Calculus (I), Final Exam 2022/01/17

Note that e is euler constant in all the following questions

1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) $\lim_{n\to\infty}\frac{1}{n}\left[\frac{\ln(\frac{2n+1}{n})}{\frac{2n+1}{n}}+\frac{\ln(\frac{2n+2}{n})}{\frac{2n+2}{n}}+\cdots+\frac{\ln(\frac{2n+n}{n})}{\frac{2n+n}{n}}\right]$

(b) $\lim_{x\to\infty} xtan(\frac{1}{x})$

(c) $\lim_{x\to 0^+}\frac{\int_0^x \sin(t^2)dt}{xsin(x^2)}$

(d) $\lim_{x\to 1^+}\frac{3}{\ln x}-\frac{2}{x-1}$

Ans:

(a) $\lim_{n\to\infty}\frac{1}{n}\left[\frac{\ln(\frac{2n+1}{n})}{\frac{2n+1}{n}}+\frac{\ln(\frac{2n+2}{n})}{\frac{2n+2}{n}}+\cdots+\frac{\ln(\frac{2n+n}{n})}{\frac{2n+n}{n}}\right]=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\ln(2+\frac{k}{n})}{2+\frac{k}{n}}=\int_2^3\frac{\ln x}{x}dx=$

$\int_{\ln 2}^{\ln 3} udu$ (let $u=\ln x,\ \mathrm{du}=\frac{1}{x}dx$) $=\frac{1}{2}u^2\Big|_{\ln 2}^{\ln 3}=\frac{1}{2}((\ln 3)^2-(\ln 2)^2)$

(b) $\lim_{x\to\infty} xtan(\frac{1}{x})=\lim_{x\to\infty}\frac{tan(\frac{1}{x})}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{-1}{x^2}sec^2(\frac{1}{x})}{\frac{-1}{x^2}}=\lim_{x\to\infty}sec^2(\frac{1}{x})=1$

(c) $\lim_{x\to 0^+}\frac{\int_0^x \sin(t^2)dt}{xsin(x^2)}=\lim_{x\to 0^+}\frac{\sin(x^2)}{sin(x^2)+2x^2\cos(x^2)}$ (By the fundamental theorem of calculus and L'Hôpital's rule) $=\lim_{x\to 0^+}\frac{1}{1+2\frac{x^2}{\sin(x^2)}\cos(x^2)}=\frac{1}{3}$

(d) $\lim_{x\to 1^+}\frac{3}{\ln x}-\frac{2}{x-1}=\lim_{x\to 1^+}\frac{3x-3-2\ln x}{(\ln x)(x-1)}=\lim_{x\to 1^+}\frac{3-\frac{2}{x}}{\frac{x-1}{x}+\ln x}=\infty$

2. (6%) (a) Show that $f(\mathrm{x})=6-x^3$ has an inverse function (Note that you should show that it is one to one)

(b) Find $(f^{-1})'(7)$

Ans:

(a) Since $f'(x)=-3x^2$ is monotonic decreasing on $(-\infty,\infty)$. Therefore, f has an inverse.

(b) $f^{-1}(7)=-1$ (Since $f(-1)=7$)

$$(f^{-1})'(7)=\frac{1}{(\mathrm{f})'(f^{-1}(7))}=\frac{1}{-3(-1)^2}=\frac{-1}{3}$$

3. (6%) Given the function $f(\mathrm{x}) = \mathrm{x} - \mathrm{elnx}$, prove that if $a > \mathrm{e}$, then $\mathrm{f}(a) > 0$.
   (Hunt: use the mean value theorem in the interval (e, a))

Ans:

Since $f(\mathrm{x})$ is continuous on $[\mathrm{e}, a]$ and differentiable on (e, a). By the mean value theorem, there exist at least one number c in (e, a) such that

$$f'(c) = 1 - \frac{e}{c} = \frac{f(a) - f(e)}{a - e} = \frac{f(a)}{a - e}$$

Since $\mathrm{c} > \mathrm{e}$, $\frac{e}{c} < 1$, therefore, $\frac{f(a)}{a-e} > 1$.

If $a > \mathrm{e}, a - e > 0$, then $\mathrm{f}(a) > \mathrm{a} - \mathrm{e} > 0$.

4. (15%) Evaluate the following integral.

   (a) $\int_0^1 \frac{1}{x+\sqrt{x}} dx$

   (b) $\int_0^1 \frac{x^2+4x}{x^3+6x^2+5} dx$

   (c) $\int \frac{e^x}{e^{2x}+2\times e^x+2} dx$

Ans:

(a) Let $\mathrm{u} = \sqrt{x}$, $\mathrm{dx} = 2\mathrm{udu}$

$$\int_0^1 \frac{1}{x+\sqrt{x}} dx = \lim_{b\to0^+} \int_b^1 \frac{1}{x+\sqrt{x}} = \lim_{b\to0^+} \int_{\sqrt{b}}^1 \frac{2}{u+1} du = \lim_{b\to0^+} 2\ln|u+1|\,\Big|_{\sqrt{b}}^1$$
$$= 2\ln 2$$

(b) $\mathrm{u} = x^3 + 6x^2 + 5, du = 3(x^2 + 4x)\mathrm{dx}$

$$\int_0^1 \frac{x^2+4x}{x^3+6x^2+5} dx = \frac{1}{3}\int_5^{12} \frac{1}{u} dx = \frac{1}{3}\ln|u|\,\Big|_5^{12} = \frac{1}{3}(\ln 12 - \ln 5)$$

(c) $\mathrm{u} = e^x, du = e^x\mathrm{dx}$

$$\int \frac{e^x}{e^{2x}+2\times e^x+2} dx = \int \frac{1}{u^2+2u+2} du = \int \frac{1}{(u+1)^2+1} du = \arctan(u+1)$$
$$+ C = \arctan(e^x+1) + \mathrm{C}$$

5. (15%) Evaluate the following expression.

   (a) Given $f(\mathrm{x}) = x^{sinx}$ find $f'(x)$

   (b) $\int \frac{\sin x cos x}{sin^4 x+cos^4 x} dx$

   (c) $\int 4\arccos(x)dx$

Ans:

(a) $\ln f(x) = sinx \ln x$

$$\frac{f'(x)}{f(x)} = cosx \ln x + sinx\frac{1}{x} \rightarrow f'(x) = x^{sinx}(cosx \ln x + \frac{sinx}{x})$$

(b) $\int \frac{\sin x cosx}{sin^4x+cos^4x} dx = \int \frac{\frac{1}{2}\sin(2x)}{(\frac{1-cos2x}{2})^2+(\frac{1+cos2x}{2})^2} dx = \int \frac{\sin(2x)}{1+cos^2(2x)} dx$

Let $\mathrm{u} = \cos 2\mathrm{x}, \mathrm{du} = -2\sin 2\mathrm{x}\mathrm{dx}$

$$= \frac{-1}{2}\int \frac{1}{1+u^2} du = \frac{-1}{2}\arctan(\cos(2\mathrm{x})) + \mathrm{C}$$

(c) Let $\mathrm{u} = \arccos(x), \mathrm{dv} = \mathrm{dx},\ \mathrm{du} = \frac{-1}{\sqrt{1-x^2}}, \mathrm{v} = \mathrm{x}$

$$\int 4\arccos(x)dx = 4\left(xarccos(x) + \int \frac{x}{\sqrt{1-x^2}}dx\right)$$

$$= 4\left(xarccos(x) - \sqrt{1-x^2}\right) + C$$

6. (20%) Evaluate the following integral. (If the integral diverges, you should point it out)

   (a) $\int \frac{2x^2+3x+4}{x^3+3x^2+3x+1} \mathrm{dx}$

   (b) $\int \frac{sec^2x}{tan^2x+5tanx+6} \mathrm{dx}$

   (c) $\int_0^\infty x^3 e^{-x^2}\, dx$

   (d) $\int_0^1 \frac{1}{\sqrt[3]{x^4}} dx$

Ans:

(a) $\int \frac{2x^2+3x+4}{x^3+3x^2+3x+1} \mathrm{dx} = \int \frac{2x^2+3x+4}{(x+1)^3} \mathrm{dx} = \int \frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{(x+1)^3} \mathrm{dx} = 2\ln|x+1| +$

$\frac{1}{x+1} - \frac{3}{2(x+1)^2} + \mathrm{C}$

(b)

Let $\mathrm{u} = \tan\mathrm{x}, \mathrm{du} = sec^2xdx$

$$\int \frac{sec^2x}{tan^2x + 5tanx + 6}\mathrm{dx} = \int \frac{1}{u^2+5u+6} du = \int \frac{-1}{u+3} du + \int \frac{1}{u+2} du$$

$$= -\ln|u+3| + \ln|u+2| + C = \ln\left|\frac{tanx+2}{tanx+3}\right| + C$$

(c) Let $u = x^2,\ du = 2xdx$

$$\int x^3 e^{-x^2}\,dx = \frac{1}{2}\int ue^{-u}du = \frac{-1}{2}(ue^{-u}) + \frac{1}{2}\int e^{-u}\,du$$

$$\int_0^{\infty} x^3 e^{-x^2}\,dx = \lim_{b\to\infty}\frac{-1}{2}(ue^{-u})\Big|_0^b + \frac{1}{2}\int_0^b e^{-u}du$$

$$= \frac{-1}{2}\lim_{b\to\infty}\frac{b}{e^b} - \frac{1}{2}\lim_{b\to\infty} e^{-u}\Big|_0^b = \frac{1}{2}$$

(d)

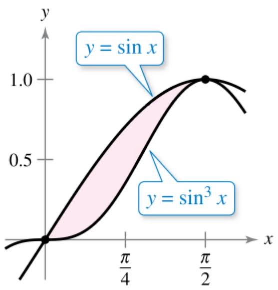
If $p = 1, \int_0^1 \frac{1}{x^p}dx = \int_0^1 \frac{1}{x}dx = \lim_{a\to 0^+} \ln x \,|_a^1 = \infty$ diverges

If $p \neq 1,\ \int_0^1 \frac{1}{x^p}dx = \lim_{a\to 0^+}\frac{x^{1-p}}{1-p}|_a^1 = \lim_{a\to 0^+}\left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right) = \frac{1}{1-p} - \frac{1}{1-p}\lim_{a\to 0^+}\frac{1}{a^{p-1}}$

Which converges to $\frac{1}{1-p}$ if $p - 1 < 0$

Therefore, $\int_0^1 \frac{1}{\sqrt[3]{x^4}}dx$ diverges

7. (6%) Find the following area of the region bounded by $y = \sin(x)$ and $y = sin^3(x)$



Ans:

Let $u = \cos x,\ du = -\sin x dx$

$$\int sin^3 x\,dx = \int(1 - cos^2 x)sinxdx = \int u^2 - 1du = \frac{1}{3}u^3 - u + C$$

$$A = \int_0^{\frac{\pi}{2}}(sinx - sin^3 x)dx = -cosx\Big|_0^{\frac{\pi}{2}} - \left(\frac{1}{3}cos^3 x - cosx\right)\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}$$

8. (6%) Find the arc length of the graph $y = \ln(x)$ over the interval [1,5]

Ans:

$$\mathrm{s} = \int_1^5 \sqrt{1 + y'^2}\, dx = \int_1^5 \frac{\sqrt{x^2+1}}{x} dx$$

Let $\mathrm{x} = \tan\theta$, $\mathrm{dx} = sec^2\theta \mathrm{d}\theta$

$$\int \frac{\sqrt{x^2+1}}{x} dx = \int \frac{sec\theta}{tan\theta} sec^2\theta d\theta = \int \frac{sec\theta}{tan\theta}(1 + tan^2\theta) d\theta = -\ln|csc\theta + cot\theta|$$

$$+ sec\theta + C = -\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + \sqrt{x^2+1} + C$$

$$\mathrm{s} = \int_1^5 \sqrt{1 + y'^2}\, dx = \int_1^5 \frac{\sqrt{x^2+1}}{x} dx = \left[-\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + \sqrt{x^2+1}\right]\Big|_1^5$$

$$= \ln\left(\frac{\sqrt{26}-1}{5(\sqrt{2}-1)}\right) + \sqrt{26} - \sqrt{2}$$

9. (6%) Find the area of the surface generated by revolving the graph of $f(\mathrm{x}) = 9 - x^2$ on the interval $[0,3]$ about the line y-axis.

Ans:

$$\mathrm{S} = 2\pi \int_0^3 x\sqrt{1 + f'(x)^2} dx = 2\pi \int_0^3 x\sqrt{1 + 4x^2} dx$$

Let $\mathrm{u} = 1 + 4x^2, du = 8xdx$

$$\mathrm{S} = \frac{\pi}{4}\int_1^{37} \sqrt{u} du = \frac{\pi}{6} u^{\frac{3}{2}}\Big|_1^{37} = \frac{\pi}{6}(37^{\frac{3}{2}} - 1)$$