1. (20%) Find the following limit. (If the limit does not exist, you should point it out).

Hint: Change of variables may be useful here

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$

(b) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2}{4x^2 y-2y^3}$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

Definition 13.1 (A function of two variables)

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a function of x and y. The set D is the domain of f, and the corresponding set of values for $f(x, y)$ is the range of f.

Surface: $z = f(x, y)$

(x, y, z)

$f(x, y)$

Domain: D

(x, y)
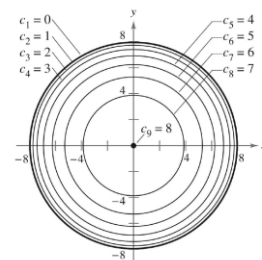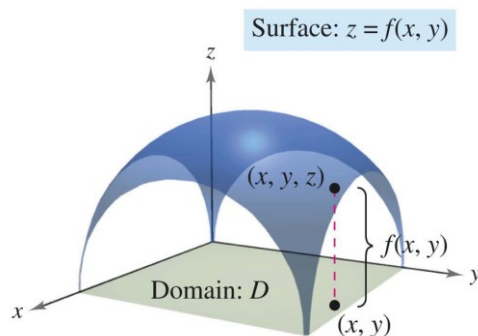
(b) Contour map of
$f(x, y) = \sqrt{64 - x^2 - y^2}$.

Definition 13.2 (Limit of a function of two variables)

Let f be a function of two variables defined, except possible at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for each ε there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Finding a Limit Using Polar Coordinates In Exercises 51–56, use polar coordinates to find the limit. [*Hint:* Let $x = r \cos\theta$ and $y = r \sin\theta$, and note that $(x, y) \to (0, 0)$ implies $r \to 0$.]

Finding a Limit Using Spherical Coordinates In Exercises 77 and 78, use spherical coordinates to find the limit. [*Hint:* Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$, and note that $(x, y, z) \to (0, 0, 0)$ implies $\rho \to 0^+$.]

Ans:

(a) Let $y = kx^2$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{kx^4}{x^4 + k^2 x^4} = \lim_{x\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2}$ which means that

if we follow the trajectory of different parabola $y = kx^2$ to approach (0,0) we will get different value, therefore, the limit does not exist.

(b)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} = \lim_{\rho\to 0^+} \frac{(\rho\sin(\varPhi)\cos(\theta))(\rho\sin(\varPhi)\sin(\theta))\rho\cos(\varPhi)}{\rho^2}$$

$$= \lim_{\rho\to 0^+} \rho\sin^2(\varPhi)\cos(\theta)\sin(\theta)\cos(\varPhi) = 0$$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{1-\cos(r^2)}{r^2}$. By L'Hospital's Rule, $\displaystyle\lim_{r\to 0} \frac{1-\cos(r^2)}{r^2} =$

$\displaystyle\lim_{r\to 0} \frac{-\sin(r^2)2r}{2r} = \lim_{r\to 0} -\sin(r^2) = 0$

(d) Let $y = mx$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2}{4x^2y-2y^3} = \lim_{x\to 0} \frac{1+m^2}{4m-2m^3} = \frac{1+m^2}{4m-2m^3}$. which means that if we

follow the trajectory of different line $y = mx$ to approach (0,0) we will get different value, therefore, the limit does not exist.

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos(\theta)\sin(\theta)}{r} = \lim_{r\to 0} r\cos(\theta)\sin(\theta) = 0.$

2. (16%)

(a) Let $f(x, y) = \int_y^x \sin(t^2)\,dt$, evaluate f_x and f_y

(b) Let $f(x,y) = x\sin(y) + ye^{xy}$, find the four second partial derivatives

(c) Let $z = f(x,y) = x^2 + y^2, x = s + t, y = s - t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

(d) Find an equation of the tangent plance of the surface $9x^2 + y^2 + 4z^2 = 25$ at $(0, -3, 2)$

Definition 13.5 (Partial derivatives of a function of two variables)

If $z = f(x,y)$, then the first partial derivatives of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

Partial derivative with respect to x and

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

Partial derivative with respect to y , provided the limits exist.

① Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

② Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

③ Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

④ Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

Definition 13.6 (Total differential)

If $z = f(x,y)$ and Δx and Δy are increments of x and y, then the differentials of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the total differential of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

Ans:

(a) Using the Fundamental theorem of calculus, $D_x f = \sin(x^2)$ and $D_y f =$

$$D_y\left(-\int_x^y \sin(t^2)\, dt\right) = -\sin(y^2)$$

(b) $\dfrac{\partial f}{\partial x} = \sin(y) + y^2 e^{xy}$, $\dfrac{\partial f}{\partial y} = x\cos(y) + e^{xy} + xye^{xy} = x\cos(y) + (1 + xy)e^{xy}$

$$\frac{\partial^2 f}{\partial x^2} = y^3 e^{xy}, \frac{\partial^2 f}{\partial x\, \partial y} = \cos(y) + 2ye^{xy} + y(1 + xy)e^{xy}$$

$$= \cos(y) + (2y + xy^2)e^{xy}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \cos(y) + 2ye^{xy} + xy^2 e^{xy} = \cos(y) + (2y + xy^2)e^{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = -x\sin(y) + xe^{xy} + x(1 + xy)e^{xy} = -x\sin(y) + (2x + x^2 y)e^{xy}$$

(c) Using chain rule

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2x \cdot 1 + 2y \cdot 1 = 2(s + t) + 2(s - t) = 4s$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 2x \cdot 1 + 2y \cdot (-1) = 2(s + t) - 2(s - t) = 4t$$

Normal line: $\quad \dfrac{x - x_0}{F_x(x_0, y_0, z_0)} = \dfrac{y - y_0}{F_y(x_0, y_0, z_0)} = \dfrac{z - z_0}{F_z(x_0, y_0, z_0)}$

(d) $\nabla F = 18x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$, $\nabla F(0, -3, 2) = -6\mathbf{j} + 16\mathbf{k}$

Tangent plane:

$$0(x - 0) - 6(y + 3) + 16(z - 2) = 0 \rightarrow -3y + 8z = 25$$

3. (6%) Let $f(x,y) = 2022 - \dfrac{x^2}{4} - \dfrac{y^2}{2}$, express the limit $\displaystyle\lim_{t\to 0}\dfrac{f(1+2t, 2+t) - f(1,2)}{t}$ as the directional derivative of f and evaluate the value of the limit.

> **Definition 13.8 (Directional derivative)**
>
> Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the directional derivative of f in the direction of $\mathbf{u}$, denoted by $D_{\mathbf{u}}f$, is
>
> $$D_{\mathbf{u}}f(x,y) = \lim_{t\to 0}\frac{f(x + t\cos\theta, y + t\sin\theta) - f(x,y)}{t}$$
>
> provided this limit exists.

> **Theorem 13.9 (Directional derivative)**
>
> If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is
>
> $$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$

> **Theorem 13.10 (Alternative form of the directional derivative)**
>
> If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is
>
> $$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

> **Theorem 13.11 (Properties of the gradient)**
>
> Let f be differentiable at the point (x,y).
> 1. If $\nabla f(x,y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x,y) = 0$ for all $\mathbf{u}$.
> 2. The direction of maximum increase of f is given by $\nabla f(x,y)$. The maximum value of $D_{\mathbf{u}}f(x,y)$ is $\|\nabla f(x,y)\|$.
> 3. The direction of minimum increase of f is given by $-\nabla f(x,y)$. The minimum value of $D_{\mathbf{u}}f(x,y)$ is $-\|\nabla f(x,y)\|$.

> **Theorem 13.12 (Gradient is normal to level curves)**
>
> If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).
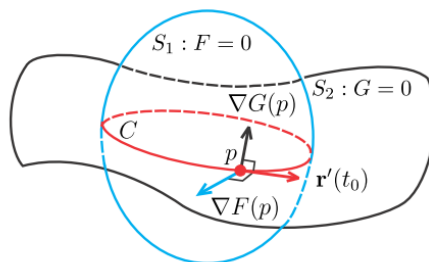
Intersection of two surfaces.

Ans:

First of all, let $(u,v) = (2,1)/\sqrt{5}$ and rewrite the limit as $\lim\limits_{t\to 0}\dfrac{f(1+2t,2+t)-f(1,2)}{t} =$

$\lim\limits_{t\to 0}\dfrac{f(1+\sqrt{5}tu,2+\sqrt{5}tv)-f(1,2)}{t} = \sqrt{5}\lim\limits_{t\to 0}\dfrac{f(1+\sqrt{5}tu,2+\sqrt{5}tv)-f(1,2)}{\sqrt{5}t}$. Let $t' = \sqrt{5}t$

Which can be expressed as $\sqrt{5}\lim\limits_{t'\to 0}\dfrac{f(1+t'u,2+t'v)-f(1,2)}{t'} = \sqrt{5}(D_{(u,v)}f)(1,2)$

$$D_{(u,v)}f = \nabla f \cdot (u,v) = \left(\frac{-x}{2},-y\right)\cdot\frac{1}{\sqrt{5}}(2,1) = \frac{-1}{\sqrt{5}}(x+y)$$

Finally, we have $\sqrt{5}(D_{(u,v)}f)(1,2) = -(1+2) = -3$

4. (8%) Find the critical points of $f(x,y) = x^3 + y^2 - 2xy + 7x - 8y + 2$. Which of them give rise to maximum values, minimum values and saddle points?

Definition 13.13 (Critical point)

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a critical point of f if one of the following is true.
1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Theorem 13.16 (Relative extrema occur only at critical points)

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Theorem 13.17 (Second Partials Test)

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity
$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$
1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a saddle point.
4. The test is inconclusive if $d = 0$.

Ans: $f_x = 3x^2 - 2y + 7, f_y = 2y - 2x - 8$. Let $f_x = 0$ and $f_y = 0$, we have

the critical points $\left(\frac{-1}{3},\frac{11}{3}\right), (1,5)$. Furthermore, since $f_{xx} = 6x, f_{xy} = f_{yx} = $

$-2, f_{yy} = 2$. We have $d = f_{xx}f_{yy} - f_{xy}f_{yx} = 12x - 4$. Therefore, $d\left(\frac{-1}{3},\frac{11}{3}\right)<0,$

$d(1,5) > 0$. Finally, we know that $(1,5)$ is absolute minimum point since

$f_{xx}(1,5) > 0$ and $\left(\frac{-1}{3}, \frac{11}{3}\right)$ is saddle point.

5. (6%) Find the minimum and maximum distance from the curve $x^2 + xy + y^2 = 1$ to the origin point $(0,0)$.

Theorem 13.18 (Lagrange's Theorem)

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

For optimization problems involving two constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Ans:

The objective function can be change from $\sqrt{x^2 + y^2}$ to $x^2 + y^2$. Use the Langrange multiplier, we have the following equations

$$\begin{cases} 2x = \lambda(2x + y) \\ 2y = \lambda(x + 2y) \\ x^2 + xy + y^2 = 1 \end{cases}$$

Cross product the first and second equation yields

$$2x\lambda(x + 2y) = 2y\lambda(2x + y)$$

If $\lambda = 0$, we have $x = y = 0$ which does not satisfy the third equation. If $\lambda \neq 0$. We have

$$x^2 + 2xy = 2xy + y^2$$

Which means $x = \pm y$

If $x = y$, from the third equation, we have $3y^2 = 1$, therefore $(x, y) = (\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3})$. The distance is therefore $\sqrt{x^2 + y^2} = \sqrt{\frac{2}{3}}$

If $x = -y$, from the third equation, we have $y^2 = 1$, therefore $(x, y) = (\pm 1, \mp 1)$. The distance is therefore $\sqrt{x^2 + y^2} = \sqrt{2}$

Finally, we know that the maximum and minimum distance are $\sqrt{2}$ and $\sqrt{\frac{2}{3}}$, respectively.

6. (20%) Evaluate the following expression

(a) $\int_0^1 \int_y^1 \frac{\sin(x)}{x} dx dy$

(b) $\int_0^2 \int_0^{\sqrt{4-x^2}} \sin(\sqrt{x^2 + y^2}) \, dy dx$

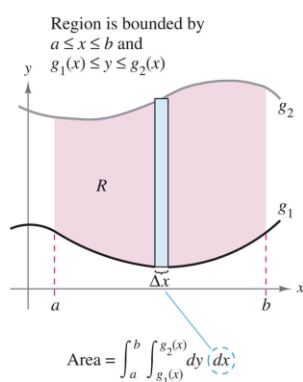(c) $\int_0^{\frac{\pi}{2}} \int_1^2 x^2 \sin(y) \, dx dy$

(d) $\int_0^1 \int_0^{1+\sqrt{y}} \int_0^{xy} y \, dz dx dy$

(e) $\int_1^2 \int_{2u-2}^u e^{(v-u+1)^2} \, dv du$

Iterated integrals

(1) $\int_{h_1(y)}^{h_2(y)} f_x(x, y) \, dx = f(x, y) \Big|_{h_1(y)}^{h_2(y)} =$
   $f(h_2(y), y) - f(h_1(y), y)$ With respect to x

(2) $\int_{g_1(x)}^{g_2(x)} f_y(x, y) \, dy = f(x, y) \Big|_{g_1(x)}^{g_2(x)} =$
   $f(x, g_2(x)) - f(x, g_1(x))$ With respect to y

Region is bounded by
$a \le x \le b$ and
$g_1(x) \le y \le g_2(x)$

Area $= \int_a^b \int_{g_1(x)}^{g_2(x)} dy \, (dx)$

(a) Vertically simple region.

Region is bounded by
$c \le y \le d$ and
$h_1(y) \le x \le h_2(y)$

Area $= \int_c^d \int_{h_1(y)}^{h_2(y)} dx \, (dy)$
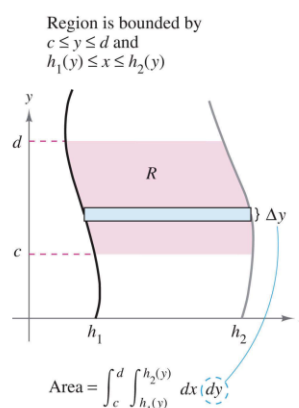
(b) Horizontally simple region.

Definition 14.1 (Area of a region in the plane)

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy \, dx. \qquad \text{Figure 2(a) (vertically simple)}$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx \, dy. \qquad \text{Figure 2(b) (horizontally simple)}$$

Definition 14.2 (Double integral)

If f defined on a closed, bounded region R in the xy-plane, then the double integral of f over R is given by

$$\iint_R f(x,y)\,\mathrm{d}A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i,y_i)\Delta A_i$$

provided the limit exists. If the limit exists, then f is integrable over R.

Volume of a solid region If f is integrable over a plane region R and $f(x,y) \geq 0$ for all (x,y) in R, then the volume of the solid region that lies above R and below the graph of f defined as

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

Definition 14.5 (Triple integral)

If f is continuous over a bounded solid region Q, then the triple integral of f over Q is defined as

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta V_i$$

provided the limit exists. The volume of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q \mathrm{d}V.$$

Theorem 14.2 (Fubini's Theorem)

Let f be continuous on a plane region R.

❶ *If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

❷ *If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Theorem 14.4 (Evaluation by iterated integrals)

Let f be continuous on a solid region Q defined by

$$a \leq x \leq b, \quad h_1(x) \leq y \leq h_2(x), \quad g_1(x,y) \leq z \leq g_2(x,y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Ans:

(a) $\int_0^1 \int_y^1 \frac{\sin(x)}{x} dxdy = \int_0^1 \int_0^x \frac{\sin(x)}{x} dydx = \int_0^1 \left[y \frac{\sin(x)}{x} \right]_0^x dx = \int_0^1 \sin(x)\, dx = 1 - \cos(1)$

Theorem 14.3 (Change of variables to polar form)

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

(b) $R = \{(x, y) | 0 \le x \le 2, 0 \le y \le \sqrt{4 - x^2}\} = \{(r, \theta) | 0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$

$\int_0^2 \int_0^{\sqrt{4-x^2}} \sin\left(\sqrt{x^2 + y^2}\right) dydx = \int_0^{\frac{\pi}{2}} \int_0^2 \sin(r) r\, drd\theta$ (Let $r = u$, $\sin(r) = dv \to$

$dr = du$, $-\cos(r) = v$) and use integration by parts, we have

$= \int_0^{\frac{\pi}{2}} [\sin(r) - r\cos(r)]_0^2 \, d\theta = \int_0^{\frac{\pi}{2}} \sin(2) - 2\cos(2)\, d\theta = \frac{\pi}{2}(\sin(2) - 2\cos(2))$

(c) $\int_0^{\frac{\pi}{2}} \int_1^2 x^2 \sin(y)\, dxdy = \left(\int_0^{\frac{\pi}{2}} \sin(y)dy \right) \times \left(\int_1^2 x^2 dx \right) = [-\cos(y)]_0^{\frac{\pi}{2}} \times \left[\frac{x^3}{3} \right]_1^2 = \frac{7}{3}$

(d) $\int_0^1 \int_0^{1+\sqrt{y}} \int_0^{xy} ydzdxdy = \int_0^1 \int_0^{1+\sqrt{y}} [yz]_0^{xy} \, dxdy = \int_0^1 \int_0^{1+\sqrt{y}} xy^2 \, dxdy =$

$\int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_0^{1+\sqrt{y}} dy = \int_0^1 \frac{1}{2}(1 + \sqrt{y})^2 y^2 dy = \frac{1}{2}\int_0^1 [y^2 + 2y^{\frac{5}{2}} + y^3]dy =$

$\frac{1}{2}[\frac{y^3}{3} + \frac{4}{7}y^{\frac{7}{2}} + \frac{1}{4}y^4]_0^1 = \frac{97}{168}$

Definition 14.6 (Jacobian)

If $x = g(u, v)$ and $y = h(u, v)$, then the Jacobian of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Theorem 14.5 (Change of variables for double integrals)

Let R be a vertically or horizontally simple region in the xy-plane, and let S be a vertically or horizontally simple region in the uv-plane. Let T from S to R be given by $T(u, v) = (x, y) = (g(u, v), h(u, v))$, where g and h have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x, y)/\partial(u, v)$ is nonzero on S, then

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

(e) $x = v - u + 1, y = u \rightarrow v = x + y - 1, u = y$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = -1$$

$$v = u \rightarrow x = 1, v = 2u - 2 \rightarrow x = y - 1$$

$$\int_1^2 \int_{2u-2}^u e^{(v-u+1)^2} \, dv \, du = \int_1^2 \int_{y-1}^1 e^{x^2} \, dx \, dy = \int_0^1 \int_1^{x+1} e^{x^2} \, dy \, dx$$

$$= \int_0^1 [ye^{x^2}]_1^{x+1} \, dx = \int_0^1 xe^{x^2} \, dx = \frac{1}{2}(e - 1)$$

7. (6%) Find the area of the surface given by $z = f(x, y) = 9 - y^2$ that lies above
   the region R where R is a triagle with vertices (-3,3),(0,0),(3,3)

   ### Definition 14.4 (Surface area)

   If f and its partial derivatives are continuous on the closed region R in the
   xy-plane, then the area of the surface S given by $z = f(x, y)$ over R is
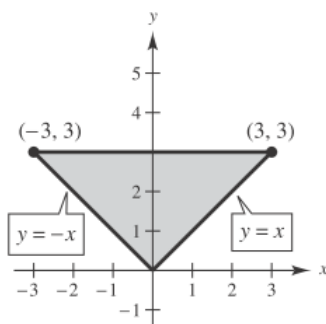   defined as

   $$\text{Surface area} = \iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA.$$
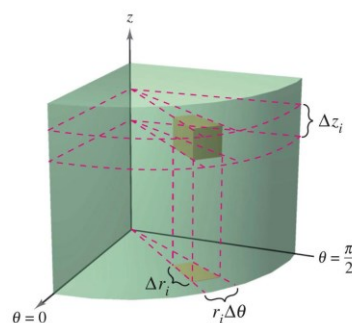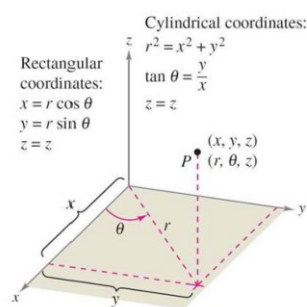
Ans:

$$f_x = 0, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$$

$$S = \int_0^3 \int_{-y}^y \sqrt{1 + 4y^2} \, dx \, dy = \int_0^3 \left[\sqrt{1 + 4y^2}x\right]_{-y}^y \, dy = \int_0^3 2y\sqrt{4y^2 + 1} \, dy$$

$$= \left[\frac{1}{6}(4y^2 + 1)^{\frac{3}{2}}\right]_0^3 = \frac{37\sqrt{37} - 1}{6}$$

8. (6%) Find the volume of the solid inside both $x^2 + y^2 + z^2 = 36$ and
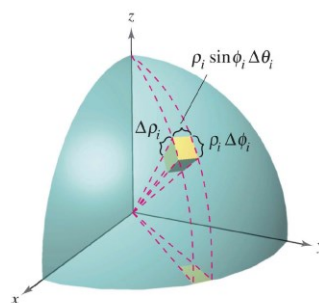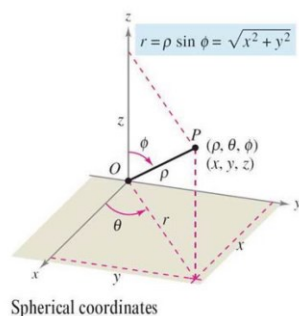
$$(x - 3)^2 + y^2 = 9$$

Cylindrical coordinates:
$$r^2 = x^2 + y^2$$
$$\tan \theta = \frac{y}{x}$$
$$z = z$$

Rectangular coordinates:
$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$z = z$$

$$\iiint_Q f(x, y, z)\, dV$$

$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r \cos \theta, r \sin \theta)}^{h_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r\, dz\, dr\, d\theta$$

Ans:

Note that $(x - 3)^2 + y^2 = 9$ is equivalent to $r = 6 \cos(\theta), \ 0 \le \theta \le \pi$

$$V = 2 \int_0^\pi \int_0^{6\cos(\theta)} \int_0^{\sqrt{36-r^2}} r\, dz\, dr\, d\theta = 2 \int_0^\pi \int_0^{6\cos(\theta)} r\sqrt{36 - r^2}\, dr\, d\theta$$

$$= 2 \int_0^\pi \left[\frac{-1}{3}(36 - r^2)^{\frac{3}{2}} \right]_0^{6\cos(\theta)} d\theta$$

$$= \frac{-2}{3} \int_0^\pi (216\sin^3\theta - 216)\, d\theta$$

$$= -144 \int_0^\pi [(1 - \cos^2\theta)\sin\theta - 1]\, d\theta$$

$$= -144 \left[-\cos(\theta) + \frac{\cos^3\theta}{3} - \theta \right]_0^\pi = 48(3\pi - 4)$$

9. (6%) Evaluate $\iiint_Q \frac{z}{\sqrt{x^2+y^2+z^2}}\, dV$ where Q is a solid region inside the sphere $x^2 + y^2 + z^2 = 9$ and above xy-plane.

$$r = \rho \sin \phi = \sqrt{x^2 + y^2}$$

$P\ (\rho, \theta, \phi)$
(x, y, z)

Spherical coordinates

$\rho_i \sin\phi_i\, \Delta\theta_i$
$\Delta\rho_i$
$\rho_i\, \Delta\phi_i$

$$\iiint_Q f(x, y, z)\, dV$$

$$= \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$$

Ans:

$$\iiint_Q \frac{z}{\sqrt{x^2 + y^2 + z^2}}\, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \frac{\rho \cos(\Phi)}{\rho} \rho^2 \sin(\Phi)\, d\rho\, d\Phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin(\Phi)\cos(\Phi)\, d\Phi \int_0^2 \rho^2\, d\rho = 2\pi \frac{1}{2} \frac{8}{3} = \frac{8}{3}\pi$$

10. (6%) Use a change of variables to find the volume of the solid region lying below the surface $z = f(x, y) = \frac{x}{1+x^2y^2}$ and above the plane region R wher R is a region bounded by $xy = 5, xy = 1, x = 1, x = 5$.

Ans:

Let $u = x, v = xy \rightarrow x = u, y = \frac{v}{u}$

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \frac{1}{u}$$

$$\iint_R \frac{x}{1 + x^2 y^2}\, dA$$

$$= \int_1^5 \int_1^5 \frac{u}{1 + u^2(v/u)^2} \frac{1}{u}\, du\, dv = \int_1^5 \int_1^5 \frac{1}{1 + v^2}\, du\, dv$$

$$= \int_1^5 \frac{4}{1 + v^2}\, dv = 4 \arctan(v) \, |_1^5 = 4 \arctan(5) - \pi$$