

1. (20%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)}{3n+4}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$

(c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+1}$

(d) $\sum_{n=3}^{\infty} \frac{(-1)^n}{n(\ln n)[\ln(\ln n)]^2}$

(e) $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)$

Theorem 9.1 (Limit of a sequence)

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n , then

$$\lim_{n \rightarrow \infty} a_n = L.$$

Definition 9.4 (Convergent and divergent series)

For the infinite series $\sum_{n=1}^{\infty} a_n$ the n th partial sum is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S , then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

$$S = a_1 + a_2 + \cdots + a_n + \cdots \quad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series diverges.

Definition 9.5 (Absolute and conditional convergence)

- ① $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
- ② $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series ($r \neq 0$)	$\sum_{n=0}^{\infty} ar^n$	$ r < 1$	$ r \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series ($a_n > 0$)	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\lim_{n \rightarrow \infty} a_n = 0$		Remainder: $ R_N \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x) dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } > 1$ or $= \infty$	Test is inconclusive when $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1$ or $= \infty$	Test is inconclusive when $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Ans:

(a) Diverges because of $\lim_{n \rightarrow \infty} \frac{(-1)^n(2n-1)}{3n+4} = \frac{2}{3} \lim_{n \rightarrow \infty} (-1)^n \neq 0$ (by the n-th term test)

(b) $\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n^3}}}{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}} = 2$, since $\frac{1}{\sqrt{n^3}}$ is a p-series with $p > 1$ which means that

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$ is convergent by the limit comparison test. Therefore,

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$ is absolute convergent.

(c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$. The given series converges by the alternating series

test since $\frac{1}{(n+1)+1} < \frac{1}{n+1}$ and $\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$, but $\sum_{n=1}^{\infty} \left| \frac{\cos(n\pi)}{n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n+1}$

diverges by a limit comparison test to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Therefore, the series converges conditionally.

(d) It is absolute convergent because of $\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n(\ln n)[\ln(\ln n)]^2} \right| =$

$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^2}$. Let $f(x) = \frac{1}{x(\ln x)[\ln(\ln x)]^2}$, since $\frac{1}{(x+1)(\ln(x+1))[\ln(\ln(x+1))]^2} <$

$\frac{1}{(x)(\ln(x))[\ln(\ln(x))]^2}$. $f(x)$ is positive, continuous and decreasing.

And $\int_3^{\infty} \frac{1}{x(\ln x)[\ln(\ln x)]^2} dx = \lim_{b \rightarrow \infty} \frac{-1}{\ln(\ln x)} \Big|_3^b < \infty$ (by integral test, it is convergent)

(e) $n^{\frac{1}{n}} - 1 = e^{\frac{\ln n}{n}} - 1 = \left[1 + \frac{\ln n}{n} + \frac{1}{2} \left(\frac{\ln n}{n} \right)^2 + \dots \right] - 1 = \frac{\ln n}{n} + \frac{1}{2} \left(\frac{\ln n}{n} \right)^2 + \dots >$

$\frac{\ln n}{n} > \frac{1}{n}$ ($n \geq 3$). By the direct comparison test, the series diverges.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function

Interval of Convergence

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \dots + (-1)^n (x-1)^n + \dots$$

$$0 < x < 2$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^n x^n + \dots$$

$$-1 < x < 1$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots + \frac{(-1)^{n-1} (x-1)^n}{n} + \dots$$

$$0 < x \leq 2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$$

$$-\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$-\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$-\infty < x < \infty$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$$

$$-1 \leq x \leq 1$$

$$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots + \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} + \dots$$

$$-1 \leq x \leq 1$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \dots + \frac{k(k-1) \dots (k-n+1)x^n}{n!} + \dots$$

$$-1 < x < 1^*$$

* The convergence at $x = \pm 1$ depends on the value of k .

Theorem 9.19 (Taylor's Theorem)

If a function f is differentiable through order $n + 1$ in an interval I containing c , then, for each x in I , there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

Definition 9.7 (Power series)

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a power series. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + a_3 (x - c)^3 + \cdots + a_n (x - c)^n + \cdots$$

is called a power series centered at c , where c is a constant.

Theorem 9.20 (Convergence of a power series)

For a power series centered at c , precisely one of the following is true.

1. The series converges only at c .
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x .

The number R is the radius of convergence. If the series converges only at c , the radius of convergence is $R = 0$, and if the series converges for all x , the radius of convergence is $R = \infty$. The set of all values of x for which it converges is the interval of convergence of the power series.

Theorem 9.21 (Properties of functions defined by power series)

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + a_3 (x - c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval $(c - R, c + R)$, f is differentiable (and therefore continuous).

Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} = a_1 + 2a_2 (x - c) + 3a_3 (x - c)^2 + \cdots$

2.

$$\int f(x) dx = C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n+1} = C + a_0 (x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + \cdots$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series.

The interval of convergence, however, may differ as a result of the behavior at the endpoints.

2. (8%) Consider the function.

$$f(x) = \frac{1}{2x-1}, x \neq \frac{1}{2}$$

(a) Find the power series expansion of $p(x)$ of f expand at the point $\frac{1}{3}$ and determine its interval of convergence.

(b) Write $p(x) = \sum_{n=0}^{\infty} a_n(x - \frac{1}{3})^n$. Is $\sum_{n=0}^{\infty} a_n(\frac{2}{3})^n = f(1) = 1$? and why?

Ans:

$$(a) f(x) = \frac{-1}{1-2x} = \frac{-1}{\frac{1}{3}-2(x-\frac{1}{3})} = \frac{-3}{1-6(x-\frac{1}{3})}$$

Using the geometric series allows us to obtain

$$p(x) = -3 \sum_{n=0}^{\infty} \left(6\left(x - \frac{1}{3}\right)\right)^n = \sum_{n=0}^{\infty} (-3)6^n \left(x - \frac{1}{3}\right)^n,$$

which converges when $\left|6\left(x - \frac{1}{3}\right)\right| < 1$ or equivalently, the interval of

convergence of the power series p is $\frac{1}{6} < x < \frac{1}{2}$

(b) No, because the point 1 does not belong to the interval of convergence of p .

Definition 9.8 (Taylor and Maclaurin series)

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n = f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \cdots$$

is called the Taylor series for $f(x)$ at c . Moreover, if $c = 0$, then the series is the Maclaurin series for f .

3. (10%)

(a) Find the Maclaurin series for $\arccos(x)$

(b) Find the radius and interval of convergence of the Maclaurin series for $\arccos(x)$.

(You can ignore examine about the endpoint of the interval)

Ans:

$$(a) \frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}} = -\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right) (-x^2)^n$$

$$\arccos(x) = \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right) (-1)^{n+1} \frac{x^{2n+1}}{2n+1} + C$$

Substitute 0 into the equation we have $C = \frac{\pi}{2}$. Therefore,

$$\arccos(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

$$(b) \text{ Use ratio test, } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\left(\frac{-1}{2}\right)^{n+1} (-1)^{n+2} \frac{x^{2n+3}}{2n+3}}{\left(\frac{-1}{2}\right)^n (-1)^{n+1} \frac{x^{2n+1}}{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(2n+1)}{(n+1)(2n+3)} \right| x^2 =$$

$$x^2$$

When $x^2 < 1$, it is converge, therefore $R = 1$ and the interval of convergence is $[-1, 1]$. For the boundary at 1 and -1, see

https://proofwiki.org/wiki/Power_Series_Expansion_for_Real_Arccosine_Function for more details.

4. (8%) Use a power series to approximate $\int_0^1 \sin(x^2) dx$ with an error of less than 0.001

$$\text{Ans: } \int_0^1 \sin(x^2) dx = \int_0^1 (x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \dots) dx = \left(\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \dots \right) \Big|_0^1 = \frac{1}{3} -$$

$$\frac{1}{42} + \frac{1}{1320} - \dots$$

It is an alternating series, therefore we know that $\int_0^1 \sin(x^2) dx = \frac{1}{3} - \frac{1}{42} \approx \frac{13}{42}$ and

the error is smaller than $\frac{1}{1320}$ which is also smaller than $\frac{1}{1000}$

5. (9%) Evaluate the following expression (Try to use the Basic series of Taylor series and notice that the power series is a continuous function)

$$(a) 1 - \frac{\pi^2}{4^2 \times 2!} + \frac{\pi^4}{4^4 \times 4!} - \frac{\pi^6}{4^6 \times 6!} + \dots$$

$$(b) \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \dots$$

$$(c) \lim_{x \rightarrow 0} \frac{\tan(x) - \sin(x)}{x^2}$$

Ans:

$$(a) 1 - \frac{\pi^2}{4^2 \times 2!} + \frac{\pi^4}{4^4 \times 4!} - \frac{\pi^6}{4^6 \times 6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(\frac{\pi}{4}\right)^{2n} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$(b) \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \left(\frac{1}{\sqrt{3}}\right)^{2n+1} = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$(c) \lim_{x \rightarrow 0} \frac{\tan(x) - \sin(x)}{x^2} = \lim_{x \rightarrow 0} \frac{\left(x + \frac{x^3}{3} - \dots\right) - \left(x - \frac{x^3}{3!} + \dots\right)}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{2}}{x^2} = 0$$

(9.10 Ex8)

6. (8%) Let $f(x) = x^6 e^{x^3}$. Try to evaluate the high order derivative $f^{(60)}(0)$
Ans:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \\ e^{x^3} &= 1 + x^3 + \frac{x^6}{2!} + \dots + \frac{x^{3n}}{n!} + \dots \\ x^6 e^{x^3} &= x^6 + x^9 + \frac{x^{12}}{2!} + \dots + \frac{x^{3n+6}}{n!} + \dots \\ f^{(60)}(0) &= \frac{1}{18!} \times 60! \end{aligned}$$

Theorem 10.11 (Slope in polar form)

If f is a differentiable function of θ , then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ) . (See Figure 14.)

Theorem 10.13 (Area in polar coordinates)

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta, \quad 0 < \beta - \alpha \leq 2\pi.$$

Theorem 10.14 (Arc length of a polar curve)

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

① $S = 2\pi \int_{\alpha}^{\beta} y ds = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$
About the polar axis

② $S = 2\pi \int_{\alpha}^{\beta} x ds = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$
About the line $\theta = \frac{\pi}{2}$

7. (8%) Find the area of the region which is inside the circle $r = 6\cos\theta$ and outside the cardioid $r = 2(1 + \cos\theta)$. (Both are represented in polar coordinates)

Ans:

Since $(r, -\theta)$ also lies on both graph, both of them are symmetric with respect to polar axis.

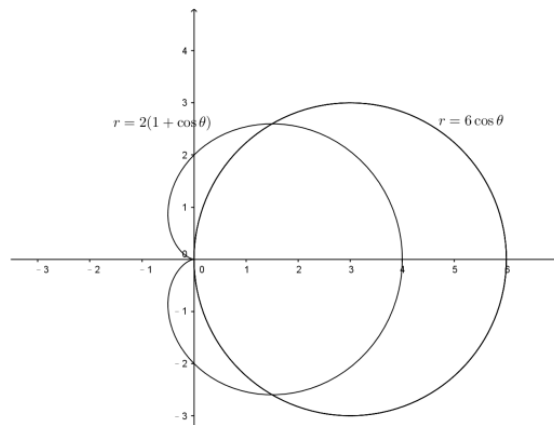
The intersection is $2(1 + \cos\theta) = 6\cos\theta \rightarrow \cos\theta = \frac{1}{2}$, therefore, $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

The area can be calculated as $A = 2 \times \frac{1}{2} \left[\int_0^{\frac{\pi}{3}} (6\cos\theta)^2 d\theta - \int_0^{\frac{\pi}{3}} 4(1 + \cos\theta)^2 d\theta \right] =$

$$36 \int_0^{\frac{\pi}{3}} \cos^2\theta d\theta - 4 \int_0^{\frac{\pi}{3}} (1 + \cos\theta)^2 d\theta = 36 \int_0^{\frac{\pi}{3}} \frac{1 + \cos(2\theta)}{2} d\theta - 4 \int_0^{\frac{\pi}{3}} (1 + \cos\theta)^2 d\theta =$$

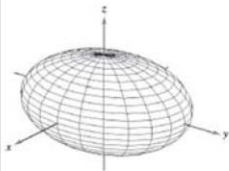
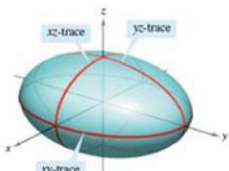
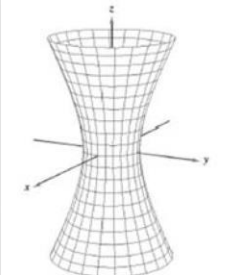
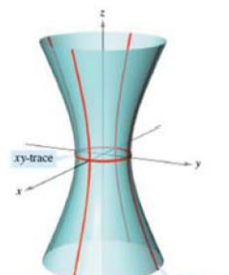
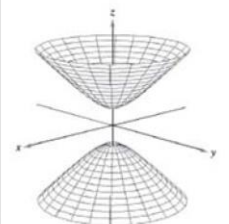
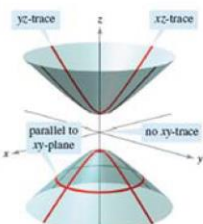
$$\int_0^{\frac{\pi}{3}} (18\cos 2\theta + 18 - 4 - 8\cos\theta - 2\cos 2\theta - 2) d\theta = (9\sin 2\theta + 18\theta - 4\theta -$$

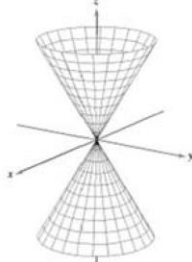
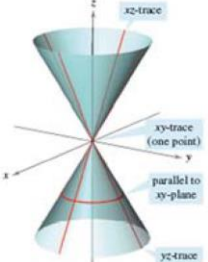
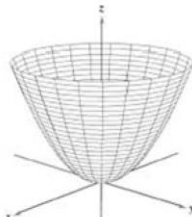
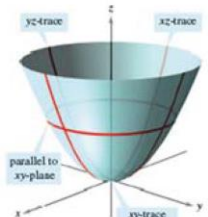
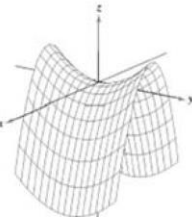
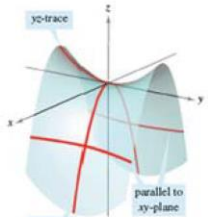
$$8\sin\theta - \sin 2\theta - 2\theta) \Big|_0^{\frac{\pi}{3}} = 4\pi$$



8. (5%) Find the arc length of $r = e^\theta$ from $\theta = 0$ to $\theta = 2\pi$

Ans: $S = \int_0^{2\pi} \sqrt{(e^\theta)^2 + (e^\theta)^2} d\theta = \sqrt{2} \int_0^{2\pi} e^\theta d\theta = \sqrt{2}(e^{2\pi} - 1)$

	<p>Ellipsoid</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <table><tr><th>Trace</th><th>Plane</th></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Ellipse</td><td>Parallel to xz-plane</td></tr><tr><td>Ellipse</td><td>Parallel to yz-plane</td></tr></table> <p>The surface is a sphere if $a = b = c \neq 0$.</p>	Trace	Plane	Ellipse	Parallel to xy -plane	Ellipse	Parallel to xz -plane	Ellipse	Parallel to yz -plane	
Trace	Plane									
Ellipse	Parallel to xy -plane									
Ellipse	Parallel to xz -plane									
Ellipse	Parallel to yz -plane									
	<p>Hyperboloid of One Sheet</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <table><tr><th>Trace</th><th>Plane</th></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to xz-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to yz-plane</td></tr></table> <p>The axis of the hyperboloid corresponds to the variable whose coefficient is negative.</p>	Trace	Plane	Ellipse	Parallel to xy -plane	Hyperbola	Parallel to xz -plane	Hyperbola	Parallel to yz -plane	
Trace	Plane									
Ellipse	Parallel to xy -plane									
Hyperbola	Parallel to xz -plane									
Hyperbola	Parallel to yz -plane									
	<p>Hyperboloid of Two Sheets</p> $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <table><tr><th>Trace</th><th>Plane</th></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to xz-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to yz-plane</td></tr></table> <p>The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.</p>	Trace	Plane	Ellipse	Parallel to xy -plane	Hyperbola	Parallel to xz -plane	Hyperbola	Parallel to yz -plane	
Trace	Plane									
Ellipse	Parallel to xy -plane									
Hyperbola	Parallel to xz -plane									
Hyperbola	Parallel to yz -plane									

	<p>Elliptic Cone</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <table> <tr> <th>Trace</th> <th>Plane</th> </tr> <tr> <td>Ellipse</td> <td>Parallel to xy-plane</td> </tr> <tr> <td>Hyperbola</td> <td>Parallel to xz-plane</td> </tr> <tr> <td>Hyperbola</td> <td>Parallel to yz-plane</td> </tr> </table> <p>The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.</p>	Trace	Plane	Ellipse	Parallel to xy -plane	Hyperbola	Parallel to xz -plane	Hyperbola	Parallel to yz -plane	
Trace	Plane									
Ellipse	Parallel to xy -plane									
Hyperbola	Parallel to xz -plane									
Hyperbola	Parallel to yz -plane									
	<p>Elliptic Paraboloid</p> $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <table> <tr> <th>Trace</th> <th>Plane</th> </tr> <tr> <td>Ellipse</td> <td>Parallel to xy-plane</td> </tr> <tr> <td>Parabola</td> <td>Parallel to xz-plane</td> </tr> <tr> <td>Parabola</td> <td>Parallel to yz-plane</td> </tr> </table> <p>The axis of the paraboloid corresponds to the variable raised to the first power.</p>	Trace	Plane	Ellipse	Parallel to xy -plane	Parabola	Parallel to xz -plane	Parabola	Parallel to yz -plane	
Trace	Plane									
Ellipse	Parallel to xy -plane									
Parabola	Parallel to xz -plane									
Parabola	Parallel to yz -plane									
	<p>Hyperbolic Paraboloid</p> $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ <table> <tr> <th>Trace</th> <th>Plane</th> </tr> <tr> <td>Hyperbola</td> <td>Parallel to xy-plane</td> </tr> <tr> <td>Parabola</td> <td>Parallel to xz-plane</td> </tr> <tr> <td>Parabola</td> <td>Parallel to yz-plane</td> </tr> </table> <p>The axis of the paraboloid corresponds to the variable raised to the first power.</p>	Trace	Plane	Hyperbola	Parallel to xy -plane	Parabola	Parallel to xz -plane	Parabola	Parallel to yz -plane	
Trace	Plane									
Hyperbola	Parallel to xy -plane									
Parabola	Parallel to xz -plane									
Parabola	Parallel to yz -plane									

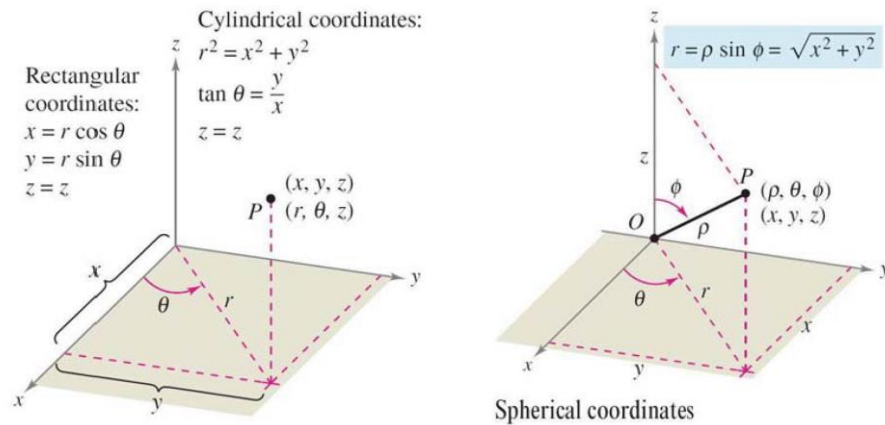
Definition 11.2 (Equation of cylinders)

The equation of a cylinder whose ruling are parallel to one of the coordinate axes contain only the variables corresponding to the other two axes.

Definition 11.4 (Surface of revolution)

If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

- ① Revolved about the x -axis: $y^2 + z^2 = [r(x)]^2$
- ② Revolved about the y -axis: $x^2 + z^2 = [r(y)]^2$
- ③ Revolved about the z -axis: $x^2 + y^2 = [r(z)]^2$



9. (12%) Classify the following surface, if it is quadratic surface you should further classify it into six basic types of surface

(a) $z = x^2 + 3y^2$

(b) $x^2 + y^2 - 2z = 0$

(c) $r^2 = z^2 + 2$ (this representation is in cylindrical coordinates)

(d) $\rho = 4\sec(\Phi)$ (this representation is in spherical coordinates)

Ans:

(a) $z = x^2 + 3y^2 = x^2 + \frac{y^2}{(\frac{1}{\sqrt{3}})^2}$ which is an elliptic paraboloid

(b) $x^2 + y^2 - 2z = 0 \rightarrow x^2 + y^2 = 2z$ which is a surface of revolution or elliptic paraboloid

(c) $r^2 = z^2 + 2 \rightarrow x^2 + y^2 - z^2 = 2 \rightarrow \frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{2})^2} - \frac{z^2}{(\sqrt{2})^2} = 1$ which is a hyperboloid of one sheet

(d) $\rho = 4\sec(\Phi) \rightarrow z = \rho \cos(\Phi) = 4$ which is a plane

Definition 12.2 (The limit of a vector-valued function)

1. If \mathbf{r} is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left[\lim_{t \rightarrow a} f(t) \right] \mathbf{i} + \left[\lim_{t \rightarrow a} g(t) \right] \mathbf{j} \quad \text{Plane}$$

provided f and g have limits as $t \rightarrow a$.

2. If \mathbf{r} is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left[\lim_{t \rightarrow a} f(t) \right] \mathbf{i} + \left[\lim_{t \rightarrow a} g(t) \right] \mathbf{j} + \left[\lim_{t \rightarrow a} h(t) \right] \mathbf{k} \quad \text{Space}$$

provided f , g , and h have limits as $t \rightarrow a$.

Theorem 12.1 (Differentiation of vector-valued functions)

- ① If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t , then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}. \quad \text{Plane}$$

- ② If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are differentiable functions of t , then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \quad \text{Space}$$

Theorem 12.2 (Properties of the derivative)

Let \mathbf{r} and \mathbf{u} be differentiable vector-valued functions of t , let w be a differentiable real-valued function of t , and let c be scalar.

- ① $D_t [c \mathbf{r}(t)] = c \mathbf{r}'(t)$
- ② $D_t [\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
- ③ $D_t [w(t) \mathbf{r}(t)] = w(t) \mathbf{r}'(t) + w'(t) \mathbf{r}(t)$
- ④ $D_t [\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
- ⑤ $D_t [\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
- ⑥ $D_t [\mathbf{r}(w(t))] = \mathbf{r}'(w(t)) w'(t)$
- ⑦ If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Definition 12.5 (Integration of vector-valued functions)

- If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, then the indefinite integral (antiderivative) of \mathbf{r} is

$$\int \mathbf{r}(t) dt = \left[\int f(t) dt \right] \mathbf{i} + \left[\int g(t) dt \right] \mathbf{j} \quad \text{Plane}$$

and its definite integral over the interval $a \leq t \leq b$ is

$$\int_a^b \mathbf{r}(t) dt = \left[\int_a^b f(t) dt \right] \mathbf{i} + \left[\int_a^b g(t) dt \right] \mathbf{j}.$$

Definition 12.5 (continue)

- If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are continuous on $[a, b]$, then the indefinite integral (antiderivative) of \mathbf{r} is

$$\int \mathbf{r}(t) dt = \left[\int f(t) dt \right] \mathbf{i} + \left[\int g(t) dt \right] \mathbf{j} + \left[\int h(t) dt \right] \mathbf{k} \quad \text{Space}$$

and its definite integral over the interval $a \leq t \leq b$ is

$$\int_a^b \mathbf{r}(t) dt = \left[\int_a^b f(t) dt \right] \mathbf{i} + \left[\int_a^b g(t) dt \right] \mathbf{j} + \left[\int_a^b h(t) dt \right] \mathbf{k}.$$

10. (12%) Evaluate the following expression

(a) $\lim_{t \rightarrow 1} \sqrt{t} \mathbf{i} + \frac{\ln t}{t^2 - 1} \mathbf{j} + \frac{1}{t - 1} \mathbf{k}$

(b) $\lim_{t \rightarrow 0} \frac{\sin 2t}{t} \mathbf{i} + e^{-t} \mathbf{j} + 5 \mathbf{k}$

(c) Let $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$, $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$, find $\frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{u}(t)]$

(d) $\int (3\sqrt{t} \mathbf{i} + \frac{2}{t} \mathbf{j} + \mathbf{k}) dt$

Ans:

(a) Does not exist, since $\lim_{t \rightarrow 1} \frac{1}{t-1}$ does not exist

$$(b) \lim_{t \rightarrow 0} \frac{\sin 2t}{t} \mathbf{i} + e^{-t} \mathbf{j} + 5\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

$$\text{Since } \lim_{t \rightarrow 0} \frac{\sin 2t}{t} = \lim_{t \rightarrow 0} \frac{\sin 2t}{2t} 2 = 2$$

$$(c) \mathbf{u}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}, \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$$

$$\begin{aligned} \frac{d}{dt} [\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{u}(t) \cdot \mathbf{r}'(t) \\ &= [(3t) + (t-1)(2t)] + [(3t) + (t^2)] = 4t + 3t^2 \end{aligned}$$

$$(d) \int (3\sqrt{t}\mathbf{i} + \frac{2}{t}\mathbf{j} + \mathbf{k}) dt = 2t^{\frac{3}{2}}\mathbf{i} + 2 \ln t \mathbf{j} + t\mathbf{k} + \mathbf{C}$$