1. (20%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)}{3n+4}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$

(c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+1}$

(d) $\sum_{n=3}^{\infty} \frac{(-1)^n}{n(\ln n)[\ln(\ln n)]^2}$

(e) $\sum_{n=1}^{\infty}(\sqrt[n]{n}-1)$

Theorem 9.1 (Limit of a sequence)

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

Definition 9.4 (Convergent and divergent series)

For the infinite series $\sum_{n=1}^{\infty} a_n$ the nth partial sum is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series diverges.

Definition 9.5 (Absolute and conditional convergence)

1. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
2. $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series ($r \neq 0$)	$\sum_{n=0}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series ($a_n > 0$)	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\|R_N\| \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} < 1$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Ans:

(a) Diverges because of $\lim_{n\to\infty} \frac{(-1)^n(2n-1)}{3n+4} = \frac{2}{3}\lim_{n\to\infty} (-1)^n \neq 0$ (by the n-th term test)

(b) $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3}}}{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}} = 2$, since $\frac{1}{\sqrt{n^3}}$ is a p-series with $p > 1$ which means that

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$ is convergent by the limit comparison test. Therefore,

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3+1}+\sqrt{n^3}}$ is absolute convergent.

(c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$. The given series converges by the alternating series test since $\frac{1}{(n+1)+1} < \frac{1}{n+1}$ and $\lim_{n\to\infty} \frac{1}{n+1} = 0$, but $\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi)}{n+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by a limit comparison test to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Therefore, the series converges conditionally.

(d) It is absolute convergent because of $\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{n(\ln n)[\ln(\ln n)]^2}\right| = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^2}$. Let $f(x) = \frac{1}{x(\ln x)[\ln(\ln x)]^2}$, since $\frac{1}{(x+1)(\ln(x+1))[\ln(\ln(x+1))]^2} < \frac{1}{(x)(\ln(x))[\ln(\ln(x))]^2}$. $f(x)$ is positive, continuous and decreasing.

And $\int_3^{\infty} \frac{1}{x(\ln x)[\ln(\ln x)]^2} dx = \lim_{b\to\infty} \frac{-1}{\ln(\ln x)} \Big|_3^b < \infty$ (by integral test, it is convergent)

(e) $n^{\frac{1}{n}} - 1 = e^{\frac{\ln n}{n}} - 1 = \left[1 + \frac{\ln n}{n} + \frac{1}{2}\left(\frac{\ln n}{n}\right)^2 + \cdots\right] - 1 = \frac{\ln n}{n} + \frac{1}{2}\left(\frac{\ln n}{n}\right)^2 + \cdots > \frac{\ln n}{n} > \frac{1}{n}$ $(n \geq 3)$. By the direct comparison test, the series diverges.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots + \frac{k(k-1)\cdots(k-n+1)x^n}{n!} + \cdots$	$-1 < x < 1$*

* The convergence at $x = \pm 1$ depends on the value of k.

Theorem 9.19 (Taylor's Theorem)

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Definition 9.7 (Power series)

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a power series. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at c, where c is a constant.

Theorem 9.20 (Convergence of a power series)

For a power series centered at c, precisely one of the following is true.

1. *The series converges only at c.*
2. *There exists a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.*
3. *The series converges absolutely for all x.*

The number R is the radius of convergence. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which it converges is the interval of convergence of the power series.

Theorem 9.21 (Properties of functions defined by power series)

If the function given by
$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$
has a radius of convergence of $R > 0$, then, on the interval
$(c-R,\ c+R)$*, f is differentiable (and therefore continuous).*
Moreover, the derivative and antiderivative of f are as follows.
1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$
2.
$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$
The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behavior at the endpoints.

2. (8%) Consider the function.

$$f(x) = \frac{1}{2x-1}, x \neq \frac{1}{2}$$

(a) Find the power series expansion of $p(x)$ of f expand at the point $\frac{1}{3}$ and determine its interval of convergence.

(b) Write $p(x) = \sum_{n=0}^{\infty} a_n (x - \frac{1}{3})^n$. Is $\sum_{n=0}^{\infty} a_n (\frac{2}{3})^n = f(1) = 1$? and why?

Ans:

(a) $f(x) = \frac{-1}{1-2x} = \frac{-1}{\frac{1}{3}-2(x-\frac{1}{3})} = \frac{-3}{1-6(x-\frac{1}{3})}$

Using the geometric series allows us to obtain

$$p(x) = -3\sum_{n=0}^{\infty}\left(6\left(x-\frac{1}{3}\right)\right)^n = \sum_{n=0}^{\infty}(-3)6^n\left(x-\frac{1}{3}\right)^n,$$

which converges when $\left|6(x-\frac{1}{3})\right| < 1$ or equivalently, the interval of convergence of the power series p is $\frac{1}{6} < x < \frac{1}{2}$

(b) No, because the point 1 does not belong to the interval of convergence of p.

Definition 9.8 (Taylor and Maclaurin series)

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the Taylor series for $f(x)$ at c. Moreover, if $c = 0$, then the series is the Maclaurin series for f.

3. (10%)

(a) Find the Maclaurin series for $\arccos(x)$

(b) Find the radius and interval of convergence of the Maclaurin series for $\arccos(x)$. (You can ignore examine about the endpoint of the interval)

Ans:

(a) $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} = -\sum_{n=0}^{\infty}\binom{\frac{-1}{2}}{n}(-x^2)^n$

$$\arccos(x) = \sum_{n=0}^{\infty}\binom{\frac{-1}{2}}{n}(-1)^{n+1}\frac{x^{2n+1}}{2n+1} + C$$

Substitute 0 into the equation we have $C = \frac{\pi}{2}$. Therefore,

$$\arccos(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

(b) Use ratio test, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{\binom{\frac{-1}{2}}{n+1}(-1)^{n+2}\frac{x^{2n+3}}{2n+3}}{\binom{\frac{-1}{2}}{n}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}}| = \lim_{n\to\infty} |\frac{(n+\frac{1}{2})(2n+1)}{(n+1)(2n+3)}|x^2 = x^2$

When $x^2 < 1$, it is converge, therefore $\mathrm{R} = 1$ and the interval of convergence is $[-1,1]$. For the boundary at 1 and -1, see https://proofwiki.org/wiki/Power_Series_Expansion_for_Real_Arccosine_Function for more details.

4. (8%) Use a power series to approximate $\int_0^1 \sin(x^2)dx$ with an error of less than 0.001

Ans: $\int_0^1 \sin(x^2)dx = \int_0^1 (x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots)dx = \left(\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \cdots\right)|_0^1 = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \cdots$

It is an alternating series, therefore we know that $\int_0^1 \sin(x^2)dx = \frac{1}{3} - \frac{1}{42} \approx \frac{13}{42}$ and the error is smaller than $\frac{1}{1320}$ which is also smaller than $\frac{1}{1000}$

5. (9%) Evaluate the following expression (Try to use the Basic series of Taylor series and notice that the power series is a continuous function)

(a) $1 - \frac{\pi^2}{4^2 \times 2!} + \frac{\pi^4}{4^4 \times 4!} - \frac{\pi^6}{4^6 \times 6!} + \cdots$

(b) $\frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \cdots$

(c) $\lim_{x\to 0} \frac{\tan(x) - \sin(x)}{x^2}$

Ans:

(a) $1 - \frac{\pi^2}{4^2 \times 2!} + \frac{\pi^4}{4^4 \times 4!} - \frac{\pi^6}{4^6 \times 6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(\frac{\pi}{4}\right)^{2n} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

(b) $\frac{1}{\sqrt{3}}-\frac{1}{3(\sqrt{3})^3}+\frac{1}{5(\sqrt{3})^5}-\frac{1}{7(\sqrt{3})^7}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left(\frac{1}{\sqrt{3}}\right)^{2n+1}=\arctan\left(\frac{1}{\sqrt{3}}\right)=$
$\frac{\pi}{6}$

(c) $\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{x^2}=\lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}-\cdots\right)-\left(x-\frac{x^3}{3!}+\cdots\right)}{x^2}=\lim_{x\to 0}\frac{\frac{x^3}{2}}{x^2}=0$
(9.10 Ex8)

6. (8%) Let $f(\mathrm{x})=x^6e^{x^3}$. Try to evaluate the high order derivative $f^{(60)}(0)$

Ans:

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$e^{x^3}=1+x^3+\frac{x^6}{2!}+\cdots+\frac{x^{3n}}{n!}+\cdots$$

$$x^6e^{x^3}=x^6+x^9+\frac{x^{12}}{2!}+\cdots+\frac{x^{3n+6}}{n!}+\cdots$$

$$f^{(60)}(0)=\frac{1}{18!}\times 60!$$

Theorem 10.11 (Slope in polar form)

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r=f(\theta)$ at the point (r,θ) is

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}=\frac{f(\theta)\cos\theta+f'(\theta)\sin\theta}{-f(\theta)\sin\theta+f'(\theta)\cos\theta}$$

provided that $\mathrm{d}x/\mathrm{d}\theta\neq 0$ at (r,θ). (See Figure 14.)

Theorem 10.13 (Area in polar coordinates)

If f is continuous and nonnegative on the interval $[\alpha,\beta]$, $0<\beta-\alpha\leq 2\pi$, then the area of the region bounded by the graph of $r=f(\theta)$ between the radial lines $\theta=\alpha$ and $\theta=\beta$ is given by

$$A=\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,\mathrm{d}\theta=\frac{1}{2}\int_{\alpha}^{\beta}r^2\,\mathrm{d}\theta,\quad 0<\beta-\alpha\leq 2\pi.$$

Theorem 10.14 (Arc length of a polar curve)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta.$$

Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

1. $S = 2\pi \int_{\alpha}^{\beta} y\, \mathrm{d}s = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta$
   About the polar axis
2. $S = 2\pi \int_{\alpha}^{\beta} x\, \mathrm{d}s = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta$
   About the line $\theta = \frac{\pi}{2}$

7. (8%) Find the area of the region which is inside the circle $r = 6\cos\theta$ and outside the cardioid $r = 2(1 + \cos\theta)$. (Both are represented in polar coordinates)

Ans:

Since $(r, -\theta)$ also lies on both graph, both of them are symmetric with respect to polar axis.

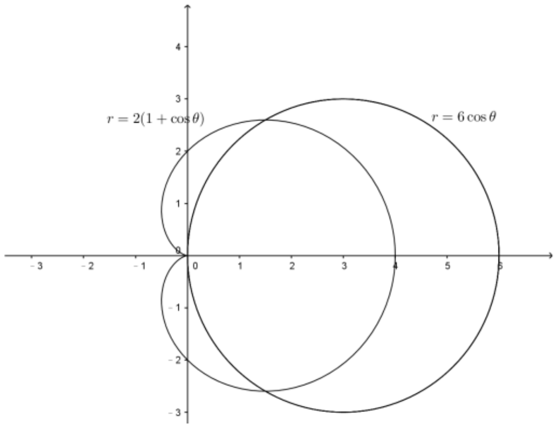
The intersection is $2(1 + \cos\theta) = 6\cos\theta \rightarrow \cos\theta = \frac{1}{2}$, therefore, $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

The area can be calculated as $A = 2 \times \frac{1}{2}\left[\int_0^{\frac{\pi}{3}} (6cos\theta)^2 d\theta - \int_0^{\frac{\pi}{3}} 4(1 + cos\theta)^2 d\theta\right] =$

$36\int_0^{\frac{\pi}{3}} cos^2\theta d\theta - 4\int_0^{\frac{\pi}{3}} (1 + cos\theta)^2 d\theta = 36\int_0^{\frac{\pi}{3}} \frac{1+\cos(2\theta)}{2} d\theta - 4\int_0^{\frac{\pi}{3}} (1 + cos\theta)^2 d\theta =$

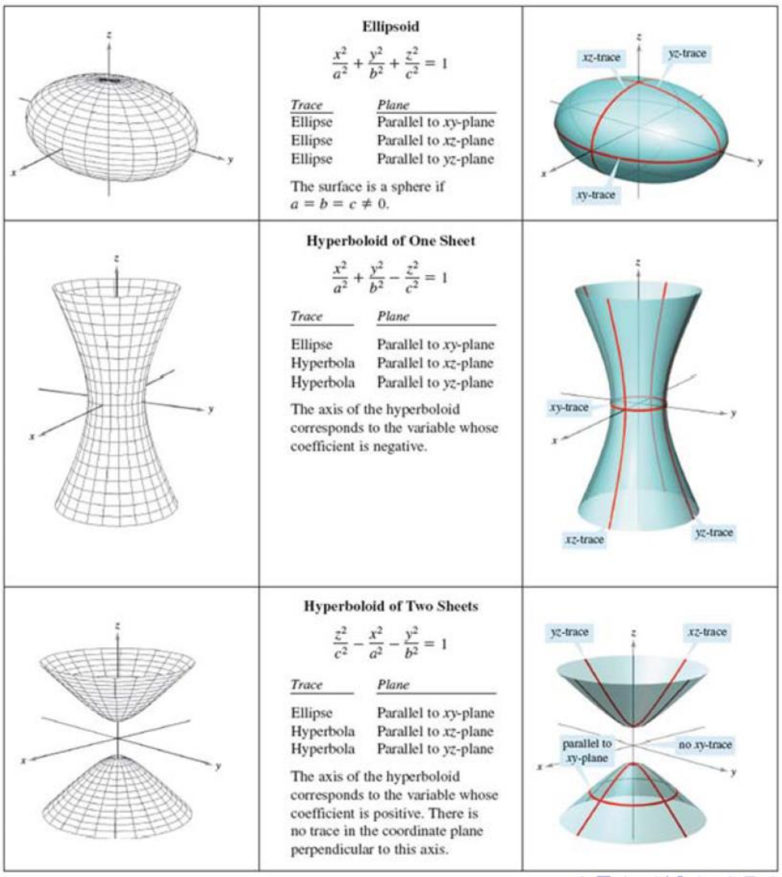
$\int_0^{\frac{\pi}{3}} (18cos2\theta + 18 - 4 - 8cos\theta - 2cos2\theta - 2) d\theta = (9sin2\theta + 18\theta - 4\theta -$

$8sin\theta - sin2\theta - 2\theta)\Big|_0^{\frac{\pi}{3}} = 4\pi$



8. (5%) Find the arc length of $r = e^{\theta}$ from $\theta = 0$ to $\theta = 2\pi$

Ans: $S = \int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta = \sqrt{2}\int_0^{2\pi} e^{\theta}\, d\theta = \sqrt{2}(e^{2\pi} - 1)$



Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.

Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

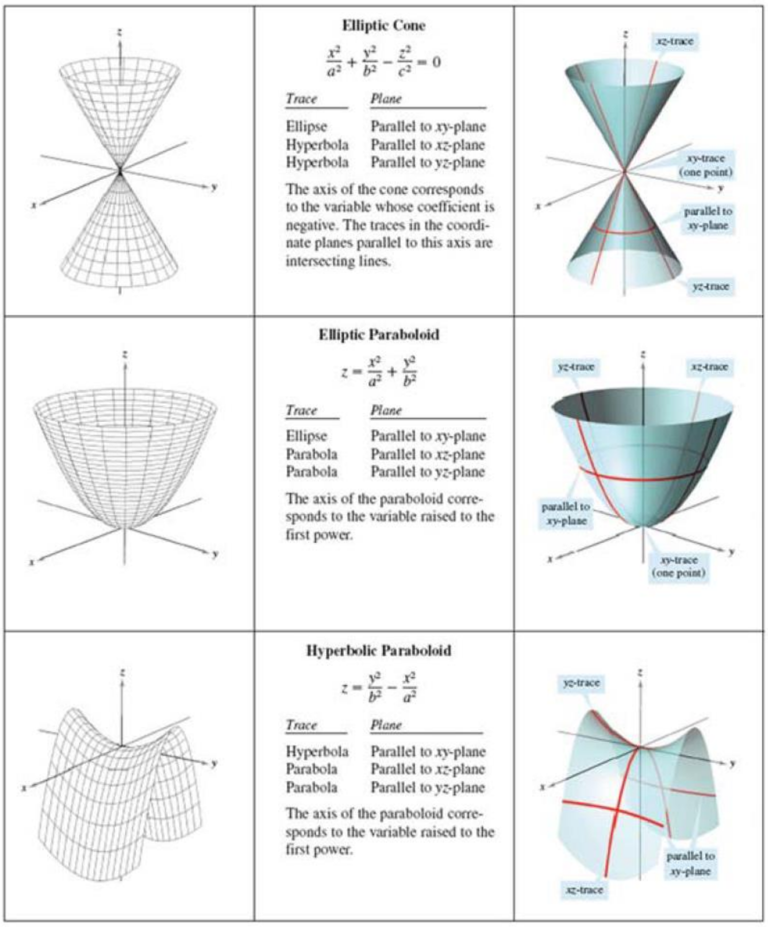
The axis of the hyperboloid corresponds to the variable whose coefficient is negative.

Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



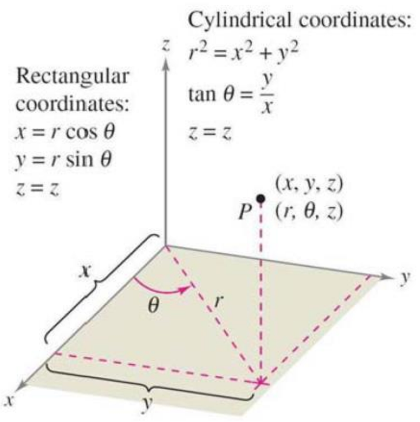
Definition 11.2 (Equation of cylinders)

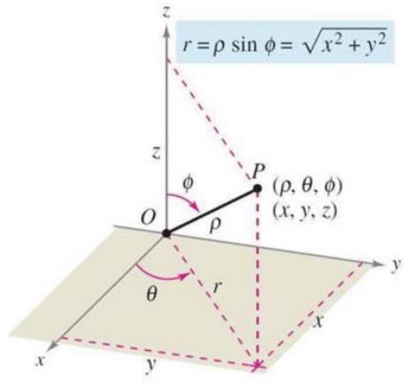
The equation of a cylinder whose ruling are parallel to one of the coordinate axes contain only the variables corresponding to the other two axes.

Definition 11.4 (Surface of revolution)

If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

1. Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$





9. (12%) Classify the following surface, if it is quadratic surface you should further classify it into six basic types of surface
   (a) $z = x^2 + 3y^2$
   (b) $x^2 + y^2 - 2z = 0$
   (c) $r^2 = z^2 + 2$ (this representation is in cylindrical coordinates)
   (d) $\rho = 4\sec(\Phi)$ (this representation is in spherical coordinates)

Ans:

(a) $z = x^2 + 3y^2 = x^2 + \frac{y^2}{(\frac{1}{\sqrt{3}})^2}$ which is an elliptic paraboloid

(b) $x^2 + y^2 - 2z = 0 \to x^2 + y^2 = 2z$ which is a surface of revolution or elliptic paraboloid

(c) $r^2 = z^2 + 2 \to x^2 + y^2 - z^2 = 2 \to \frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{2})^2} - \frac{z^2}{(\sqrt{2})^2} = 1$ which is a hyperboloid of one sheet

(d) $\rho = 4\sec(\Phi) \to z = \rho\cos(\Phi) = 4$ which is a plane

Definition 12.2 (The limit of a vector-valued function)

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} \qquad \text{Plane}$$

   provided f and g have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k} \qquad \text{Space}$$

   provided f, g, and h have limits as $t \to a$.

Theorem 12.1 (Differentiation of vector-valued functions)

1. *If* $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, *where* f *and* g *are differentiable functions of* t, *then*
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}. \qquad \textit{Plane}$$
2. *If* $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, *where* f, g, *and* h *are differentiable functions of* t, *then*
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}. \qquad \textit{Space}$$

Theorem 12.2 (Properties of the derivative)

Let $\mathbf{r}$ *and* $\mathbf{u}$ *be differentiable vector-valued functions of* t, *let* w *be a differentiable real-valued function of* t, *and let* c *be scalar.*

1. $D_t\,[c\,\mathbf{r}(t)] = c\,\mathbf{r}'(t)$
2. $D_t\,[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t\,[w(t)\,\mathbf{r}(t)] = w(t)\,\mathbf{r}'(t) + w'(t)\,\mathbf{r}(t)$
4. $D_t\,[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t\,[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t\,[\mathbf{r}\,(w(t))] = \mathbf{r}'\,(w(t))\,w'(t)$
7. *If* $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, *then* $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Definition 12.5 (Integration of vector-valued functions)

- If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where f and g are continuous on $[a, b]$, then the indefinite integral(antiderivative) of $\mathbf{r}$ is
$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j} \qquad \text{Plane}$$
and its definite integral over the interval $a \le t \le b$ is
$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j}.$$

Definition 12.5 (continue)

- If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the indefinite integral (antiderivative) of $\mathbf{r}$ is
$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int h(t)\,\mathrm{d}t\right]\mathbf{k} \qquad \text{Space}$$
and its definite integral over the interval $a \le t \le b$ is
$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int_a^b h(t)\,\mathrm{d}t\right]\mathbf{k}.$$

10. (12%) Evaluate the following expression

(a) $\lim\limits_{t\to 1} \sqrt{t}\boldsymbol{i} + \frac{\ln t}{t^2-1}\boldsymbol{j} + \frac{1}{t-1}\boldsymbol{k}$

(b) $\lim\limits_{t\to 0} \frac{sin2t}{t}\boldsymbol{i} + e^{-t}\boldsymbol{j} + 5\boldsymbol{k}$

(c) Let $\boldsymbol{r}(\mathrm{t}) = 3\mathrm{t}\boldsymbol{i} + (\mathrm{t}-1)\boldsymbol{j}$, $\boldsymbol{u}(\mathrm{t}) = \mathrm{t}\boldsymbol{i} + t^2\boldsymbol{j} + \frac{2}{3}t^3\boldsymbol{k}$, find $\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{u}(t)]$

(d) $\int(3\sqrt{t}\boldsymbol{i} + \frac{2}{t}\boldsymbol{j} + \boldsymbol{k})dt$

Ans:

(a) Does not exit, since $\lim_{t\to 1}\frac{1}{t-1}$ does not exist

(b) $\lim_{t\to 0}\frac{sin2t}{t}\boldsymbol{i} + e^{-t}\boldsymbol{j} + 5\boldsymbol{k} = 2\mathbf{t} + \boldsymbol{j} + 5\boldsymbol{k}$

Since $\lim_{t\to 0}\frac{sin2t}{t} = \lim_{t\to 0}\frac{sin2t}{2t}2 = 2$

(c) $\boldsymbol{u}'(t) = \boldsymbol{i} + 2\text{t}\boldsymbol{j} + 2\text{t}^2\boldsymbol{k}$, $\boldsymbol{r}'(t) = 3\boldsymbol{i} + \boldsymbol{j}$

$$\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{u}(t)] = \boldsymbol{r}(t)\cdot\boldsymbol{u}'(t) + \boldsymbol{u}(t)\cdot\boldsymbol{r}'(t)$$

$$= [(3t) + (t-1)(2t)] + [(3t) + (t^2)] = 4t + 3t^2$$

(d) $\int(3\sqrt{t}\boldsymbol{i} + \frac{2}{t}\boldsymbol{j} + \boldsymbol{k})dt = 2t^{\frac{3}{2}}\boldsymbol{i} + 2\ln t\,\boldsymbol{j} + t\boldsymbol{k} + \boldsymbol{C}$