# Technical Report

## Introduction

Given a graph *G* = (*V*, *E*), where *V* is the set of vertices (nodes) in *G* and *E* is the set of edges in *G*, the ***vertex coloring problem*** is to assign a color from the set K = {1, ..., *c*} to each *v* in *V* such that the endpoints of each edge are assigned different colors, using as few colors as possible. This report describes an implementation of two 0-1 integer programming (IP) formulations of the vertex coloring problem and explores the effects of adding particular classes of facet-defining constraints to the solution times of these formulations.

### IP Formulations

The following sections describe the 0-1 IP formulations implemented and explored in this report.

#### "Assignment" formulation, with symmetry breaking

Mendez-Diaz and Zabala describe this straightforward formulation in "A Branch-and-Cut Algorithm for Graph Coloring".

Let variable *xik* = 1 if node i is assigned color *k*, 0 else, ∀ *i* ∈ V, ∀ *k* ∈ K. Let variable *wk* = 1 if color *k* is used on at least one node, 0 else, ∀ *k* ∈ K. Then the vertex coloring problem becomes:

Minimize Σ*k*∈K *wk*

Subject to: Σ*k*∈K *xik* = 1, ∀ *i* ∈ V

*xik* + *xjk* ≤ *wk*, ∀ (*i*, *j*) in E, ∀ *k* ∈ K

*xik* ∈ {0, 1}, ∀ *i* ∈ V, ∀ *k* ∈ K

*wk* ∈ {0, 1}, ∀ *k* ∈ K

These constraints ensure that, respectively: each node is assigned exactly one color, and endpoints of an edge cannot receive the same color.

The assignment formulation suffers from inherent symmetry: given a particular solution, one can obtain many equivalent solutions by performing a one-to-one switch of one color for another: e.g. those nodes colored with color 1 get color 2 instead, and vice versa. To eliminate symmetrical equivalent solutions from consideration, add the following constraints:

*wk* ≤ Σ*i*∈V *xik*, ∀ *k* ∈ K

*wk* ≥ *wk+1*, ∀ *k* ∈ K \ {*c*}

These constraints ensure that, respectively: a color is not considered used unless at least one node is marked with it, and a greater-numbered color is not used unless all the lesser-numbered colors are used.

##### Clique constraints

Mendez-Diaz and Zabala identified numerous categories of valid inequalities for the polytope that this assignment formulation represents, including the facet-defining ***clique inequalities***. Let Q be a maximal clique of G; then the following inequalities define facets:

Σ*i*∈Q *xik* ≤ *wk*, ∀ *k* ∈ K \ {c}

We can view such constraints as stronger statements about adjacent nodes not receiving the same color, summing all members of the clique at once rather than leaning on the pairwise sums from the original formulation.

#### "Representative" formulation

Since no adjacent nodes can receive the same color in a vertex coloring, all nodes in a feasible coloring that have received the same color are an independent set. Then a feasible coloring can be considered a partition of the graph's nodes into some number of independent sets, and an optimal coloring as such a partition of the nodes into a minimal number of independent sets. The representative formulation of Campelo, Correa, and Frota [] builds on the work of Mehrotra and Trick [] to model the vertex coloring problem using the notion of independent sets without using one variable per maximal independent set.

Let every node in the graph that is assigned the same color be considered members of a color class. Suppose every color class has exactly one node designated as the ***representative*** of that color class. Let N-(*i*) denote the *anti-neighborhood* of node *i*: {*v* ∈ V : (*i*, *v*) ∉ E}, and let N-[*i*] denote N-(*i*) ∪ {*i*}. Let G[S] be the subgraph induced by some S ⊂ V, and let E[S] be the edge set of G[S]. Let variable *xij* = 1 iff node *i* represents the color of node *j*, 0 else. Then the vertex coloring problem becomes:

Minimize Σ*i*∈*V* *xii*

Subject to: Σ*j*∈*N-[i]* *xij* ≥ 1, ∀ *i* ∈ V

*xij* + *xik* <= *xii*, ∀ *i* ∈ V, ∀ (*j*, *k*) ∈ E[N-[*u*]]

These constraints ensure that, respectively: either a node represents its color class or some node not adjacent to it does, and that adjacent nodes cannot share a representative (and hence a color class).

##### Clique constraints

Campelo et al. identified numerous categories of valid inequalities for the polytope that this representative formulation describes, including the facet-defining ***clique inequalities***. Let *i* ∈ V and let Q ⊆ N-(i) so that G[Q] is a maximal clique of G[N-(i)]; then the following inequality defines a facet:

Σ*j*∈Q *xij* ≤ *xii*, ∀ *k* ∈ K \ {c}

We can view such constraints as stronger statements about adjacent nodes not sharing a color representative, summing all members of the clique at once rather than leaning on the pairwise sums from the original formulation.

## Assumptions

We assume that there are as many colors available for use in the coloring of a graph as there are vertices in the graph; that is, *k* = ⏐*V*⏐.

We assume, in the case of the representative formulation, that no node is universal (i.e. its anti-neighborhood is the empty set) and that no node's anti-neighborhood has isolated nodes (with no edges incident). Note that the assignment formulation has no such restriction.

## Methods and Analysis

This implementation was developed and tested using:

* Python[[1]](#footnote-1) 2.7.14
* IBM ILOG CPLEX Optimization Studio[[2]](#footnote-2) 12.8, and its Python 2.7 bindings
* NetworkX[[3]](#footnote-3) 2.1, for modeling graph structures and performing algorithms on them
* pytest[[4]](#footnote-4) 3.5.0, for writing tests
* matplotlib[[5]](#footnote-5) 2.2.2, for plotting coloring solutions

### Running the Program

### Program Structure

### Results

### Analysis

1. <https://www.python.org> [↑](#footnote-ref-1)
2. <https://www.ibm.com/support/knowledgecenter/SSSA5P_12.8.0/ilog.odms.studio.help/Optimization_Studio/topics/COS_relnotes_intro.html> [↑](#footnote-ref-2)
3. <http://networkx.github.io> [↑](#footnote-ref-3)
4. <https://docs.pytest.org/en/latest/> [↑](#footnote-ref-4)
5. <https://matplotlib.org> [↑](#footnote-ref-5)