1. (20%) Find the following limit. (If the limit does not exist, you should point it out).

Hint: Change of variables may be useful here

(a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

(b) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

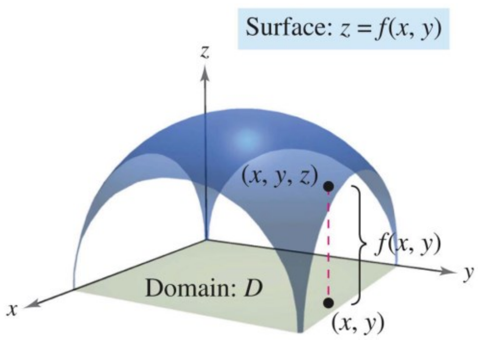
(c) $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

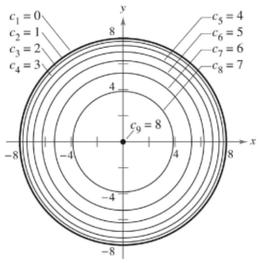
(d) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+xy^2}{4x^2y-2y^3}$

(e) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

Definition 13.1 (A function of two variables)

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a function of x and y. The set D is the domain of f, and the corresponding set of values for $f(x, y)$ is the range of f.





(b) Contour map of $f(x, y) = \sqrt{64 - x^2 - y^2}$.

Definition 13.2 (Limit of a function of two variables)

Let f be a function of two variables defined, except possible at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for each ε there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Definition 13.3 (Continuity of a function of two variables)

A function f of two variables is continuous at a point (x_0, y_0) in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0).$$

The function f is continuous in the open region R if it is continuous at every point in R.

Finding a Limit Using Polar Coordinates In Exercises 51–56, use polar coordinates to find the limit. [*Hint:* Let $x = r\cos\theta$ and $y = r\sin\theta$, and note that $(x, y) \to (0, 0)$ implies $r \to 0$.]

Finding a Limit Using Spherical Coordinates In Exercises 77 and 78, use spherical coordinates to find the limit. [*Hint:* Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, and note that $(x, y, z) \to (0, 0, 0)$ implies $\rho \to 0^+$.]

Ans:

(a) Let $y = kx^2$, $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{kx^4}{x^4+k^2x^4} = \lim_{x\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2}$ which means that if we follow the trajectory of different parabola $y = kx^2$ to approach (0,0) we will get different value, therefore, the limit does not exist.

(b)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0^+} \frac{(\rho\sin(\Phi)\cos(\theta))(\rho\sin(\Phi)sin(\theta))\rho\cos(\Phi)}{\rho^2}$$

$$= \lim_{\rho\to 0^+} \rho\sin^2(\Phi)\cos(\theta)sin(\theta)\cos(\Phi) = 0$$

(c) $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{1-\cos(r^2)}{r^2}$. By L'Hospital's Rule, $\lim_{r\to 0} \frac{1-\cos(r^2)}{r^2} =$

$\lim_{r\to 0} \frac{-\sin(r^2)2r}{2r} = \lim_{r\to 0} -\sin(r^2) = 0$

(d) Let $y = mx$, $\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2}{4x^2y-2y^3} = \lim_{x\to 0} \frac{1+m^2}{4m-2m^3} = \frac{1+m^2}{4m-2m^3}$. which means that if we follow the trajectory of different line $y = mx$ to approach (0,0) we will get different value, therefore, the limit does not exist.

(e) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2cos(\theta)sin(\theta)}{r} = \lim_{r\to 0} r\,cos(\theta)\,sin(\theta) = 0.$

2. (16%)

(a) Let $f(x,y) = \int_y^x sin(t^2)dt$, evaluate f_x and f_y

(b) Let $f(x,y) = x sin(y) + ye^{xy}$, find the four second partial derivatives

(c) Let $z = f(x,y) = x^2 + y^2, x = s + t, y = s - t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

(d) Find an equation of the tangent plance of the surface $9x^2 + y^2 + 4z^2 = 25$ at $(0,-3,2)$

Definition 13.5 (Partial derivatives of a function of two variables)

If $z = f(x,y)$, then the first partial derivatives of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Partial derivative with respect to x and

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Partial derivative with respect to y , provided the limits exist.

1. Differentiate twice with respect to x:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$
2. Differentiate twice with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$
3. Differentiate first with respect to x and then with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$
4. Differentiate first with respect to y and then with respect to x:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

Definition 13.6 (Total differential)

If $z = f(x,y)$ and Δx and Δy are increments of x and y, then the differentials of the independent variables x and y are

$$\mathrm{d}x = \Delta x \quad \text{and} \quad \mathrm{d}y = \Delta y$$

and the total differential of the dependent variable z is

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = f_x(x,y)\,\mathrm{d}x + f_y(x,y)\,\mathrm{d}y.$$

Definition 13.7 (Differentiability)

A function f given by $z = f(x, y)$ is differentiable at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where both ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is differentiable in a region R if it is differentiable at each point in R.

Theorem 13.4 (Sufficient condition for differentiability)

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Theorem 13.5 (Differentiability implies continuity)

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Ans:

(a) Using the Fundamental theorem of calculus, $D_x f = \sin(x^2)$ and $D_y f =$

$D_y\left(-\int_x^y \sin(t^2)\, dt\right) = -\sin(y^2)$

(b) $\frac{\partial f}{\partial x} = sin(y) + y^2 e^{xy}$, $\frac{\partial f}{\partial y} = xcos(y) + e^{xy} + xye^{xy} = xcos(y) + (1 + xy)e^{xy}$

$$\frac{\partial^2 f}{\partial x^2} = y^3 e^{xy}, \frac{\partial^2 f}{\partial x\, \partial y} = \cos(y) + 2ye^{xy} + y(1 + xy)e^{xy}$$
$$= \cos(y) + (2y + xy^2)e^{xy}$$
$$\frac{\partial^2 f}{\partial y\, \partial x} = \cos(y) + 2ye^{xy} + xy^2 e^{xy} = \cos(y) + (2y + xy^2)e^{xy}$$
$$\frac{\partial^2 f}{\partial y^2} = -xsin(y) + xe^{xy} + x(1 + xy)e^{xy} = -xsin(y) + (2x + x^2 y)e^{xy}$$

Theorem 13.8 (Chain Rule: implicit differentiation)

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

Theorem 13.7 (Chain Rule: two independent variables)

Let $w = f(x,y)$, where f is a differentiable function of x and y. If $x = g(s,t)$ and $y = h(s,t)$ such that the first partial $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist, then $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

(c) Using chain rule

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2x \cdot 1 + 2y \cdot 1 = 2(s+t) + 2(s-t) = 4s$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 2x \cdot 1 + 2y \cdot (-1) = 2(s+t) - 2(s-t) = 4t$$

Definition 13.9 (Gradient of a function of two variables)

Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. Then the gradient of f, denoted by $\nabla f(x,y)$, is the vector

$$\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}.$$

∇f is read as "del f". Another notation for the gradient is **grad** $f(x,y)$. In Figure 32, note that for each (x,y), the gradient $\nabla f(x,y)$ is a vector in the plane (not a vector in space).

Definition 13.11 (Tangent plane and normal line)

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x,y,z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the tangent plane to S at P.
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the normal line to S at P.

Theorem 13.13 (Equation of tangent plane)

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x,y,z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$

Normal line: $$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

(d) $\nabla F = 18x\boldsymbol{i} + 2y\boldsymbol{j} + 8z\boldsymbol{k}$, $\nabla F(0,-3,2) = -6\boldsymbol{j} + 16\boldsymbol{k}$

Tangent plane:

$$0(x-0) - 6(y+3) + 16(z-2) = 0 \rightarrow -3y + 8z = 25$$

3. (6%) Let $f(x,y) = 2022 - \frac{x^2}{4} - \frac{y^2}{2}$, express the limit $\lim_{t\to 0} \frac{f(1+2t,2+t)-f(1,2)}{t}$ as the directional derivative of f and evaluate the value of the limit.

Definition 13.8 (Directional derivative)

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the directional derivative of f in the direction of $\mathbf{u}$, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x,y) = \lim_{t\to 0} \frac{f(x+t\cos\theta, y+t\sin\theta) - f(x,y)}{t}$$

provided this limit exists.

Theorem 13.9 (Directional derivative)

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$

Theorem 13.10 (Alternative form of the directional derivative)

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

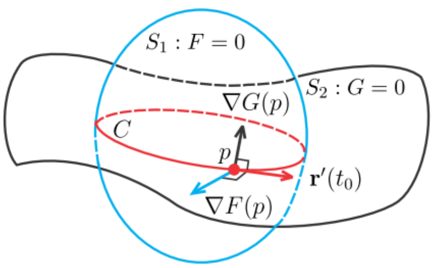
Theorem 13.11 (Properties of the gradient)

Let f be differentiable at the point (x,y).

1. *If $\nabla f(x,y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x,y) = 0$ for all $\mathbf{u}$.*
2. *The direction of maximum increase of f is given by $\nabla f(x,y)$. The maximum value of $D_{\mathbf{u}}f(x,y)$ is $\|\nabla f(x,y)\|$.*
3. *The direction of minimum increase of f is given by $-\nabla f(x,y)$. The minimum value of $D_{\mathbf{u}}f(x,y)$ is $-\|\nabla f(x,y)\|$.*

Theorem 13.12 (Gradient is normal to level curves)

If f is differentiable at (x_0,y_0) and $\nabla f(x_0,y_0) \neq \mathbf{0}$, then $\nabla f(x_0,y_0)$ is normal to the level curve through (x_0,y_0).



Intersection of two surfaces.

Ans:

First of all, let $(\mathrm{u}, \mathrm{v}) = (2,1)/\sqrt{5}$ and rewrite the limit as $\lim_{t\to 0}\frac{f(1+2t,2+t)-f(1,2)}{t} =$

$\lim_{t\to 0}\frac{f(1+\sqrt{5}tu,2+\sqrt{5}tv)-f(1,2)}{t} = \sqrt{5}\lim_{t\to 0}\frac{f(1+\sqrt{5}tu,2+\sqrt{5}tv)-f(1,2)}{\sqrt{5}t}$. Let $t' = \sqrt{5}t$

Which can be expressed as $\sqrt{5}\lim_{t'\to 0}\frac{f(1+t'u,2+t'v)-f(1,2)}{t'} = \sqrt{5}\left(D_{(u,v)}f\right)(1,2)$

$$D_{(u,v)}f = \nabla f \cdot (u, v) = \left(\frac{-x}{2}, -y\right) \cdot \frac{1}{\sqrt{5}}(2,1) = \frac{-1}{\sqrt{5}}(x + y)$$

Finally, we have $\sqrt{5}\left(D_{(u,v)}f\right)(1,2) = -(1 + 2) = -3$

4. (8%) Find the critical points of $f(x, y) = x^3 + y^2 - 2xy + 7x - 8y + 2$. Which of them give rise to maximum values, minimum values and saddle points?

Definition 13.13 (Critical point)

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a critical point of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Theorem 13.16 (Relative extrema occur only at critical points)

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Theorem 13.17 (Second Partials Test)

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity $d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

1. *If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b).*
2. *If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b).*
3. *If $d < 0$, then $(a, b, f(a, b))$ is a saddle point.*
4. *The test is inconclusive if $d = 0$.*

Ans: $f_x = 3x^2 - 2y + 7, f_y = 2y - 2x - 8$. Let $f_x = 0$ and $f_y = 0$, we have the critical points $\left(\frac{-1}{3}, \frac{11}{3}\right), (1,5)$. Furthermore, since $f_{xx} = 6x, f_{xy} = f_{yx} = -2, f_{yy} = 2$. We have $d = f_{xx}f_{yy} - f_{xy}f_{yx} = 12x - 4$. Therefore, $d\left(\frac{-1}{3}, \frac{11}{3}\right)<0$,

$d(1,5) > 0$. Finally, we know that $(1,5)$ is absolute minimum point since $f_{xx}(1,5) > 0$ and $\left(\frac{-1}{3}, \frac{11}{3}\right)$ is saddle point.

5. (6%) Find the minimum and maximum distance from the curve $x^2 + xy + y^2 = 1$ to the origin point $(0,0)$.

Theorem 13.18 (Lagrange's Theorem)

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

For optimization problems involving two constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Ans:

The objective function can be change from $\sqrt{x^2 + y^2}$ to $x^2 + y^2$. Use the Langrange multiplier, we have the following equations

$$\begin{cases} 2x = \lambda(2x + y) \\ 2y = \lambda(x + 2y) \\ x^2 + xy + y^2 = 1 \end{cases}$$

Cross product the first and second equation yields

$$2x\lambda(x + 2y) = 2y\lambda(2x + y)$$

If $\lambda = 0$, we have $x = y = 0$ which does not satisfy the third equation. If $\lambda \neq 0$. We have

$$x^2 + 2xy = 2xy + y^2$$

Which means $x = \pm y$

If $x = y$, from the third equation, we have $3y^2 = 1$, therefore $(x, y) = (\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3})$. The distance is therefore $\sqrt{x^2 + y^2} = \sqrt{\frac{2}{3}}$

If $x = -y$, from the third equation, we have $y^2 = 1$, therefore $(x, y) = (\pm 1, \mp 1)$. The distance is therefore $\sqrt{x^2 + y^2} = \sqrt{2}$

Finally, we know that the maximum and minimum distance are $\sqrt{2}$ and $\sqrt{\frac{2}{3}}$, respectively.

6. (20%) Evaluate the following expression

(a) $\int_0^1 \int_y^1 \frac{\sin(x)}{x} dxdy$

(b) $\int_0^2 \int_0^{\sqrt{4-x^2}} \sin(\sqrt{x^2+y^2})\, dydx$

(c) $\int_0^{\frac{\pi}{2}} \int_1^2 x^2 \sin(y)\, dxdy$

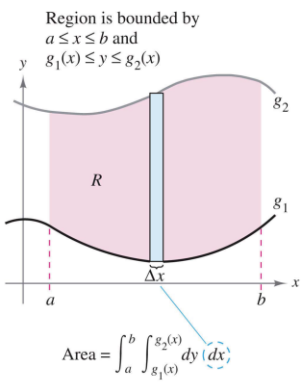
(d) $\int_0^1 \int_0^{1+\sqrt{y}} \int_0^{xy} ydzdxdy$

(e) $\int_1^2 \int_{2u-2}^{u} e^{(v-u+1)^2}\, dvdu$

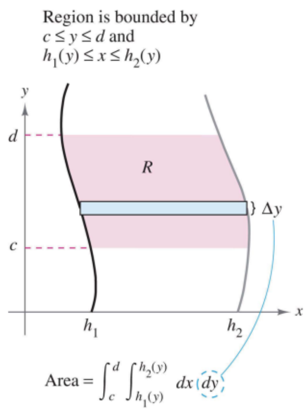
Iterated integrals

(1) $\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,\mathrm{d}x = f(x,y)\Big|_{h_1(y)}^{h_2(y)} =$
$f(h_2(y),y) - f(h_1(y),y)$ With respect to x

(2) $\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,\mathrm{d}y = f(x,y)\Big|_{g_1(x)}^{g_2(x)} =$
$f(x,g_2(x)) - f(x,g_1(x))$ With respect to y



(a) Vertically simple region.



(b) Horizontally simple region.

Definition 14.1 (Area of a region in the plane)

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} \mathrm{d}y\,\mathrm{d}x. \quad \text{Figure 2}(a) \text{ (vertically simple)}$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} \mathrm{d}x\,\mathrm{d}y. \quad \text{Figure 2}(b) \text{ (horizontally simple)}$$

Definition 14.2 (Double integral)

If f defined on a closed, bounded region R in the xy-plane, then the double integral of f over R is given by

$$\iint_R f(x,y)\,\mathrm{d}A = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

provided the limit exists. If the limit exists, then f is integrable over R.

Volume of a solid region If f is integrable over a plane region R and $f(x,y) \geq 0$ for all (x,y) in R, then the volume of the solid region that lies above R and below the graph of f defined as

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

Definition 14.5 (Triple integral)

If f is continuous over a bounded solid region Q, then the triple integral of f over Q is defined as

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

provided the limit exists. The volume of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q \mathrm{d}V.$$

Theorem 14.2 (Fubini's Theorem)

Let f be continuous on a plane region R.

1. *If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

2. *If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Theorem 14.4 (Evaluation by iterated integrals)

Let f be continuous on a solid region Q defined by

$$a \leq x \leq b, \quad h_1(x) \leq y \leq h_2(x), \quad g_1(x,y) \leq z \leq g_2(x,y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Ans:

(a) $\int_0^1\int_y^1 \frac{\sin(x)}{x}\,dxdy = \int_0^1\int_0^x \frac{\sin(x)}{x}\,dydx = \int_0^1\left[y\frac{\sin(x)}{x}\right]_0^x dx = \int_0^1 \sin(x)\,dx = 1 - \cos(1)$

Theorem 14.3 (Change of variables to polar form)

Let R be a plane region consisting of all points $(x,y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

(b) $R = \left\{(x,y)\middle|0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\right\} = \left\{(r,\theta)\middle|0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\right\}$

$\int_0^2\int_0^{\sqrt{4-x^2}} \sin\left(\sqrt{x^2+y^2}\right)dydx = \int_0^{\frac{\pi}{2}}\int_0^2 \sin(r)\,r\,drd\theta$ $\quad$ ($Let\ \ r = u, \sin(r) = dv \rightarrow dr = du, -\cos(r) = v$) and use integration by parts, we have

$$= \int_0^{\frac{\pi}{2}} [\sin(r) - r\cos(r)]_0^2\,d\theta = \int_0^{\frac{\pi}{2}} \sin(2) - 2\cos(2)\,d\theta = \frac{\pi}{2}(\sin(2) - 2\cos(2))$$

(c) $\int_0^{\frac{\pi}{2}}\int_1^2 x^2\sin(y)\,dxdy = \left(\int_0^{\frac{\pi}{2}}\sin(y)dy\right) \times \left(\int_1^2 x^2dx\right) = [-\cos(y)]_0^{\frac{\pi}{2}} \times \left[\frac{x^3}{3}\right]_1^2 = \frac{7}{3}$

(d) $\int_0^1\int_0^{1+\sqrt{y}}\int_0^{xy} y\,dzdxdy = \int_0^1\int_0^{1+\sqrt{y}} [yz]_0^{xy}\,dxdy = \int_0^1\int_0^{1+\sqrt{y}} xy^2\,dxdy =$

$\int_0^1\left[\frac{1}{2}x^2y^2\right]_0^{1+\sqrt{y}} dy = \int_0^1 \frac{1}{2}(1+\sqrt{y})^2y^2dy = \frac{1}{2}\int_0^1 [y^2 + 2y^{\frac{5}{2}} + y^3]dy =$

$\frac{1}{2}\left[\frac{y^3}{3} + \frac{4}{7}y^{\frac{7}{2}} + \frac{1}{4}y^4\right]_0^1 = \frac{97}{168}$

Definition 14.6 (Jacobian)

If $x = g(u,v)$ and $y = h(u,v)$, then the Jacobian of x and y with respect to u and v, denoted by $\partial(x,y)/\partial(u,v)$, is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Theorem 14.5 (Change of variables for double integrals)

Let R be a vertically or horizontally simple region in the xy-plane, and let S be a vertically or horizontally simple region in the uv-plane. Let T from S to R be given by $T(u,v) = (x,y) = (g(u,v), h(u,v))$, where g and h have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x,y)/\partial(u,v)$ is nonzero on S, then

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_S f(g(u,v), h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v.$$

(e) $x = v - u + 1, y = u \rightarrow v = x + y - 1, u = y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = -1$$

$$v = u \rightarrow x = 1, v = 2u - 2 \rightarrow x = y - 1$$

$$\int_1^2 \int_{2u-2}^{u} e^{(v-u+1)^2}\, dvdu = \int_1^2 \int_{y-1}^{1} e^{x^2}\, dxdy = \int_0^1 \int_1^{x+1} e^{x^2}\, dydx$$

$$= \int_0^1 \left[ye^{x^2}\right]_1^{x+1} dx = \int_0^1 xe^{x^2}\, dx = \frac{1}{2}(e-1)$$

7. (6%) Find the area of the surface given by $z = f(x,y) = 9 - y^2$ that lies above the region R where R is a triagle with vertices (-3,3),(0,0),(3,3)

Definition 14.4 (Surface area)

If f and its partial derivatives are continuous on the closed region R in the xy-plane, then the area of the surface S given by $z = f(x,y)$ over R is defined as

$$\text{Surface area} = \iint_R \mathrm{d}S = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, \mathrm{d}A.$$

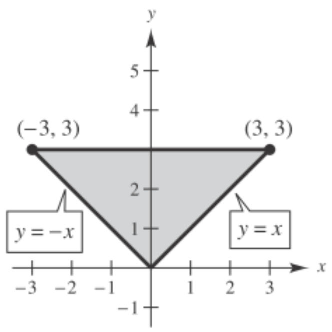
Ans:

$$f_x = 0, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$$

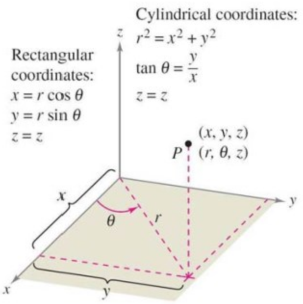
$$S = \int_0^3 \int_{-y}^{y} \sqrt{1+4y^2}\, dxdy = \int_0^3 \left[\sqrt{1+4y^2}x\right]_{-y}^{y} dy = \int_0^3 2y\sqrt{4y^2+1} dy$$

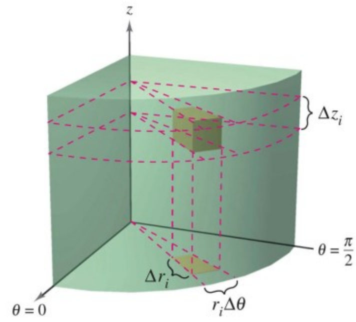
$$= \left[\frac{1}{6}(4y^2+1)^{\frac{3}{2}}\right]_0^3 = \frac{37\sqrt{37} - 1}{6}$$



8. (6%) Find the volume of the solid inside both $x^2 + y^2 + z^2 = 36$ and

$(x-3)^2 + y^2 = 9$





$$\iiint_Q f(x,y,z)\,\mathrm{d}V$$

$$= \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

Ans:

Note that $(x-3)^2 + y^2 = 9$ is equivalent to $r = 6\,cos(\theta),\ 0 \le \theta \le \pi$

$$\mathrm{V} = 2\int_0^{\pi}\int_0^{6\cos(\theta)}\int_0^{\sqrt{36-r^2}} rdzdrd\theta = 2\int_0^{\pi}\int_0^{6\cos(\theta)} r\sqrt{36-r^2}drd\theta$$

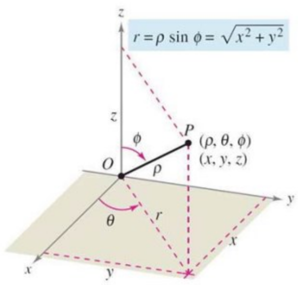
$$= 2\int_0^{\pi}\left[\frac{-1}{3}(36-r^2)^{\frac{3}{2}}\right]_0^{6\cos(\theta)} d\theta$$

$$= \frac{-2}{3}\int_0^{\pi}(216sin^3\theta - 216)\,d\theta$$

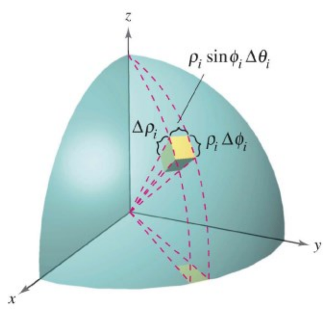
$$= -144\int_0^{\pi}[(1-cos^2\theta)sin\theta - 1]d\theta$$

$$= -144\left[-\cos(\theta) + \frac{cos^3\theta}{3} - \theta\right]_0^{\pi} = 48(3\pi - 4)$$

9. (6%) Evaluate $\int\int\int_Q \frac{z}{\sqrt{x^2+y^2+z^2}}dV$ where Q is a solid region inside the sphere $x^2 + y^2 + z^2 = 9$ and above xy-plane.



Spherical coordinates



$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Ans:

$$\int\int\int_Q \frac{z}{\sqrt{x^2+y^2+z^2}}dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^3 \frac{\rho\cos(\Phi)}{\rho}\rho^2\sin(\Phi)d\rho d\Phi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin(\Phi)\cos(\Phi)d\Phi \int_0^3 \rho^2\, d\rho = 2\pi\frac{1}{2}\frac{27}{3} = 9\pi$$

10. (6%) Use a change of variables to find the volume of the solid region lying below the surface $z = f(x,y) = \frac{x}{1+x^2y^2}$ and above the plane region R wher R is a region bounded by $xy = 5, xy = 1, x = 1, x = 5$.

Ans:

Let $u = x, v = xy \to x = u, y = \frac{v}{u}$

$$\frac{\partial(\mathrm{x},\mathrm{y})}{\partial(\mathrm{u},\mathrm{v})} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{u}$$

$$\int\int_R \frac{x}{1+x^2y^2}\ dA$$

$$= \int_1^5\int_1^5 \frac{u}{1+u^2(v/u)^2}\frac{1}{u}dudv = \int_1^5\int_1^5 \frac{1}{1+v^2}dudv$$

$$= \int_1^5 \frac{4}{1+v^2}dv = 4\arctan(v)\,|_1^5 = 4\arctan(5) - \pi$$