

# Chapter 14 Multiple Integration

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

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# Iterated integrals

## Iterated integrals

$$(1) \int_{h_1(y)}^{h_2(y)} f_x(x, y) dx = f(x, y) \Big|_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \quad \text{With respect to } x$$

$$(2) \int_{g_1(x)}^{g_2(x)} f_y(x, y) dy = f(x, y) \Big|_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \quad \text{With respect to } y$$

- Note that the variable of integration cannot appear in either limit of integration.
- For instance, it makes no sense to write

$$\int_0^x y dx.$$

## Example 1 (Integrating with respect to $y$ )

Evaluate

$$\int_1^x (2xy + 3y^2) dy.$$

- Considering  $x$  to be constant and integrating with respect to  $y$  produces

$$\begin{aligned}\int_1^x (2xy + 3y^2) dy &= [xy^2 + y^3]_1^x \\ &= (2x^3) - (x + 1) = 2x^3 - x - 1. \quad \blacksquare\end{aligned}$$

## Example 2 (The integral of an integral)

Evaluate

$$\int_1^2 \left[ \int_1^x (2xy + 3y^2) dy \right] dx.$$

- Using the result of Example 1, you have

$$\begin{aligned} \int_1^2 \left[ \int_1^x (2xy + 3y^2) dy \right] dx &= \int_1^2 (2x^3 - x - 1) dx \\ &= \left[ \frac{x^4}{2} - \frac{x^2}{2} - x \right]_1^2 = 4 - (-1) = 5. \blacksquare \end{aligned}$$

- The integral in Example 2 is an iterated integral. The brackets used in Example 2 are normally not written.
- Instead, iterated integrals are usually written simply as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad \text{and} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

- The inside limits of integration can be variable with respect to the outer variable of integration. However, the outside limits of integration must be constant with respect to both variables of integration.
- For instance, in Example 2, the outside limits indicate that  $x$  lies in the interval  $1 \leq x \leq 2$  and the inside limits indicate that  $y$  lies in the interval  $1 \leq y \leq x$ .
- Together, these two intervals determine the region of integration  $R$  of the iterated integral, as shown in Figure 1.

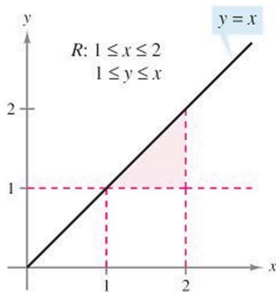
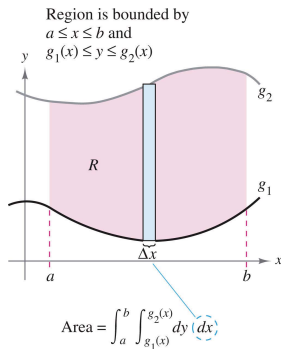


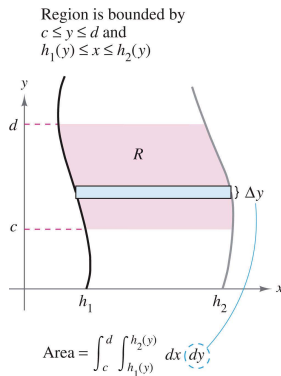
Figure 1: The region of integration for  $\int_1^2 \int_1^x f(x,y) dy dx$ .

# Area of a plane region

- Consider the plane region  $R$  bounded by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , as shown in Figure 2(a).



(a) Vertically simple region.



(b) Horizontally simple region.

Figure 2: Vertically and horizontally simple regions.



- The area of  $R$  is given by the definite integral

$$\int_a^b [g_2(x) - g_1(x)] dx. \quad \text{Area of } R$$

- Using the Fundamental Theorem of Calculus, you can rewrite the integrand  $g_2(x) - g_1(x)$  as a definite integral.
- Specifically, if you consider  $x$  to be fixed and let  $y$  vary from  $g_1(x)$  to  $g_2(x)$ , you can write

$$\int_{g_1(x)}^{g_2(x)} dy = y \Big|_{g_1(x)}^{g_2(x)} = g_2(x) - g_1(x).$$

- Combining these two integrals, you can write the area of the region  $R$  as an iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy dx = \int_a^b y \Big|_{g_1(x)}^{g_2(x)} dx = \int_a^b [g_2(x) - g_1(x)] dx.$$

- Placing a representative rectangle in the region  $R$  helps determine both the order and the limits of integration.
- A vertical rectangle implies the order  $dy\,dx$ , with the inside limits corresponding to the upper and lower bounds of the rectangle, as shown in Figure 2(a).
- This type of region is called vertically simple, because the outside limits of integration represent the vertical lines  $x = a$  and  $x = b$ .
- Similarly, a horizontal rectangle implies the order  $dx\,dy$ , with the inside limits determined by the left and right bounds of the rectangle, as shown in Figure 2(b).
- This type of region is called horizontally simple, because the outside limits represent the horizontal lines  $y = c$  and  $y = d$ .

## Definition 14.1 (Area of a region in the plane)

1. If  $R$  is defined by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , where  $g_1$  and  $g_2$  are continuous on  $[a, b]$ , then the area of  $R$  is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy \, dx. \quad \text{Figure 2(a) (vertically simple)}$$

2. If  $R$  is defined by  $c \leq y \leq d$  and  $h_1(y) \leq x \leq h_2(y)$ , where  $h_1$  and  $h_2$  are continuous on  $[c, d]$ , then the area of  $R$  is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx \, dy. \quad \text{Figure 2(b) (horizontally simple)}$$

### Example 3 (The area of a rectangular region)

Use an iterated integral to represent the area of the rectangle shown in Figure 3.

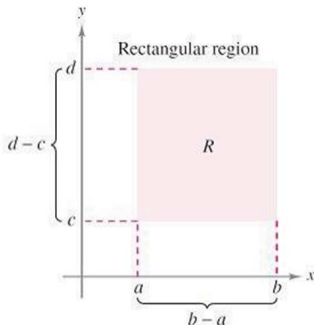


Figure 3: The area of a rectangular region.

- The region shown in Figure 3 is both vertically simple and horizontally simple, so you can use either order of integration.

- By choosing the order  $dy dx$ , you obtain the following.

$$\begin{aligned}\int_a^b \int_c^d dy dx &= \int_a^b y \Big|_c^d dx = \int_a^b (d - c) dx \\ &= [(d - c)x]_a^b = (d - c)(b - a)\end{aligned}$$

#### Example 4 (Finding area by an iterated integral)

Use an iterated integral to find the area of the region bounded by the graphs of

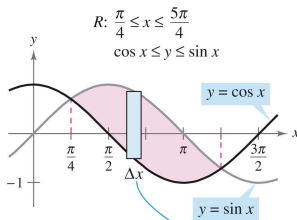
$$f(x) = \sin x$$

$$g(x) = \cos x$$

between  $x = \pi/4$  and  $x = 5\pi/4$ .

- Because  $f$  and  $g$  are given as functions of  $x$ , a vertical representative rectangle is convenient, and you can choose  $dy\,dx$  as the order of integration, as shown in the following figure.
- The outside limits of integration are  $\pi/4 \leq x \leq 5\pi/4$ .
- Moreover, because the rectangle is bounded above by  $f(x) = \sin x$  and below by  $g(x) = \cos x$ , you have

$$\begin{aligned}\text{Area of } R &= \int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} dy\,dx = \int_{\pi/4}^{5\pi/4} y \Big|_{\cos x}^{\sin x} dx \\ &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx = -\cos x - \sin x \Big|_{\pi/4}^{5\pi/4} = 2\sqrt{2}.\end{aligned}$$



$$\text{Area} = \int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} dy\,dx$$

The order of integration affects the ease of integration, but not the value of the integral.

### Example 5 (Comparing different orders of integration)

Sketch the region whose area is represented by the integral

$$\int_0^2 \int_{y^2}^4 dx dy.$$

Then find another iterated integral using the order  $dy dx$  to represent the same area and show that both integrals yield the same value.

- From the given limits of integration, you know that

$$y^2 \leq x \leq 4$$

which means that the region  $R$  is bounded on the left by the parabola  $x = y^2$  and on the right by the line  $x = 4$ .

- Furthermore, because

$$0 \leq y \leq 2$$

you know that  $R$  is bounded below by the  $x$ -axis, as shown in Figure 5(a).

- The value of this integral is

$$\int_0^2 \int_{y^2}^4 dx dy = \int_0^2 x \Big|_{y^2}^4 dy = \int_0^2 (4 - y^2) dy = \left[ 4y - \frac{y^3}{3} \right]_0^2 = \frac{16}{3}.$$

- To change the order of integration to  $dy dx$ , place a vertical rectangle in the region, as shown in Figure 5(b).
- From this you can see that the constant bounds  $0 \leq x \leq 4$  serve as the outer limits of integration.
- By solving for  $y$  in the equation  $x = y^2$ , you can conclude that the inner bounds are  $0 \leq y \leq \sqrt{x}$ .



- So, the area of the region can also be represented by

$$\int_0^4 \int_0^{\sqrt{x}} dy \, dx.$$

- By evaluating this integral, you can see that it has the same value as the original integral.

$$\int_0^4 \int_0^{\sqrt{x}} dy \, dx = \int_0^4 y \Big|_0^{\sqrt{x}} dx = \int_0^4 \sqrt{x} \, dx = \left. \frac{2}{3} x^{3/2} \right|_0^4 = \frac{16}{3} \quad \blacksquare$$

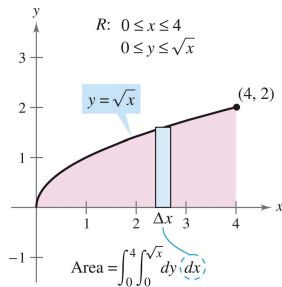
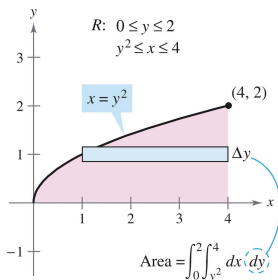


Figure 5: Comparing different orders of integration.

## Example 6 (An area represented by two iterated integrals)

Find the area of the region  $R$  that lies below the parabola

$$y = 4x - x^2 \quad \text{Parabola forms upper boundary}$$

above the  $x$ -axis, and above the line

$$y = -3x + 6. \quad \text{Line and } x\text{-axis form lower boundary}$$

- Begin by dividing  $R$  into the two subregions  $R_1$  and  $R_2$  shown below.

$$\begin{aligned} \text{Area} &= \int_1^2 \int_{-3x+6}^{4x-x^2} dy \, dx + \int_2^4 \int_0^{4x-x^2} dy \, dx \\ &= \int_1^2 (4x - x^2 + 3x - 6) \, dx + \int_2^4 (4x - x^2) \, dx \\ &= \left[ \frac{7x^2}{2} - \frac{x^3}{3} - 6x \right]_1^2 + \left[ 2x^2 - \frac{x^3}{3} \right]_2^4 \\ &= \left( 14 - \frac{8}{3} - 12 - \frac{7}{2} + \frac{1}{3} + 6 \right) + \left( 32 - \frac{64}{3} - 8 + \frac{8}{3} \right) = \frac{15}{2}. \end{aligned}$$

- The area of the region is  $15/2$  square units. Try checking this using the procedure for finding the area between two curves, as presented in Section 7.1.

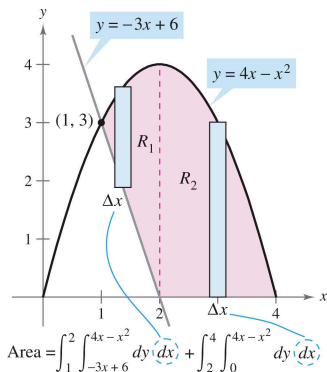


Figure 6: An area represented by two iterated integrals.

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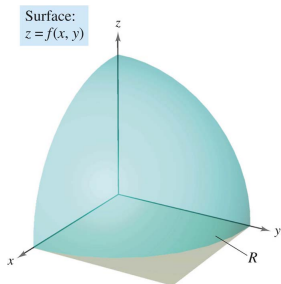
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# Double integrals and volume of a solid region

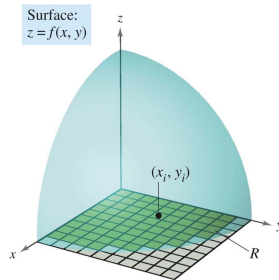
- You know that a definite integral over an interval uses a limit process to assign measures to quantities such as area, volume, arc length, and mass.
- In this section, you will use a similar process to define the double integral of a function of two variables over a region in the plane.
- Consider a continuous function  $f$  such that  $f(x, y) \geq 0$  for all  $(x, y)$  in a region  $R$  in the  $xy$ -plane. The goal is to find the volume of the solid region lying between the surface given by

$$z = f(x, y) \quad \text{Surface lying above the } xy\text{-plane}$$

and the  $xy$ -plane, as shown in Figure 7(a).



(a) Volume of the solid region lying between the surface given by  $z = f(x, y)$  and the  $xy$ -plane.



(b) The rectangles lying within  $R$  form an inner partition of  $R$ .

Figure 7: Volume of the solid region and rectangles.

- You can begin by superimposing a rectangular grid over the region, as shown in Figure 7(b).

- The rectangles lying entirely within  $R$  form an inner partition  $\Delta$ , whose norm  $\|\Delta\|$  is defined as the length of the longest diagonal of the  $n$  rectangles. Next, choose a point  $(x_i, y_i)$  in each rectangle and form the rectangular prism whose height is  $f(x_i, y_i)$ , as shown in Figure 8(a).
- Because the area of the  $i$ th rectangle is

$$\Delta A_i \quad \text{Area of } i\text{th rectangle}$$

it follows that the volume of the  $i$ th prism is

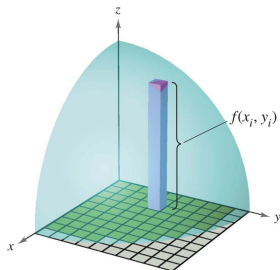
$$f(x_i, y_i)\Delta A_i. \quad \text{Volume of } i\text{th prism}$$

- You can approximate the volume of the solid region by the Riemann sum of the volumes of all  $n$  prisms,

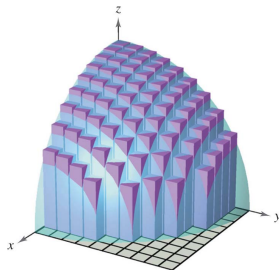
$$\sum_{i=1}^n f(x_i, y_i)\Delta A_i \quad \text{Riemann sum}$$

as shown in Figure 8(b).

- This approximation can be improved by tightening the mesh of the grid to form smaller and smaller rectangles.



(a) Rectangular prism whose base has an area  $\Delta A_i$ , and whose height is  $f(x_i, y_i)$ .



(b) Volume approximated by rectangular prisms.

Figure 8: Rectangular prism and volume approximated by rectangular prisms.



## Example 1 (Approximating the volume of a solid)

Approximate the volume of the solid lying between the paraboloid

$$f(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

and the square region  $R$  given by  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ . Use a partition made up of squares whose sides have a length of  $\frac{1}{4}$ .

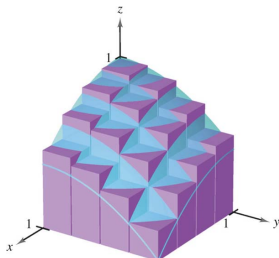
- Begin by forming the specified partition of  $R$ .
- For this partition, it is convenient to choose the centers of the subregions as the points at which to evaluate  $f(x, y)$ .

$$\begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ \frac{5}{8} \\ \frac{7}{8} \end{pmatrix}, \begin{pmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{3}{8} \\ \frac{5}{8} \end{pmatrix}, \begin{pmatrix} \frac{1}{8} \\ \frac{5}{8} \\ \frac{9}{8} \\ \frac{13}{8} \end{pmatrix}, \begin{pmatrix} \frac{1}{8} \\ \frac{9}{8} \\ \frac{13}{8} \\ \frac{17}{8} \end{pmatrix}$$

- Because the area of each square is  $\Delta A_i = 1/16$ , you can approximate the volume by the sum

$$\sum_{i=1}^{16} f(x_i, y_i) \Delta A_i = \sum_{i=1}^{16} \left[ 1 - \frac{1}{2} x_i^2 - \frac{1}{2} y_i^2 \right] \left[ \frac{1}{16} \right] \approx 0.672.$$

- This approximation is shown graphically in Figure 9.
- The exact volume of the solid is  $2/3$ .



Surface:

$$f(x, y) = 1 - \frac{1}{2} x^2 - \frac{1}{2} y^2$$

Figure 9: Volume approximated by rectangular prisms:  $f(x, y) = 1 - \frac{1}{2} x^2 - \frac{1}{2} y^2$ .

- You can obtain a better approximation by using a finer partition. For example, with a partition of squares with sides of length  $1/10$  the approximation is 0.668. ■
- In Example 1, note that by using finer partitions, you obtain better approximations of the volume. This observation suggests that you could obtain the exact volume by taking a limit. That is,

$$\text{Volume} = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i) \Delta A_i.$$

- The precise meaning of this limit is that the limit is equal to  $L$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that

$$\left| L - \sum_{i=1}^n f(x_i, y_i) \Delta A_i \right| < \varepsilon$$

for all partitions  $\Delta$  of the plane region  $R$  (that satisfy  $\|\Delta\| < \delta$ ) and for all possible choices of  $x_i$  and  $y_i$  in the  $i$ th region.

- Using the limit of a Riemann sum to define volume is a special case of using the limit to define a double integral.
- The general case, however, does not require that the function be positive or continuous.

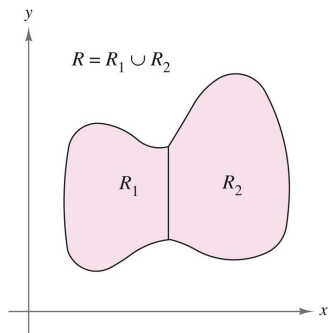
### Definition 14.2 (Double integral)

If  $f$  defined on a closed, bounded region  $R$  in the  $xy$ -plane, then the double integral of  $f$  over  $R$  is given by

$$\iint_R f(x, y) \, dA = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i) \Delta A_i$$

provided the limit exists. If the limit exists, then  $f$  is integrable over  $R$ .

- Having defined a double integral, you will see that a definite integral is occasionally referred to as a single integral.
- Sufficient conditions for the double integral of  $f$  on the region  $R$  to exist are that  $R$  can be written as a union of a finite number of nonoverlapping subregions (see Figure 10) that are vertically or horizontally simple and that  $f$  is continuous on the region  $R$ .



**Figure 10:** The regions are nonoverlapping if their intersection is a set that has an area of 0. In this figure, the area of the line segment that is common to  $R_1$  and  $R_2$  is 0.

- A double integral can be used to find the volume of a solid region that lies between the  $xy$ -plane and the surface given by  $z = f(x, y)$ .

Volume of a solid region    If  $f$  is integrable over a plane region  $R$  and  $f(x, y) \geq 0$  for all  $(x, y)$  in  $R$ , then the volume of the solid region that lies above  $R$  and below the graph of  $f$  defined as

$$V = \iint_R f(x, y) \, dA.$$

- Double integrals share many properties of single integrals.

## Theorem 14.1 (Properties of double integrals)

*Let  $f$  and  $g$  be continuous over a closed, bounded plane region  $R$ , and let  $c$  be a constant.*

- ①  $\iint_R cf(x, y) \, dA = c \iint_R f(x, y) \, dA$
- ②  $\iint_R [f(x, y) \pm g(x, y)] \, dA = \iint_R f(x, y) \, dA \pm \iint_R g(x, y) \, dA$
- ③  $\iint_R f(x, y) \, dA \geq 0$ , if  $f(x, y) \geq 0$
- ④  $\iint_R f(x, y) \, dA \geq \iint_R g(x, y) \, dA$ , if  $f(x, y) \geq g(x, y)$
- ⑤  $\iint_R f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA$ , where  $R$  is the union of two nonoverlapping subregions  $R_1$  and  $R_2$ .

# Evaluation of double integrals

- Consider the solid region bounded by the plane  $z = f(x, y) = 2 - x - 2y$  and the three coordinate planes, as shown in Figure 11.

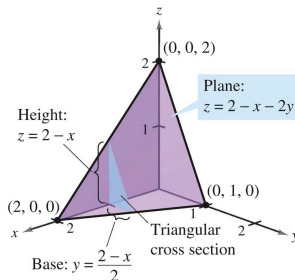


Figure 11: Volume:  $\int_0^2 A(x) dx$

- Each vertical cross section taken parallel to the  $yz$ -plane is a triangular region whose base has a length of  $y = (2 - x)/2$  and whose height is  $z = 2 - x$ .



- This implies that for a fixed value of  $x$ , the area of the triangular cross section is

$$A(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \left( \frac{2-x}{2} \right) (2-x) = \frac{(2-x)^2}{4}.$$

- By the formula for the volume of a solid with known cross sections, the volume of the solid is

$$\text{Volume} = \int_a^b A(x) dx = \int_0^2 \frac{(2-x)^2}{4} dx = -\frac{(2-x)^3}{12} \Big|_0^2 = \frac{2}{3}.$$

- This procedure works no matter how  $A(x)$  is obtained as below

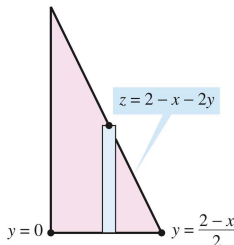


Figure 12: Triangular cross section  $z = 2 - x - 2y$  where  $0 \leq y \leq \frac{2-x}{2}$ .

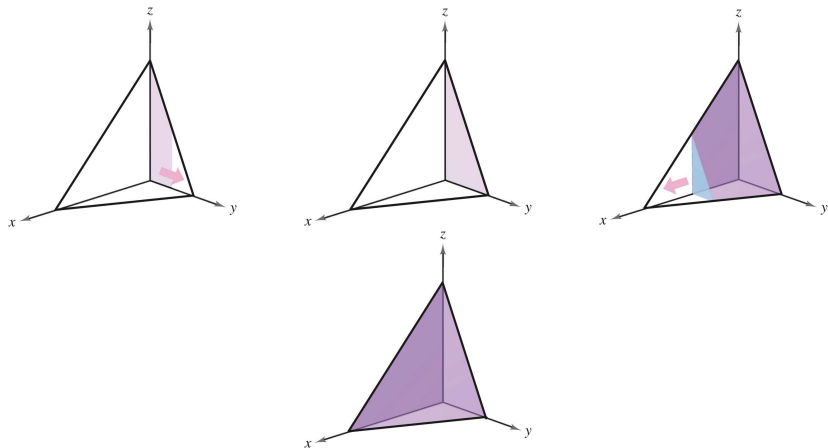
- That is, you consider  $x$  to be constant, and integrate  $z = 2 - x - 2y$  from 0 to  $(2 - x)/2$  to obtain

$$A(x) = \int_0^{(2-x)/2} (2 - x - 2y) dy = [(2 - x)y - y^2]_0^{(2-x)/2} = \frac{(2 - x)^2}{4}.$$

- Combining these results, you have the iterated integral

$$\text{Volume} = \iint_R f(x, y) dA = \int_0^2 \int_0^{(2-x)/2} (2 - x - 2y) dy dx.$$

- To understand this procedure better, it helps to imagine the integration as two sweeping motions. For the inner integration, a vertical line sweeps out the area of a cross section. For the outer integration, the triangular cross section sweeps out the volume, as shown in Figure 13.



**Figure 13:** Integrate with respect to  $y$  to obtain the area of the cross section; integrate with respect to  $x$  to obtain the volume of the cross section.

## Theorem 14.2 (Fubini's Theorem)

Let  $f$  be continuous on a plane region  $R$ .

- ① If  $R$  is defined by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , where  $g_1$  and  $g_2$  are continuous on  $[a, b]$ , then

$$\iint_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.$$

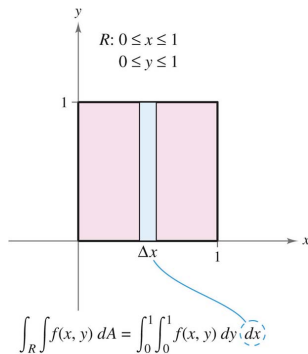
- ② If  $R$  is defined by  $c \leq y \leq d$  and  $h_1(y) \leq x \leq h_2(y)$ , where  $h_1$  and  $h_2$  are continuous on  $[c, d]$ , then

$$\iint_R f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.$$

## Example 2 (Evaluating a double integral as an iterated integral)

Evaluate  $\iint_R (1 - \frac{1}{2}x^2 - \frac{1}{2}y^2) \, dA$  where  $R$  is the region given by  $0 \leq x \leq 1, 0 \leq y \leq 1$ .

- Because the region  $R$  is a square, it is both vertically and horizontally simple, and you can use either order of integration.
- Choose  $dy \, dx$  by placing a vertical representative rectangle in the region, as shown below.



- This produces the following.

$$\begin{aligned}\iint_R \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dA &= \int_0^1 \int_0^1 \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dy dx \\&= \int_0^1 \left[ \left(1 - \frac{1}{2}x^2\right)y - \frac{y^3}{6} \right]_0^1 dx \\&= \int_0^1 \left(\frac{5}{6} - \frac{1}{2}x^2\right) dx = \left[\frac{5}{6}x - \frac{x^3}{6}\right]_0^1 = \frac{2}{3} \blacksquare\end{aligned}$$

### Example 3 (Finding volume by a double integral)

Find the volume of the solid region bounded by the paraboloid  $z = 4 - x^2 - 2y^2$  and the  $xy$ -plane, as shown in Figure 15(a).

- By letting  $z = 0$ , you can see that the base of the region in the  $xy$ -plane is the ellipse  $x^2 + 2y^2 = 4$ , as shown in Figure 15(b).

- This plane region is both vertically and horizontally simple, so the order  $dy dx$  is appropriate.

$$\text{Variable bounds for } y: \quad -\sqrt{\frac{(4-x^2)}{2}} \leq y \leq \sqrt{\frac{(4-x^2)}{2}}$$

$$\text{Constant bounds for } x: \quad -2 \leq x \leq 2$$

- The volume is given by

$$\begin{aligned} V &= \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4-x^2-2y^2) dy dx && \text{See Figure 15(b)} \\ &= \int_{-2}^2 \left[ (4-x^2)y - \frac{2y^3}{3} \right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx = \frac{4}{3\sqrt{2}} \int_{-2}^2 (4-x^2)^{3/2} dx \\ &= \frac{4}{3\sqrt{2}} \int_{-\pi/2}^{\pi/2} 16 \cos^4 \theta d\theta = \frac{64}{3\sqrt{2}} (2) \int_0^{\pi/2} \cos^4 \theta d\theta \\ &= \frac{128}{3\sqrt{2}} \left( \frac{3\pi}{16} \right) = 4\sqrt{2}\pi. \end{aligned}$$



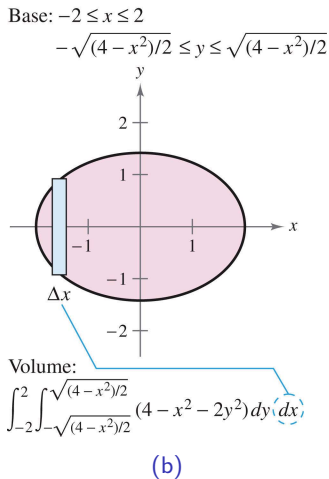
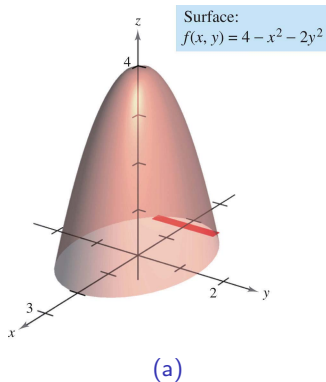


Figure 15: Finding volume by a double integral.



## Example 4 (Comparing different orders of integration)

Find the volume of the solid region  $R$  bounded by the surface

$$f(x, y) = e^{-x^2} \quad \text{Surface}$$

and the planes  $z = 0$ ,  $y = 0$ ,  $y = x$ , and  $x = 1$ , as shown in Figure 16.

- The base of  $R$  in the  $xy$ -plane is bounded by the lines  $y = 0$ ,  $x = 1$ , and  $y = x$ . The two possible orders of integration are shown in next slide. By setting up the corresponding iterated integrals, you can see that the order  $dx dy$  requires the antiderivative  $\int e^{-x^2} dx$ , which is not an elementary function.
- On the other hand, the order  $dy dx$  produces the integral

$$\begin{aligned} \int_0^1 \int_0^x e^{-x^2} dy dx &= \int_0^1 e^{-x^2} y \Big|_0^x dx = \int_0^1 x e^{-x^2} dx \\ &= -\frac{1}{2} e^{-x^2} \Big|_0^1 = -\frac{1}{2} \left( \frac{1}{e} - 1 \right) = \frac{e-1}{2e} \approx 0.316. \end{aligned}$$

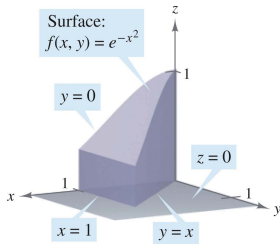
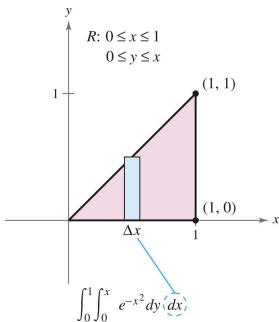
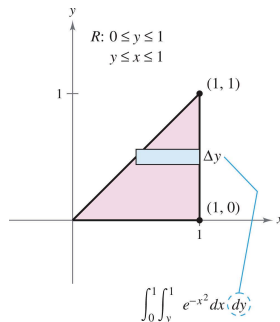


Figure 16: Base is bounded by  $y = 0$ ,  $y = x$ , and  $x = 1$ .



(a)  $dy \, dx$



(b)  $dx \, dy$

## Example 5 (Volume of a region bounded by two surfaces)

Find the volume of the solid region  $R$  bounded above by the paraboloid  $z = 1 - x^2 - y^2$  and below by the plane  $z = 1 - y$ , as shown in Figure 18.

- Equating  $z$ -values, you can determine that the intersection of the two surfaces occurs on the right circular cylinder given by

$$1 - y = 1 - x^2 - y^2 \implies x^2 = y - y^2.$$

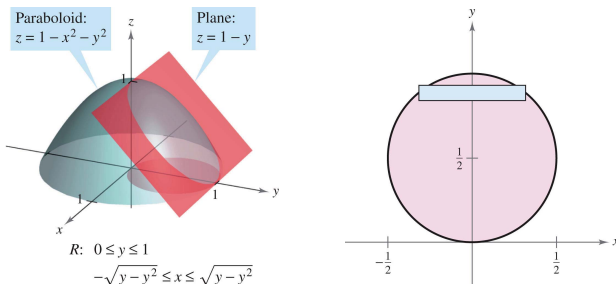


Figure 18: Volume of a region bounded by two surfaces.

- Because the volume of  $R$  is the difference between the volume under the paraboloid and the volume under the plane, you have

$$\begin{aligned}
 \text{Volume} &= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1 - x^2 - y^2) \, dx \, dy - \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1 - y) \, dx \, dy \\
 &= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (y - y^2 - x^2) \, dx \, dy \\
 &= \int_0^1 \left[ (y - y^2)x - \frac{x^3}{3} \right]_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} dy = \frac{4}{3} \int_0^1 (y - y^2)^{3/2} dy \\
 &= \left( \frac{4}{3} \right) \left( \frac{1}{8} \right) \int_0^1 [1 - (2y - 1)^2]^{3/2} dy \\
 &= \frac{1}{6} \int_{-\pi/2}^{\pi/2} \frac{\cos^4 \theta}{2} d\theta = \frac{1}{6} \int_0^{\pi/2} \cos^4 \theta d\theta \\
 &= \left( \frac{1}{6} \right) \left( \frac{3\pi}{16} \right) = \frac{\pi}{32}.
 \end{aligned}$$

# Average value of a function

- For a function  $f$  in one variable, the average value of  $f$  on  $[a, b]$  is

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

- Given a function  $f$  in two variables, you can find the average value of  $f$  over the region  $R$  as shown in the following definition.

## Definition 14.3 (The average value of a function over a region)

If  $f$  is integrable over the plane region  $R$ , then the average value of  $f$  over  $R$  is

$$\frac{1}{A} \iint_R f(x, y) dA$$

where  $A$  is the area of  $R$ .

### Example 6 (Finding the average value of a function)

Find the average value of  $f(x, y) = \frac{1}{2}xy$  over the region  $R$ , where  $R$  is a rectangle with vertices  $(0, 0)$ ,  $(4, 0)$ ,  $(4, 3)$ , and  $(0, 3)$ .

- The area of the rectangular region  $R$  is  $A = 12$  (see Figure 19).
- The average value is given by

$$\begin{aligned}\frac{1}{A} \iint_R f(x, y) \, dA &= \frac{1}{12} \int_0^4 \int_0^3 \frac{1}{2} xy \, dy \, dx \\ &= \frac{1}{12} \int_0^4 \left[ \frac{1}{4} xy^2 \right]_0^3 dx = \left( \frac{1}{12} \right) \left( \frac{9}{4} \right) \int_0^4 x \, dx \\ &= \frac{3}{16} \left[ \frac{1}{2} x^2 \right]_0^4 = \left( \frac{3}{16} \right) (8) = \frac{3}{2}. \quad \blacksquare\end{aligned}$$

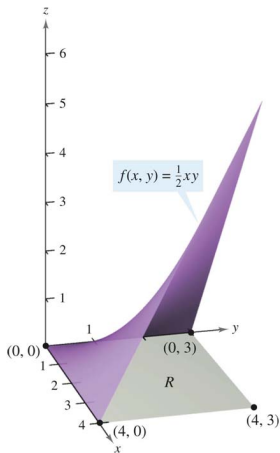


Figure 19: The average value of  $f(x, y) = \frac{1}{2}xy$ .

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- 7 Change of variables: Jacobians



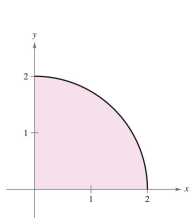
# Double integrals in polar coordinates

- The polar coordinates  $(r, \theta)$  of a point are related to the rectangular coordinates  $(x, y)$  of the point as follows.

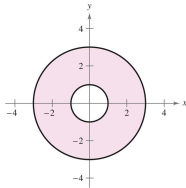
$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x}$$

## Example 1 (Using polar coordinates to describe a region)

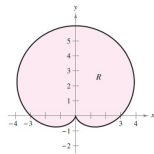
Use polar coordinates to describe each region shown in Figure 20.



(a) A quarter circle with radius 2.



(b) Region  $R$  consists of all points between concentric circles of radii 1 and 3.



(c) Region  $R$  is a cardioid

Figure 20: Using polar coordinates to describe a region.

- a. The region  $R$  is a quarter circle of radius 2. It can be described in polar coordinates as

$$R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq \pi/2\}.$$

- b. The region  $R$  consists of all points between concentric circles of radii 1 and 3. It can be described in polar coordinates as

$$R = \{(r, \theta) : 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}.$$

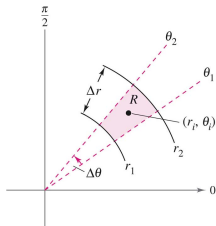
- c. The region  $R$  is a cardioid with  $a = b = 3$ . It can be described in polar coordinates as

$$R = \{(r, \theta) : 0 \leq r \leq 3 + 3 \sin \theta, 0 \leq \theta \leq 2\pi\}.$$

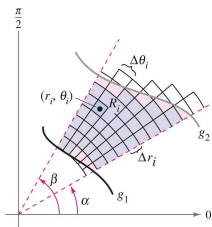
- The regions in Example 1 are special cases of polar sectors

$$R = \{(r, \theta) : r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\} \quad \text{Polar sector}$$

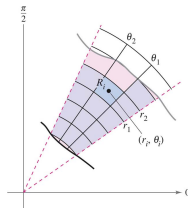
as shown in Figure 21(a).



(a) Polar sector  
 $R = \{(r, \theta) : r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\}.$



(b) Polar grid superimposed over region  $R$ .



(c) The polar sector  $R_i$  is the set of all points  $(r, \theta)$  such that  $r_1 \leq r \leq r_2$  and  $\theta_1 \leq \theta \leq \theta_2$ .

- To define a double integral of a continuous function  $z = f(x, y)$  in polar coordinates, consider a region  $R$  bounded by the graphs of  $r = g_1(\theta)$  and  $r = g_2(\theta)$  and the lines  $\theta = \alpha$  and  $\theta = \beta$ .
- Instead of partitioning  $R$  into small rectangles, use a partition of small polar sectors. On  $R$ , superimpose a polar grid made of rays and circular arcs, as shown in Figure 21(b).
- The polar sectors  $R_i$  lying entirely within  $R$  form an inner polar partition  $\Delta$ , whose norm  $\|\Delta\|$  is the length of the longest diagonal of the  $n$  polar sectors.
- Consider a specific polar sector  $R_i$ , as shown in Figure 21(c).

- It can be shown that the area of  $R_i$  is

$$\begin{aligned}\Delta A_i &= \left( \frac{\Delta \theta_i}{2\pi} \right) (\pi r_2^2 - \pi r_1^2) \\ &= \frac{r_2 + r_1}{2} (r_2 - r_1) \Delta \theta_i = r_i \Delta r_i \Delta \theta_i \quad \text{Area of } R_i\end{aligned}$$

where  $r_i = \frac{r_2 + r_1}{2}$ ,  $\Delta r_i = r_2 - r_1$  and  $\Delta \theta_i = \theta_2 - \theta_1$ . This implies that the volume of the solid of height  $f(r_i \cos \theta_i, r_i \sin \theta_i)$  above  $R_i$  is approximately

$$f(r_i \cos \theta_i, r_i \sin \theta_i) r_i \Delta r_i \Delta \theta_i$$

and you have

$$\iint_R f(x, y) \, dA \approx \sum_{i=1}^n f(r_i \cos \theta_i, r_i \sin \theta_i) r_i \Delta r_i \Delta \theta_i.$$

- The sum on the right can be interpreted as a Riemann sum for  $f(r \cos \theta, r \sin \theta) r$ .

- The region  $R$  corresponds to a horizontally simple region  $S$  in the  $r\theta$ -plane, as shown in Figure 22.

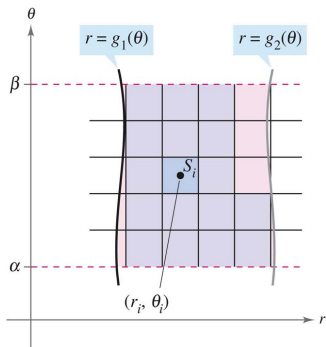


Figure 22:  $S = \{(r, \theta) \mid g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta\}$ .

- The polar sectors  $R_i$  correspond to rectangles  $S_i$ , and the area  $\Delta A_i$  of  $S_i$  is  $\Delta r_i \Delta \theta_i$ .

- So, the right-hand side of the equation corresponds to the double integral

$$\iint_S f(r \cos \theta, r \sin \theta) r \, dA.$$

- From this, you can write

$$\begin{aligned} \iint_R f(x, y) \, dA &= \iint_S f(r \cos \theta, r \sin \theta) r \, dA \\ &= \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta. \end{aligned}$$

- This suggests the following Theorem 14.3.

## Theorem 14.3 (Change of variables to polar form)

Let  $R$  be a plane region consisting of all points  $(x, y) = (r \cos \theta, r \sin \theta)$  satisfying the conditions  $0 \leq g_1(\theta) \leq r \leq g_2(\theta)$ ,  $\alpha \leq \theta \leq \beta$ , where  $0 \leq (\beta - \alpha) \leq 2\pi$ . If  $g_1$  and  $g_2$  are continuous on  $[\alpha, \beta]$  and  $f$  is continuous on  $R$ , then

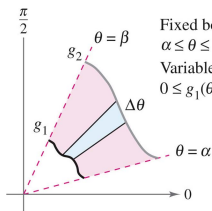
$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

The region  $R$  is restricted to two basic types,

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta \quad r\text{-simple region}$$

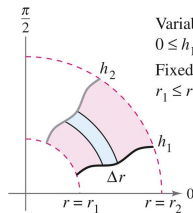
$$\iint_R f(x, y) \, dA = \int_{r_1}^{r_2} \int_{h_1(r)}^{h_2(r)} f(r \cos \theta, r \sin \theta) r \, d\theta \, dr \quad \theta\text{-simple region}$$





(a)  $r$ -simple region.

Fixed bounds for  $\theta$ :  
 $\alpha \leq \theta \leq \beta$   
 Variable bounds for  $r$ :  
 $0 \leq g_1(\theta) \leq r \leq g_2(\theta)$



(b)  $\theta$ -simple region.

Variable bounds for  $\theta$ :  
 $0 \leq h_1(r) \leq \theta \leq h_2(r)$   
 Fixed bounds for  $r$ :  
 $r_1 \leq r \leq r_2$

Figure 23:  $r$ -simple region and  $\theta$ -simple region.

## Example 2 (Evaluating a double polar integral)

Let  $R$  be the annular region lying between the two circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 5$ . Evaluate the integral  $\iint_R (x^2 + y) \, dA$ .

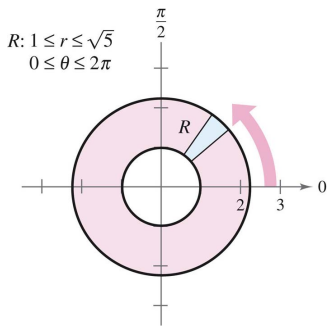


Figure 24:  $r$ -simple region,  $R: 1 \leq r \leq \sqrt{5}, 0 \leq \theta \leq 2\pi$ .

- The polar boundaries are  $1 \leq r \leq \sqrt{5}$  and  $0 \leq \theta \leq 2\pi$  as shown in Figure 24.

- Furthermore,  $x^2 = (r \cos \theta)^2$  and  $y = r \sin \theta$ . So, you have

$$\begin{aligned}\iint_R (x^2 + y) \, dA &= \int_0^{2\pi} \int_1^{\sqrt{5}} (r^2 \cos^2 \theta + r \sin \theta) r \, dr \, d\theta \\&= \int_0^{2\pi} \int_1^{\sqrt{5}} (r^3 \cos^2 \theta + r^2 \sin \theta) \, dr \, d\theta \\&= \int_0^{2\pi} \left[ \frac{r^4}{4} \cos^2 \theta + \frac{r^3}{3} \sin \theta \right]_1^{\sqrt{5}} d\theta \\&= \int_0^{2\pi} \left( 6 \cos^2 \theta + \frac{5\sqrt{5}-1}{3} \sin \theta \right) d\theta \\&= \int_0^{2\pi} \left( 3 + 3 \cos 2\theta + \frac{5\sqrt{5}-1}{3} \sin \theta \right) d\theta \\&= \left[ 3\theta + \frac{3 \sin 2\theta}{2} - \frac{5\sqrt{5}-1}{3} \cos \theta \right]_0^{2\pi} \\&= 6\pi.\end{aligned}$$

### Example 3 (Change of variables to polar coordinates)

Use polar coordinates to find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2} \quad \text{Hemisphere forms upper surface}$$

and below by the circular region  $R$  given by

$$x^2 + y^2 \leq 4 \quad \text{Circular region forms lower surface}$$

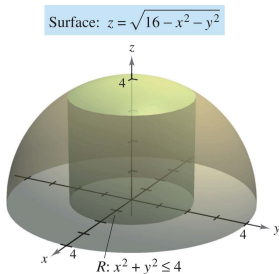


Figure 25: Change of variables to polar coordinates.

- In Figure 25, you can see that  $R$  has the bounds

$$-\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}, \quad -2 \leq y \leq 2$$

and that  $0 \leq z \leq \sqrt{16-x^2-y^2}$ .

- In polar coordinates, the bounds are

$$0 \leq r \leq 2 \quad \text{and} \quad 0 \leq \theta \leq 2\pi$$

with height  $z = \sqrt{16-x^2-y^2} = \sqrt{16-r^2}$ .

- Consequently, the volume  $V$  is given by

$$\begin{aligned} V &= \iint_R f(x, y) \, dA = \int_0^{2\pi} \int_0^2 \sqrt{16-r^2} \, r \, dr \, d\theta \\ &= -\frac{1}{3} \int_0^{2\pi} (16-r^2)^{3/2} \Big|_0^2 \, d\theta \\ &= -\frac{1}{3} \int_0^{2\pi} (24\sqrt{3} - 64) \, d\theta = -\frac{8}{3} \left[ (3\sqrt{3} - 8)\theta \right]_0^{2\pi} \\ &= \frac{16\pi}{3} (8 - 3\sqrt{3}) \approx 46.979. \end{aligned}$$



## Example 4 (Finding areas of polar regions)

Use a double integral to find the area enclosed by the graph of  $r = 3 \cos 3\theta$ .

- Let  $R$  be one petal of the curve shown in Figure 26.
- This region is  $r$ -simple, and the boundaries are as follows.

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6} \quad \text{and} \quad 0 \leq r \leq 3 \cos 3\theta$$

- So, the area of one petal is

$$\begin{aligned} \frac{1}{3} A &= \iint_R dA = \int_{-\pi/6}^{\pi/6} \int_0^{3 \cos 3\theta} r \, dr \, d\theta \\ &= \int_{-\pi/6}^{\pi/6} \left[ \frac{r^2}{2} \right]_0^{3 \cos 3\theta} d\theta = \frac{9}{2} \int_{-\pi/6}^{\pi/6} \cos^2 3\theta \, d\theta \\ &= \frac{9}{4} \int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta) \, d\theta = \frac{9}{4} \left[ \theta + \frac{1}{6} \sin 6\theta \right]_{-\pi/6}^{\pi/6} = \frac{3\pi}{4}. \end{aligned}$$

- So, the total area is  $A = 9\pi/4$ .

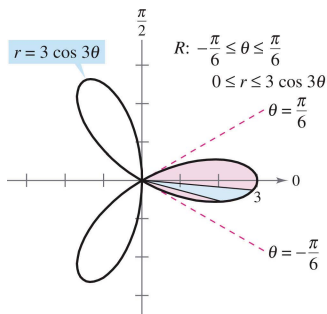


Figure 26: Find area of the polar region: rose curve with three petals.

### Example 5 (Changing the order of integration)

Find the area of the region bounded above by the spiral  $r = \pi/(3\theta)$  and below by the polar axis, between  $r = 1$  and  $r = 2$ .

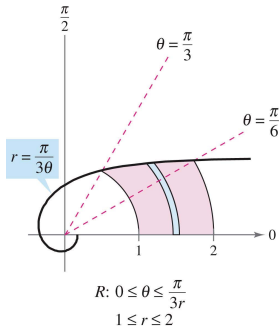


Figure 27:  $\theta$ -simple region.

- The region is shown above. The polar boundaries for the region are

$$1 \leq r \leq 2 \quad \text{and} \quad 0 \leq \theta \leq \frac{\pi}{3}.$$

- So, the area of the region can be evaluated as follows.

$$A = \int_1^2 \int_0^{\pi/(3r)} r \, d\theta \, dr = \int_1^2 r\theta \Big|_0^{\pi/(3r)} \, dr = \int_1^2 \frac{\pi}{3} \, dr = \frac{\pi r}{3} \Big|_1^2 = \frac{\pi}{3}$$



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# Surface area

- You know about the solid region lying between a surface and a closed and bounded region  $R$  in the  $xy$ -plane, as shown in Figure 28(a).

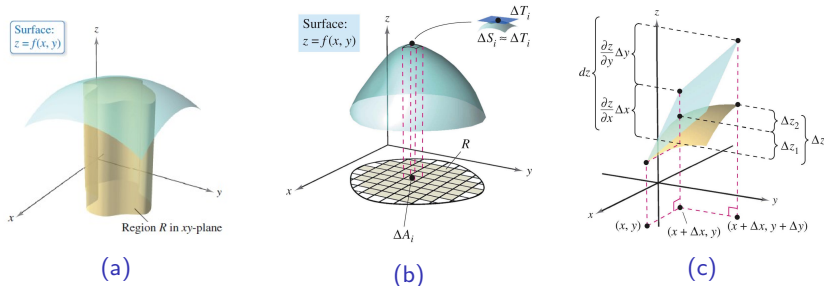


Figure 28: Region  $R$  in  $xy$  plane with surface:  $z = f(x, y)$ .

- For example, you know how to find the extrema of  $f$  on  $R$ .

- In this section, you will learn how to find the upper surface area of the solid. To begin, consider a surface  $S$  given by

$$z = f(x, y) \quad \text{Surface defined over a region } R$$

defined over a region  $R$ .

- Assume that  $R$  is closed and bounded and that  $f$  has continuous first partial derivatives.
- To find the surface area, construct an inner partition of  $R$  consisting of  $n$  rectangles, where the area of the  $i$ th rectangle  $R_i$  is  $\Delta A_i = \Delta x_i \Delta y_i$ , as shown in Figure 28(b).
- In each  $R_i$  let  $(x_i, y_i)$  be the point that is closest to the origin. At the point  $(x_i, y_i, z_i) = (x_i, y_i, f(x_i, y_i))$  on the surface  $S$ , construct a tangent plane  $T_i$ .

- The area of the portion of the tangent plane that lies directly above  $R_i$  is approximately equal to the area of the surface lying directly above  $R_i$ .
- That is,  $\Delta T_i \approx \Delta S_i$ . So, the surface area of  $S$  is given by

$$\sum_{i=1}^n \Delta S_i \approx \sum_{i=1}^n \Delta T_i.$$

- To find the area of the parallelogram  $\Delta T_i$ , note that its sides are given by the vectors

$$\mathbf{u} = \Delta x_i \mathbf{i} + f_x(x_i, y_i) \Delta x_i \mathbf{k} \quad \text{and} \quad \mathbf{v} = \Delta y_i \mathbf{j} + f_y(x_i, y_i) \Delta y_i \mathbf{k}.$$

- The area of  $\Delta T_i$  is given by  $\|\mathbf{u} \times \mathbf{v}\|$ , where

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_i) \Delta x_i \\ 0 & \Delta y_i & f_y(x_i, y_i) \Delta y_i \end{vmatrix} \\ &= -f_x(x_i, y_i) \Delta x_i \Delta y_i \mathbf{i} - f_y(x_i, y_i) \Delta x_i \Delta y_i \mathbf{j} + \Delta x_i \Delta y_i \mathbf{k} \\ &= (-f_x(x_i, y_i) \mathbf{i} - f_y(x_i, y_i) \mathbf{j} + \mathbf{k}) \Delta A_i. \end{aligned}$$

- So, the area of  $\Delta T_i$  is  $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{[f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2 + 1} \Delta A_i$  and

$$\text{Surface area of } S \approx \sum_{i=1}^n \Delta S_i \approx \sum_{i=1}^n \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2} \Delta A_i.$$

### Definition 14.4 (Surface area)

If  $f$  and its partial derivatives are continuous on the closed region  $R$  in the  $xy$ -plane, then the area of the surface  $S$  given by  $z = f(x, y)$  over  $R$  is defined as

$$\text{Surface area} = \iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA.$$

- As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

Length on  $x$ -axis:  $\int_a^b dx$

Arc length in  $xy$ -plane:  $\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2} dx$

Area in  $xy$ -plane:  $\iint_R dA$

Surface area in space:  $\iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$

### Example 1 (The surface area of a plane region)

Find the surface area of the portion of the plane

$$z = 2 - x - y$$

that lies above the circle  $x^2 + y^2 \leq 1$  in the first quadrant, as shown in Figure 29.

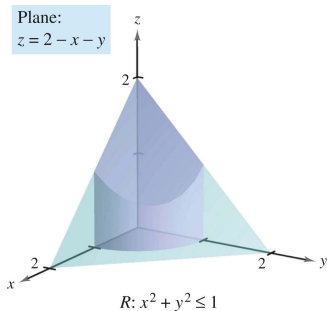


Figure 29: Plane:  $z = 2 - x - y$  and  $R: x^2 + y^2 \leq 1$ .

- Because  $f(x, y) = 2 - x - y$ ,  $f_x(x, y) = -1$  and  $f_y(x, y) = -1$ , the surface area is given by

$$\begin{aligned} S &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA \\ &= \iint_R \sqrt{1 + (-1)^2 + (-1)^2} \, dA = \iint_R \sqrt{3} \, dA = \sqrt{3} \iint_R dA. \end{aligned}$$

- Note that the last integral is simply  $\sqrt{3}$  times the area of the region  $R$ .
- $R$  is a quarter circle of radius 1, with an area of  $\frac{1}{4}\pi(1^2)$  or  $\pi/4$ .
- So, the area of  $S$  is

$$S = \sqrt{3}(\text{area of } R) = \sqrt{3}\left(\frac{\pi}{4}\right) = \frac{\sqrt{3}\pi}{4}.$$

## Example 2 (Finding surface area)

Find the area of the portion of the surface

$$f(x, y) = 1 - x^2 + y$$

that lies above the triangular region with vertices  $(1, 0, 0)$ ,  $(0, -1, 0)$ , and  $(0, 1, 0)$ , as shown in Figure 30(a).



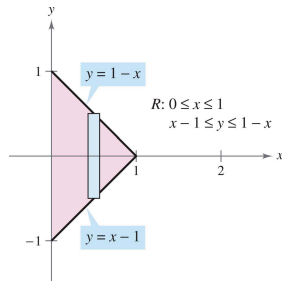
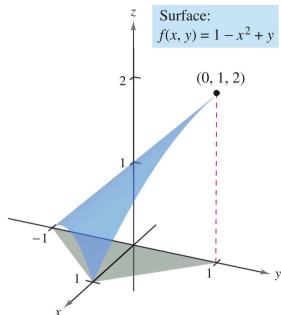


Figure 30: Finding surface area.

- Because  $f_x(x, y) = -2x$  and  $f_y(x, y) = 1$ , you have

$$S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA = \iint_R \sqrt{1 + 4x^2 + 1} \, dA.$$

- In Figure 30(b), you can see that the bounds for  $R$  are  $0 \leq x \leq 1$  and  $x - 1 \leq y \leq 1 - x$ . So, the integral becomes

$$\begin{aligned} S &= \int_0^1 \int_{x-1}^{1-x} \sqrt{2+4x^2} \, dy \, dx = \int_0^1 y \sqrt{2+4x^2} \Big|_{x-1}^{1-x} \, dx \\ &= \int_0^1 \left[ (1-x) \sqrt{2+4x^2} - (x-1) \sqrt{2+4x^2} \right] \, dx \\ &= \int_0^1 \left( 2\sqrt{2+4x^2} - 2x\sqrt{2+4x^2} \right) \, dx \\ &= \left[ x\sqrt{2+4x^2} + \ln(2x + \sqrt{2+4x^2}) - \frac{(2+4x^2)^{3/2}}{6} \right]_0^1 \\ &= \sqrt{6} + \ln(2 + \sqrt{6}) - \sqrt{6} - \ln \sqrt{2} + \frac{1}{3}\sqrt{2} \approx 1.618. \quad \blacksquare \end{aligned}$$



### Example 3 (Change of variables to polar coordinate)

Find the surface area of the paraboloid  $z = 1 + x^2 + y^2$  that lies above the unit circle, as shown in Figure 31.

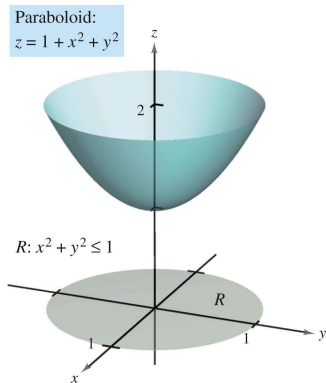


Figure 31: Surface area of the paraboloid  $z = 1 + x^2 + y^2$ .

- Because  $f_x(x, y) = 2x$  and  $f_y(x, y) = 2y$ , you have

$$S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA = \iint_R \sqrt{1 + 4x^2 + 4y^2} \, dA.$$

- You can convert to polar coordinates by letting  $x = r \cos \theta$  and  $y = r \sin \theta$ . Then, because the region  $R$  is bounded by  $0 \leq r \leq 1$  and  $0 \leq \theta \leq 2\pi$ , you have

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \frac{1}{12} (1 + 4r^2)^{3/2} \Big|_0^1 \, d\theta = \int_0^{2\pi} \frac{5\sqrt{5} - 1}{12} \, d\theta \\ &= \frac{5\sqrt{5} - 1}{12} \theta \Big|_0^{2\pi} = \frac{\pi (5\sqrt{5} - 1)}{6} \approx 5.33. \end{aligned}$$



## Example 4 (Change of variables to polar coordinate)

Find the surface area of the portion of the hemisphere

$$f(x, y) = \sqrt{25 - x^2 - y^2}$$

that lies above the region  $R$  bounded by the circle  $x^2 + y^2 \leq 9$ , as shown in Figure 32.

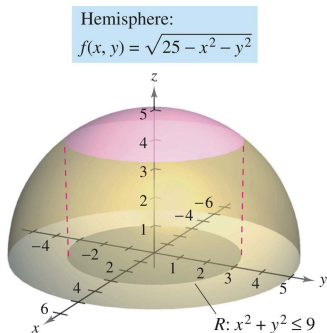


Figure 32: Surface area of the portion of a hemisphere.

- The first partial derivatives of  $f$  are

$$f_x(x, y) = \frac{-x}{\sqrt{25 - x^2 - y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

and, from the formula for surface area, you have

$$\begin{aligned} dS &= \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA \\ &= \sqrt{1 + \left( \frac{-x}{\sqrt{25 - x^2 - y^2}} \right)^2 + \left( \frac{-y}{\sqrt{25 - x^2 - y^2}} \right)^2} dA \\ &= \frac{5}{\sqrt{25 - x^2 - y^2}} dA. \end{aligned}$$

- So, the surface area is

$$S = \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}} dA.$$

- You can convert to polar coordinates by letting  $x = r \cos \theta$  and  $y = r \sin \theta$ . Then, because the region  $R$  is bounded by  $0 \leq r \leq 3$  and  $0 \leq \theta \leq 2\pi$ , you obtain

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25-r^2}} r \, dr \, d\theta \\ &= 5 \int_0^{2\pi} -\sqrt{25-r^2} \Big|_0^3 \, d\theta = 5 \int_0^{2\pi} d\theta = 10\pi. \end{aligned}$$



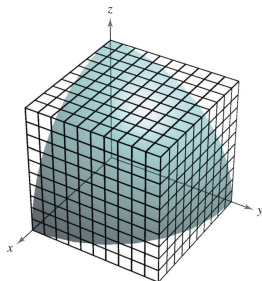


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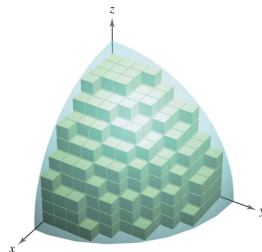
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# Triple integrals

- The procedure used to define a triple integral follows that used for double integrals.
- Consider a function  $f$  of three variables that is continuous over a bounded solid region  $Q$ . Then, encompass  $Q$  with a network of boxes and form the inner partition consisting of all boxes lying entirely within  $Q$ , as shown in Figure 33.



(a) Solid region  $Q$ .



(b) Volume of  
 $Q \approx \sum_{i=1}^n \Delta V_i.$

Figure 33: Volume of solid region  $Q \approx \sum_{i=1}^n \Delta V_i.$

- The volume of the  $i$ th box is

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i. \quad \text{Volume of } i\text{th box}$$

- The norm  $\|\Delta\|$  of the partition is the length of the longest diagonal of the  $n$  boxes in the partition.
- Choose a point  $(x_i, y_i, z_i)$  in each box and form the Riemann sum

$$\sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i.$$

- Taking the limit as  $\|\Delta\| \rightarrow 0$  leads to the following definition.

## Definition 14.5 (Triple integral)

If  $f$  is continuous over a bounded solid region  $Q$ , then the triple integral of  $f$  over  $Q$  is defined as

$$\iiint_Q f(x, y, z) \, dV = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i$$

provided the limit exists. The volume of the solid region  $Q$  is given by

$$\text{Volume of } Q = \iiint_Q dV.$$

Some of the properties of double integrals can be restated in terms of triple integrals.

1.  $\iiint_Q c f(x, y, z) \, dV = c \iiint_Q f(x, y, z) \, dV$
2.  $\iiint_Q [f(x, y, z) \pm g(x, y, z)] \, dV = \iiint_Q f(x, y, z) \, dV \pm \iiint_Q g(x, y, z) \, dV$
3.  $\iiint_Q f(x, y, z) \, dV = \iiint_{Q_1} f(x, y, z) \, dV + \iiint_{Q_2} f(x, y, z) \, dV$

## Theorem 14.4 (Evaluation by iterated integrals)

Let  $f$  be continuous on a solid region  $Q$  defined by

$$a \leq x \leq b, \quad h_1(x) \leq y \leq h_2(x), \quad g_1(x, y) \leq z \leq g_2(x, y)$$

where  $h_1$ ,  $h_2$ ,  $g_1$ , and  $g_2$  are continuous functions. Then,

$$\iiint_Q f(x, y, z) \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) \, dz \, dy \, dx.$$

## Example 1 (Evaluating a triple iterated integral)

Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x (y + 2z) \, dz \, dy \, dx.$$

- For the first integration, hold  $x$  and  $y$  constant and integrate with respect to  $z$ .

$$\begin{aligned}\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z) dz dy dx &= \int_0^2 \int_0^x e^x(yz + z^2) \Big|_0^{x+y} dy dx \\ &= \int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2) dy dx\end{aligned}$$

- For the second integration, hold  $x$  constant and integrate with respect to  $y$ .

$$\begin{aligned}\int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2) dy dx &= \int_0^2 e^x \left( x^2y + \frac{3xy^2}{2} + \frac{2y^3}{3} \right) \Big|_0^x dx \\ &= \frac{19}{6} \int_0^2 x^3 e^x dx\end{aligned}$$

- Finally, integrate with respect to  $x$ .

$$\frac{19}{6} \int_0^2 x^3 e^x dx = \frac{19}{6} e^x(x^3 - 3x^2 + 6x - 6) \Big|_0^2 = 19 \left( \frac{e^2}{3} + 1 \right) \approx 65.797 \blacksquare$$

- To find the limits for a particular order of integration, it is generally advisable first to determine the innermost limits, which may be functions of the outer two variables.
- Then, by projecting the solid  $Q$  onto the coordinate plane of the outer two variables, you can determine their limits of integration by the methods used for double integrals.
- For instance, to evaluate

$$\iiint_Q f(x, y, z) \, dz \, dy \, dx$$

first determine the limits for  $z$ , and then the integral has the form

$$\iint \left[ \int_{g_1(x,y)}^{g_2(x,y)} dz \right] dy \, dx.$$

- By projecting the solid  $Q$  onto the  $xy$ -plane, you can determine the limits for  $x$  and  $y$  as you did for double integrals, as shown in Figure 34.

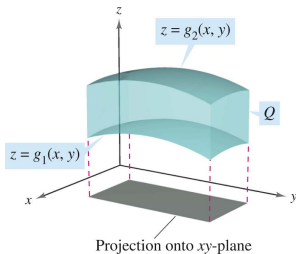


Figure 34: Solid region  $Q$  lies between  $z = g_1(x, y)$  and  $z = g_2(x, y)$ .

### Example 2 (Using a triple integral to find volume)

Find the volume of the ellipsoid given by  $4x^2 + 4y^2 + z^2 = 16$ .

- Because  $x$ ,  $y$ , and  $z$  play similar roles in the equation, the order of integration is probably immaterial, and you can arbitrarily choose  $dz \, dy \, dx$ .
- Moreover, you can simplify the calculation by considering only the portion of the ellipsoid lying in the first octant, as shown in Figure 35(a).



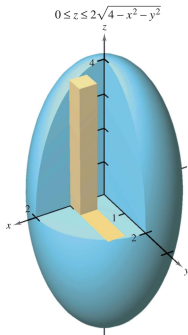
- From the order  $dz \, dy \, dx$ , you first determine the bounds for  $z$ .

$$0 \leq z \leq 2\sqrt{4 - x^2 - y^2}$$

- In Figure 35(b), you can see that the boundaries for  $x$  and  $y$  are  $0 \leq x \leq 2$  and  $0 \leq y \leq \sqrt{4 - x^2}$ , so the volume of the ellipsoid is

$$\begin{aligned} V &= \iiint_Q dV = 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2\sqrt{4-x^2-y^2}} dz \, dy \, dx \\ &= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} z \Big|_0^{2\sqrt{4-x^2-y^2}} dy \, dx \\ &= 16 \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{(4-x^2)-y^2} \, dy \, dx \\ &= 8 \int_0^2 \left[ y\sqrt{4-x^2-y^2} + (4-x^2) \sin^{-1} \left( \frac{y}{\sqrt{4-x^2}} \right) \right]_0^{\sqrt{4-x^2}} dx \\ &= 8 \int_0^2 [0 + (4-x^2) \sin^{-1}(1) - 0 - 0] \, dx \\ &= 8 \int_0^2 (4-x^2) \left( \frac{\pi}{2} \right) dx = 4\pi \left[ 4x - \frac{x^3}{3} \right]_0^2 = \frac{64\pi}{3}. \end{aligned}$$





Ellipsoid:  $4x^2 + 4y^2 + z^2 = 16$

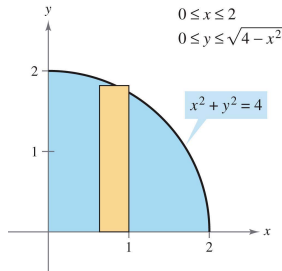


Figure 35: Using a triple integral to find volume.

### Example 3 (Changing the order of integration)

Evaluate

$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2) \, dz \, dy \, dx.$$

- Note that after one integration in the given order, you would encounter the integral  $2 \int \sin(y^2) dy$ , which is not an elementary function.
- To avoid this problem, change the order of integration to  $dz dx dy$ , so that  $y$  is the outer variable.
- From Figure 36, you can see that the solid region  $Q$  is given by

$$0 \leq x \leq \sqrt{\frac{\pi}{2}}, \quad x \leq y \leq \sqrt{\frac{\pi}{2}}, \quad 1 \leq z \leq 3$$

and the projection of  $Q$  in the  $xy$ -plane yields the bounds

$$0 \leq y \leq \sqrt{\frac{\pi}{2}}, \quad \text{and} \quad 0 \leq x \leq y.$$

- So, evaluating the triple integral using the order  $dz \, dx \, dy$  produces

$$\begin{aligned}\int_0^{\sqrt{\pi/2}} \int_0^y \int_1^3 \sin(y^2) \, dz \, dx \, dy &= \int_0^{\sqrt{\pi/2}} \int_0^y z \sin(y^2) \Big|_1^3 \, dx \, dy \\&= 2 \int_0^{\sqrt{\pi/2}} \int_0^y \sin(y^2) \, dx \, dy \\&= 2 \int_0^{\sqrt{\pi/2}} x \sin(y^2) \Big|_0^y \, dy \\&= 2 \int_0^{\sqrt{\pi/2}} y \sin(y^2) \, dy \\&= -\cos(y^2) \Big|_0^{\sqrt{\pi/2}} = 1. \quad \blacksquare\end{aligned}$$

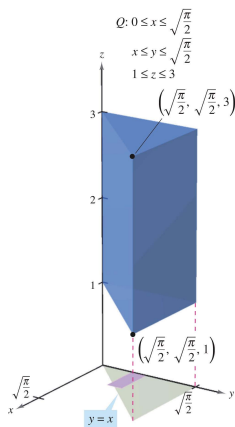


Figure 36: Changing the order of integration.

## Example 4 (Determining the limits of integration)

Set up a triple integral for the volume of each solid region.

- a. The region in the first octant bounded above by the cylinder  $z = 1 - y^2$  and lying between the vertical planes  $x + y = 1$  and  $x + y = 3$ .
  - b. The upper hemisphere given by  $z = \sqrt{1 - x^2 - y^2}$ .
  - c. The region bounded below by the paraboloid  $z = x^2 + y^2$  and above by the sphere  $x^2 + y^2 + z^2 = 6$ .
- a. In Figure 37(a), note that the solid is bounded below by the  $xy$ -plane ( $z = 0$ ) and above by the cylinder  $z = 1 - y^2$ . So,

$$0 \leq z \leq 1 - y^2. \quad \text{Bounds for } z$$

Projecting the region onto the  $xy$ -plane produces a parallelogram. Because two sides of the parallelogram are parallel to the  $x$ -axis, you have the following bounds:

$$1 - y \leq x \leq 3 - y \quad \text{and} \quad 0 \leq y \leq 1.$$

So, the volume of the region is given by

$$V = \iiint_Q dV = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz \, dx \, dy.$$

**b.** For the upper hemisphere given by  $z = \sqrt{1 - x^2 - y^2}$ , you have

$$0 \leq z \leq \sqrt{1 - x^2 - y^2}. \quad \text{Bounds for } z$$

In Figure 37(b), note that the projection of the hemisphere onto the  $xy$ -plane is the circle given by  $x^2 + y^2 = 1$ , and you can use either order  $dx \, dy$  or  $dy \, dx$ . Choosing the first produces

$$-\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2} \quad \text{and} \quad -1 \leq y \leq 1$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz \, dx \, dy.$$



- c. For the region bounded below by the paraboloid  $z = x^2 + y^2$  and above by the sphere  $x^2 + y^2 + z^2 = 6$ , you have

$$x^2 + y^2 \leq z \leq \sqrt{6 - x^2 - y^2}. \quad \text{Bounds for } z$$

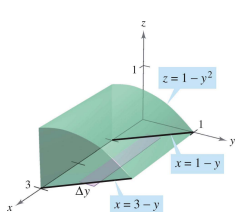
The sphere and the paraboloid intersect at  $z = 2$ . Moreover, you can see in Figure 37(c) that the projection of the solid region onto the  $xy$ -plane is the circle given by  $x^2 + y^2 = 2$ . Using the order  $dy \, dx$  produces

$$-\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2} \quad \text{and} \quad -\sqrt{2} \leq x \leq \sqrt{2}$$

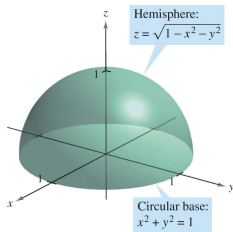
which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz \, dy \, dx.$$

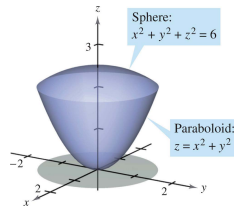




$$Q: \begin{aligned} 0 &\leq z \leq 1 - y^2 \\ 1 - y &\leq x \leq 3 - y \\ 0 &\leq y \leq 1 \end{aligned}$$



$$Q: \begin{aligned} 0 &\leq z \leq \sqrt{1 - x^2 - y^2} \\ -\sqrt{1 - y^2} &\leq x \leq \sqrt{1 - y^2} \\ -1 &\leq y \leq 1 \end{aligned}$$



$$Q: \begin{aligned} x^2 + y^2 &\leq z \leq \sqrt{6 - x^2 - y^2} \\ -\sqrt{2 - x^2} &\leq y \leq \sqrt{2 - x^2} \\ -\sqrt{2} &\leq x \leq \sqrt{2} \end{aligned}$$

Figure 37: Determining the limits of integration.

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# Triple integrals in cylindrical coordinates

- The rectangular conversion equations for cylindrical coordinates are

$$x = r \cos \theta \qquad y = r \sin \theta \qquad z = z.$$

- In this coordinate system, the simplest solid region is a cylindrical block determined by

$$r_1 \leq r \leq r_2, \quad \theta_1 \leq \theta \leq \theta_2, \quad z_1 \leq z \leq z_2$$

as shown in Figure 38.

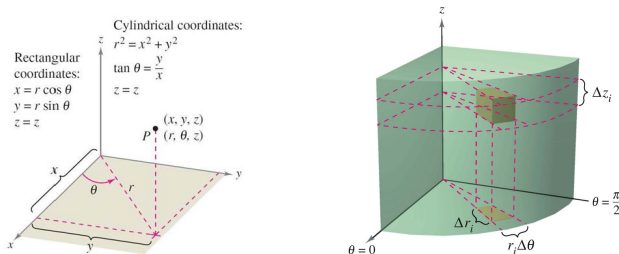


Figure 38: Volume of cylindrical block:  $\Delta V_i = r_i \Delta r_i \Delta \theta_i \Delta z_i$ .

- To obtain the cylindrical coordinate form of a triple integral, suppose that  $Q$  is a solid region whose projection  $R$  onto the  $xy$ -plane can be described in polar coordinates.
- That is,

$$Q = \{(x, y, z) : (x, y) \text{ is in } R, h_1(x, y) \leq z \leq h_2(x, y)\}$$

and

$$R = \{(r, \theta) : \theta_1 \leq \theta \leq \theta_2, g_1(\theta) \leq r \leq g_2(\theta)\}.$$

- If  $f$  is a continuous function on the solid  $Q$ , you can write the triple integral of  $f$  over  $Q$  as

$$\iiint_Q f(x, y, z) \, dV = \iint_R \left[ \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) \, dz \right] \, dA$$

where the double integral over  $R$  is evaluated in polar coordinates.

- That is,  $R$  is a plane region that is either  $r$ -simple or  $\theta$ -simple. If  $R$  is  $r$ -simple, the iterated form of the triple integral in cylindrical form is triple integral in cylindrical coordinates

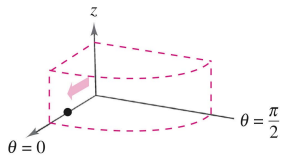
$$\begin{aligned} & \iiint_Q f(x, y, z) \, dV \\ &= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r \cos \theta, r \sin \theta)}^{h_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta. \end{aligned}$$

- To visualize a particular order of integration, it helps to view the iterated integral in terms of three sweeping motions—each adding another dimension to the solid.
- For instance, in the order

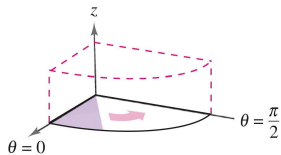
$$dr \, d\theta \, dz$$

the first integration occurs in the  $r$ -direction as a point sweeps out a ray.

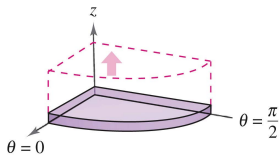
- Then, as  $\theta$  increases, the line sweeps out a sector. Finally, as  $z$  increases, the sector sweeps out a solid wedge, as shown in Figure 39.



(a) Integrate with respect to  $r$ .



(b) Integrate with respect to  $\theta$ .



(c) Integrate with respect to  $z$ .

Figure 39: Triple integrals in cylindrical coordinates.

## Example 1 (Finding volume in cylindrical coordinates)

Find the volume of the solid region  $Q$  cut from the sphere  $x^2 + y^2 + z^2 = 4$  by the cylinder  $r = 2 \sin \theta$ , as shown in Figure 40.

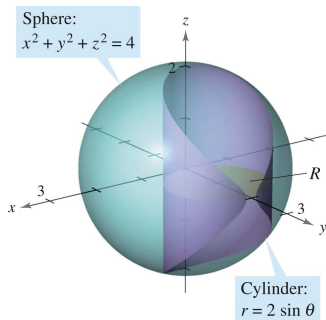


Figure 40: Sphere:  $x^2 + y^2 + z^2 = 4$  and cylinder:  $r = 2 \sin \theta$ .



- Because  $x^2 + y^2 + z^2 = r^2 + z^2 = 4$ , the bounds on  $z$  are

$$-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}.$$

- Let  $R$  be the circular projection of the solid onto the  $r\theta$ -plane.
- Then the bounds on  $R$  are  $0 \leq r \leq 2 \sin \theta$  and  $0 \leq \theta \leq \pi$ . So, the volume of  $Q$  is

$$\begin{aligned} V &= \int_0^\pi \int_0^{2 \sin \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta \\ &= 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} 2r\sqrt{4-r^2} \, dr \, d\theta \\ &= 2 \int_0^{\pi/2} \left. -\frac{2}{3}(4-r^2)^{3/2} \right|_0^{2 \sin \theta} d\theta = \frac{4}{3} \int_0^{\pi/2} (8-8\cos^3 \theta) \, d\theta \\ &= \frac{32}{3} \int_0^{\pi/2} [1 - (\cos \theta)(1 - \sin^2 \theta)] \, d\theta \\ &= \frac{32}{3} \left[ \theta - \sin \theta + \frac{\sin^3 \theta}{3} \right]_0^{\pi/2} = \frac{16}{9}(3\pi - 4) \approx 9.644. \end{aligned}$$



# Triple integrals in spherical coordinates

- The rectangular conversion equations for spherical coordinates are

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi.$$

- In this coordinate system, the simplest region is a spherical block determined by

$$\{(\rho, \theta, \phi) : \rho_1 \leq \rho \leq \rho_2, \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2\}$$

where  $\rho_1 \geq 0$ ,  $\theta_2 - \theta_1 \leq 2\pi$ , and  $0 \leq \phi_1 \leq \phi_2 \leq \pi$ , as below.

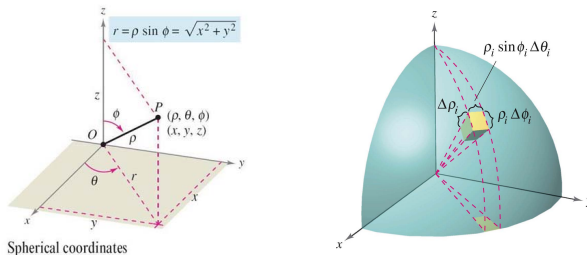


Figure 41: Spherical block:  $\Delta V_i \approx \rho_i^2 \sin \phi_i \Delta \rho_i \Delta \phi_i \Delta \theta_i$ .

- If  $(\rho, \theta, \phi)$  is a point in the interior of such a block, then the volume of the block can be approximated by

$$\Delta V \approx \rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta.$$

- The volume of this spherical block can be determined as follows. One side is length  $\Delta \rho$ .
- Another side is  $\rho \Delta \phi$ . Finally, the third side is given by the length of an arc of angle  $\Delta \theta$  in a circle of radius  $\rho \sin \phi$ . Thus:

$$\Delta V \approx (\Delta \rho)(\rho \Delta \phi)(\rho \sin \phi \Delta \theta) = \rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta. \quad \blacksquare$$

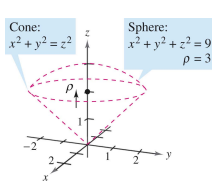
- Using the usual process involving an inner partition, summation, and a limit, you can develop the following version of a triple integral in spherical coordinates for a continuous function  $f$  defined on the solid region  $Q$ . triple integral in spherical coordinates

$$\begin{aligned} & \iiint_Q f(x, y, z) \, dV \\ &= \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

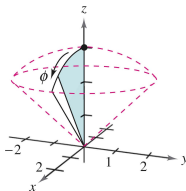
- Triple integrals in spherical coordinates are evaluated with iterated integrals. You can visualize a particular order of integration by viewing the iterated integral in terms of three sweeping motions—each adding another dimension to the solid.
- For instance, the iterated integral

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

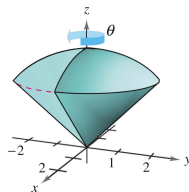
is illustrated in Figure 42.



(a)  $\rho$  varies from 0 to 3 with  $\phi$  and  $\theta$  held constant.



(b)  $\phi$  varies from 0 to  $\pi/4$  with  $\theta$  held constant.



(c)  $\theta$  varies from 0 to  $2\pi$ .

Figure 42: Cone:  $x^2 + y^2 = z^2$  and sphere:  $x^2 + y^2 + z^2 = 9$ ,  $\rho = 3$ .

## Example 2 (Finding volume in spherical coordinates)

Find the volume of the solid region  $Q$  bounded below by the upper nappe of the cone  $z^2 = x^2 + y^2$  and above by the sphere  $x^2 + y^2 + z^2 = 9$ , as shown in Figure 43.

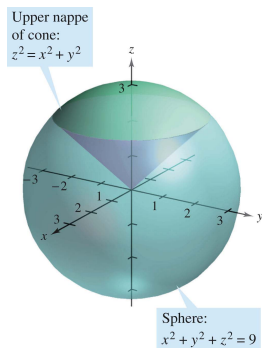


Figure 43: Cone:  $x^2 + y^2 = z^2$  and sphere:  $x^2 + y^2 + z^2 = 9$ ,  $\rho = 3$ .

- In spherical coordinates, the equation of the sphere is

$$\rho^2 = x^2 + y^2 + z^2 = 9 \implies \rho = 3.$$

- Furthermore, the sphere and cone intersect when

$$(x^2 + y^2) + z^2 = (z^2) + z^2 = 9 \implies z = \frac{3}{\sqrt{2}}$$

and, because  $z = \rho \cos \phi$ , it follows that

$$\left(\frac{3}{\sqrt{2}}\right) \left(\frac{1}{3}\right) = \cos \phi \implies \phi = \frac{\pi}{4}.$$

- Consequently, you can use the integration order  $d\rho d\phi d\theta$ , where  $0 \leq \rho \leq 3$ ,  $0 \leq \phi \leq \pi/4$ , and  $0 \leq \theta \leq 2\pi$ . The volume is

$$\begin{aligned} V &= \iiint_Q dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} 9 \sin \phi d\phi d\theta = 9 \int_0^{2\pi} -\cos \phi \Big|_0^{\pi/4} d\theta \\ &= 9 \int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right) d\theta = 9\pi(2 - \sqrt{2}) \approx 16.563. \end{aligned}$$



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- 2 Double integrals and volume
- 3 Change of variables: Polar coordinates
- 4 Surface area
- 5 Triple integrals and applications
- 6 Triple integrals in cylindrical and spherical coordinates
- 7 Change of variables: Jacobians**

# Jacobians

- For the single integral

$$\int_a^b f(x) dx$$

you can change variables by letting  $x = g(u)$ , so that  $dx = g'(u) du$ , and obtain

$$\int_a^b f(x) dx = \int_c^d f(g(u))g'(u) du$$

where  $a = g(c)$  and  $b = g(d)$ .

- The change of variables process introduces an additional factor  $g'(u)$  into the integrand. This also occurs in the case of double integrals

$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \underbrace{\left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right|}_{\text{Jacobian}} du dv$$

where the change of variables  $x = g(u, v)$  and  $y = h(u, v)$  introduces a factor called the Jacobian of  $x$  and  $y$  with respect to  $u$  and  $v$ .



- In defining the Jacobian, it is convenient to use the following determinant notation.

### Definition 14.6 (Jacobian)

If  $x = g(u, v)$  and  $y = h(u, v)$ , then the Jacobian of  $x$  and  $y$  with respect to  $u$  and  $v$ , denoted by  $\partial(x, y)/\partial(u, v)$ , is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}.$$

### Example 1 (The Jacobian for rectangular-to-polar conversion)

Find the Jacobian for the change of variables defined by

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

- From the definition of the Jacobian, you obtain

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r. \quad \blacksquare$$

- Example 1 points out that the change of variables from rectangular to polar coordinates for a double integral can be written as

$$\begin{aligned} \iint_R f(x, y) \, dA &= \iint_S f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \\ &= \iint_S f(r \cos \theta, r \sin \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr \, d\theta \end{aligned}$$

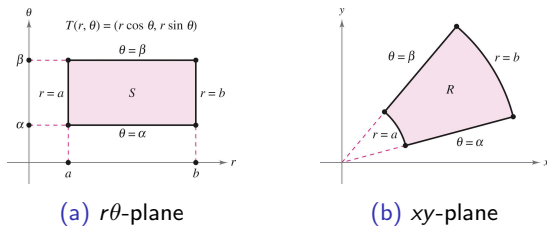
where  $S$  is the region in the  $r\theta$ -plane that corresponds to the region  $R$  in the  $xy$ -plane, as shown in Figure 44.

- In general, a change of variables is given by a one-to-one transformation  $T$  from a region  $S$  in the  $uv$ -plane to a region  $R$  in the  $xy$ -plane, to be given by

$$T(u, v) = (x, y) = (g(u, v), h(u, v))$$

where  $g$  and  $h$  have continuous first partial derivatives in the region  $S$ .

- Note that the point  $(u, v)$  lies in  $S$  and the point  $(x, y)$  lies in  $R$ .



**Figure 44:**  $S$  is the region in the  $r\theta$ -plane that corresponds to the region  $R$  in the  $xy$ -plane where  $\alpha \leq \theta \leq \beta$  and  $a \leq r \leq b$ .

## Example 2 (Finding a change of variables to simplify a region)

Let  $R$  be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in Figure 45. Find a transformation  $T$  from a region  $S$  to  $R$  such that  $S$  is a rectangular region (with sides parallel to the  $u$ - or  $v$ -axis).

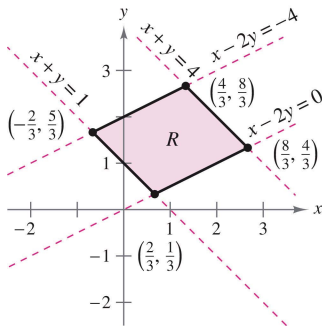


Figure 45: Region  $R$  in the  $xy$ -plane.

- To begin, let  $u = x + y$  and  $v = x - 2y$ . Solving this system of equations for  $x$  and  $y$  produces  $T(u, v) = (x, y)$ , where

$$x = \frac{1}{3}(2u + v) \quad \text{and} \quad y = \frac{1}{3}(u - v).$$

- The four boundaries for  $R$  in the  $xy$ -plane give rise to the following bounds for  $S$  in the  $uv$ -plane

Bounds in the  $xy$ -plane

$$x + y = 1$$

$\implies$

$$x + y = 4$$

$\implies$

$$x - 2y = 0$$

$\implies$

$$x - 2y = -4$$

$\implies$

Bounds in the  $uv$ -plane

$$u = 1$$

$$u = 4$$

$$v = 0$$

$$v = -4$$

- The region  $S$  is shown in Figure 46. Note that the transformation

$$T(u, v) = (x, y) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \left( \frac{1}{3}[2u + v], \frac{1}{3}[u - v] \right)$$

maps the vertices of the region  $S$  onto the vertices of the region  $R$ .

- For instance,

$$T(1, 0) = \left( \frac{1}{3}[2(1) + 0], \frac{1}{3}[1 - 0] \right) = \left( \frac{2}{3}, \frac{1}{3} \right)$$

$$T(4, 0) = \left( \frac{1}{3}[2(4) + 0], \frac{1}{3}[4 - 0] \right) = \left( \frac{8}{3}, \frac{4}{3} \right)$$

$$T(4, -4) = \left( \frac{1}{3}[2(4) - 4], \frac{1}{3}[4 - (-4)] \right) = \left( \frac{4}{3}, \frac{8}{3} \right)$$

$$T(1, -4) = \left( \frac{1}{3}[2(1) - 4], \frac{1}{3}[1 - (-4)] \right) = \left( -\frac{2}{3}, \frac{5}{3} \right).$$

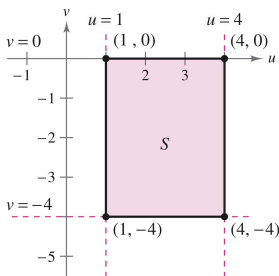


Figure 46: Region  $S$  in the  $uv$ -plane.

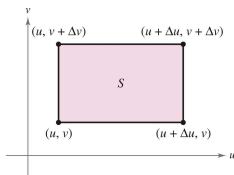
# Change of variables for double integrals

## Theorem 14.5 (Change of variables for double integrals)

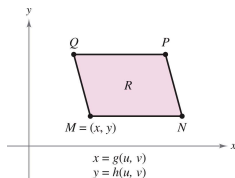
*Let  $R$  be a vertically or horizontally simple region in the  $xy$ -plane, and let  $S$  be a vertically or horizontally simple region in the  $uv$ -plane. Let  $T$  from  $S$  to  $R$  be given by  $T(u, v) = (x, y) = (g(u, v), h(u, v))$ , where  $g$  and  $h$  have continuous first partial derivatives. Assume that  $T$  is one-to-one except possibly on the boundary of  $S$ . If  $f$  is continuous on  $R$ , and  $\partial(x, y)/\partial(u, v)$  is nonzero on  $S$ , then*

$$\iint_R f(x, y) \, dx \, dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$

- Consider the case in which  $S$  is a rectangular region in the  $uv$ -plane with vertices  $(u, v)$ ,  $(u + \Delta u, v)$ ,  $(u + \Delta u, v + \Delta v)$ , and  $(u, v + \Delta v)$ , as shown in Figure 47(a). The images of these vertices in the  $xy$ -plane are shown in Figure 47(b).



(a) Area of  $S = \Delta u \Delta v$ ,  $\Delta u > 0$ ,  $\Delta v > 0$ .



(b) The vertices in the  $xy$ -plane are  $M(g(u, v), h(u, v))$ ,  $N(g(u + \Delta u, v), h(u + \Delta u, v))$ ,  $P(g(u + \Delta u, v + \Delta v), h(u + \Delta u, v + \Delta v))$ , and  $Q(g(u, v + \Delta v), h(u, v + \Delta v))$ .

Figure 47: Change of variables for double integrals.



- If  $\Delta u$  and  $\Delta v$  are small, the continuity of  $g$  and  $h$  implies that  $R$  is approximately a parallelogram determined by the vectors  $\overrightarrow{MN}$  and  $\overrightarrow{MQ}$ . So, the area of  $R$  is

$$\Delta A \approx \left\| \overrightarrow{MN} \times \overrightarrow{MQ} \right\|.$$

- Moreover, for small  $\Delta u$  and  $\Delta v$ , the partial derivatives of  $g$  and  $h$  with respect to  $u$  can be approximated by

$$g_u(u, v) = \frac{g(u + \Delta u, v) - g(u, v)}{\Delta u} \quad \text{and} \quad h_u(u, v) = \frac{h(u + \Delta u, v) - h(u, v)}{\Delta u}.$$

- Consequently,

$$\begin{aligned} \overrightarrow{MN} &= [g(u + \Delta u, v) - g(u, v)]\mathbf{i} + [h(u + \Delta u, v) - h(u, v)]\mathbf{j} \\ &\approx [g_u(u, v)\Delta u]\mathbf{i} + [h_u(u, v)\Delta u]\mathbf{j} = \frac{\partial x}{\partial u}\Delta u\mathbf{i} + \frac{\partial y}{\partial u}\Delta u\mathbf{j}. \end{aligned}$$

- Similarly, you can approximate  $\overrightarrow{MQ}$  by  $\frac{\partial x}{\partial v}\Delta v\mathbf{i} + \frac{\partial y}{\partial v}\Delta v\mathbf{j}$ , which implies that

$$\overrightarrow{MN} \times \overrightarrow{MQ} \approx \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v \mathbf{k}.$$

- It follows that, in Jacobian notation,

$$\Delta A \approx \left\| \overrightarrow{MN} \times \overrightarrow{MQ} \right\| \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v.$$

- Because this approximation improves as  $\Delta u$  and  $\Delta v$  approach 0, the limiting case can be written as

$$dA \approx \left\| \overrightarrow{MN} \times \overrightarrow{MQ} \right\| \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

- So,

$$\iint_R f(x, y) dx dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$



Views of linear transformation (Taylor Theorem):

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} g(u + \Delta u) - g(u) \\ h(v + \Delta v) - h(v) \end{bmatrix} \approx \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

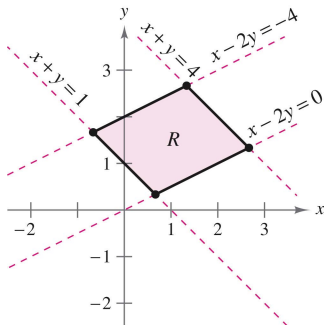
### Example 3 (Using a change of variables to simplify a region)

Let  $R$  be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in below. Evaluate the double integral

$$\iint_R 3xy \, dA.$$



- You can use the following change of variables.

$$x = \frac{1}{3}(2u + v) \quad \text{and} \quad y = \frac{1}{3}(u - v), \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

- The partial derivatives of  $x$  and  $y$  are

$$\frac{\partial x}{\partial u} = \frac{2}{3}, \quad \frac{\partial x}{\partial v} = \frac{1}{3}, \quad \frac{\partial y}{\partial u} = \frac{1}{3}, \quad \text{and} \quad \frac{\partial y}{\partial v} = -\frac{1}{3}$$

which implies that the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{1}{3}.$$

- So, by Theorem 14.5, you obtain

$$\begin{aligned}\iint_R 3xy \, dA &= \iint_S 3 \left[ \frac{1}{3}(2u + v) \frac{1}{3}(u - v) \right] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dv \, du \\&= \int_1^4 \int_{-4}^0 \frac{1}{9}(2u^2 - uv - v^2) dv \, du \\&= \frac{1}{9} \int_1^4 \left[ 2u^2 v - \frac{uv^2}{2} - \frac{v^3}{3} \right]_{-4}^0 du \\&= \frac{1}{9} \int_1^4 \left[ 8u^2 + 8u - \frac{64}{3} \right] du \\&= \frac{1}{9} \left[ \frac{8u^3}{3} + 4u^2 - \frac{64}{3}u \right]_1^4 = \frac{164}{9}.\end{aligned}$$

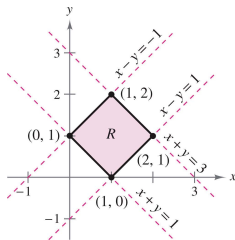


### Example 4 (Using a change of variables to simplify an integrand)

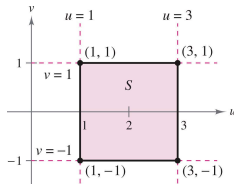
Let  $R$  be the region bounded by the square with vertices  $(0, 1)$ ,  $(1, 2)$ ,  $(2, 1)$ , and  $(1, 0)$ . Evaluate the integral

$$\iint_R (x + y)^2 \sin^2(x - y) \, dA.$$

- Note that the sides of  $R$  lie on the lines  $x + y = 1$ ,  $x - y = 1$ ,  $x + y = 3$ , and  $x - y = -1$ , as shown in Figure 49(a).



(a) Region  $R$  in the  $xy$ -plane.



(b) Region  $S$  in the  $uv$ -plane.

Figure 49: Using a change of variables to simplify an integrand. ▶

- Letting  $u = x + y$  and  $v = x - y$ , you can determine the bounds for region  $S$  in the  $uv$ -plane to be

$$1 \leq u \leq 3 \quad \text{and} \quad -1 \leq v \leq 1$$

as shown in Figure 49(b).

- Solving for  $x$  and  $y$  in terms of  $u$  and  $v$  produces

$$x = \frac{1}{2}(u + v) \quad \text{and} \quad y = \frac{1}{2}(u - v), \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

- The partial derivatives of  $x$  and  $y$  are

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{2}, \quad \text{and} \quad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

which implies that the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

- By Theorem 14.5, it follows that

$$\begin{aligned}\iint_R (x+y)^2 \sin^2(x-y) \, dA &= \int_{-1}^1 \int_1^3 u^2 \sin^2 v \left(\frac{1}{2}\right) \, du \, dv \\&= \frac{1}{2} \int_{-1}^1 (\sin^2 v) \left. \frac{u^3}{3} \right|_1^3 \, dv \\&= \frac{13}{3} \int_{-1}^1 \sin^2 v \, dv \\&= \frac{13}{6} \int_{-1}^1 (1 - \cos 2v) \, dv \\&= \frac{13}{6} \left[ v - \frac{1}{2} \sin 2v \right]_{-1}^1 \\&= \frac{13}{6} \left[ 2 - \frac{1}{2} \sin 2 + \frac{1}{2} \sin(-2) \right] \\&= \frac{13}{6} (2 - \sin 2) = 2.363. \quad \blacksquare\end{aligned}$$