Chapter 12 Vector-Valued Functions

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March 22, 2022

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Space curves and vector-valued functions

- A plane curve is defined as the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

  where f and g are continuous functions of t on an interval I.
- This definition can be extended naturally to three-dimensional space as follows.
- A space curve C is the set of all ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations

$$x = f(t), \quad y = g(t), \quad \text{and} \quad z = h(t)$$

  where f, g, and h are continuous functions of t on an interval I.
- A new type of function, called a vector-valued function, is introduced.
- This type of function maps real numbers to vectors.

Definition 12.1 (Vector-valued function)

A function of the form

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} \quad \text{(Plane)}$$

or

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \quad \text{(Space)}$$

is a vector-valued function, where the component functions f, g, and h are real-valued functions of the parameter t. Vector-valued functions are sometimes denoted as $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.

- Technically, a curve in the plane or in space consists of a collection of points and the defining parametric equations. Two different curves can have the same graph.
- For instance, each of the curves given by

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} \quad \text{and} \quad \mathbf{r}(t) = \sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}$$

  has the unit circle as its graph, but these equations do not represent the same curve—because the circle is traced out in different ways.
- Be sure you see the distinction between the vector-valued function $\mathbf{r}$ and the real-valued functions f, g, and h.
- All are functions of the real variable t, but $\mathbf{r}(t)$ is a vector, whereas $f(t)$, $g(t)$, and $h(t)$ are real numbers (for each specific value of t).
- Vector-valued functions serve dual roles in the representation of curves.
  - By letting the parameter t represent time, you can use a vector-valued function to represent motion along a curve.
  - Or, in the more general case, you can use a vector-valued function to trace the graph of a curve.

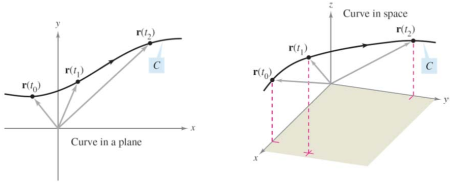

Figure 1: Curve C is traced out by the terminal point of position vector $\mathbf{r}(t)$.

- In either case, the terminal point of the position vector $\mathbf{r}(t)$ coincides with the point (x, y) or (x, y, z) on the curve given by the parametric equations, as shown in Figure 1.
- The arrowhead on the curve indicates the curve's orientation by pointing in the direction of increasing values of t.
- Unless stated otherwise, the domain of a vector-valued function $\mathbf{r}$ is considered to be the intersection of the domains of the component functions f, g, and h.
- For instance, the domain of $\mathbf{r}(t) = \ln t\,\mathbf{i} + \sqrt{1-t}\,\mathbf{j} + t\,\mathbf{k}$ is the interval $(0, 1]$.

Example 1 (Sketching a plane curve)

Sketch the plane curve represented by the vector-valued function

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi.$$

- From the position vector $\mathbf{r}(t)$, you can write the parametric equations $x = 2\cos t$ and $y = -3\sin t$.
- Solving for $\cos t$ and $\sin t$ and using the identity $\cos^2 t + \sin^2 t = 1$ produces the rectangular equation

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1. \qquad \text{Rectangular equation}$$

- The graph of this rectangular equation is the ellipse shown in Figure 2.
- The curve has a clockwise orientation.
- That is, as t increases from 0 to 2π, the position vector $\mathbf{r}(t)$ moves clockwise, and its terminal point traces the ellipse. ■

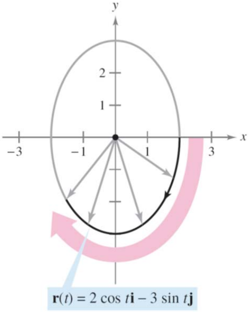

Figure 2: The ellipse $\mathbf{r}(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$ is traced clockwise as t increases from 0 to 2π.

Example 2 (Sketching a space curve)

Sketch the space curve represented by the vector-valued function

$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 4\pi.$$

- From the first two parametric equations $x = 4\cos t$ and $y = 4\sin t$, you can obtain

$$x^2 + y^2 = 16. \qquad \text{Rectangular equation}$$

- This means that the curve lies on a right circular cylinder of radius 4, centered about the z-axis.
- To locate the curve on this cylinder, you can use the third parametric equation $z = t$.
- In Figure 3, note that as t increases from 0 to 4π, the point (x, y, z) spirals up the cylinder to produce a helix.
- A real-life example of a helix is shown in the drawing at the lower left. ■

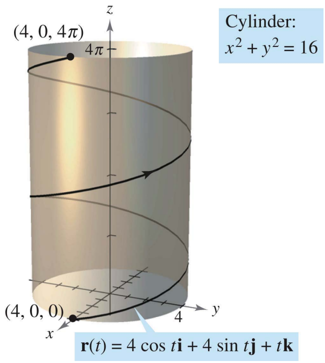

Figure 3: As t increases from 0 to 4π, two spirals on the helix are traced out.

Example 3 (Representing a graph by a vector-valued function)

Represent the parabola given by $y = x^2 + 1$ by a vector-valued function.

- Although there are many ways to choose the parameter t, a natural choice is to let $x = t$.
- Then $y = t^2 + 1$ and you have

$$\mathbf{r}(t) = t\,\mathbf{i} + (t^2 + 1)\mathbf{j}.$$

- Note in Figure 4 the orientation produced by this particular choice of parameter.
- Had you chosen $x = -t$ as the parameter, the curve would have been oriented in the opposite direction. ■

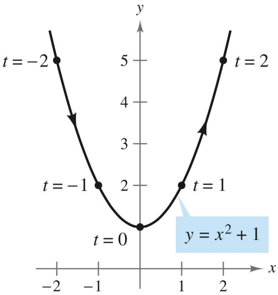

Figure 4: There are many ways to parametrize this graph. One way is to let $x = t$.

Example 4 (Representing a graph by a vector-valued function)

Sketch the space curve C represented by the intersection of the semiellipsoid

$$\frac{x^2}{12} + \frac{y^2}{24} + \frac{z^2}{4} = 1, \quad z \geq 0$$

and the parabolic cylinder $y = x^2$. Then, find a vector-valued function to represent the graph.

- The intersection of the two surfaces is shown in Figure 5.
- As in Example 3, a natural choice of parameter is $x = t$.
- For this choice, you can use the given equation $y = x^2$ to obtain $y = t^2$. Then, it follows that

$$\frac{z^2}{4} = 1 - \frac{x^2}{12} - \frac{y^2}{24} = 1 - \frac{t^2}{12} - \frac{t^4}{24} = \frac{24 - 2t^2 - t^4}{24} = \frac{(6 + t^2)(4 - t^2)}{24}.$$

- Because the curve lies above the xy-plane, you should choose the positive square root for z and obtain the following equations.

$$x = t, \quad y = t^2, \quad \text{and } z = \sqrt{\frac{(6+t^2)(4-t^2)}{6}}$$

- The resulting vector-valued function is

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \sqrt{\frac{(6+t^2)(4-t^2)}{6}}\,\mathbf{k}, \quad -2 \le t \le 2.$$

(Note that the $\mathbf{k}$-component of $\mathbf{r}(t)$ implies $-2 \le t \le 2$.)

- From the points $(-2, 4, 0)$ and $(2, 4, 0)$ shown in Figure 5, you can see that the curve is traced as t increases from -2 to 2. ■

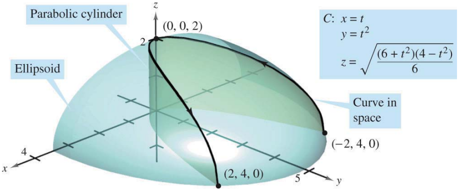

Figure 5: The curve C is the intersection of the semiellipsoid and the parabolic cylinder.

Limits and continuity

- To add or subtract two vector-valued functions (in the plane), you can write

$$\begin{aligned}
\mathbf{r}_1 + \mathbf{r}_2 &= [f_1(t)\,\mathbf{i} + g_1(t)\,\mathbf{j}] + [f_2(t)\,\mathbf{i} + g_2(t)\,\mathbf{j}] \\
&= [f_1(t) + f_2(t)]\,\mathbf{i} + [g_1(t) + g_2(t)]\,\mathbf{j} \\
\mathbf{r}_1 - \mathbf{r}_2 &= [f_1(t)\,\mathbf{i} + g_1(t)\,\mathbf{j}] - [f_2(t)\,\mathbf{i} + g_2(t)\,\mathbf{j}] \\
&= [f_1(t) - f_2(t)]\,\mathbf{i} + [g_1(t) - g_2(t)]\,\mathbf{j}.
\end{aligned}$$

- To multiply and divide a vector-valued function by a scalar, you can write

$$\begin{aligned}
c\mathbf{r}(t) &= c[f_1(t)\,\mathbf{i} + g_1(t)\,\mathbf{j}] = cf_1(t)\,\mathbf{i} + cg_1(t)\,\mathbf{j} \\
\frac{\mathbf{r}(t)}{c} &= \frac{[f_1(t)\,\mathbf{i} + g_1(t)\,\mathbf{j}]}{c} = \frac{f_1(t)}{c}\,\mathbf{i} + \frac{g_1(t)}{c}\,\mathbf{j}, \quad c \neq 0.
\end{aligned}$$

Definition 12.2 (The limit of a vector-valued function)

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right] \mathbf{i} + \left[\lim_{t\to a} g(t)\right] \mathbf{j} \qquad \text{Plane}$$

   provided f and g have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right] \mathbf{i} + \left[\lim_{t\to a} g(t)\right] \mathbf{j} + \left[\lim_{t\to a} h(t)\right] \mathbf{k} \qquad \text{Space}$$

   provided f, g, and h have limits as $t \to a$.

- If $\mathbf{r}(t)$ approaches the vector $\mathbf{L}$ as $t \to a$, the length of the vector $\mathbf{r}(t) - \mathbf{L}$ approaches 0.
- That is, $\|\mathbf{r}(t) - \mathbf{L}\| \to 0$ as $t \to a$. This is illustrated graphically in Figure 6.

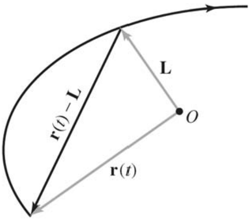

Figure 6: As t approaches a, $\mathbf{r}(t)$ approaches the limit $\mathbf{L}$. For the limit $\mathbf{L}$ to exist, it is not necessary that $\mathbf{r}(a)$ be defined or that $\mathbf{r}(a)$ be equal to $\mathbf{L}$.

Definition 12.3 (Continuity of a vector-valued function)

A vector-valued function $\mathbf{r}$ is continuous at a point given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is continuous on an interval I if it is continuous at every point in the interval.

A vector-valued function is continuous at $t = a$ if and only if each of its component function is continuous at $t = a$.

Example 5 (Continuity of vector-valued functions)

Discuss the continuity of the vector-valued function given by

$$\mathbf{r}(t) = t\,\mathbf{i} + a\,\mathbf{j} + (a^2 - t^2)\,\mathbf{k} \qquad a \text{ is a constant}$$

at $t = 0$.

- As t approaches 0, the limit is

$$\begin{aligned}\lim_{t\to 0} \mathbf{r}(t) &= \left[\lim_{t\to 0} t\right]\mathbf{i} + \left[\lim_{t\to 0} a\right]\mathbf{j} + \left[\lim_{t\to 0}(a^2 - t^2)\right]\mathbf{k} \\ &= 0\,\mathbf{i} + a\,\mathbf{j} + a^2\,\mathbf{k} = a\,\mathbf{j} + a^2\,\mathbf{k}.\end{aligned}$$

- Because

$$\mathbf{r}(0) = (0)\,\mathbf{i} + (a)\,\mathbf{j} + (a^2)\,\mathbf{k} = a\,\mathbf{j} + a^2\,\mathbf{k}$$

  you can conclude that $\mathbf{r}$ is continuous at $t = 0$.
- By similar reasoning, you can conclude that the vector-valued function $\mathbf{r}$ is continuous at all real-number values of t. ■

Example 6 (Continuity of vector-valued functions)

Determine the interval(s) on which the vector-valued function $\mathbf{r}(t) = t\,\mathbf{i} + \sqrt{t+1}\,\mathbf{j} + (t^2+1)\,\mathbf{k}$ is continuous.

- The component functions are

$$f(t) = t,\ \ g(t) = \sqrt{t+1},\ \text{and}\ h(t) = (t^2+1).$$

- Both f and h are continuous for all real-number values of t. The function g, however, is continuous only for $t \geq -1$. So, $\mathbf{r}$ is continuous the interval $[-1, \infty)$. ■

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Differentiation of vector-valued functions

- The definition of the derivative of a vector-valued function parallels the definition given for real-valued functions.

Definition 12.4 (The derivative of a vector-valued function)

The derivative of a vector-valued function $\mathbf{r}$ is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is differentiable at t. If $\mathbf{r}'(t)$ exists for all t in an open interval I, then $\mathbf{r}$ is differentiable on the interval I. Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

- Differentiation of vector-valued functions can be done on a component-by-component basis.
- To see why this is true, consider the function given by

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}.$$

- Applying the definition of the derivative produces the following.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{f(t+\Delta t)\,\mathbf{i} + g(t+\Delta t)\,\mathbf{j} - f(t)\,\mathbf{i} - g(t)\,\mathbf{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\{ \left[\frac{f(t+\Delta t) - f(t)}{\Delta t} \right] \mathbf{i} + \left[\frac{g(t+\Delta t) - g(t)}{\Delta t} \right] \mathbf{j} \right\} \\
&= \left\{ \lim_{\Delta t \to 0} \left[\frac{f(t+\Delta t) - f(t)}{\Delta t} \right] \right\} \mathbf{i} + \left\{ \lim_{\Delta t \to 0} \left[\frac{g(t+\Delta t) - g(t)}{\Delta t} \right] \right\} \mathbf{j} \\
&= f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}
\end{aligned}$$

- This important result is listed in the Theorem 12.1.
- Note that the derivative of the vector-valued function $\mathbf{r}$ is itself a vector-valued function.
- You can see from Figure 7 that $\mathbf{r}'(t)$ is a vector tangent to the curve given by $\mathbf{r}(t)$ and pointing in the direction of increasing t-values.

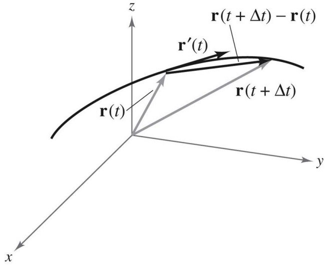

Figure 7: Definition of the derivative of a vector-valued functions.

Theorem 12.1 (Differentiation of vector-valued functions)

1. *If* $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, *where* f *and* g *are differentiable functions of* t, *then*
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}. \qquad \textit{Plane}$$
2. *If* $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, *where* f, g, *and* h *are differentiable functions of* t, *then*
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}. \qquad \textit{Space}$$

Example 1 (Differentiation of vector-valued functions)

For the vector-valued function given by $\mathbf{r}(t) = t\,\mathbf{i} + (t^2 + 2)\,\mathbf{j}$, find $\mathbf{r}'(t)$. Then sketch the plane curve represented by $\mathbf{r}(t)$, and the graphs of $\mathbf{r}(1)$ and $\mathbf{r}'(1)$.

- Differentiate on a component-by-component basis to obtain

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}.$$

- From the position vector $\mathbf{r}(t)$, you can write the parametric equations $x = t$ and $y = t^2 + 2$.
- The corresponding rectangular equation is $y = x^2 + 2$. When $t = 1$, $\mathbf{r}(1) = \mathbf{i} + 3\,\mathbf{j}$ and $\mathbf{r}'(1) = \mathbf{i} + 2\,\mathbf{j}$.
- In Figure 8, $\mathbf{r}(1)$ is drawn starting at the origin, and $\mathbf{r}'(1)$ is drawn starting at the terminal point of $\mathbf{r}(1)$. ■

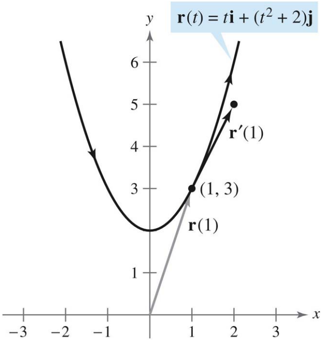

Figure 8: $\mathbf{r}(t) = t\,\mathbf{i} + (t^2 + 2)\,\mathbf{j}$

Example 2 (Higher-order differentiation)

For the vector-valued function given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$, find each of the following.

a. $\mathbf{r}'(t)$ **b.** $\mathbf{r}''(t)$ **c.** $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ **d.** $\mathbf{r}'(t) \times \mathbf{r}''(t)$

a. $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 2\,\mathbf{k}$

b. $\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + 0\,\mathbf{k} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \sin t \cos t - \sin t \cos t = 0$

d. $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \begin{vmatrix} \cos t & 2 \\ -\sin t & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -\sin t & 2 \\ -\cos t & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} \mathbf{k} = 2\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j} + \mathbf{k}$

- Note that the dot product in part (c) is a real-valued function, not a vector-valued function. ■

- The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

is smooth on an open interval if f', g', and h' are continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t in the interval I.

Example 3 (Finding intervals on which a curve is smooth)

Find the intervals on which the epicycloid C given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\,\mathbf{i} + (5\sin t - \sin 5t)\,\mathbf{j}, \qquad 0 \le t \le 2\pi$$

is smooth.

- The derivative of $\mathbf{r}$ is

$$\mathbf{r}'(t) = (-5\sin t + 5\sin 5t)\,\mathbf{i} + (5\cos t - 5\cos 5t)\,\mathbf{j}.$$

- In the interval $[0, 2\pi]$, the only values of t for which

$$\mathbf{r}'(t) = 0\,\mathbf{i} + 0\,\mathbf{j}$$

  are $t = 0$, $\pi/2$, π, $3\pi/2$, and 2π.

- Therefore, you can conclude that C is smooth in the intervals $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, $\left(\pi, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$ as shown in Figure 9. ■

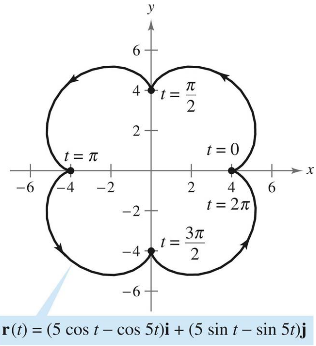

Figure 9: The epicycloid $\mathbf{r}(t) = (5\cos t - \cos 5t)\,\mathbf{i} + (5\sin t - \sin 5t)\,\mathbf{j}$ is not smooth at the points where it in intersects the axes.

- In the Figure 9, note that the curve is not smooth at points at which the curve makes abrupt changes in direction.
- Such points are called cusps or nodes.

Theorem 12.2 (Properties of the derivative)

Let $\mathbf{r}$ *and* $\mathbf{u}$ *be differentiable vector-valued functions of* t*, let* w *be a differentiable real-valued function of* t*, and let* c *be scalar.*

1. $D_t\left[c\,\mathbf{r}(t)\right] = c\,\mathbf{r}'(t)$
2. $D_t\left[\mathbf{r}(t) \pm \mathbf{u}(t)\right] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t\left[w(t)\,\mathbf{r}(t)\right] = w(t)\,\mathbf{r}'(t) + w'(t)\,\mathbf{r}(t)$
4. $D_t\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t\left[\mathbf{r}\left(w(t)\right)\right] = \mathbf{r}'\left(w(t)\right) w'(t)$
7. *If* $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$*, then* $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Example 4 (Using properties of the derivative)

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\,\mathbf{i} - \mathbf{j} + \ln t\,\mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2\,\mathbf{i} - 2t\,\mathbf{j} + \mathbf{k}$$

find **a.** $D_t\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right]$ and **b.** $D_t\left[\mathbf{u}(t)\times\mathbf{u}'(t)\right]$.

a. Because $\mathbf{r}'(t) = -\frac{1}{t^2}\,\mathbf{i} + \frac{1}{t}\,\mathbf{k}$ and $\mathbf{u}'(t) = 2t\,\mathbf{i} - 2\,\mathbf{j}$, you have

$$\begin{aligned} D_t\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right] &= \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t) \\ &= \left(\frac{1}{t}\,\mathbf{i} - \mathbf{j} + \ln t\,\mathbf{k}\right)\cdot\left(2t\,\mathbf{i} - 2\,\mathbf{j}\right) \\ &\quad + \left(-\frac{1}{t^2}\,\mathbf{i} + \frac{1}{t}\,\mathbf{k}\right)\cdot\left(t^2\,\mathbf{i} - 2t\,\mathbf{j} + \mathbf{k}\right) \\ &= 2 + 2 + (-1) + \frac{1}{t} = 3 + \frac{1}{t}. \end{aligned}$$

b. Because $\mathbf{u}'(t) = 2t\,\mathbf{i} - 2\,\mathbf{j}$ and $\mathbf{u}''(t) = 2\,\mathbf{i}$, you have

$$D_t\left[\mathbf{u}(t) \times \mathbf{u}'(t)\right] = \left[\mathbf{u}(t) \times \mathbf{u}''(t)\right] + \left[\mathbf{u}'(t) \times \mathbf{u}'(t)\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0}$$

$$= \begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix}\,\mathbf{i} - \begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix}\,\mathbf{j} + \begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix}\,\mathbf{k}$$

$$= 0\,\mathbf{i} - (-2)\,\mathbf{j} + 4t\,\mathbf{k} = 2\,\mathbf{j} + 4t\,\mathbf{k}. \qquad \blacksquare$$

The following definition is a rational consequence of the definition of the derivative of a vector-valued function.

Definition 12.5 (Integration of vector-valued functions)

- If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where f and g are continuous on $[a, b]$, then the indefinite integral(antiderivative) of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j} \qquad \text{Plane}$$

  and its definite integral over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j}.$$

Definition 12.5 (continue)

- If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the indefinite integral (antiderivative) of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int h(t)\,\mathrm{d}t\right]\mathbf{k} \qquad \text{Space}$$

and its definite integral over the interval $a \leq t \leq b$ is

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int_a^b h(t)\,\mathrm{d}t\right]\mathbf{k}.$$

- The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector $\mathbf{C}$.

- For instance, if $\mathbf{r}(t)$ is a three-dimensional vector-valued function, then for the indefinite integral $\int \mathbf{r}(t)\,\mathrm{d}t$, you obtain three constants of integration

$$\int f(t)\,\mathrm{d}t = F(t) + C_1, \int g(t)\,\mathrm{d}t = G(t) + C_2, \int h(t)\,\mathrm{d}t = H(t) + C_3$$

where $F'(t) = f(t)$, $G'(t) = g(t)$, and $H'(t) = h(t)$.

- These three scalar constants produce one vector constant of integration,

$$\begin{aligned}\int \mathbf{r}(t)\,\mathrm{d}t &= [F(t) + C_1]\,\mathbf{i} + [G(t) + C_2]\,\mathbf{j} + [H(t) + C_3]\,\mathbf{k} \\ &= [F(t)\,\mathbf{i} + G(t)\,\mathbf{j} + H(t)\,\mathbf{k}] + [C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}] \\ &= \mathbf{R}(t) + \mathbf{C}\end{aligned}$$

where $\mathbf{R}'(t) = \mathbf{r}(t)$.

Example 5 (Integrating a vector-valued function)

Find the indefinite integral $\int (t\,\mathbf{i} + 3\,\mathbf{j})\,\mathrm{d}t$.

Integrating on a component-by-component basis produces

$$\int (t\,\mathbf{i} + 3\,\mathbf{j})\,\mathrm{d}t = \frac{t^2}{2}\,\mathbf{i} + 3t\,\mathbf{j} + \mathbf{C}.$$

■

Example 6 (Definite Integral of a vector-valued function)

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,\mathrm{d}t = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\,\mathbf{j} + e^{-t}\,\mathbf{k}\right)\,\mathrm{d}t.$$

$$\begin{aligned}\int_0^1 \mathbf{r}(t)\,\mathrm{d}t &= \left(\int_0^1 t^{1/3}\,\mathrm{d}t\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t+1}\,\mathrm{d}t\right)\mathbf{j} + \left(\int_0^1 e^{-t}\,\mathrm{d}t\right)\mathbf{k} \\ &= \left[\left(\frac{3}{4}\right)t^{4/3}\right]_0^1 \mathbf{i} + \Big[\ln|t+1|\Big]_0^1 \mathbf{j} + \Big[-e^{-t}\Big]_0^1 \mathbf{k} \\ &= \frac{3}{4}\mathbf{i} + (\ln 2)\,\mathbf{j} + \left(1 - \frac{1}{e}\right)\mathbf{k} \qquad \blacksquare\end{aligned}$$

Example 7 (The antiderivative of a vector-valued function)

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t\,\mathbf{i} - 2\sin t\,\mathbf{j} + \frac{1}{1+t^2}\,\mathbf{k}$$

that satisfies the initial condition $\mathbf{r}(0) = 3\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,\mathrm{d}t = \left(\int \cos 2t\,\mathrm{d}t\right)\mathbf{i} + \left(\int -2\sin t\,\mathrm{d}t\right)\mathbf{j} + \left(\int \frac{1}{1+t^2}\,\mathrm{d}t\right)$$
$$= \left(\frac{1}{2}\sin 2t + C_1\right)\mathbf{i} + (2\cos t + C_2)\,\mathbf{j} + (\arctan t + C_3)\,\mathbf{k}$$

- Letting $t = 0$ and using the fact that $\mathbf{r}(0) = 3\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$, you have

$$\mathbf{r}(0) = (0 + C_1)\,\mathbf{i} + (2 + C_2)\,\mathbf{j} + (0 + C_3)\,\mathbf{k} = 3\,\mathbf{i} + (-2)\,\mathbf{j} + \mathbf{k}.$$

- Equating corresponding components produces

$$C_1 = 3, \quad 2 + C_2 = -2, \quad \text{and} \quad C_3 = 1.$$

So, the antiderivative that satisfies the given initial condition is

$$\mathbf{r}(t) = \left(\frac{1}{2}\sin 2t + 3\right)\mathbf{i} + (2\cos t - 4)\,\mathbf{j} + (\arctan t + 1)\,\mathbf{k}.$$