1. (16%) Find the following limit. (If the limit does not exist you should point it out.)

(a) $\lim\limits_{(x,y)\to(0,0)} \arcsin(\frac{x^3+y^3}{x^2+y^2})$

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^5+x^2y^3}{x^4+y^6}$

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{(x^2+y^2)}{4}\cdot\ln(x^2+y^2)$

(d) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{e^{xyz}-1}{x^2+y^2+z^2}$

Ans:

(a) Let $x = rcos(\theta), y = rsin(\theta)$ $\lim\limits_{(x,y)\to(0,0)} \arcsin(\frac{x^3+y^3}{x^2+y^2}) =$

$\lim\limits_{r\to 0} \arcsin(\frac{r^3(cos^3\theta+sin^2\theta)}{r^2}) = 0$

(b) When we approach the limit with $x = 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{x^5+x^2y^3}{x^4+y^6} = \lim\limits_{y\to 0}\frac{0}{y^6} = 0$

When along $x^2 = y^3$, $\lim\limits_{(x,y)\to(0,0)} \frac{x^5+x^2y^3}{x^4+y^6} = \lim\limits_{y\to 0}\frac{y^{15/2}+y^6}{y^6+y^6} = \lim\limits_{y\to 0}\frac{y^{\frac{3}{2}}+1}{2} = \frac{1}{2}$ which means that if we follow the trajectory to approach (0,0) we will get a different value, therefore, the limit does not exist.

(c) $r = x^2 + y^2$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)}{4}\cdot\ln(x^2+y^2) = \lim_{r\to 0}\frac{r^2}{4}\ln r^2 = \frac{1}{2}\lim_{r\to 0}\frac{\ln r}{r^{-2}} = \frac{1}{2}\lim_{r\to 0}\frac{r^{-1}}{-2r^{-3}}$$

$$= \frac{1}{2}\lim_{r\to 0}\frac{r^2}{-2} = 0$$

(d) Let $x = \rho\, sin(\Phi)\cos(\theta), y = \rho\, sin(\Phi)\, sin(\theta), z = \rho\, cos(\Phi)$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{e^{xyz}-1}{x^2+y^2+z^2}$$

$$= \lim_{\rho^+\to 0}\left(\frac{e^{\rho^3 sin^2(\Phi)\cos(\theta)\, sin(\theta)\, cos(\Phi)}-1}{\rho^2}\right) = \lim_{\rho^+\to 0}\left(\frac{e^{\rho^3\mu}3\rho^2\mu}{2\rho}\right) = \lim_{\rho^+\to 0}\rho\left(\frac{e^{\rho^3\mu}3\mu}{2}\right) = 0$$

Where $\mu = sin^2(\Phi)cos(\theta)\, sin(\theta)\, cos(\Phi)$

2. (12%)

(a) Let $f(x,y) = \begin{cases} sin(\frac{xy^2}{x^2+y^4}) & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$, is $f(x,y)$ continuous at

(0,0) and why?

(b) Given the equation $w - \sqrt{x-y} - \sqrt{y-z} = 0$, differentiate implicitly to find the three first partial derivatives of w

(c) Considering the level surface defined by $z^5 + (\sin(x))z^3 + yz = 4$. Find an equation of the tangent plane at (0,3,1)

Ans:

(a) Along $x^2 = y$, $\lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} sin\frac{y^4}{y^4+y^4} = sin\frac{1}{2}$. Since $sin\frac{1}{2} \neq (0,0)$. It is not continuous.

(b) $F(x,y,z,w) = w - \sqrt{x-y} - \sqrt{y-z} = 0$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{1}{2}\frac{(x-y)^{\frac{-1}{2}}}{1} = \frac{1}{2\sqrt{x-y}}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-1}{2}(x-y)^{\frac{-1}{2}} + \frac{1}{2}(y-z)^{\frac{-1}{2}} = \frac{-1}{2\sqrt{x-y}} + \frac{1}{2\sqrt{y-z}}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y-z}}$$

(c) $F(x,y,z) = z^5 + (sin(x))z^3 + yz = 4$

$$\nabla F = cos(x)\, z^3\boldsymbol{i} + z\boldsymbol{j} + (3(sin(x))z^2 + y)\boldsymbol{k}$$

$$\nabla F(0,3,1) = 1\boldsymbol{i} + 1\boldsymbol{j} + 8\boldsymbol{k}$$

$$(x-0) + (y-3) + 8(z-1) = 0$$

3. (10%) Let $f(x,y) = \int_1^{2y-x^2} e^{t^2}\, dt$

   (a) Find the directional derivative of $f(x,y)$ at (1,1) in the direction from P(1,1) to Q(6,13)

   (b) Find the direction in which $f(x,y)$ decrease most. What is the rate of decrease?

   Ans:

(a) $\nabla f(x,y) = \left(-2x \cdot e^{(2y-x^2)^2}, 2e^{(2y-x^2)^2}\right) = 2e^{(2y-x^2)^2}(-x\boldsymbol{i} + \boldsymbol{j}), \overrightarrow{PQ} =$ $(6,13) - (1,1) = 5\boldsymbol{i} + 12\boldsymbol{j}$. $u = \frac{5}{13}\boldsymbol{i} + \frac{12}{13}\boldsymbol{j}$. $D_u f = \nabla f \cdot u = (-2e, 2e) \cdot$ $\left(\frac{5}{13}, \frac{12}{13}\right) = \frac{14}{13}e$

(b) Increasing most rapidly is the direction of the gradient. That is $\frac{-\nabla f}{|\nabla f|} = -(\frac{-x}{\sqrt{x^2+1}}\boldsymbol{i} +$

$\frac{1}{\sqrt{x^2+1}}\boldsymbol{j})$

$$|\nabla f| = 2e^{(2y-x^2)^2}\sqrt{x^2+1}$$

4. (10%) Let $f(x,y) = x^4 - 2x^2 - 2xy^2 - y^2$
   (a) Find the critical points of $f(x,y)$
   (b) Determine whether they are local maximum, local minimum or saddle points

Ans:

(a) $f_x = 4x^3 - 4x - 2y^2 = 4(x^3 - x) - 2y^2,\ f_y = -(4x+2)y.$

Let $f_x = 0$ and $f_y = 0$,

From $f_y = 0$ we know $x = \frac{-1}{2}$ or $y = 0$. If $x = \frac{-1}{2}, y = \pm\frac{\sqrt{3}}{2}$. When $y = 0, x = 0, \pm 1$

Therefore, the critical points are $(0,0), (1,0), (-1,0), \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), (\frac{-1}{2}, \frac{-\sqrt{3}}{2})$

(b)

Since $f_{xx} = 4(3x^2-1), f_{xy} = f_{yx} = -4y, f_{yy} = -(4x+2)$.

(x, y)	f_{xx}	f_{xy}	f_{yy}	d	
(0,0)	-4	0	-2	8	Local maximum
(1,0)	8	0	-6	-48	Saddle point
$(-1,0)$	8	0	2	16	Local minimum
$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$	-1	$-2\sqrt{3}$	0	-12	Saddle point
$(\frac{-1}{2}, \frac{-\sqrt{3}}{2})$	-1	$2\sqrt{3}$	0	-12	Saddle point

5. (6%) Use Lagrange multipliers to find the maxima of $f = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

Ans:

Let $g = 2x - y$ and $h = y + z$

Use the Lagrange multiplier, we have the following equations

$$\begin{cases} 2x = 2\lambda \\ 2 = -\lambda + \mu \\ -2z = \mu \\ 2x - y = 0 \\ y + z = 0 \end{cases}$$

We have $\lambda = x,\ \mu = -2z \rightarrow x = -2z - 2,\ y = 2x = -4z - 4 \rightarrow z = \frac{-4}{3}, y =$

$\frac{4}{3}, x = \frac{2}{3}$

$$f\left(\frac{2}{3}, \frac{4}{3}, \frac{-4}{3}\right) = \frac{4}{3}$$

6. (20%) Evaluate the following expression

   (a) $\int_0^4 \int_{\sqrt{x}}^2 \sin(\frac{x}{y})\, dydx$

   (b) $\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x+y}{x^2+y^2} dxdy + \int_0^1 \int_{1-y}^1 \frac{x+y}{x^2+y^2} dxdy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \frac{x+y}{x^2+y^2} dxdy$

   (Hint: draw the region of integration, you may find it easier to calculate the area using polar coordinates)

   (c) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{y}} \int_0^{\frac{1}{y}} \sin(y)\, dzdxdy$

   (d) $\int_0^{\frac{\pi}{4}} \int_0^6 \int_0^{6-r} rz \;\; dzdrd\theta$

Ans:

(a) $\int_0^4 \int_{\sqrt{x}}^2 \sin(\frac{x}{y})\, dydx = \int_0^2 \int_0^{y^2} \sin(\frac{x}{y})\, dxdy = \int_0^2 -ycos(\frac{x}{y})|_0^{y^2}\, dy =$

$\int_0^2 -ycos(y) + y\, dy = -ysin(y)\Big|_0^2 + \int_0^2 \sin(y)\, dy + \frac{y^2}{2}\Big|_0^2 = 3 - 2\sin(2) -$

$\cos(2)$

(b) $\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x+y}{x^2+y^2} dxdy + \int_0^1 \int_{1-y}^1 \frac{x+y}{x^2+y^2} dxdy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \frac{x+y}{x^2+y^2} dxdy =$

$\iint_R \frac{x+y}{x^2+y^2} dA$ where A is bounded by $x^2 + y^2 = 2$ and $x + y = 1$

$$\iint_A \frac{x+y}{x^2+y^2} dA = \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos(\theta)+\sin(\theta)}}^{\sqrt{2}} \frac{rcos(\theta) + rsin(\theta)}{r^2} rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos(\theta)+\sin(\theta)}}^{\sqrt{2}} (\cos(\theta) + \sin(\theta))drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\,(\cos(\theta) + \sin(\theta)) - 1 \;\; \mathrm{d}\theta$$

$$= \sqrt{2}(\sin(\theta) - \cos(\theta))\Big|_0^{\frac{\pi}{2}} - \frac{\pi}{2} = 2\sqrt{2} - \frac{\pi}{2}$$

(c) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{2}{y}}\int_0^{\frac{1}{y}} \sin(y)\, dzdxdy = \int_0^{\frac{\pi}{2}}\int_0^{\frac{2}{y}} \frac{\sin(y)}{y} dxdy = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(y)dy = \frac{-1}{2}\cos(y)\,\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$

(d) $\int_0^{\frac{\pi}{4}}\int_0^{6}\int_0^{6-r} rz\ \ dzdrd\theta = \int_0^{\frac{\pi}{4}}\int_0^{6} \frac{rz^2}{2}\Big|_0^{6-r} drd\theta = \int_0^{\frac{\pi}{4}}\int_0^{6} \frac{1}{2}(r^3 - 12r^2 +$

$36r)\, drd\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}[\frac{r^4}{4} - 4r^3 + 18r^2]_0^6\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(108)d\theta = \frac{27\pi}{2}$

7. (8%) Find the area of the surface given by $z = f(x,y) = xy$ that lies above the region R where $R = \{(x,y): x^2 + y^2 \le 16\}$

Ans:

$$f_x = y, f_y = x$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + x^2 + y^2}$$

$$S = \int_0^{2\pi}\int_0^{4} \sqrt{1 + r^2}\, rdrd\theta = \frac{2\pi}{3}(17\sqrt{17} - 1)$$

8. (8%) Evaluate the triple integral $\int\int\int_Q x^2 + y^2 dV$ where $Q = \{-2 \le x \le 2, -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, \sqrt{x^2 + y^2} \le z \le 2\}$

Ans:

Use cylindrical coordinates

$$\int\int\int_Q x^2 + y^2\ \ dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} x^2 + y^2\ \ dzdydx$$

$$= \int_0^{2\pi}\int_0^{2}\int_r^{2} r^2 r\ \ dzdrd\theta = 2\pi\int_0^{2} r^3(2 - r)\ \ dr = \frac{16}{5}\pi$$

9. (10%) Use the change of variables to find the volume of the solid region lying below the surface $z = f(x,y) = \ln(x^2 y + x)$ and above the plane region R wher R is a region bounded by $xy = 1, xy = 3, x = 1, x = e$.

Ans:

Let $u = xy, v = x$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{-1}{v}$$

$$\int\int_R \ln(x^2y+x) \quad \mathrm{dA}$$

$$= \int_1^e \int_1^3 \ln(uv+v)\frac{1}{v}dudv = \int_1^e \int_1^3 \frac{\ln(u+1)+\ln v}{v}dudv =$$

$$= \int_1^e [(u+1)\ln(u+1)-(u+1)]_1^3 \frac{1}{v} + (3-1)\frac{\ln v}{v}dv =$$

$$= \ln e\,[(3+1)\ln(3+1)-(3+1)-2\ln 2+2] + \frac{(3-1)}{2}(\ln e)^2$$

$$= 8\ln 2 - 4 - 2\ln 2 + 2 + 1 = 6\ln 2 - 1$$