor\right)x_3 + \left(\tfrac{1}{3} - \lfloor \tfrac{1}{3} \rfloor\right)x_4 + (1 - \lfloor 1 \rfloor)x_5 \ge \tfrac{2}{3} - \lfloor \tfrac{2}{3} \rfloor$$

$$\to \left(-\tfrac{1}{3} - (-1)\right)x_3 + \left(\tfrac{1}{3} - 0\right)x_4 + (1-1)x_5 \ge \tfrac{2}{3} - \emptyset$$

$$\to \tfrac{2}{3}x_3 + \tfrac{1}{3}x_4 \ge \tfrac{2}{3}.$$

Adding to tableau: $\tfrac{2}{3}x_3 + \tfrac{1}{3}x_4 - x_6 = \tfrac{2}{3}$

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
	1	0	0	1/5	6/5	0	0	38
	0	1	0	4/15	-1/15	0	0	17/3
	0	0	1	-1/15	4/15	0	0	16/3
	0	0	0	-1/3	1/3	1	0	2/3
(mult by -1 to get a basis)	0	0	0	-2/3	-1/3	0	1	-2/3

$\left|\dfrac{1/5}{-2/3}\right| = \dfrac{3}{10} \qquad \left|\dfrac{6/5}{-1/3}\right| = \dfrac{18}{5}$

(cont'd) ⇒

II.9.13 (Cont'd)

Re-solving:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
row 0 + row 4 $\cdot \left(\frac{3}{10}\right)$	1	0	0	0	$11/10$	0	$3/10$	$38 - \frac{6}{30} = 37\,4/5$
row 1 + row 4 $\cdot \left(\frac{2}{5}\right)$	0	1	0	0	$-1/5$	0	$2/5$	$17/3 - \frac{4}{15} = 81/15 = 27/5$
row 2 + row 4 $\cdot \left(-\frac{1}{10}\right)$	0	0	0	0	$9/30$	0	$-1/10$	$81/15 = 27/5$
row 3 + row 4 $\cdot \left(-\frac{1}{2}\right)$	0	0	0	0	$1/2$	1	$-1/2$	1
row 4 $\cdot \left(-3/2\right)$	0	0	0	1	$1/2$	0	$-3/2$	1

■

$\frac{6}{5} - \frac{1}{10}$

II.9.14

For $S = \{\vec{x} \in \mathbb{Z}_+^2 : 4x_1 + x_2 \le 28$ (A)

$x_1 + 4x_2 \le 27$, (B)

$x_1 - x_2 \le 1\}$ (C)

show that the following are superadditive inequalities:

(i) $x_1 \le 5$

(A) and (C) are valid linear inequalities, so their sum is also:

$$5x_1 \le 29 \rightarrow x_1 \le \frac{29}{5}$$

Take $F(x) = \lfloor x \rfloor$, a superadditive function.

Then $\lfloor 1 \rfloor x_1 \le \lfloor \frac{29}{5} \rfloor$

$\rightarrow x_1 \le 5.$

(ii) $x_1 + 2x_2 \le 15$

As shown in II.9.1, we arrived at this inequality using composition of superadditive functions

- floor
- multiplication by constants
- addition of functions (linear)

(cont'd) $\Rightarrow$

II.9.14 (cont'd)

iii) $2x_1 + 5x_2 \le 36$

$$14(x_1 + 4x_2) \le 14(27)$$
$$\rightarrow 14x_1 + 56x_2 \le 378 \qquad (a)$$

$$6(x_1 - x_2) \le 6(1)$$
$$\rightarrow 6x_1 - 6x_2 \le 6 \qquad (b)$$

(a)+(b) : $20x_1 + 50x_2 \le 384$
$$\rightarrow 2x_1 + 5x_2 \le 384/10$$
$$\rightarrow 2x_1 + 5x_2 \le \lfloor 384/10 \rfloor$$

$$\rightarrow 2x_1 + 5x_2 \le 38 \qquad (c)$$

$$4(x_1 + 4x_2) \le 4(27)$$
$$\rightarrow 4x_1 + 16x_2 \le 108 \qquad (d)$$

(c)+(d) : $6x_1 + 21x_2 \le 146 \qquad (e)$

$$2(x_1 - x_2) \le 2(1) \qquad (f)$$

(e)+(f) : $8x_1 + 19x_2 \le 148$
$$\rightarrow 2x_1 + 19/4\, x_2 \le 37$$

$$\rightarrow 2x_1 + 4x_2 \le 37 \qquad (g)$$

... ugh. Not seeing next moves.

$\boxtimes$

II.2.6.3

$$T = \{\vec{x} \in \mathbb{B}^n, \vec{y} \in \mathbb{R}_+^{mn} : \sum_{j \in N} y_{ij} = 1 \ \forall i \in M$$

$$y_{ij} \le x_j \quad \forall i \in M, \forall j \in N$$

(i) $\mathrm{conv}(T) \subseteq \mathbb{R}^{(mn+n)}$. Therefore

$$\dim(\mathrm{conv}(T)) = mn + n - \mathrm{rank}(A^=, b^=),$$

where $(A^=, b^=)$ is that part of the constraint matrix with equality constraints.

$(A^=, b^=)$ looks like this:

$$\begin{array}{c|c|c|c|c|c}
x_1 \cdots x_n & y_{11} \cdots y_{1n} & y_{21} \cdots y_{2n} & \cdots & y_{m1} \cdots y_{mn} & b \\
 & 1 \cdots 1 & 0 & & & 1 \\
 & & 1 \cdots 1 & & & 1 \\
\emptyset & \emptyset & & \cdots & \emptyset & \cdots \\
 & & \emptyset & & & \vdots \\
 & & & & 1 \cdots 1 & 1
\end{array}$$

This has m LI rows, so its rank is m.

$$\text{So } \dim(\mathrm{conv}(T)) = (mn + n) - m$$

$$= mn - m + n.$$

$\boxtimes$

(cont'd)

II.2.6.3 (cont'd)

(ii) Show that $y_{ij} \le x_j$ define facets of conv(G).

Choose $i \in M$, $j \in N$ arbitrarily.

The induced face F of $y_{ij} \le x_j$
is $F = \{ y_{ij} \in \mathbb{R}_+, x_j \in \mathbb{B} : y_{ij} = x_j \}$.

F isn't the whole space, since feasible point
$\underbrace{x_{ij} = 1,\ y_{ij} = \emptyset}_{\text{all others } \emptyset}$ doesn't meet F at equality
$\Rightarrow \dim(F) < (mn - m + n)$.

Exhibit the following points:

- $x_j = 1$, $y_{ij} = 1$, all others $\emptyset$ $\Big\}\ 1$
- $x_k = 1$, all others zero $\forall k \in N$, $k <> j$ $\Big\}\ n-1$
- $x_k = 1$, $y_{pk} = 1$, all others zero $\forall k \in N$, $k <> j$ $\forall p \in M$ $\Big\}\ m(n-1)$

e.g. for $n=2$, $m=3$; $i=1$, $j=1$:

x_1	x_2	y_{11}	y_{21}	y_{31}	y_{12}	y_{22}	y_{32}
1	0	1	0	0	0	0	0
0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	0	1

$m(n-1) + (n-1) + 1 = mn - m + n.$

These points are affinely independent, and i and j are arbitrary
$\Longrightarrow$ (Cont'd)

II.2.6.3 (cont'd)

Therefore $\dim(F) \geq mn - m + n - 1$
and $< mn - m + n$

$\rightarrow \dim(F) = mn - m + n - 1$
$\rightarrow F$ is a facet of $\mathrm{conv}(T)$.

$\blacksquare$

II.2.6.6

For $S = \{\vec{x} \in \mathbb{B}^n : \sum_{j \in N} a_j x_j \le b\}$

show that $x_j \ge \emptyset$ and $x_j \le 1$ define facets of $\mathrm{conv}(S)$ when $\vec{a} \in \mathbb{Z}_+^n$ and $a_j + a_k \le b$ $\forall j, k \in N$, $j \ne k$.

S is a knapsack polytope $\rightarrow \dim(\mathrm{conv}(S)) = n$.

Consider the inequality $x_j \ge \emptyset$ for an arbitrary $j \in N$.
Feasible point $x_j = 1$, all others zero does not meet its induced face, $\hat{F}$ at equality

$\rightarrow \dim(F) < \dim(\mathrm{conv}(S))$.

Assume w/o loss of generality that $j = 1$.
For $n = 1$, point $[\emptyset]$ trivially defines a facet.
For $n=2$, exhibit points $(0,1)$ and $(0,0)$ – these are two affinely independent points.
For $n=3$ exhibit points $(0,1,0)$, $(0,0,1)$, $(0,0,0)$ – these are three affinely independent points.
For $n > 3$ exhibit these points:

- $x_1 = \emptyset$, $x_k = 1$, $x_{k+1} = 1$ $\forall k \in [2, n-1]$
- $x_{\lceil n/2 \rceil} = 1$, others zero
- $x_{\lceil n/2 \rceil + 1} = 1$, others zero

These n points are affinely independent

(cont'd) $\Longrightarrow$

II.2.6.6 (cont'd)

e.g. for $n = 7$:

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} \end{array}$$

So $\dim(F) \geq n-1$
$\dim(F) < n$
$\rightarrow \dim(F) = n-1$
$\rightarrow F$ is a facet.

For an inequality $x_j \leq 1$, feasible point $(0, 0, \ldots, 0)$ doesn't meet the inequality's induced face F $\rightarrow \dim(F) < \dim(\mathrm{conv}(S))$

Exhibit these n points:

- $x_j = 1$, $x_k = 1$, others zero $\forall k \in N$, $k \neq j$
- $x_j = 1$, all others zero

e.g. for $n = 5$, $j = 1$:

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} 1 & 1 & 1 & 1 & 1 \end{matrix} \end{array}$$

(cont'd) $\Rightarrow$

II.2.6.6 (cont'd)

These n points are affinely independent

$\Rightarrow \dim(F) \geq n-1$

$\dim(F) < n$

$\Rightarrow \dim(F) = n-1$

$\Rightarrow F$ is a facet. ∎