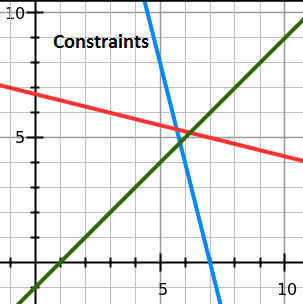
II.1.9 #1. Let . Determine the facets of conv(S) graphically (See Exercise 10 of Section I.4.8). Then derive each of the facets of conv(S) as a C-G inequality.

Facets x1 ≥ 0

x2 ≥ 0

x2 ≤ 6

x1 ≤ 5

x1 – x2 ≤ 1

x1 + 2x2 ≤ 15

Original Constraints

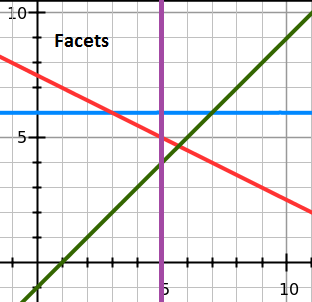
4x1 + x2 ≤ 28

x1 + 4x2 ≤ 27

x1 – x2 ≤ 1

x1 ≥ 0

x2 ≥ 0



(4x1 + x2 ≤ 28) + (x1 – x2 ≤ 1) => 5x1 ≤ 29

1/5(5x1 ≤ 29) => x1 ≤ 29/5

floor(x1 ≤ 29/5) => x1 ≤ 5

¼(x1 + 4x2 ≤ 27) => ¼ x1 + x2 ≤ 6¾

floor(¼ x1 + x2 ≤ 6¾) => x2 ≤ 6

½ (x1 + 4x2 ≤ 27) => ½ x1 + 2x2 ≤ 13.5

½(x1 ≤ 5) => ½ x1 ≤ 2.5

(½ x1 + 2x2 ≤ 13.5) + (½ x1 ≤ 2.5) => x1 + 2x2 ≤ 16

½( x1 + 2x2 ≤ 16) => ½ x1 + x­2 ≤ 8

½(x1 ≤ 5) => ½ x1 ≤ 2.5

(½ x1 + x­2 ≤ 8) + (½ x1 ≤ 2.5) => x1 + x2 ≤ 10.5

floor(x1 + x2 ≤ 10.5) => x1 + x2 ≤ 10

2/3(x1 + x2 ≤ 10) => 2/3 x1 + 2/3 x2 ≤ 6 and 2/3

1/3(x1 + 4x2 ≤ 27) => 1/3 x1+ 4/3 x2 ≤ 9

(2/3 x1 + 2/3 x2 ≤ 6 and 2/3) + (1/3 x1+ 4/3 x2 ≤ 9) => x1 + 2x2 ≤ 15 and 2/3

floor(x1 + 2x2 ≤ 15 and 2/3) => x1 + 2x2 ≤ 15

1. II.1.9 #2. Let . Derive the valid inequality x1 + x2 + 5x3 ≥ 8 using modular arithmetic.

19x1 + 28x2 – 184x3 = 8 d = 18

(19 mod 9)x1 + (28 mod 9)x2 – (184 mod 9)x3 ≥ 8 mod 9

x1 + x2 + 5x3 ≥ 8

1. II.1.9 #3. For Show that 4x1 + 5x2 – 2x3 – 4x4 ≤ 12 is a valid inequality by disjunctive arguments.

Make both inequalities weaker so that x2, x3, and x4 all have the same coefficient.

Now apply the disjunctive procedure with δ =0

Want to solve for , λ.

= 2, and λ =5. Both are positive so the following inequality is valid

7x1 + 5x2 – 2x3 – 4x4 ≤ 10

Making this inequality weaker in x1 results in

4x1 + 5x2 – 2x3 – 4x4 ≤ 10

Making this inequality weaker in rhs results in

4x1 + 5x2 – 2x3 – 4x4 ≤ 12

1. II.1.9 #13. Consider the integer program max {2x1 + 5x2 : x ϵ S}, where S is given in Exercise 1. Using the optimal basis of the corresponding linear program, the problem can be rewritten as

max z

Derive a Gomory fractional cut from each equation. Express each out in terms of the original variables (x1, x2). Derive each cut a s a rank 1 C-G inequality.

Add the final Gomory cut to the LP and resolve with dual simplex

Making it a basis (multiplying bottom row by -1). The red element is the pivot element.

1. II.1.9 #14. For S = P ∩ Z2 as given in Exercise 1 show that
   1. x1 ≤ 5

(4x1 + x2 ≤ 28) + (x1 – x2 ≤ 1) => 5x1 ≤ 29

f(x) = 1/5 x

x1 ≤ 5.8

f(x) = floor(x)

x1 ≤ 5

* 1. x1 + 2x2 ≤ 15

x1 ≤ 5

f(x) = ½ x

½ x1 ≤ 2.5

x1 + 4x­2 ≤ 27

f(x) = ½ x

½ x1 + 2x2 ≤ 13.5

(½ x1 ≤ 2.5) + (½ x1 + 2x2 ≤ 13.5) => x1 + 2x­2 ≤ 16

f(x) = ½ x

½ x1 + x2 ≤ 8

(½ x1 + x2 ≤ 8) + (½ x1 ≤ 2.5) => x1+ x2 ≤ 10.5

f(x) = floor(x)

x1 + x2 ≤ 10

f(x) = (2/3) x

(2/3)x1 + (2/3)x­­2 ≤ 6 and 2/3

x1 + 4x2 ≤ 27

f(x) = (1/3)x

(1/3)x1 + (4/3)x2 ≤ 9

((2/3)x1 + (2/3)x­­2 ≤ 6 and 2/3) + ((1/3)x1 + (4/3)x2 ≤ 9) => x1 + 2x2 ≤ 15 and 2/3

f(x) floor(x)

x1 + 2x2 ≤ 15

* 1. 2x1 + 5x2 ≤ 36

x1 + 4x2 ≤ 27

f(x) = ¼

¼ x1 + x2 ≤ 6 and ¾

f(x) = floor(x)

x2 ≤ 6

x1 + 2x2 ≤ 15

f(x) = 2

2x1 + 4x­2 ≤ 30

(x2 ≤ 6) + (2x1 + 4x­2 ≤ 30) => 2x1 + 5x2 ≤ 36

**Pg 291 Problem 3**

There are n2 variables. To calculate the rank of (A=,b=) observe that the final constraint can be written as the sum of the first n constraints minus the next n-1 constraints. Thus this equation in redundant. So the rank of (A=,b=) ≥ 2n-1. Using one of the fundamental theorems we get n2= dim(P) + rank(A=,b=). Thus dim(P)≤ n2-2n+1.

We need to n2-2n+2 affinely independent points. This will be done by induction on n. The base case n=2 has exactly 2 solutions x11=1, x22=1 or x12=1, x21=1. These points are affinely independent and so the base case holds.

Assume true for n=k and show true for n=k+1. That is there needs to be (k+1)2-2(k+1)+2 = k2+1 affinely independent points. The following points are affinely independent. Notice that everything with an k+1 is on the bottom. Note any item not shown is a 0.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1,1 | With the xk+1,k+1=1 the problem now reduces to an identical problem with only n. Thus, by induction assumption there are n2-2n+2 affinely independent points. | | | |  | | | | | | | | | |
| . |
| . |
| 1,k |
| 2,1 |
| . |
| 2,k |
| . |
| . |
| k,1 |
| . |
| . |
| k,k |
| 1,k+1 |  |  |  |  | 1 |  |  |  |  | 1 | 1 | 1 | 1 | 1 |
| 2,k+1 |  |  |  |  |  | 1 |  |  |  |  |  |  |  |  |
| . |  |  |  |  |  |  | . |  |  |  |  |  |  |  |
| . |  |  |  |  |  |  |  | . |  |  |  |  |  |  |
| k,k+1 |  |  |  |  |  |  |  |  | 1 |  |  |  |  |  |
| k+1,1 |  |  |  |  |  |  |  |  |  | 1 |  |  |  |  |
| k+1,2 |  |  |  |  |  |  |  |  |  |  | 1 |  |  |  |
| . |  |  |  |  |  |  |  |  |  |  |  | . |  |  |
| . |  |  |  |  |  |  |  |  |  |  |  |  | . |  |
| k+1,k-1 |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |
| k+1,k |  |  |  |  | 1 | 1 | 1 | 1 | 1 |  |  |  |  |  |
| k+1,k+1 | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  |  |  |  |

There are a total on k2-2k+2 points in the first section, and there are 2k-1 points in the second half of the matrix. Observe that all of these rows would have had to have been 0 in the first half. So there are a total of k2+1 total pointsand so the dim(Pch) ≥ k2-2k+1 and the result follows.

b. Facets of this are xij ≥0for each i,j ∈{1,...,n}. By symmetry

we only need to show that the inequality x11 ≥0 is facet defining. Clearly x11 ≥0 is valid and there exists a feasible solution with x11=1 and so the face induced by the inequality has dimension ≤k2-2k. We need to find k2-2k+1 affinely independent points

Show this by induction. Let n=2, which has exactly 2 solutions x11=1, x22=1 or x12=1, x21=1. Clearly only one of these points has x11=0 and so the dimension x11≥0 is a facet defining inequality.

Assume true for n=k and show true for n=k+1. That is I need to show that there exists there needs to be (k+1)2-2(k+1)+1 = k2 affinely independent points that satisfy x11=0.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1,1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1,2 |  |  |  |  | 1 |  |  |  |  |  |  |  |  |
| 1,3 |  |  |  |  |  | 1 |  |  |  |  |  |  |  |
| . |  |  |  |  |  |  | . |  |  |  |  |  |  |
| 1,k |  |  |  |  |  |  |  | 1 |  |  |  |  |  |
| 1,k+1 | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  |  |  |
| 2,1 | From the above induction assumption, this is equivalent to a problem with n=k. Thus there exists (k)2 –2(k-1) +2 such points that are  affinely independent | | | |  |  |  |  | 1 |  |  |  |  |
| . |  |  |  |  |  |  |  |  |  |
| 2,k+1 |  |  |  |  |  |  |  |  |  |
| 3,1 |  |  |  |  |  | 1 |  |  |  |
| . . |  |  |  |  |  |  |  |  |  |
| 3,k+1 |  |  |  |  |  |  | . |  |  |
| . . |  |  |  |  |  |  |  | . |  |
| k+1,1 |  |  |  |  |  |  |  |  | 1 |
| . |  |  |  |  |  |  |  |  |  |
| . |  |  |  |  |  |  |  |  |  |
| k+1,k+1 |  |  |  |  |  |  |  |  |  |  |  |  |  |

On the second half of this matrix there are a total of k-2+k+1 ones. Observe that in all previous points, these rows would have had to have been 1 and so they are affinely independent. Thus there are (k)2 –2(k) +2 + 2k-1 =k2+1 and the result follows.

Observe that x11≥0 is equivalent to Σi=2..n x1,j ≥1. As the above points also satisfy this inequality at equality. This follows the theorem that states that if the polyhedron is not full dimensional, then there is not a unique description of the facet defining inequalities.

II. Since there has to be at least 1 in each j, by the pigeon hole principle, the only set of feasible points will again satisfy both classes of inequalities at equality. Thus A=b= hasn’t changed. So all of the above analysis is still valid. You should observe that it is not ok to just take the equality constraints and find the rank. That is, the intersection of constraints can create more equality constraints.

Problem 3.

The following n points are feasible

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **1,2** |  | **1** | **1** | **.** | **1** | **1** | **1** |
| **2,3** | **1** |  | **1** | **.** | **1** | **1** | **1** |
| **3,4** | **1** | **1** |  | **.** | **1** | **1** | **1** |
| **.** | **1** | **1** | **1** | **.** | **1** | **1** | **1** |
| **n-2,n-1** | **1** | **1** | **1** | **.** |  | **1** | **1** |
| **n-1,n** | **1** | **1** | **1** | **.** | **1** |  | **1** |
| **n,1** | **1** | **1** | **1** | **.** | **1** | **1** |  |
| **0,1** | **1** |  |  |  |  |  | **1** |
| **0,2** | **1** | **1** |  |  |  |  |  |
| **0,3** |  | **1** | **1** |  |  |  |  |
| **0,4** |  |  | **1** | **.** |  |  |  |
| **.** |  |  |  |  |  |  |  |
| **.** |  |  |  |  |  |  |  |
| **0,n-2** |  |  |  |  | **1** |  |  |
| **0,n-1** |  |  |  |  | **1** | **1** |  |
| **0,n** |  |  |  |  |  | **1** | **1** |

By adding the row of ones on the bottom and then minusing this row from each of the first n rows results in the negative of the identity matrix. Thus each of these points is affinely independent. So the dim(P) is at least n-1. In addition, there are n+1 equality constraints and 2n variables, so the dim(P)≤ 2n-(n+1) and so the dimension of P= n-1.

II. Show that xe≥0 is redundant. There at most n-2 feasible points with xi,j=0. So it isn’t a facet and thus it is a redundant.

III There are at most 2 points with x0,1=1and so it is not a facet either and it is redundant.

IV Σj:{i,j}∈E xij=2 for all i. xi,(i mod n)+1≤1 for i = 1, ..., n.

V The extreme points are above.

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This problem is a knapsack problem. The points 0, ej for j ϵN={1,…,n} are feasible because aj+ak≤b for all j,kϵN. Thus, the dimension of PCH is at least n. There are n variables, so the dimension of PCH is less than or equal to n. The dimension of PCH is n.

Clearly xj≥0 is a valid inequality. The points 0, ek for k ϵN\{j} are all feasible, affinely independent and satisfy the inequality at equality. So they are in the face. The point ej does not meet the inequality at equality. Thus, the dimension of F is n-1 and it is a facet defining equality.

Clearly xj≤1 is a valid inequality. The points ej, and ej + ek for k ϵN\{j} are all feasible (aj+ak≤b assumption), affinely independent and satisfy the inequality at equality. So they are in the face. The point 0 does not meet the inequality at equality. Thus, the dimension of F is n-1 and it is a facet defining equality.