1. (16%) Find the following limit. (If the limit does not exist you should point it out.) Hint: Change of variables may be useful here

(a) $\lim_{(x,y)\to(1,1)} \frac{xy-x-y+1}{x^2+y^2-2x-2y+2}$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

(c) $\lim_{(x,y,z)\to(0,0,0)} \arccos(\frac{x^3+z^3}{x^2+y^2+z^2})$

(d) $\lim_{(x,y)\to(0,0)} \frac{3(x^2+y^2)}{\tan(x^2+y^2)}$

Ans:

(a) $\lim_{(x,y)\to(1,1)} \frac{xy-x-y+1}{x^2+y^2-2x-2y+2} = \lim_{(x,y)\to(1,1)} \frac{(x-1)(y-1)}{(x-1)^2+(y-1)^2} = \lim_{(u,v)\to(0,0)} \frac{uv}{u^2+v}$

When $u = mv$, $\underset{(u,v)\to(0,0)}{lim} \frac{uv}{u^2+v^2} = \frac{m}{(m^2+1)}$ which means that if we follow the trajectory of different lines $u = mv$ to approach (0,0) we will get different values. Therefore, the limit does not exist.

(b) Let $x = rcos(\theta)$, $y = rsin(\theta)$. We have

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = \lim_{r\to0} \frac{r^2(cos^2\theta - sin^2\theta)}{r} = \lim_{r\to0} r(cos^2\theta - sin^2\theta) = 0$$

(c) Let $x = \rho\, sin(\Phi)\cos(\theta)\,, y = \rho\, sin(\Phi)\, sin(\theta)\,, z = \rho\, cos(\Phi)$. We have

$$\lim_{(x,y,z)\to(0,0,0)} \arccos\left(\frac{x^3+z^3}{x^2+y^2+z^2}\right)$$
$$= \lim_{\rho^+\to0} \arccos\left(\frac{\rho^3(sin^3(\Phi)cos^3(\theta) + cos^3(\Phi))}{\rho^2}\right) = \frac{\pi}{2}$$

(d) Let $x = rcos(\theta)$, $y = rsin(\theta)$. We have

$$\lim_{(x,y)\to(0,0)} \frac{3(x^2+y^2)}{\tan(x^2+y^2)} = \lim_{r\to0} \frac{3r^2}{\tan(r^2)} = \lim_{r\to0} \frac{6r}{\sec^2(r^2)2r} = 3$$

2. (12%) Solve the following problems

(a) Let $\mathrm{f(x,y)} = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$, find f_x and f_y when $(x,y) \neq$

(0,0) and when $(x, y) = (0,0)$, respectively

(b) Let $w = x\cos(yz), x = s^2, y = t^2, z = s - 2t$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$

(c) Find a set of parametric equations for the tangent line to the curve of intersection of the surfaces $z = x^2 + y^2$, $x + y + 6z = 33$ at the point (1,2,5)

Ans:

(a) For $(x, y) \neq (0,0)$:

$$f_x(x, y) = \frac{3x^2(x^2 + y^2) - 2x(x^3 - y^3)}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2 + y^2)^2},$$

$$f_y(x, y) = \frac{-3y^2(x^2 + y^2) - 2y(x^3 - y^3)}{(x^2 + y^2)^2} = \frac{-y^4 - 3x^2y^2 - 2yx^3}{(x^2 + y^2)^2}$$

For $(x, y) = (0,0)$:

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^3}{(\Delta x)^2} \frac{1}{\Delta x} = 1,$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(0,0 + \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} -\frac{(\Delta y)^3}{(\Delta y)^2} \frac{1}{\Delta y} = -1$$

(b) Using the chain rule

$$\frac{\partial w}{\partial s} = cos(yz)\,(2s) - xzsin(yz)(0) - xysin(yz)(1)$$

$$= cos(st^2 - 2t^3)\,2s - s^2t^2\,sin(st^2 - 2t^3)$$

$$\frac{\partial w}{\partial t} = cos(yz)\,(0) - xzsin(yz)(2t) - xysin(yz)(-2)$$

$$= -2s^2t(s - 2t)\,sin(st^2 - 2t^3) + 2s^2t^2\,sin(st^2 - 2t^3)$$

$$= (6s^2t^2 - 2s^3t)sin(st^2 - 2t^3)$$

(c) Let $F(x, y, z) = x^2 + y^2 - z, G(x, y, z) = x + y + 6z - 33$

$$\nabla F = 2x\boldsymbol{i} + 2y\boldsymbol{j} - \boldsymbol{k}, \nabla G = \boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}$$

$$\nabla F(1,2,5) = 2\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}, \nabla G(1,2,5) = \boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\boldsymbol{i} - 13\boldsymbol{j} - 2\boldsymbol{k}$$

$$x = 1 + 25t, y = 2 - 13t, z = 5 - 2t$$

3. (10%) Considering the function $f(x, y) = xe^y + cos(xy)$
   (a) Calculate the gradient of $f(x, y)$
   (b) Find the direction in which $f(x, y)$ decrease most rapidly at (2,0) and the directions where the change of f at (2,0) is zero

   Ans:

(a) $\nabla f(x, y) = e^y - y\sin(xy)\boldsymbol{i} + xe^y - x\sin(xy)\boldsymbol{j}$

(b) The direction that decreases most rapidly is the inverse direction of the gradient.

That is $\frac{-\nabla f(2,0)}{|\nabla f(2,0)|} = \frac{-1}{\sqrt{5}}\boldsymbol{i} + \frac{-2}{\sqrt{5}}\boldsymbol{j}$.

The directions where the change of f at (2,0) is zero can be found as follows

Let $u = a\boldsymbol{i} + b\boldsymbol{j}$ and $\sqrt{a^2+b^2} = 1 \rightarrow a^2 + b^2 = 1$

$$u \cdot \nabla f(2,0) = 0 \rightarrow \frac{1}{\sqrt{5}}a + \frac{2}{\sqrt{5}}b = 0$$

$$a = -2b \rightarrow 5b^2 = 1 \rightarrow b = \frac{\pm 1}{\sqrt{5}}, a = \frac{\mp 2}{\sqrt{5}}$$

$u = \frac{2}{\sqrt{5}}\boldsymbol{i} + \frac{-1}{\sqrt{5}}\boldsymbol{j}$ or $u = \frac{-2}{\sqrt{5}}\boldsymbol{i} + \frac{1}{\sqrt{5}}\boldsymbol{j}$

4. (8%) Let $f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 1$
   (a) Find the critical points of $f(x, y)$
   (b) Determine whether they are local maximum, local minimum or saddle points

Ans:

(a) $f_x = 6xy - 12x = 6x(y-2), f_y = 3y^2 + 3x^2 - 12y = 3(x^2 + y^2 - 4y)$.

Let $f_x = 0$ and $f_y = 0$,

For $x = 0$: $y = 0$, or 4

For $x \neq 0$: $y = 2$ and $x = \pm 2$.

Therefore, the critical points are (0,0), (0,4), (2,2) and (-2,2)

(b)

Since $f_{xx} = 6y - 12, f_{xy} = f_{yx} = 6x, f_{yy} = 6y - 12$.

For (2,2) and (-2,2)

$D = f_{xx}f_{yy} - f_{xy}f_{yx} = -144 < 0 \rightarrow$ (2,2) and (-2,2) are saddle points.

For (0,0)

$D = f_{xx}f_{yy} - f_{xy}f_{yx} = 144 > 0$ and $f_{xx}(0,0) < 0 \rightarrow$ (0,0) is local maximum

For (0,4)

$D = f_{xx}f_{yy} - f_{xy}f_{yx} = 144 > 0$ and $f_{xx}(0,4) > 0 \rightarrow$ (0,4) is local minimum

5. (8%) Find the extreme values of $xy + z^2$ subject to the constraint $x^2 + y^2 + (z - \frac{1}{2})^2 \leq 1$

Ans:

Let $f = xy + z^2$ and $g = x^2 + y^2 + (z - \frac{1}{2})^2 - 1$. We can break the constraint into two cases.

For the points inside $x^2 + y^2 + (z - \frac{1}{2})^2 < 1$, we have $\nabla f = y\boldsymbol{i} + x\boldsymbol{j} + 2z\boldsymbol{k}$.

The critical points are $(x, y, z) = (0,0,0) \to f(0,0,0) = 0$

For the points on the boundary $x^2 + y^2 + (z - \frac{1}{2})^2 = 1$. Use the Langrange multiplier, we have the following equations

$$\begin{cases} y = \lambda 2x \\ x = \lambda 2y \\ 2z = \lambda(2z - 1) \\ x^2 + y^2 + (z - \frac{1}{2})^2 = 1 \end{cases}$$

From the first two equations, we have $x = \lambda\big(2\lambda(2x)\big) \to x(4\lambda^2 - 1) = 0 \to x = 0$ or $\lambda = \pm\frac{1}{2}$

(i) If $x = 0 \to y = 0 \to z = \frac{-1}{2}$ or $\frac{3}{2}$, $f\left(0,0,\frac{-1}{2}\right) = \frac{1}{4}, f\left(0,0,\frac{3}{2}\right) = \frac{9}{4}$

(ii) If $\lambda = \frac{1}{2} \to x = y$ and $z = \frac{-1}{2} \to x = y = 0$. $f\left(0,0,\frac{-1}{2}\right) = \frac{1}{4}$

(iii) If $\lambda = \frac{-1}{2} \to x = -y$ and $z = \frac{1}{6} \to x = \pm\frac{2}{3}, y = \mp\frac{2}{3}$. $f\left(\frac{2}{3},\frac{-2}{3},\frac{1}{6}\right) = f\left(\frac{-2}{3},\frac{2}{3},\frac{1}{6}\right) = \frac{-15}{36} = \frac{-5}{12}$

Therefore, the maximum is $\frac{9}{4}$ and the minimum is $\frac{-5}{12}$

6. (20%) Evaluate the following expressions

(a) $\int_0^1 \int_y^1 \tan(x^2)\, dxdy$

(b) $\int_{\frac{1}{\sqrt{2}}}^1 \int_{\sqrt{1-x^2}}^x 1\, dydx + \int_1^{\sqrt{2}} \int_0^x 1\, dydx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} 1\, dydx$

(Hint: draw the region of integration first and you may find it easier to calculate the area using polar coordinates)

(c) $\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} dydxdz$

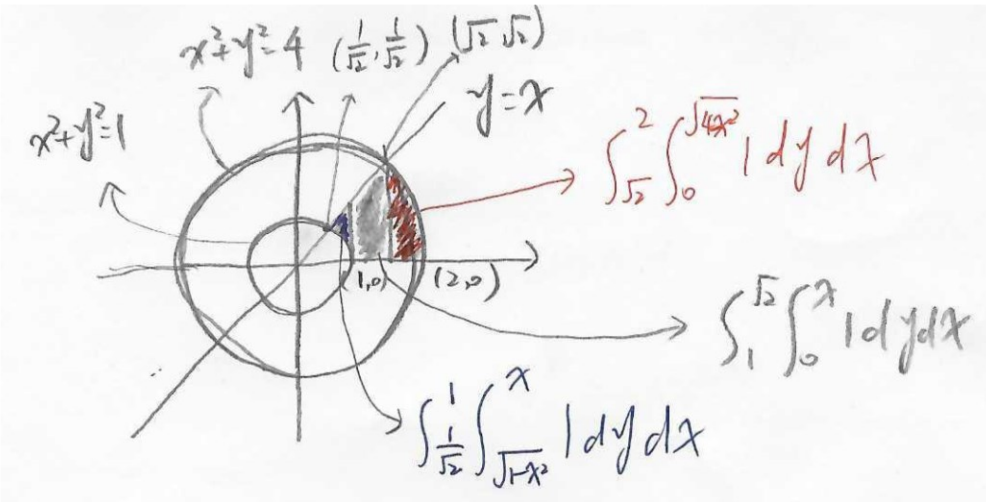
(d) Evaluate $\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$ (Hint: Express it as an iterated integral)

Ans:

(a) $\int_0^1 \int_y^1 \tan(x^2)\, dxdy = \int_0^1 \int_0^x \tan(x^2)\, dydx = \int_0^1 x\tan(x^2)\, dx =$

$\frac{1}{2}\ln|\sec(x^2)|\Big|_0^1 = \frac{1}{2}\ln(\sec(1))$

(b) We can interpret the integral as the area of the following region



Which can be described using polar coordinates:

$$\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} 1\, dydx + \int_1^{\sqrt{2}}\int_0^x 1\, dydx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} 1\, dydx$$

$$= \int_0^{\frac{\pi}{4}}\int_1^2 r dr d\theta = \int_1^2 r dr \int_0^{\frac{\pi}{4}} d\theta = \frac{1}{2}r^2\Big|_1^2 \theta\Big|_0^{\frac{\pi}{4}} = \frac{3}{8}\pi$$

(c) $\int_1^4 \int_0^1 2ze^{-x^2}y\Big|_0^x dxdz = \int_1^4 \int_0^1 2zxe^{-x^2}\, dxdz = \int_1^4 -ze^{-x^2}\Big|_0^1 dz =$

$\int_1^4 z(1-e^{-1})dz = (1-e^{-1})\frac{z^2}{2}\Big|_1^4 = \frac{15}{2}(1-\frac{1}{e})$

(d) $\int_0^\infty \frac{\arctan(\pi x)-\arctan(x)}{x} dx = \int_0^\infty \int_{\arctan(x)}^{\arctan(\pi x)} \frac{1}{x} dydx = \int_0^{\frac{\pi}{2}} \int_{\frac{\tan(y)}{\pi}}^{\tan(y)} \frac{1}{x} dxdy =$

$\int_0^{\frac{\pi}{2}} \ln x\Big|_{\frac{\tan(y)}{\pi}}^{\tan(y)} = \frac{\pi}{2}\ln\pi$

7. (8%) Find the area of the surface given by $z = f(x,y) = 13 + x^2 - y^2$ that lies above the region R where $R = \{(x,y): x^2 + y^2 \le 4\}$

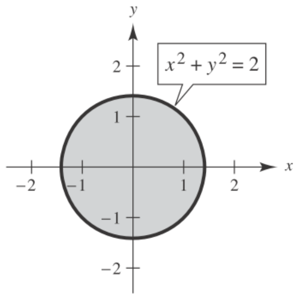
Ans:

$$f_x = 2x, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, dr d\theta = \int_0^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{\frac{3}{2}}\right]_0^2 d\theta = \int_0^{2\pi} \frac{1}{12}(17^{\frac{3}{2}} - 1) d\theta =$$

$$= \frac{\pi}{6}(17\sqrt{17} - 1)$$



8. (8%) Evaluate the triple integral $\int\int\int_Q xyzdV$ where $Q = \{0 \leq \mathrm{x} \leq \sqrt{4 - y^2}, 0 \leq \mathrm{y} \leq 2, \sqrt{x^2 + y^2} \leq \mathrm{z} \leq \sqrt{8 - x^2 - y^2}\}$

Ans:

1. Use rectangular coordinates

$$\int\int\int_Q xyzdV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} xyz\ \ dzdxdy$$

$$= \int_0^2 \int_0^{\sqrt{4-y^2}} 4xy - x^3y - xy^3\, dxdy$$

$$= \int_0^2 2y(4 - y^2) - y\frac{(4 - y^2)^2}{4} - y^3\frac{4 - y^2}{2}\ \ dy = \frac{8}{3}$$

2. Use cylindrical coordinates

$$\int\int\int_Q xyzdV = \int_0^{\frac{\pi}{2}} \int_0^2 \int_r^{\sqrt{8-r^2}} r^2 sin\theta cos\theta zr\ \ dzdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^3 \sin\theta cos\theta(4 - r^2)\ \ drd\theta$$

$$= \int_0^{\frac{\pi}{2}} 16\sin\theta cos\theta - \frac{32}{3}\sin\theta cos\theta\ \ d\theta = \int_0^{\frac{\pi}{2}} \frac{8}{3}\sin 2\theta\ \ d\theta = \frac{8}{3}$$

3. Use spherical coordinates

$$\iiint_Q xyzdV = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{2\sqrt{2}} \rho^3 sin^2\Phi sin\theta cos\theta cos\Phi \rho^2 sin\Phi \ \ d\rho d\theta d\Phi$$

$$= \int_0^{2\sqrt{2}} \rho^5 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta \ \ d\theta \times \int_0^{\frac{\pi}{4}} sin^3\Phi \ \ cos\Phi d\Phi$$

$$= \frac{(2\sqrt{2})^6}{6} \times \frac{1}{4}(-cos2\theta)|_0^{\frac{\pi}{2}} \times \frac{1}{4} sin^4\Phi|_0^{\frac{\pi}{4}} = \frac{8}{3}$$

9. (10%) Use the change of variables to find the volume of the solid region lying below the surface $z = f(x,y) = \sin(\frac{3x+y}{x-2y})$ and above the plane region R wher R is a region bounded by $2x + 3y = 0, 3x + y = 0, x - 2y = 1, x - 2y = 2.$

Ans:

Let $u = 3x + y, v = x - 2y$

$$\frac{\partial(\mathrm{x,y})}{\partial(\mathrm{u,v})} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = -\frac{1}{7}$$

$$\iint_R \sin(\frac{3x+y}{x-2y}) \ \ \mathrm{dA} = \int_1^2\int_0^v \sin(\frac{u}{v})\frac{1}{7} dudv = \frac{1}{7}\int_1^2 \left[\frac{-\cos(\frac{u}{v})}{\frac{1}{v}}\right]_0^v dv$$

$$= \frac{1}{7}\int_1^2 v(\cos(0) - \cos(1))dv = \frac{1}{7}(1-\cos(1))\frac{v^2}{2}|_1^2$$

$$= \frac{3(1-\cos(1))}{14}$$