1. (12%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) $\lim_{x\to 3}\frac{x^2+14x-51}{x^3-5x^2+4x+6}$

(b) $\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}$

(c) $\lim_{x\to 0} x\left(\cos 2x+\cos\frac{1}{x}\right)$

(d) $\lim_{x\to 2}\frac{2x+6}{x-2}$

Ans:

(a) $\lim_{x\to 3}\frac{x^2+14x-51}{x^3-5x^2+4x+6}=\lim_{x\to 3}\frac{(x-3)(x+17)}{(x-3)(x^2-2x-2)}=\lim_{x\to 3}\frac{(x+17)}{(x^2-2x-2)}=\lim_{x\to 3}\frac{(3+17)}{(3^2-2\cdot 3-2)}=20$

(Dividing out technique)

(b) $\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}=\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}=\lim_{x\to 0}\frac{3+x-3}{x(\sqrt{3+x}+\sqrt{3})}$

$$=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}=\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}$$

(Rationalizing technique)

(c) For any x, $-2\le \cos 2x+\cos\frac{1}{x}\le 2 \Rightarrow -2|x|\le x\left(\cos 2x+\cos\frac{1}{x}\right)\le 2|x|$,

According to Squeeze theorem , $\lim_{x\to 0} x\left(\cos 2x+\cos\frac{1}{x}\right)=0$

Notice	$\lim_{x\to 0}\frac{\sin x}{x}=1$	$\lim_{x\to 0}\frac{1-\cos x}{x}=0$	$\lim_{x\to 0}\frac{2x}{\sin(5x)}=\frac{2}{5}$

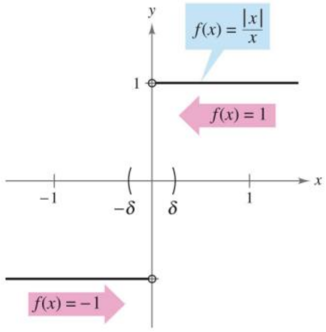
Theorem 1.8 (The Squeeze Theorem)

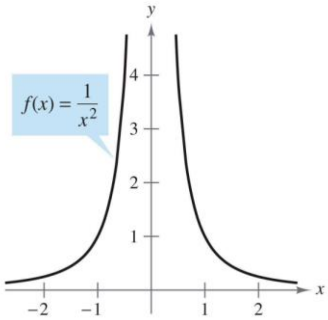
If $h(x)\le f(x)\le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

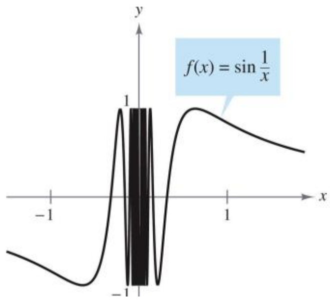
$$\lim_{x\to c}h(x)=L=\lim_{x\to c}g(x)$$

then $\lim_{x\to c}f(x)$ exists and is equal to L.

(d) $\lim_{x\to2^+}\frac{2x+6}{x-2}=\infty$ and $\lim_{x\to2^-}\frac{2x+6}{x-2}=-\infty$. Therefore, the limit does not exist



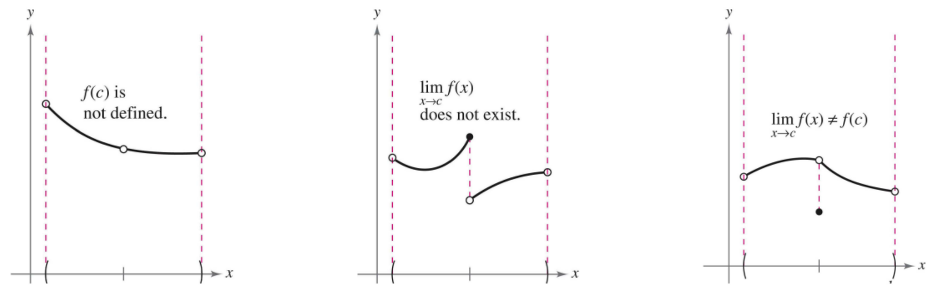




2. (8%) Determine all values of the constant a such that the following function is continuous at $x=0$. (感謝同學提醒，原題因x = 0未定義故送分，應改為以下型式)

$$f(x)=\begin{cases} a^2-2, & x\leq 0 \\ \dfrac{ax}{\tan x}, & x>0 \end{cases}$$

Not Continuous



Ans:

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^-}(a^2-2)=a^2-2$$

$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}\frac{ax}{\tan x}=\lim_{x\to0^+}\frac{x}{\tan x}\lim_{x\to0^+} a=a$ (Remember the proof at 69)

Thus, $a^2-2=a\rightarrow(a-2)(a+1)=0\rightarrow a=-1,2$

Definition 2.2 (The derivative of a function)

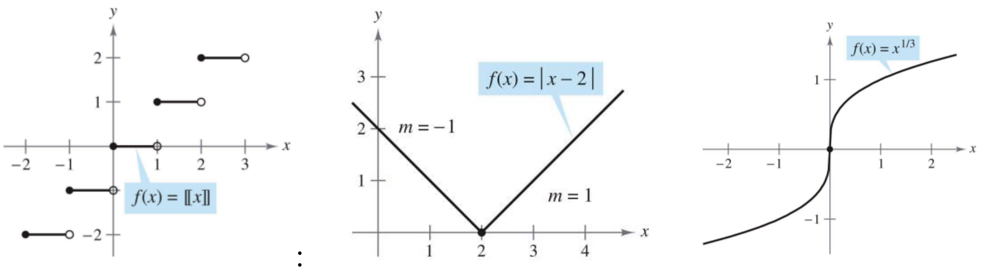
The derivative of f at x is given by

$$f'(x)=\lim_{\Delta x\to0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Not differentiable



3. (10%) Proof that there is only one intersect point between $f(x) = 5x^3 + 2x^2 + 4x + 1$ and $g(x) = 2x^2 + \cos x$. (Hint: use the mean value theorem)

Theorem 1.13 (The Intermediate Value Theorem)

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

Theorem 3.3 (Rolle's Theorem)

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b)$$

then there is at least one number c in (a, b) such that $f'(c) = 0$.

Theorem 3.4 (The Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Ans:

Let $F(x) = f(x) - g(x)$, clearly $F(0) = 0$. Using proof by contradiction, assume there exist a such that $F(a) = 0, a \neq 0$. According to Mean value theorem, $\exists c \in (a, 0)$ such that $F'(c) = \frac{F(a)-F(0)}{a-0} = 0$, contradict. ($F'(x) = 15x^2 + 4 + \sin x > 0$). Therefore, there is only one intersect point.

Related questions:

Exercise in 1.4

Exercise in 3.2

4. (12%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Given $f(x) = \frac{x}{(x+1)(x+2)...(x+2021)}$, what is the value of $f'(0)$?

(b) Find the derivative of $f(x) = \frac{x^3+5x+3}{x^2-1}$

(c) Find the derivative of $f(x) = \sin(\sqrt{cot(5\pi x)}\ )$

(d) Find the following limit. $\lim\limits_{x\to 0} \frac{sin(\sqrt{3+x})-sin(\sqrt{3})}{x}$

Ans:

(a) $f'(0) = \lim\limits_{\Delta x->0} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim\limits_{\Delta x->0} \frac{\frac{\Delta x}{(\Delta x+1)(\Delta x+2)...(\Delta x+2021)}}{\Delta x} = \frac{1}{2021!}$

(b) $f'(x) = \frac{(x^2-1)(3x^2+5)-(x^3+5x+3)(2x)}{(x^2-1)^2} = \frac{x^4-8x^2-6x-5}{(x^2-1)^2}$

(c) $y = \sin((cot(5\pi x))^{\frac{1}{2}}),\ y' =$

$cos(cot(5\pi x)^{\frac{1}{2}})[\frac{1}{2}cot(5\pi x)^{\frac{-1}{2}}(-csc^2(5\pi x))(5\pi)] = -\frac{5\pi cos(\sqrt{cot(5\pi x)})csc^2(5\pi x)}{2\sqrt{cot(5\pi x)}}$

(d) Let $f(x) = sin(\sqrt{x})$, then the limit is the derivative of $f(x)$ at $x = 3$. Which is $\frac{\cos(\sqrt{3})}{2\sqrt{3}}$

Theorem 2.10 (The Chain Rule)

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

5. (10%) Given $x^2y^3 - 5xy^2 - 4y = 4$, find the tangent line at (3,2)

Ans:

$$(x^2y^3 - 5xy^2 - 4y = 4)' = (4)'$$

$$y' = \frac{-7}{11}$$

The tangent line is $(y-2) = \frac{-7}{11}(x-3)$

6. (10%) Let $f(x) = \frac{2x^2}{x^2-1}$

(a) Find the critical numbers and the possible points of inflection of $f(x)$

(b) Find the open intervals on which f is increasing or decreasing

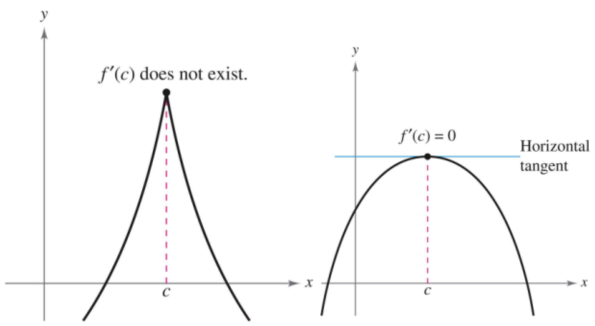
(c) Find the open intervals of concavity

(d) Find all the asymptotes (Both vertical and horizontal)

(e) Sketch the graph of $f(x)$

Definition 3.3 (Critical number)

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f.

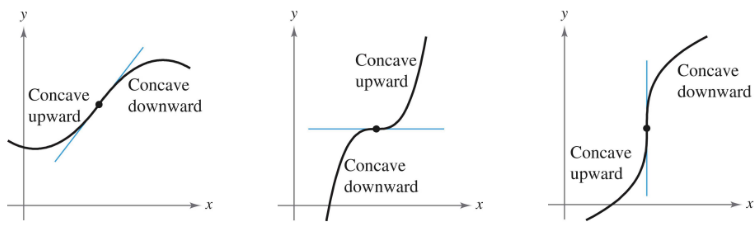


Definition 3.6 (Point of inflection)

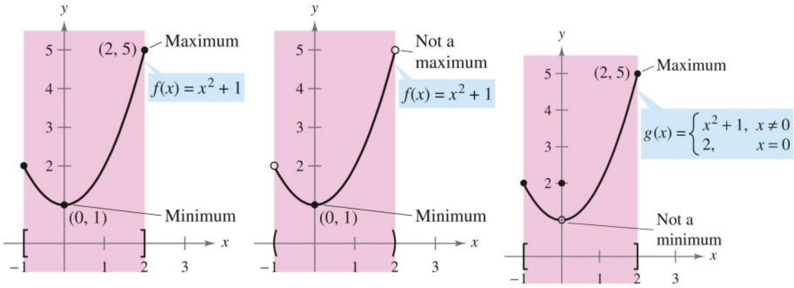
Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Theorem 3.8 (Points of inflection)

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.



Ans: $f(x)=\frac{2x^2}{x^2-1}, f'(x)=\frac{-4x}{(x^2-1)^2}, f''(x)=\frac{12x^2+4}{(x^2-1)^3}$



(a) The critical numbers are $x = 0$

x	$f(x)$	$f'(x)$	$f''(x)$	Graph
$-\infty < x < -1$		$+$	$+$	Increasing, upward
-1	Undef.	Undef.	Undef.	Vertical asymptote
$-1 < x < 0$		$+$	$-$	Increasing, downward
0	0	0	-4	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, downward
1	Undef.	Undef.	Undef.	Vertical asymptote
$1 < x < \infty$		$-$	$+$	Decreasing, upward

Possible points of inflection: None

(b) Increasing $(-\infty,-1),(-1,0)$. Decreasing $(0,1),(1,\infty)$.

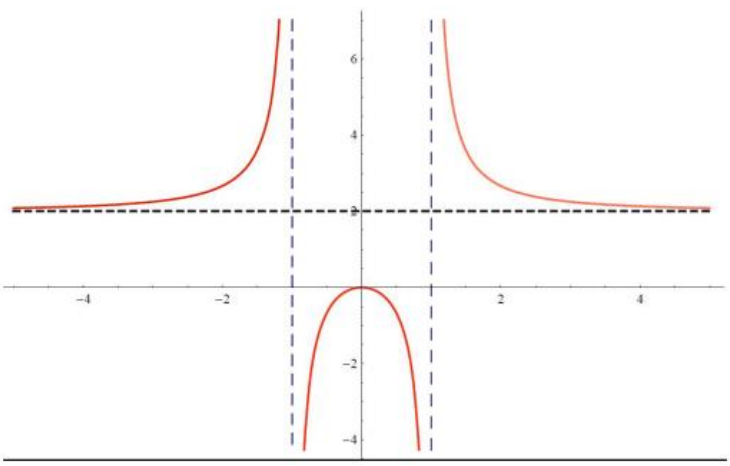
(c) Upward: $(-\infty,-1),(1,\infty)$. Downward $(-1,1)$

(d) Since $\lim_{x\to\pm\infty} f(x) = 2$ -> Horizontal asymptote $y = 2$

Since $\lim_{x\to 1^+} f(x) = \infty$ (or $\lim_{x\to 1^-} f(x) = -\infty$), vertical asymptote $x = 1$

Since $\lim_{x\to -1^+} f(x) = -\infty$ (or $\lim_{x\to -1^-} f(x) = \infty$), vertical asymptote $x = -1$

(e) Graph



7. (10%) Prove that $|tanx - tany| \geq |x - y|$ for all $x, y \in (\frac{-\pi}{2}, \frac{\pi}{2})$ (Hint use mean value theorem)

> **Theorem 3.4 (The Mean Value Theorem)**
>
> *If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that*
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Ans: Note the inequality holds for $x = y$

WLOG, assume $x < y$. (absolute value)

Let $f(t) = tan(t)$. By the mean value theorem, $\exists c \in (x, y)$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$.

Since $f'(t) = sec^2(t) \to f'(c) = sec^2(c) \to \frac{|tanx-tany|}{|x-y|} = |sec^2(c)|$

$|tanx - tany| = |sec^2(c)|\ |x - y|$, since $|sec^2(c)| \geq 1$, we have $|tanx - tany| \geq |x - y|$

8. (12%)

(a) $\int \frac{1+x+x^2}{\sqrt{x}} dx$

(b) $\int 6t - csc^2 t\ \ dt$

(c) Evaluate $\lim_{n\to\infty} \frac{1}{\sqrt{n}}(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}})$ (Hint: use Riemann sum and the definition of the definite integral)

(d) $\int_{-\pi}^{\pi} \frac{x^3 cosx}{1+x^4} dx$

Ans:

(a) $\int \frac{1+x+x^2}{\sqrt{x}} dx = 2\sqrt{x} + \frac{2x^{3/2}}{3} + \frac{2x^{5/2}}{5} + C$

(b) $\int 6t - csc^2 t\ \ dt = 3t^2 + cot(t) + C$

Definition 4.4 (Riemann sum)

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is any point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a Riemann sum of f for the partition Δ.

Definition 4.5 (Definite integral)

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists (as described above), then f is said to be integrable on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x.$$

The limit is called the definite integral of f from a to b. The number a is the lower limit of integration, and the number b is the upper limit of integration.

(c) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}) = \lim_{n\to\infty} \frac{1}{n}(\sqrt{n} + \frac{\sqrt{n}}{\sqrt{2}} + \frac{\sqrt{n}}{\sqrt{3}} + \ldots + \frac{\sqrt{n}}{\sqrt{n}}) =$
$\lim_{n\to\infty} \frac{1}{n}(\frac{1}{\sqrt{\frac{1}{n}}} + \frac{1}{\sqrt{\frac{2}{n}}} + \frac{1}{\sqrt{\frac{3}{n}}} + \ldots + \frac{1}{\sqrt{\frac{n}{n}}}) = \int_0^1 \frac{1}{\sqrt{x}} dx = 2x^{\frac{1}{2}}\Big|_0^1 = 2$

Theorem 4.16 (Integration of even and odd functions)

Let f be integrable on the closed interval $[-a, a]$.

1. *If f is an even function, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$.*
2. *If f is an odd function, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.*

(d) Since it is an odd function, $\int_{-\pi}^{\pi} \frac{x^3 cosx}{1+x^4} dx = 0$

9. (8%) Find $\frac{d}{dx}\int_{2x}^{x^2} cos\sqrt{t}\,dt$ when $x > 0$. (Hint: Let Let $F(x) = \int_1^x cos\sqrt{t}dt$ and use the fundamental theorem of calculus)

Theorem 4.9 (The Fundamental Theorem of Calculus)

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Theorem 4.11 (The Second Fundamental Theorem of Calculus)

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Ans: Let $F(x) = \int_1^x cos\sqrt{t}dt$, since $cos\sqrt{t}$ is continuous, by the fundamental theorem of calculus, $F'(x) = cos\sqrt{x}$. Also $F(b) - F(a) = \int_a^b cos\sqrt{t}dt\,, a, b \in R$, therefore, when $x > 0$,

$$\frac{d}{dx}\int_{2x}^{x^2} cos\sqrt{t}\,dt = \frac{d}{dx}[F(x^2) - F(2x)] = F'(x^2)2x - F'(2x)2$$
$$= 2xcosx - 2cos\sqrt{2x}$$

Related questions:

Exercise in chapter 4.4

10. (8%) Evaluate $\int_1^2 2x^2\sqrt{x^3+1}dx$

Theorem 4.13 (Antidifferentiation of a composite function)

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

Letting $u = g(x)$ gives $\mathrm{d}u = g'(x)\,\mathrm{d}x$ and

$$\int f(u)\,\mathrm{d}u = F(u) + C.$$

Guidelines for making a change of variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2. Compute $\mathrm{d}u = g'(x)\,\mathrm{d}x$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Theorem 4.15 (Change of variables for definite integrals)

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

Ans:

Let $u = x^3 + 1, du = 3x^2dx$

$$\int_1^2 2x^2\sqrt{x^3+1}dx = 2\frac{1}{3}\int_1^2 (x^3+1)^{1/2}(3x^2)dx = \frac{4}{9}[(x^3+1)^{3/2}]_1^2 = 12 - \frac{8}{9}\sqrt{2}$$