Chapter 11 Vectors and the Geometry of Space

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

March 23, 2022

Table of Contents

Table of Contents

Cylindrical surfaces

- You have already known two special types of surfaces.
  1. Spheres: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$
  2. Planes: $ax + by + cz + d = 0$
- A third type of surface in space is called a cylindrical surface, or simply a cylinder.
- To define a cylinder, consider the familiar right circular cylinder shown in Figure 1.

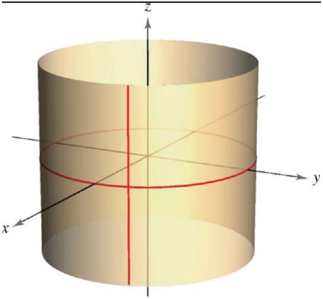


Figure 1: Right circular cylinder: $x^2 + y^2 = a^2$. Rulings are parallel to the z-axis.

- You can imagine that this cylinder is generated by a vertical line moving around the circle $x^2 + y^2 = a^2$ in the xy-plane.
- This circle is called a generating curve for the cylinder, as indicated in the following definition

Definition 11.1 (Cylinder)

Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a cylinder. C is called the generating curve (or **directrix**) of the cylinder, and the parallel lines are called rulings.

- For the right circular cylinder shown in Figure 1, the equation of the generating curve is

$$x^2 + y^2 = a^2. \qquad \text{Equation of generating curve in } xy\text{-plane}$$

- To find an equation of the cylinder, note that you can generate any one of the rulings by fixing the values of x and y and then allowing z to take on all real values.
- In this sense, the value of z is arbitrary and is, therefore, not included in the equation.
- In other words, the equation of this cylinder is simply the equation of its generating curve.

$$x^2 + y^2 = a^2 \qquad \text{Equation of cylinder in space}$$

Definition 11.2 (Equation of cylinders)

The equation of a cylinder whose ruling are parallel to one of the coordinate axes contain only the variables corresponding to the other two axes.

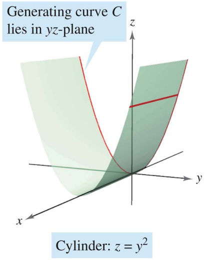
Example 1 (Sketching a cylinder)

Sketch the surface represented by each equation.

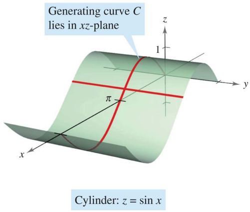
a. $z = y^2$ **b.** $z = \sin x,\ 0 \le x \le 2\pi$.

a.
- The graph is a cylinder whose generating curve, $z = y^2$, is a parabola in the yz-plane.
- The rulings of the cylinder are parallel to the x-axis.

b.
- The graph is a cylinder generated by the sine curve in the xz-plane.
- The rulings are parallel to the y-axis. ■



(a) Rulings are parallel to the x-axis.



(b) Rulings are parallel to the y-axis.

Quadric surfaces

- The fourth basic type of surface in space is a quadric surface.
- Quadric surfaces are the three-dimensional analogs of conic sections.

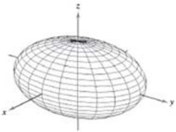
Definition 11.3 (Quadric surface)

The equation of a quadric surface in space is a second-degree equation in three variables. The general form of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

There are six basic types of quadric surfaces: ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.

- The intersection of a surface with a plane is called the trace of the surface in the plane.
- To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes.
- The traces of quadric surfaces are conics.
- These traces, together with the standard form of the equation of each quadric surface, are shown in the following table.

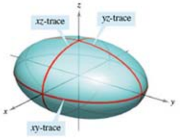


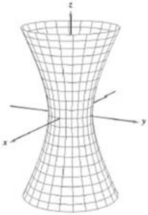
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.



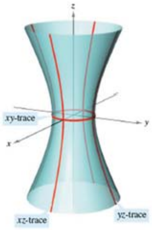


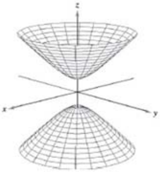
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



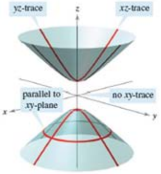


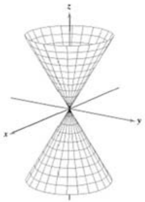
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



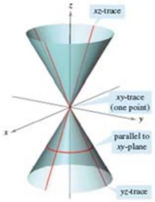


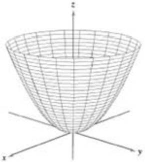
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



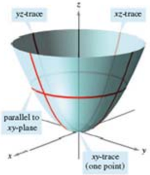


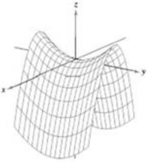
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



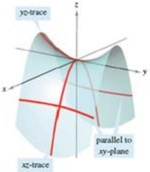


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



Example 2 (Sketching a quadric surface)

Classify and sketch the surface given by

$$4x^2 - 3y^2 + 12z^2 + 12 = 0.$$

- Begin by writing the equation in standard form.

$$\begin{aligned} 4x^2 - 3y^2 + 12z^2 + 12 &= 0 \\ \frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 &= 0 \\ \frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} &= 1 \end{aligned}$$

- You can conclude that the surface is a hyperboloid of two sheets with the y-axis as its axis.

- To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

$$xy\text{-trace } (z=0): \quad \frac{y^2}{4} - \frac{x^2}{3} = 1 \qquad \text{Hyperbola}$$

$$xz\text{-trace } (y=0): \quad \frac{x^2}{3} + \frac{z^2}{1} = -1 \qquad \text{No trace}$$

$$yz\text{-trace } (x=0): \quad \frac{y^2}{4} - \frac{z^2}{1} = 1 \qquad \text{Hyperbola}$$

■

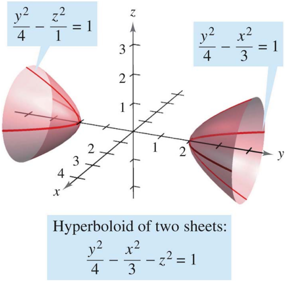


Figure 3: Hyperboloid of two sheets: $\frac{y^2}{4} - \frac{x^2}{3} - z^2 = 1$.

Example 3 (Sketching a quadric surface)

Classify and sketch the surface given by $x - y^2 - 4z^2 = 0$.

- Because x is raised only to the first power, the surface is a paraboloid.
- The axis of the paraboloid is the x-axis. In the standard form, the equation is

$$x = y^2 + 4z^2.$$

- Some convenient traces are as follows.

$$\begin{aligned}
xy\text{-trace } (z = 0): &\quad x = y^2 && \text{Parabola}\\
xz\text{-trace } (y = 0): &\quad x = 4z^2 && \text{Parabola}\\
\text{parallel to } yz\text{-plane } (x = 4): &\quad \frac{y^2}{4} + \frac{z^2}{1} = 1 && \text{Ellips}
\end{aligned}$$

- The surface is an elliptic paraboloid, as shown in Figure 4. ■

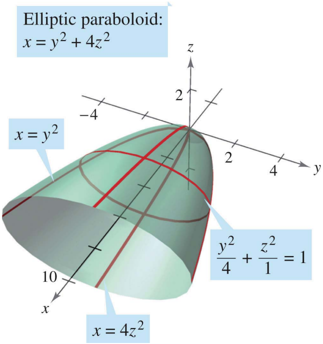


Figure 4: Elliptic paraboloid.

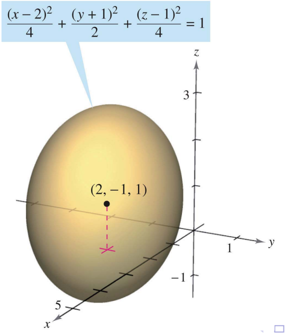
Example 4 (A quadric surface not centered at the origin)

Classify and sketch the surface given by
$x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0$.

- Completing the square for each variable produces the following.

$$
\begin{aligned}
(x^2-4x+\quad)+2(y^2+2y+\quad)+(z^2-2z+\quad)&=-3\\
(x^2-4x+4)+2(y^2+2y+1)+(z^2-2z+1)&=-3+4+2+1\\
(x-2)^2+2(y+1)^2+(z-1)^2&=4\\
\frac{(x-2)^2}{4}+\frac{(y+1)^2}{2}+\frac{(z-1)^2}{4}&=1
\end{aligned}
$$

- From this equation, you can see that the quadric surface is an ellipsoid that is centered at $(2,-1,1)$. ■

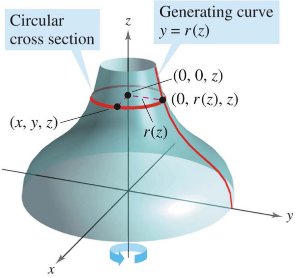


Surfaces of revolution

- The fifth special type of surface you will study is called a surface of revolution.
- You will now look at a procedure for finding its equation.
- Consider the graph of the radius function

$$y = r(z) \qquad \text{Generating curve}$$

  in the yz-plane.
- If this graph is revolved about the z-axis, it forms a surface of revolution.



- The trace of the surface in the plane $z = z_0$ is a circle whose radius is $r(z_0)$ and whose equation is

$$x^2 + y^2 = [r(z_0)]^2 . \qquad \text{Circular trace in plane: } z = z_0$$

- Replacing z_0 with z produces equation that is valid for all values of z.
- You can obtain equations for surfaces of revolution for the other two axes, and the results are summarized as follows.

Definition 11.4 (Surface of revolution)

If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

1. Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$

Example 5 (Finding an equation for a surface of revolution)

Find an equation for the surface of revolution formed by revolving (a) the graph of $y = 1/z$ about the z-axis and (b) the graph of $9x^2 = y^3$ about the y-axis.

a. An equation for the surface of revolution formed by revolving the graph of $y = \frac{1}{z}$ (radius function) about the z-axis is

$$x^2 + y^2 = [r(z)]^2 \qquad\qquad x^2 + y^2 = \left(\frac{1}{z}\right)^2.$$

b. To find an equation for the surface formed by revolving the graph of $9x^2 = y^3$ about the y-axis, solve for x in terms of y to obtain $x = \frac{1}{3}y^{3/2} = r(y)$ (radius function). So, the equation for surface is

$$x^2 + z^2 = [r(y)]^2 \qquad x^2 + z^2 = \left(\frac{1}{3}y^{3/2}\right)^2 \qquad x^2 + z^2 = \frac{1}{9}y^3.$$

The graph is shown in Figure 5. ■

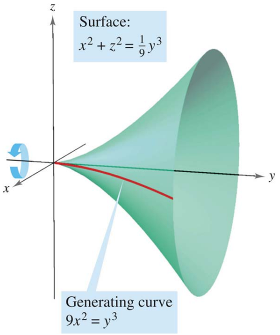


Figure 5: Surface of revolution: $x^2 + z^2 = \frac{1}{9}y^3$ with generating curve $9x^2 = y^3$ about the y-axis.

Example 6 (Finding a generating curve for a surface of revolution)

Find a generating curve and the axis of revolution for the surface given by

$$x^2 + 3y^2 + z^2 = 9.$$

- You now know that the equation has one of the following forms.

$x^2 + y^2 = [r(z)]^2$	Revolved about z-axis
$y^2 + z^2 = [r(x)]^2$	Revolved about x-axis
$x^2 + z^2 = [r(y)]^2$	Revolved about y-axis

- Because the coefficients of x^2 and z^2 are equal, you should choose the third form and write

$$x^2 + z^2 = 9 - 3y^2.$$

- The y-axis is the axis of revolution.
- You can choose a generating curve from either of the following traces.

$x^2 = 9 - 3y^2$	Trace in xy-plane
$z^2 = 9 - 3y^2$	Trace in yz-plane

- For example, using the first trace, the generating curve is the semiellipse given by

$$x = \sqrt{9 - 3y^2}.$$

- The graph of this surface is shown in Figure 6. ■

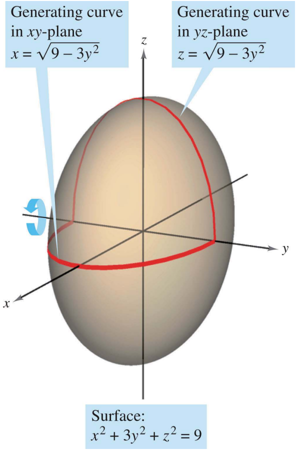


Figure 6: Finding a generating curve for a surface of revolution: not unique.

Table of Contents

Cylindrical coordinates

- The cylindrical coordinate system, is an extension of polar coordinates in the plane to three-dimensional space.

Definition 11.5 (The cylindrical coordinate system)

In a cylindrical coordinate system, a point P in space is represented by an ordered triple (r, θ, z).

1. (r, θ) is a polar representation of the projection of P in the xy-plane.
2. z is the directed distance from (r, θ) to P.

- To convert from rectangular to cylindrical coordinates (or vice versa), use the following conversion guidelines for polar coordinates, as illustrated in Figure 7.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

- The point $(0, 0, 0)$ is called the pole.

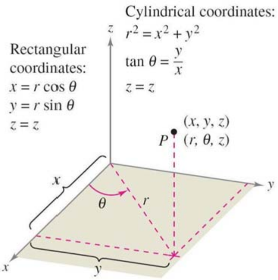


Figure 7: The relationship between cylindrical and rectangular coordinates.

- Moreover, because the representation of a point in the polar coordinate system is not unique, it follows that the representation in the cylindrical coordinate system is also not unique.

Example 1 (Converting from cylindrical to rectangular coordinates)

Convert the point $(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$ to rectangular coordinates.

- Using the cylindrical-to-rectangular conversion equations produces

$$x = 4\cos\frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

$$y = 4\sin\frac{5\pi}{6} = 4\left(\frac{1}{2}\right) = 2$$

$$z = 3.$$

- So, in rectangular coordinates, the point is $(x, y, z) = \left(-2\sqrt{3}, 2, 3\right)$ as shown in Figure 8. ■

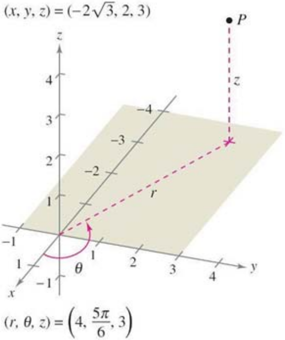


Figure 8: Converting $(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$ to $(x, y, z) = \left(-2\sqrt{3}, 2, 3\right)$.

Example 2 (Converting from rectangular to cylindrical coordinate)

Convert the point $(x, y, z) = (1, \sqrt{3}, 2)$ to cylindrical coordinates.

- Use the rectangular-to-cylindrical conversion equations.

$$
\begin{aligned}
r &= \pm\sqrt{1+3} = \pm 2 \\
\tan\theta &= \sqrt{3} \quad \Longrightarrow \quad \theta = \arctan(\sqrt{3}) + n\pi = \frac{\pi}{3} + n\pi \\
z &= 2
\end{aligned}
$$

- You have two choices for r and infinitely many choices for θ.
- As shown in Figure 9, two convenient representations of the point are

$$
\begin{aligned}
&\left(2, \frac{\pi}{3}, 2\right) && r > 0 \text{ and } \theta \text{ in Quadrant I} \\
&\left(-2, \frac{4\pi}{3}, 2\right). && r < 0 \text{ and } \theta \text{ in Quadrant III}
\end{aligned}
\qquad \blacksquare
$$

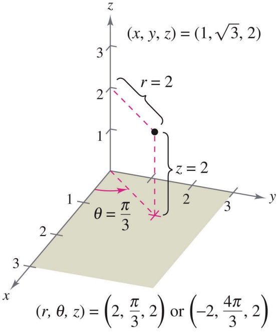


Figure 9: Converting from rectangular to cylindrical coordinates.

- Cylindrical coordinates are especially convenient for representing cylindrical surfaces and surfaces of revolution with the z-axis as the axis of symmetry, as shown in Figure 10.
- Vertical planes containing the z-axis and horizontal planes also have simple cylindrical coordinate equations, as shown in Figure 11.

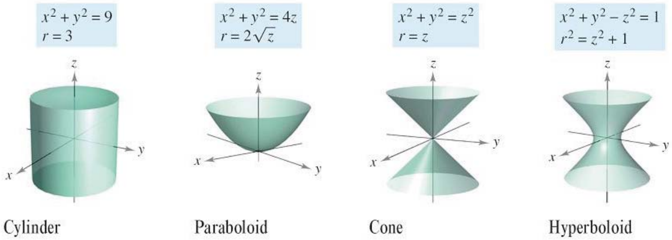


Figure 10: Different cylindrical equations.

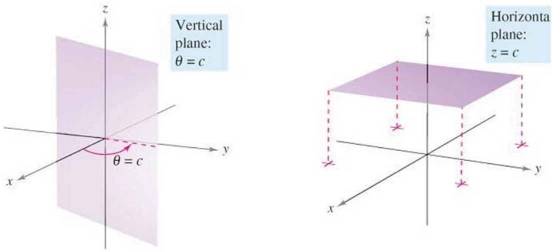


Figure 11: Vertical plane: $\theta = c$ and horizontal plane: $z = c$.

Example 3 (Rectangular-to-cylindrical conversion)

Find an equation in cylindrical coordinates for the surface represented by each rectangular equation.
a. $x^2 + y^2 = 4z^2$ **b.** $y^2 = x$

a. From the preceding section, you know that the graph $x^2 + y^2 = 4z^2$ is an elliptic cone with its axis along the z-axis, as shown in Figure 12(a). If you replace $x^2 + y^2$ with r^2, the equation in cylindrical coordinates is

$$x^2 + y^2 = 4z^2 \qquad\qquad r^2 = 4z^2.$$

b. The graph of the surface $y^2 = x$ is a parabolic cylinder with rulings parallel to the z-axis, as shown in Figure 12(b). By replacing y^2 with $r^2 \sin^2\theta$ and x with $r\cos\theta$, you obtain the following equation in cylindrical coordinates.

$$y^2 = x \qquad r^2\sin^2\theta = r\cos\theta \qquad r(r\sin^2\theta - \cos\theta) = 0$$

$$r\sin^2\theta - \cos\theta = 0 \qquad r = \frac{\cos\theta}{\sin^2\theta} \qquad r = \csc\theta\cot\theta$$

- Note that this equation includes a point for which $r = 0$, so nothing was lost by dividing each side by the factor r. ■

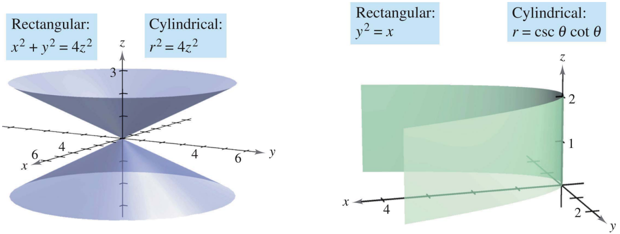


Figure 12: Rectangular-to-cylindrical conversion.

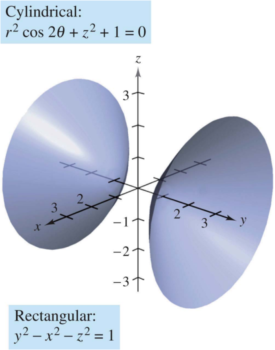
Example 4 (Cylindrical-to-rectangular conversion)

Find an equation in rectangular coordinates for the surface represented by the cylindrical equation

$$r^2 \cos 2\theta + z^2 + 1 = 0.$$

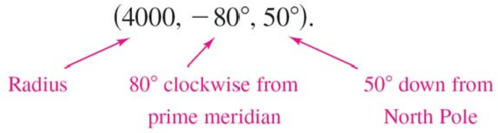
$$
\begin{aligned}
r^2 \cos 2\theta + z^2 + 1 &= 0 \\
r^2(\cos^2\theta - \sin^2\theta) + z^2 &= -1 \\
r^2\cos^2\theta - r^2\sin^2\theta + z^2 &= -1 \\
x^2 - y^2 + z^2 &= -1 \\
y^2 - x^2 - z^2 &= 1
\end{aligned}
$$

This is a hyperboloid of two sheets whose axis lies along the y-axis. ■



Spherical coordinates

- In the spherical coordinate system, each point is represented by an ordered triple: the first coordinate is a distance, and the second and third coordinates are angles.
- This system is similar to the latitude-longitude system used to identify points on the surface of Earth.
- For example, the point on the surface of Earth whose latitude is 40° North (of the equator) and whose longitude is 80° West (of the prime meridian) is shown in Figure 14. Assuming that the Earth is spherical and has a radius of 6371 kilometers, you would label this point as



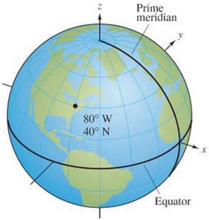


Figure 14: Spherical coordinate of 80° W 40° N is $(4000, -80^\circ, 50^\circ)$.

Definition 11.6 (The spherical coordinate system)

In a spherical coordinate system, a point P in space is represented by an ordered triple (ρ, θ, ϕ).

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the same angle used in cylindrical coordinates for $r \geq 0$.
3. ϕ is the angle between the positive z-axis and the line segment $\overrightarrow{OP}$, $0 \leq \phi \leq \pi$.

Note that the first and third coordinates, ρ and ϕ, are nonnegative. ρ is the lowercase Greek letter rho, and ϕ is the lowercase Greek letter phi.

- The relationship between rectangular and spherical coordinates is illustrated in Figure 15.

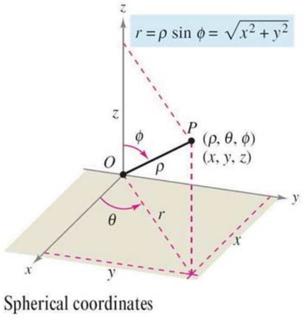


Figure 15: The relationship between rectangular coordinate (x, y, z) and spherical coordinates (ρ, θ, ϕ) where $r = \rho \sin \phi = \sqrt{x^2 + y^2}$.

- To convert from one system to the other, use the following.
- Spherical to rectangular:

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$

- Rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2, \quad \tan\theta = \frac{y}{x}, \quad \phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

- To change coordinates between the cylindrical and spherical systems, use the following.
- Spherical to cylindrical ($r \geq 0$):

$$r^2 = \rho^2 \sin^2\phi, \quad \theta = \theta, \quad z = \rho\cos\phi.$$

- Cylindrical to spherical ($r \geq 0$):

$$\rho = \sqrt{r^2+z^2}, \quad \theta = \theta, \quad \phi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right).$$

- The spherical coordinate system is useful primarily for surfaces in space that have a point or center of symmetry.
- For example, Figure 16 shows three surfaces with simple spherical equations.

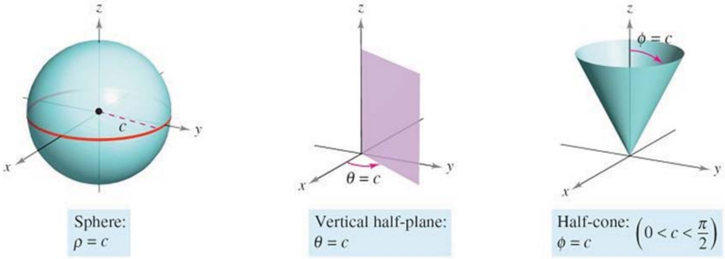


Figure 16: Three surfaces with simple spherical equations.

Example 5 (Rectangular-to-spherical conversion)

Find an equation in spherical coordinates for the surface represented by each rectangular equation.

a. Cone: $x^2 + y^2 = z^2$ **b.** Sphere: $x^2 + y^2 + z^2 - 4z = 0$

a. Making the appropriate replacements for x, y, and z in the given equation yields the following.

$$\begin{aligned}
x^2 + y^2 &= z^2 \\
\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta &= \rho^2 \cos^2\phi \\
\rho^2 \sin^2\phi (\cos^2\theta + \sin^2\theta) &= \rho^2 \cos^2\phi \\
\rho^2 \sin^2\phi &= \rho^2 \cos^2\phi \\
\frac{\sin^2\phi}{\cos^2\phi} &= 1 \qquad\qquad \rho \geq 0 \\
\tan^2\phi &= 1 \qquad\qquad \phi = \pi/4 \text{ or } \phi = 3\pi/4
\end{aligned}$$

The equation $\phi = \pi/4$ represents the upper half-cone, and the equation $\phi = 3\pi/4$ represents the lower half-cone.

b. Because $\rho^2 = x^2 + y^2 + z^2$ and $z = \rho\cos\phi$, the given equation has the following spherical form.

$$\rho^2 - 4\rho\cos\phi = 0 \implies \rho(\rho - 4\cos\phi) = 0$$

- Temporarily discarding the possibility that $\rho = 0$, you have the spherical equation

$$\rho - 4\cos\phi = 0 \quad \text{or} \quad \rho = 4\cos\phi.$$

Note that the solution set for this equation includes a point for which $\rho = 0$, so nothing is lost by discarding the factor ρ.
The sphere represented by the equation $\rho = 4\cos\phi$ is below. ■

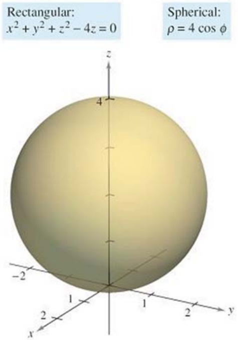


Figure 17: $x^2 + y^2 + z^2 - 4z = 0$ in rectangular coordinate is equivalent to $\rho = 4\cos\phi$ in spherical coordinate.