Paul Holser IMSE 884 HW #1

① Find an IP that has an exponential number of branches. Prove that there are an exponential number of branches. Provide a bound for the number of branches.

Jeroslow (1974) offered this IP:

$$\begin{aligned} \max\ & x_1 \\ \text{s.t.}\ & 2x_1 + \dots + 2x_n = n \\ & x_i \in \{0,1\},\ n \text{ an odd positive integer.} \end{aligned}$$

This family of problems ~~is~~ infeasible — no set of binary values for the x_i will result in a LHS that's odd.

If we take the greedy approach in constructing a solution to the linear relaxation of the initial problem, the root node of the branch-and-bound tree will look like this:

$$\boxed{\begin{array}{c} z=1 \\ (1, 1, \dots, 1/2, 0, \dots, 0) \end{array}}$$

where $x_1, \dots, x_{\frac{n-1}{2}} = 1$, $x_{\frac{n+1}{2}} = 1/2$, and remaining x_i are 0.

The first level of branches are fixing $x_{\frac{n+1}{2}}$ at either 0 or $1 \rightarrow 2$ branches, for 1 fixed variable total. From these two, fixing another variable at either 0 or $1 \rightarrow 4$ branches. We're safe to continue in this manner and have each B+B node (LP relaxation) stay feasible up to, but not including, tree level $\frac{n+1}{2}$ — at which level some nodes must be infeasible.

⟹ cont'd

① (cont'd)

As soon as a node has $\frac{n+1}{2}$ variables fixed at $\emptyset$ or 1, it becomes infeasible, since (if all $\emptyset$) the LHS will be $< n$, and if all 1 the LHS is at least $2\left(\frac{n+1}{2}\right) = n+1 > n$.

So, as a conservative bound on # of branches to evaluate, we have $2+4+\dots+2^{\frac{n-1}{2}}$

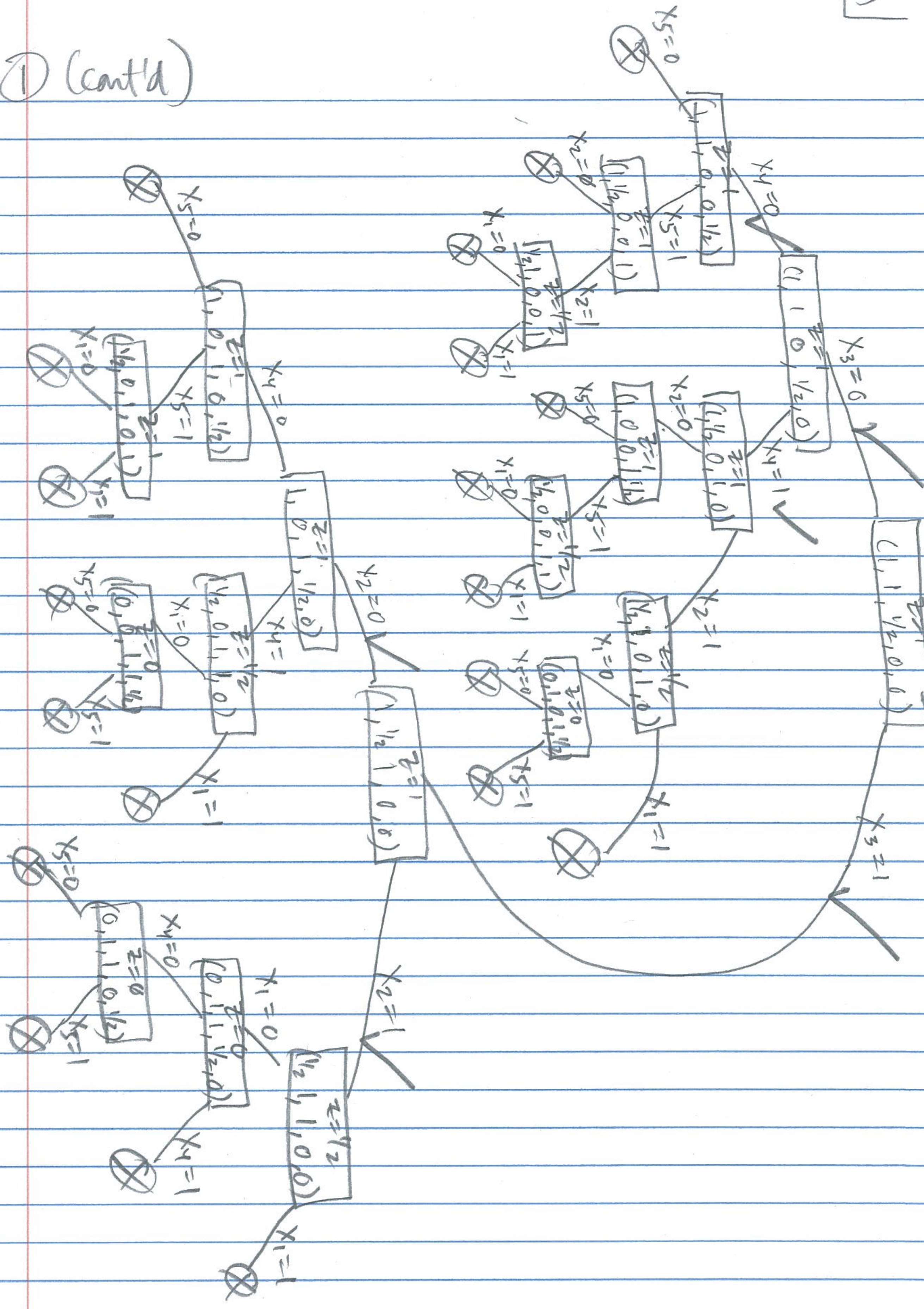
$$= \sum_{i=1}^{n-1/2} 2^i$$, exponential relative to the # of variables in the problem. ▨

Note: still more branches need to be evaluated to declare authoritatively the infeasibility of this IP.

Note also: We can declare this IP infeasible by inspection. Surely solvers can sniff this condition out beforehand without need to branch and bound.

A B+B tree for this problem for $n=5$ follows.

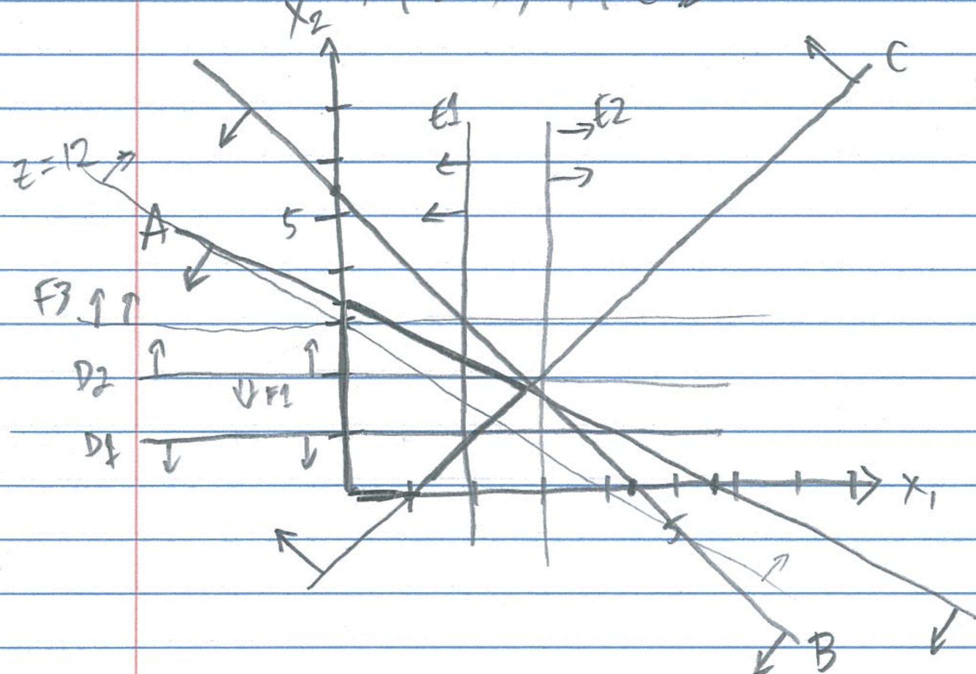
1 (cont'd)



(2) Solve using breadth-first search, L→R:

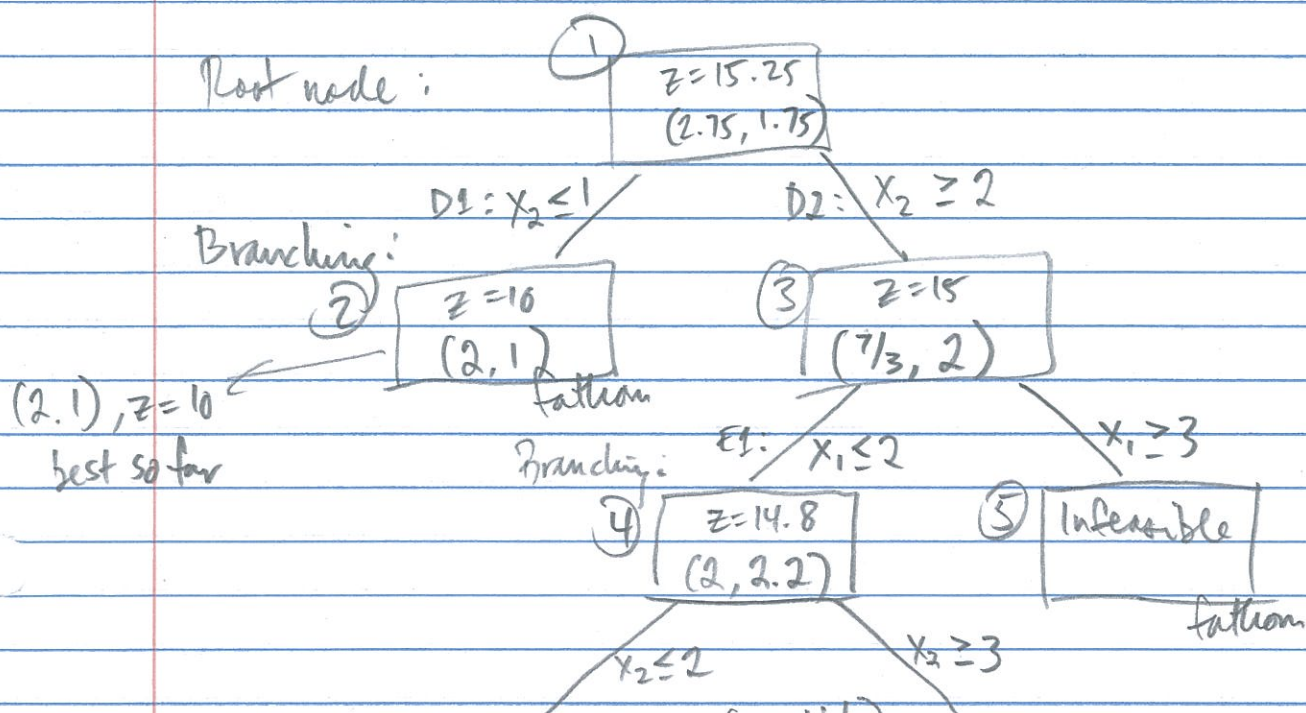
max $3x_1 + 4x_2$

s.t.	$3x_1 + 5x_2 \le 17$	(A)	$(5\frac{2}{3}, 0)$; $(0, 3\frac{2}{5})$
	$5x_1 + 4x_2 \le 22$	(B)	$(4\frac{2}{5}, 0)$; $(0, 5\frac{1}{2})$
	$x_1 - x_2 \le 1$	(C)	$(1, 0)$; $(0, -1)$
	$x_i \ge 0,\ x_i \in \mathbb{Z}^+$		



z=12 at (4, 0) and (0, 3)

Root node:



(cont'd)

2 (cont'd)

$x_2 \le 2$

6 — $z = 14$, $(2, 2)$

fathom

$(2,2)$, $z = 14$ best so far

$x_2 \ge 3$

7 — $z = 14$, $(\frac{4}{3}, 3)$

fathom: children can't do better than $z = 14$ @ 6.

$$\boxed{\begin{aligned} z^* &= 14 \\ x^* &= (2,2) \end{aligned}}$$

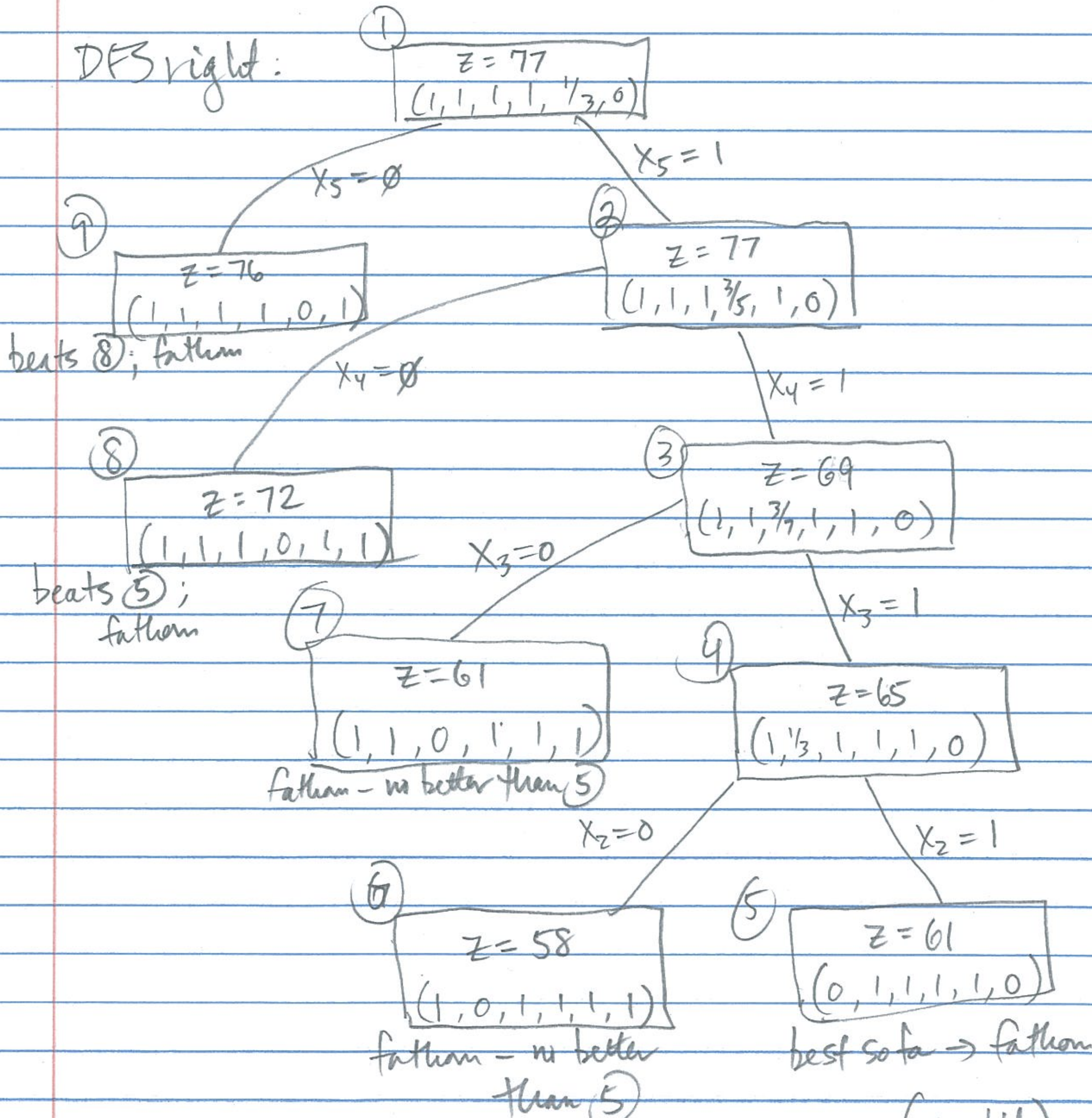
∎

Easton 1) Solve using branch-and-bound, once using DFS-right, and once using best-child.

$$\max \; 20x_1 + 24x_2 + 21x_3 + 10x_4 + 6x_5 + x_6$$
$$\text{s.t.} \; 4x_1 + 6x_2 + 7x_3 + 10x_4 + 6x_5 + 2x_6 \le 29$$
$$x_i \in \{\emptyset, 1\}$$

$$\frac{c_i}{a_i} : [5, 4, 3, 1, 1, 1/2]$$

DFS right:

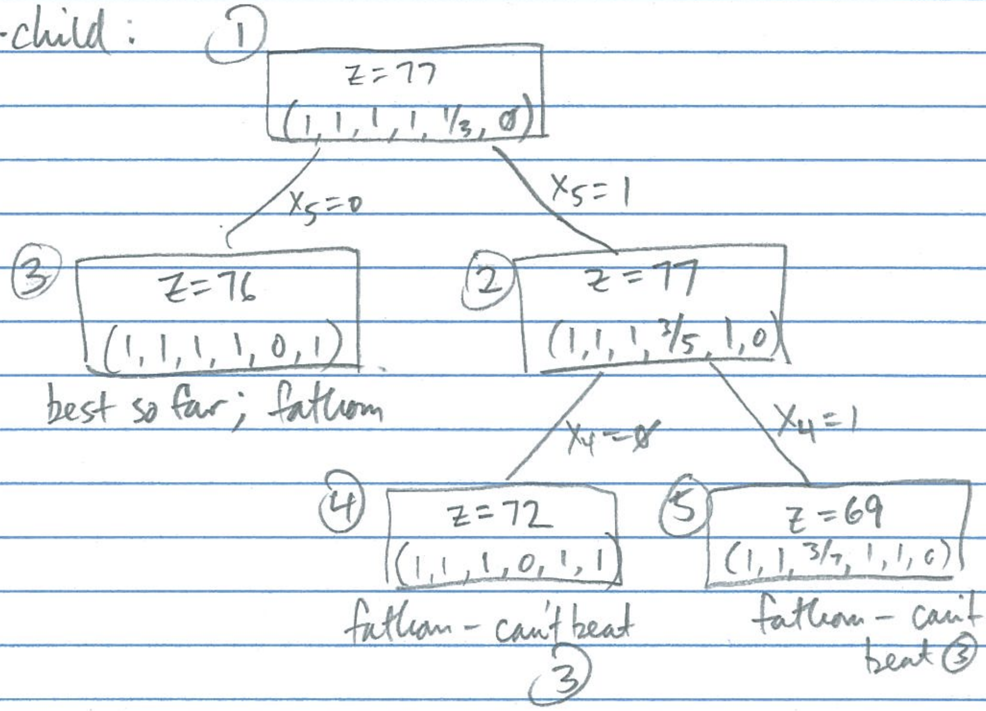


(cont'd)

Easton 1 (cont'd)

So, $z^* = 76$, $x^* = (1, 1, 1, 1, \emptyset, 1)$,
found at node 9 via DFS-right.

Best-child:



The fewer nodes evaluated before reaching a feasible integer solution, the sooner we'll be in a position to prune large sections of the search space known to carry solutions inferior to what we've found so far. Our DFS-right search strategy led us to miss what ultimately was the optimum until we had gone fairly deep into the search tree. DFS left, BFS, or best-child in this case would have found an excellent - indeed, optimum - solution early, and obviate exploring many inferior candidates.

Easton 2

(a) Assume that:

- an IP has a feasible point
- A and $\vec{b}$ are rational

Prove that if the LR of the IP is unbounded, then the IP is unbounded.

Let $\vec{x}$ be a feasible (integer) point of the IP. Then $\vec{x}$ is also in the feasible region of the LR of the IP.

Since the LR is unbounded, there exists for every feasible point $\vec{s}$ of the LR a direction $\vec{d}$ such that

★ $\vec{s} + c\vec{d}$ is also feasible, $\forall c \in \mathbb{R} \geq \emptyset$

★ $A\vec{d} = \vec{\emptyset}$

★ $\vec{d} \geq \vec{\emptyset}$

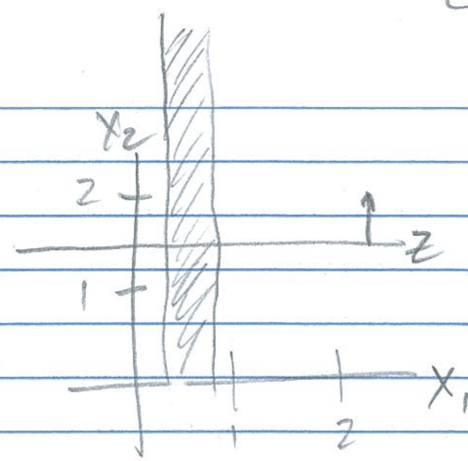
A is rational; $\vec{\emptyset}$ is rational; so $\vec{d}$ is also rational.

Thus $\vec{d} = \begin{bmatrix} p_1/q_1 \\ \cdots \\ p_n/q_n \end{bmatrix}$ where p_i and $q_i \in \mathbb{Z}^+$, $q_i <> \emptyset$ $\forall i \in \{1, \dots n\}$

Choose c = least common multiple of $\{q_1, \dots, q_n\}$. Then $\vec{x} + c\vec{d}$ is also an integer point, and is feasible as shown above. Further, the objective value at $\vec{x} + c\vec{d} \geq$ obj value at $\vec{x}$, since $c > \emptyset$ and $\vec{d} \geq \vec{\emptyset}$. So we can continue to find integer points that improve the objective value - hence the IP is unbounded.

Easton 2 a (cont'd)

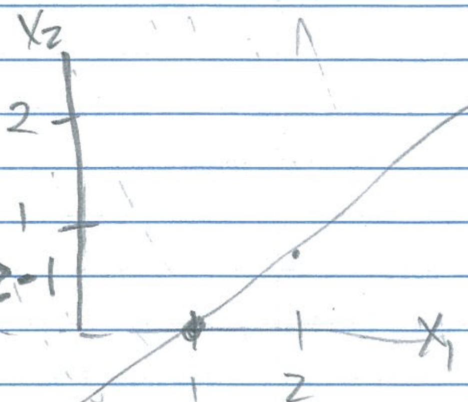
However, consider the IP:



max x_2
s.t. $x_1 \geq 1/4$
$x_1 \leq 3/4$
$x_1, x_2 \geq 0, \; x_1, x_2 \in \mathbb{Z}^+$

This IP is infeasible; but its linear relaxation is unbounded.

* Also consider the IP:



max x_1
s.t. $x_1 - \sqrt{2}\, x_2 \leq 1; \; -(x_1 - \sqrt{2}\, x_2) \geq -1$
$x_1, x_2 \geq 0; \; x_1, x_2 \in \mathbb{Z}^+$

Any rational will do; irrational * other → irrational (>0)

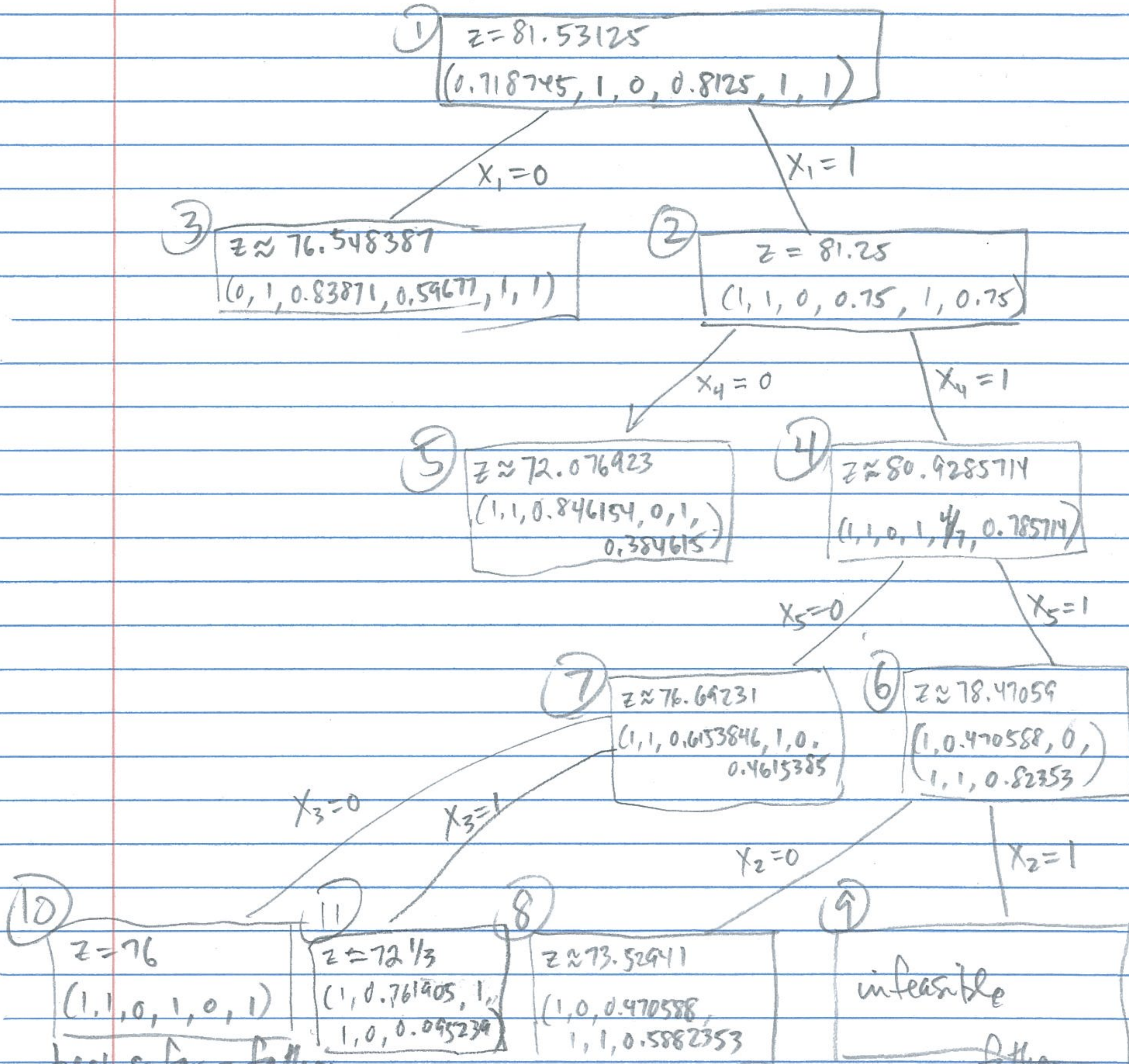
The linear relaxation of this problem is unbounded (make x_1 as large as you like), but the only feasible integer point is $(1, 0)$; hence the IP is bounded at 1. ∎

b) Example * satisfies the desired conditions — a feasible integer point exists, $((1, 0))$, $A, \vec{b}$ contain irrationals; LR unbounded; IP bounded.

c) Example of $A + \vec{b}$ rational, LR unbounded, but IP bounded → Easton 2a proved the IP must be unbounded if the IP is feasible given the other assumptions; if there is no feasible integer point, the IP certainly isn't bounded. Therefore, I believe no such IP exists. ∎

Easton 3 Solve using branch-and-bound:

$$\begin{aligned}
\max\ & 19x_1 + 18x_2 + 16x_3 + 22x_4 + 15x_5 + 17x_6 \\
\text{s.t. } & 4x_1 + 5x_2 + 6x_3 + 10x_4 + 6x_5 + 2x_6 \le 24 \\
& 9x_1 + 7x_2 + 11x_3 + 8x_4 + 6x_5 + 7x_6 \le 34 \\
& 8x_1 + 3x_2 + 7x_3 + 4x_4 + 3x_5 + 8x_6 \le 23 \\
& x_i \in \{\emptyset, 1\}
\end{aligned}$$



Commentary follows ⟹

Easton 3 (cont'd)

Generally, using best-child, branching on var furthest from an integer value.

- Node 1, root. Not an integer solution.
- Create nodes 2 and 3, branching on x_1.
- Node 2 is the better child.
- Create nodes 4 and 5, branching on x_4.
- Of nodes 3, 4, and 5, 4 is best.
- Create nodes 6 and 7, branching on x_5.
- Of nodes 3, 5, 6, and 7, 6 is best.
- Create nodes 8 and 9, branching on x_2.
- Node 9 is infeasible; fathom it right away.
- Of nodes 3, 5, 7, and 8, 7 is best.
- Create nodes 10 and 11, branching on x_3.
- Node 10 is integer - best so far.
- Nodes 3, 5, 8, and 11 can be fathomed because any integer solutions found by branching will yield z better than the 76 at node 10.

So, $z^* = 76$, $x^* = (1, 1, \emptyset, 1, \emptyset, 1)$. ▨