

1. (16%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) $\lim_{x \rightarrow (-\frac{1}{2})} \frac{6x^2+x-1}{4x^2-4x-3}$

(b) $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\cos x}{1+\sin x}$

(c) $\lim_{x \rightarrow -\infty} \sqrt{x^2 - 2x} + x$

(d) $\lim_{x \rightarrow \infty} \sqrt{2x^2 + 1} \sin \frac{1}{x}$

Ans:

(a) $\lim_{x \rightarrow (-\frac{1}{2})} \frac{6x^2+x-1}{4x^2-4x-3} = \lim_{x \rightarrow (-\frac{1}{2})} \frac{(3x-1)(2x+1)}{(2x-3)(2x+1)} = \lim_{x \rightarrow (-\frac{1}{2})} \frac{(3x-1)}{(2x-3)} = \frac{5}{8}$

(b) $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\cos x}{1+\sin x} = \lim_{x \rightarrow \frac{3\pi}{2}} \frac{\cos x(1-\sin x)}{(1+\sin x)(1-\sin x)} = \lim_{x \rightarrow \frac{3\pi}{2}} \frac{\cos x(1-\sin x)}{\cos^2 x}$

$\lim_{x \rightarrow \frac{3\pi}{2}^+} \frac{1-\sin x}{\cos x} = \infty$ $\lim_{x \rightarrow \frac{3\pi}{2}^-} \frac{1-\sin x}{\cos x} = -\infty \rightarrow$ limit does not exist

(c) $\lim_{x \rightarrow -\infty} \sqrt{x^2 - 2x} + x = \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2-2x+x})(\sqrt{x^2-2x-x})}{(\sqrt{x^2-2x-x})} = \lim_{x \rightarrow -\infty} \frac{-2x}{(\sqrt{x^2-2x-x})} = 1$

(d) $\lim_{x \rightarrow \infty} \sqrt{2x^2 + 1} \sin x \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{2x^2+1}}{x} x \sin \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{2x^2+1}}{x} \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = \sqrt{2} = \sqrt{2}$

2. (9%)

Let $f(x) = \begin{cases} x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

(a) Is $f(x)$ continuous at $x = 0$?

(b) Compute $f'(x)$ for $x \neq 0$ and $f'(0)$.

Ans:

(a)

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

$$-x^{\frac{4}{3}} \leq x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right) \leq x^{\frac{4}{3}}$$

By squeeze theorem, we have

$$\lim_{x \rightarrow 0} x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right) = 0$$

Therefore, it is continuous at $x = 0$.

$$(b) f'(0) = \lim_{x \rightarrow 0} \frac{x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^{\frac{4}{3}} \cos\left(\frac{1}{x}\right)}{x} = \lim_{x \rightarrow 0} x^{\frac{1}{3}} \cos\left(\frac{1}{x}\right) = 0$$

$$f'(x) = \frac{4}{3} x^{\frac{1}{3}} \cos\left(\frac{1}{x}\right) - x^{\frac{4}{3}} \sin\left(\frac{1}{x}\right) (-x^{-2}) = \frac{4}{3} x^{\frac{1}{3}} \cos\left(\frac{1}{x}\right) + x^{\frac{-2}{3}} \sin\left(\frac{1}{x}\right)$$

3. (6%) Proof that $f(x) = x^7 + x + \frac{1}{2}$ has exactly one real root. (Hint: use the mean value theorem)

Ans:

$$f(0) = \frac{1}{2}, f(-1)$$

$= -\frac{3}{2}$, by the intermediate value theorem it has at least one real root between -1 and 0 .

Assume the real root is a and there is a second real root b . Then by the mean value

theorem, there is a c such that $f'(c) = \frac{f(b) - f(a)}{b - a} = 0$. However, $f'(x) = 7x^6 +$

$1 > 0$. Contradict, therefore, there is only one real root.

4. (16%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Find the derivative of $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$

(b) Let $f(x) = x^5 - 2x^4 + 3x + \pi$, find $f'''(x)$

(c) Let $g(x) = \tan(x^2)$, find $g'(0)$

(d) Find the following limit. $\lim_{x \rightarrow 1} \frac{\frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}}{x-1}$

Ans:

$$(a) f(x) = x^{\frac{5}{6}} + 3x^{\frac{1}{3}} \rightarrow f'(x) = \frac{5}{6} x^{-\frac{1}{6}} + x^{\frac{-2}{3}}$$

$$(b) f'(x) = 5x^4 - 8x^3 + 3 \rightarrow f''(x) = 20x^3 - 24x^2 \rightarrow f'''(x) = 60x^2 - 48x$$

$$(c) g'(x) = 2x \sec^2(x^2) \rightarrow g'(0) = 0$$

$$(d) \text{ Let } f(x) = \frac{x}{\sqrt{x^2+1}}, \text{ then } \lim_{x \rightarrow 1} \frac{\frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}}{x-1} = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x-1} = f'(1).$$

$$f'(x) = \frac{\sqrt{x^2+1} - x(x^2+1)^{-\frac{1}{2}}}{(x^2+1)}$$

Therefore, $\lim_{x \rightarrow 1} \frac{\frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}}{x-1} = f'(1) = \frac{1}{2} \times \left(\sqrt{2} - \frac{1}{\sqrt{2}} \right) = \frac{\sqrt{2}}{4}$

5. (8%) Given the graph $2x^2y - y^3 + 1 = x + y$.

(a) Express y' in terms of x and y

(b) Find the points on the curve with $y = 1$ and the tangent lines at these points

Ans:

$$(a) \frac{d}{dx} (2x^2y - y^3 + 1) = \frac{d}{dx} (x + y)$$

$$4xy + 2x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$(2x^2 - 3y^2 - 1) \frac{dy}{dx} = 1 - 4xy$$

$$\frac{dy}{dx} = \frac{1 - 4xy}{(2x^2 - 3y^2 - 1)}$$

$$(b) 2x^2 \cdot 1 - 1 + 1 = x + 1 \rightarrow (2x + 1)(x - 1) = 0 \rightarrow x = \frac{-1}{2}, 1$$

$$\text{Slopes are } \frac{-6}{7}, \frac{3}{2} \rightarrow \frac{-6}{7} \left(x + \frac{1}{2} \right) = (y - 1), \frac{3}{2} (x - 1) = (y - 1)$$

6. (15%) Let $f(x) = \frac{(x+1)^2}{x^2+1}$

(a) Find the open intervals on which f is increasing or decreasing. Indicates the extreme values.

(b) Find the open intervals on which f is concave upward or concave downward. Indicates the points of inflection

(c) Find all the asymptotes (Both vertical and horizontal)

(d) Sketch the graph of $f(x)$

(e) What is the domain and range of $f(x)$?

Ans:

$$(a) f'(x) = \frac{-2(x+1)(x-1)}{(x^2+1)^2}. \text{ The critical numbers are } \pm 1. f \text{ is increasing on } (-1, 1)$$

since $f'(x) > 0$, f is decreasing on $(-\infty, -1)$ and $(1, \infty)$ since $f'(x) < 0$.

Local maxima is $(1, 2)$ and local minima is $(-1, 0)$.

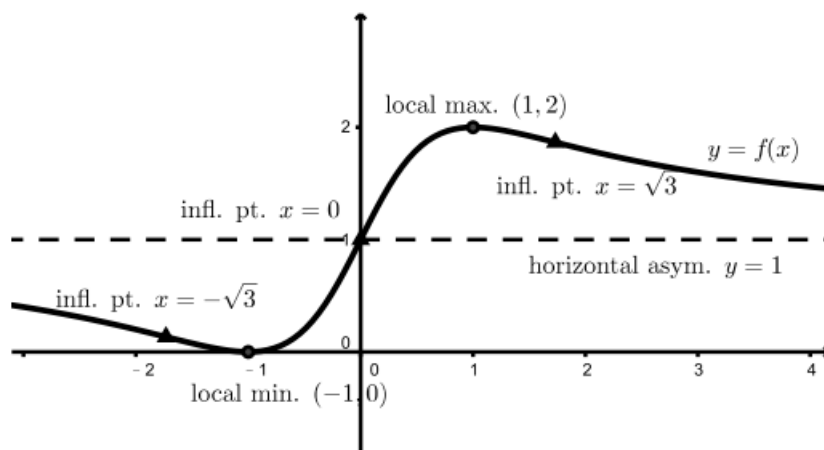
$$(b) f''(x) = \frac{4x(x^2-3)}{(x^2+1)^3}. \text{ The possible points of inflection are}$$

$$(0, 1), (-\sqrt{3}, \frac{(-\sqrt{3}+1)^2}{4}), (\sqrt{3}, \frac{(\sqrt{3}+1)^2}{4}). f \text{ is concave downward on } (-\infty, -\sqrt{3}) \text{ and}$$

$(0, \sqrt{3})$ since $f''(x) < 0$, f is concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ since $f''(x) > 0$. Points of inflection are $(0, 1)$, $(-\sqrt{3}, \frac{(-\sqrt{3}+1)^2}{4})$, $(\sqrt{3}, \frac{(\sqrt{3}+1)^2}{4})$.

(c) $y = 1$ is the horizontal asymptote. since $\lim_{x \rightarrow \pm\infty} f(x) = 1$

(d)



(e) Domain is entire real line. Range is $[0,2]$. Since the end behavior of f approaches 1 and there are no other extrema.

7. (15%) Remember the meaning and the definition of definite integral when solving the following question

(a) $\int 2x^5 + 6x^3 - \sqrt{5}x^2 + 3 \, dx$

(b) $\int \tan^2 y + 2 \, dy$

(c) $\int_{-6}^6 \sqrt{36 - x^2} \, dx$

(d) $\lim_{n \rightarrow \infty} 2 \left(\frac{1^5 + 2^5 + \dots + n^5}{n^6} \right)$

(e) $\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} \, dx$

Ans:

(a) $\frac{x^6}{3} + \frac{3x^4}{2} - \frac{\sqrt{5}}{3}x^3 + 3x + C$

(b) $\int \tan^2 y + 2 \, dy = \int (\tan^2 y + 1) + 1 \, dy = \int (\sec^2 y + 1) + 1 \, dy = \tan y + y + C$

(c) 18π (Since it is the area of a semi-circle)

(d) $\lim_{n \rightarrow \infty} 2 \left(\frac{1^5 + 2^5 + \dots + n^5}{n^6} \right) = \lim_{n \rightarrow \infty} \frac{2}{n} \left(\frac{1^5 + 2^5 + \dots + n^5}{n^5} \right) = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left(\frac{i}{n} \right)^5 = 2 \int_0^1 x^5 \, dx =$

$$2\frac{1}{6}x^6\Big|_0^1 = \frac{1}{3}$$

(e) Let $u = 1 + \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$

$$\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx = 2 \int_2^3 u^{-2} du = -2u^{-1} \Big|_2^3 = \frac{1}{3}$$

8. (9%) Show that the function $f(x) = \int_0^x \frac{2}{t^2+1} dt + \int_0^x \frac{2}{t^2+1} dt$ is constant for $x > 0$ (Hint: use the fundamental theorem of calculus)

Ans:

By the fundamental theorem of calculus

$$f'(x) = \frac{2}{\left(\frac{1}{x}\right)^2 + 1} \left(\frac{-1}{x^2}\right) + \frac{2}{x^2 + 1} = 0$$

Since the derivative of $f(x)$ is zero, $f(x)$ is a constant.

9. (6%) Find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (t^2 + t^6 \tan(t)) dt$.

Ans:

Note that $t^6 \tan(t)$ is an odd function and t^2 is an even function

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (t^2 + t^6 \tan(t)) dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (t^2) dt + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (t^2 + t^6 \tan(t)) dt = \left[\frac{1}{3} t^3 \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi^3}{96}$$