Chapter 1: Limits and Their Properties

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What is calculus?

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- Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths and a variety of other concepts that have enabled us to model real-life situations.
- Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus.
- Precalculus mathematics is more static, whereas calculus is more dynamic.

Here are some examples.

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1. The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
2. The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.
3. An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.

So, one way to answer the question "What is calculus?" is to say that calculus is a "limit machine" that involves three stages.

So, one way to answer the question "What is calculus?" is to say that calculus is a "limit machine" that involves three stages.

- The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle.
- The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.

Precalculus mathematics $\Longrightarrow$ Limit process $\Longrightarrow$ Calculus

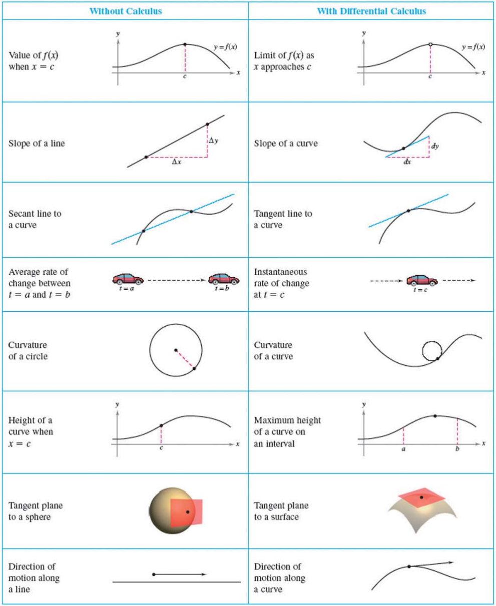
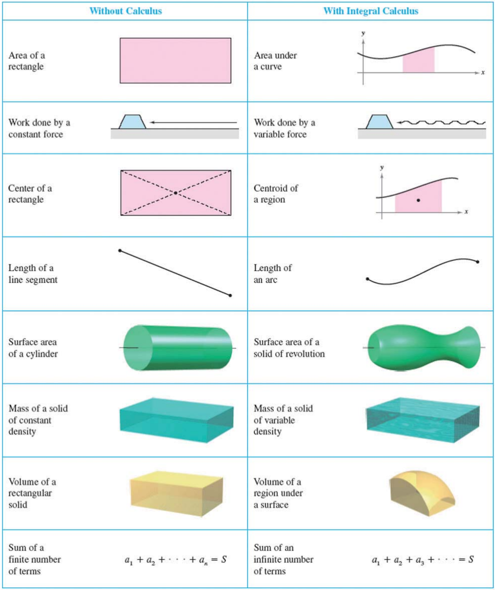


Figure 1: Without calculus versus with differential calculus.



Without Calculus		With Integral Calculus	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdots + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdots = S$

Figure 2: Without calculus versus with integral calculus.

The tangent line problem

- The notion of a limit is fundamental to the study of calculus.
- The tangent line problem and area problem should give you some idea of the way limits are used in calculus.

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- The notion of a limit is fundamental to the study of calculus.
- The tangent line problem and area problem should give you some idea of the way limits are used in calculus.
- In the tangent line problem, you are given a function f and a point P on its graph and are asked to find an equation of the tangent line to the graph at point P, as shown in Figure 3.

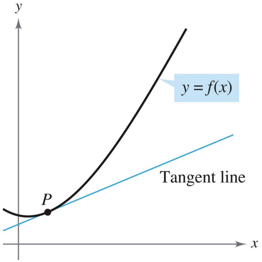
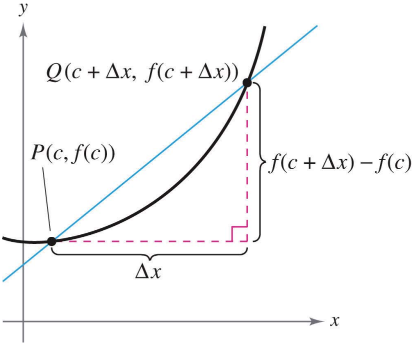
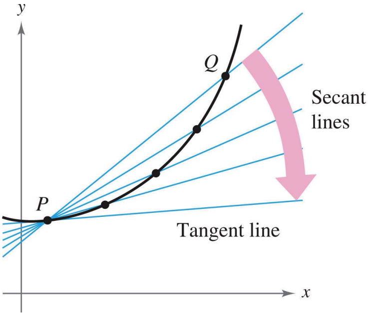


Figure 3: The tangent line to the graph of f at a point.



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$.



(b) As Q approaches P, the secant lines approach the tangent line.

Figure 4: The secant line and tangent line.

- You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 4a. Such a line is called a secant line.

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- If $P(c, f(c))$ is the point of tangency and $Q(c+\Delta x, f(c+\Delta x))$ is a second point on the graph of f, then the slope of the secant line through these two points can be found using precalculus and are given by

$$m_{\text{sec}} = \frac{f(c+\Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x}.$$

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- As point Q approaches point P, the slopes of the secant lines approach the slope of the tangent line, as shown in Figure 4b.

The area problem

- The area problem is finding the area of a plane region that is bounded by the graphs of functions.
  - In this case, the limit process is applied to the area of rectangles to find the area of a general region.

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- The area problem is finding the area of a plane region that is bounded by the graphs of functions.
  - In this case, the limit process is applied to the area of rectangles to find the area of a general region.
- As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 5.

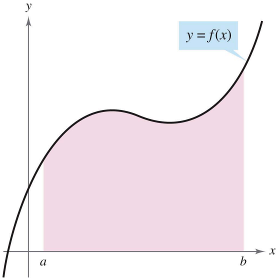
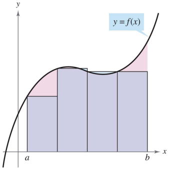
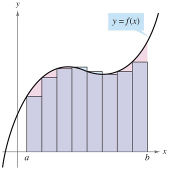


Figure 5: Area under a curve.



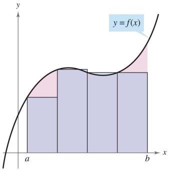
(a) Approximation using four rectangles.



(b) Approximation using eight rectangles.

Figure 6: Approximation area under a curve using rectangles.

- You can approximate the area of the region with several rectangular regions using $\sum_{j=1}^{n} f(x_j)\Delta x_j$, as shown in Figure 6.



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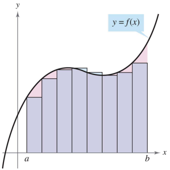

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- You can approximate the area of the region with several rectangular regions using $\sum_{j=1}^{n} f(x_j)\Delta x_j$, as shown in Figure 6.
- As you increase the number of rectangles, the approximation tends to become better and better.

(a) Approximation using four rectangles.

(b) Approximation using eight rectangles.

Figure 6: Approximation area under a curve using rectangles.

- You can approximate the area of the region with several rectangular regions using $\sum_{j=1}^{n} f(x_j)\Delta x_j$, as shown in Figure 6.
- As you increase the number of rectangles, the approximation tends to become better and better.
- Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bounds.

Notes

Remark - Tangent line problem and the area the problem

- They are close related to each other!
- $lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}$.
- $lim_{n \to \infty} \sum_{j=1}^{n} f(x_j)\Delta x_j$.

- This discovery led to the birth of calculus. You will learn about the relationship between these two problems when we study the Fundamental Theorem of Calculus in Chapter 4.

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An introduction to limits

- Suppose you are asked to sketch the graph of the function f given by

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- Suppose you are asked to sketch the graph of the function f given by

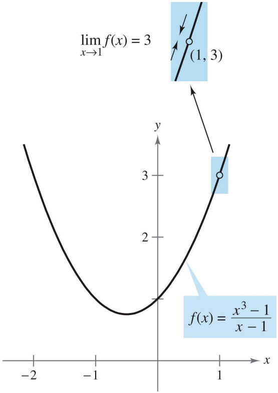
$$f(x) = \frac{x^3 - 1}{x - 1}, \quad x \neq 1.$$

- However, at $x = 1$, it is not clear what to expect.
- We can use two sets of x-values—one set that approaches 1 from the left and one set that approaches 1 from the right, as shown in the table.

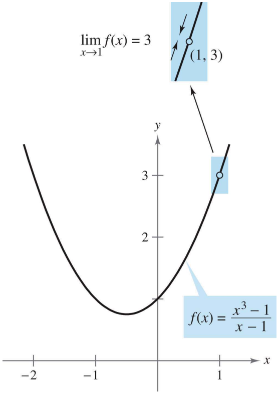
x approaches 1 from the left. → ← x approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

$f(x)$ approaches 3. → ← $f(x)$ approaches 3.



- The graph of f is a parabola that has a gap at the point $(1, 3)$.
  - Although x can not equal 1, you can move arbitrarily close to 1, and as a result, $f(x)$ moves arbitrarily close to 3.



- The graph of f is a parabola that has a gap at the point $(1, 3)$.
  - Although x can not equal 1, you can move arbitrarily close to 1, and as a result, $f(x)$ moves arbitrarily close to 3.
  - Using limit notation, you can write

$$\lim_{x\to 1} f(x) = 3.$$

Remark - informal definition of limit

- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side but not equals c, the limit of $f(x)$, as x approaches c, is L.
- This limit is written as $\boxed{\lim_{x\to c} f(x) = L}$.
- It implies that the limit exists *and* the limit is L.
- If L is ∞ or $-\infty$, it is considered in section 1.5.

Example 1 (Estimating a limit numerically)

Evaluate the function $f(x) = x/(\sqrt{x+1} - 1)$ at several points near $x = 0$ and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

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x approaches 0 from the left. x approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

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- The table lists the values of $f(x)$ for several x-values near 0.

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- The table lists the values of $f(x)$ for several x-values near 0.
- From the results shown in the table, you can estimate the limit to be 2. ■

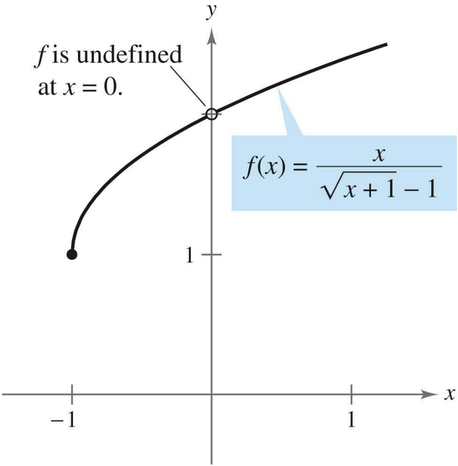

Figure 7: The limit of $f(x) = \frac{x}{\sqrt{x+1}-1}$ as x approaches 0 is 2.

Example 2 (Finding a limit)

Find the limit of $f(x)$ as x approaches 2, where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$

Example 3 (Behavior that differs from the right and from the left)

Show that the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

- Because $|x|/x$ approaches a different number from the right side of 0 than it approaches from the left side, the limit $\lim_{x\to 0} |x|/x$ does not exist. ■

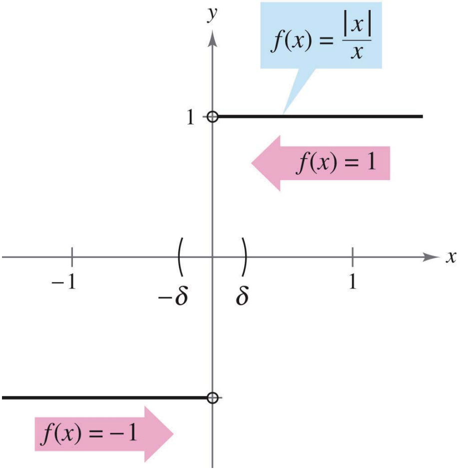

Figure 8: $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 4 (Unbounded behavior)

Discuss the existence of the limit $\lim_{x \to 0} \frac{1}{x^2}$.

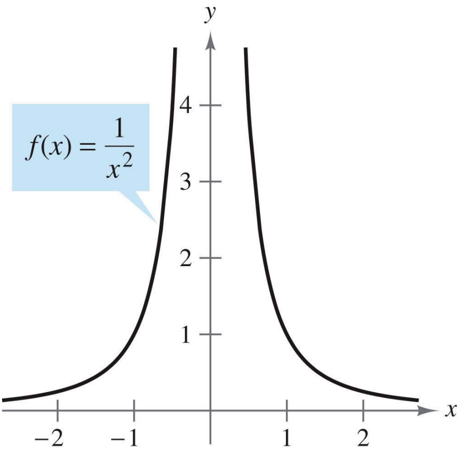

Figure 9: $\lim_{x \to 0} 1/x^2$ does not exist.

Example 5 (Oscillating behavior)

Discuss the existence of the limit $\lim_{x \to 0} \sin \frac{1}{x}$.

x	$2/\pi$	$2/3\pi$	$2/5\pi$	$2/7\pi$	$2/9\pi$	$2/11\pi$	$x \to 0$
$\sin(1/x)$	1	-1	1	-1	1	-1	Limit does not exist.

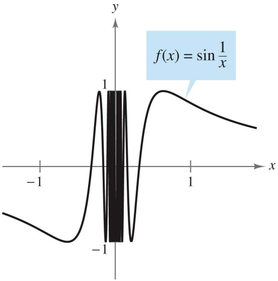

Figure 10: $\lim_{x\to 0} \sin(1/x)$ does not exist.

Common types of behavior associated with nonexistence of a limit

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
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There are many other interesting functions that have unusual limit behavior. One is the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational.} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Because this function has no limit at any real number c, it is actually not continuous at any real number c.

A formal definition of limit

- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, is written as

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A formal definition of limit

- If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L, is written as

$$\lim_{x \to c} f(x) = L.$$

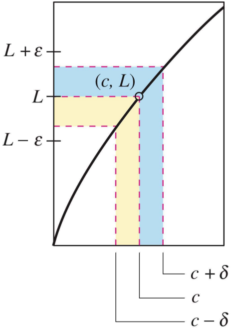
- What is "$f(x)$ becomes arbitrarily close to L" and "x approaches c."?
- In figure below, let ε represent a (small) positive number. Then the phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$. Using absolute value, you can write this as

$$|f(x) - L| < \varepsilon.$$

- Similarly, the phrase "x approaches c" means that there exists a positive number δ such that x lies in either the interval $(c-\delta, c)$ or the interval $(c, c+\delta)$. This fact can be concisely expressed by

$$0 < |x-c| < \delta.$$

  - The first inequality $0 < |x-c|$ says that the distance between x and c is more than 0 which expresses the fact that $x \neq c$. The second inequality $|x-c| < \delta$ indicate that x is within δ units of c.



Definition 1.1 (Limit)

Let f be a function defined on an <u>open interval containing c (except possibly at c)</u> and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

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Properties of limits

- The limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$.

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Such <u>well-behaved functions</u> are <u>continuous</u> at c.

Theorem 1.1 (Some basic limits)

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Example 1 (Evaluating basic limits)

a. $\lim_{x\to 2} 3$ **b.** $\lim_{x\to -4} x$ **c.** $\lim_{x\to 2} x^2$ ■

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Theorem 1.2 (Properties of limits)

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \textit{and} \quad \lim_{x\to c} g(x) = K$$

1. *Scalar multiple:* $\lim_{x\to c}[bf(x)] = bL$
2. *Sum or difference:* $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$
3. *Product:* $\lim_{x\to c}[f(x)g(x)] = LK$
4. *Quotient:* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, *provided* $K \neq 0$
5. *Power:* $\lim_{x\to c}[f(x)]^n = L^n$

Example 2 (The limit of a polynomial)

Find the limit: $\lim_{x\to 2}(4x^2+3)$.

Theorem 1.3 (Limits of polynomial and rational functions)

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 3 (The limit of a rational function)

Find the limit: $\lim_{x\to 1} \frac{x^2+x+2}{x+1}$.

Theorem 1.4 (The limit of a function involving a radical)

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem 1.5 (The limit of a composite function)

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Example 4 (The limit of a composite function)

Find the limit.

a. $\lim_{x\to 0}\sqrt{x^2+4}$ **b.** $\lim_{x\to 3}\sqrt[3]{2x^2-10}$

Theorem 1.6 (Limits of trigonometric functions)

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \cot x = \cot c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \csc x = \csc c$

Example 5 (Limits of trigonometric functions)

a. $\lim_{x\to 0} \tan x$

b. $\lim_{x\to \pi} (x \cos x)$

c. $\lim_{x\to 0} \sin^2 x$ ■

A strategy for finding limits

Theorem 1.7 (Functions that agree at all but one point)

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Example 6 (Finding the limit of a function)

Find the limit: $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

A strategy for finding limits analytically

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$.
3. Apply Theorem 1.7 to conclude analytically that
$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$
4. Use the other two approaches: a graph or table to reinforce your conclusion.

Dividing out and rationalizing techniques

Example 7 (Dividing out technique)

Find the limit: $\lim_{x \to -3} \frac{x^2+x-6}{x+3}$.

Example 8 (Rationalizing technique)

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

x	-0.25	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.4721

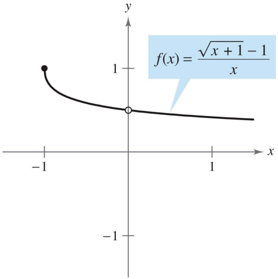

Figure 11: The limit of $f(x) = \frac{\sqrt{x+1}-1}{x}$ as x approaches 0 is $\frac{1}{2}$.

- An expression such as $0/0$ is called an indeterminate form because you cannot determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit.
- One way to do this is to divide out like factors, as shown in Example 7. A second way is to rationalize the numerator, as shown in Example 8.

- The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 12.

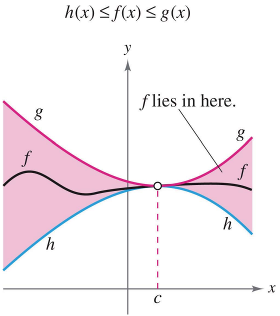

Figure 12: The Squeeze Theorem.

The Squeeze Theorem

Theorem 1.8 (The Squeeze Theorem)

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

- Squeeze Theorem is also called the Sandwich Theorem or the Pinching Theorem.

Theorem 1.9 (Two special trigonometric limits)

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

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1. The proof is presented using the variable θ, where θ is an acute positive angle measured in radians. Figure 13 shows a circular sector that is squeezed between two triangles.

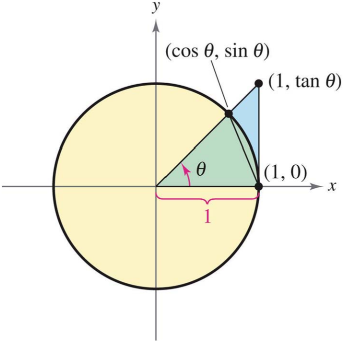
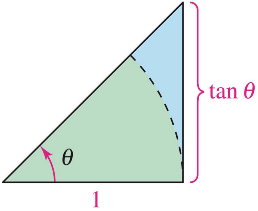
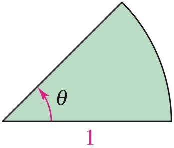
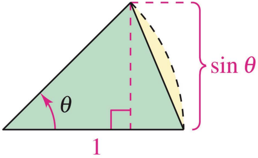


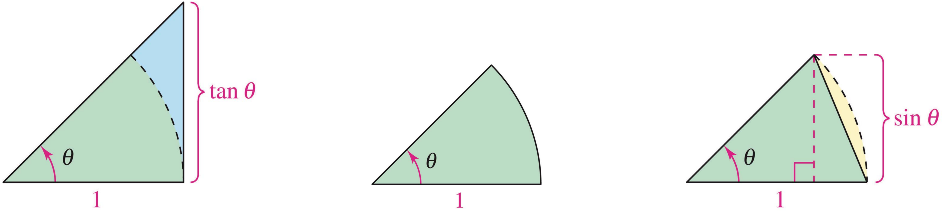
Figure 13: A circular sector is used to prove Theorem 1.9.







Area of triangle $= \frac{\tan\theta}{2} \quad \geq \quad$ Area of sector $= \frac{\theta}{2} \quad \geq \quad$ Area of triangle $= \frac{\sin\theta}{2}$



Area of triangle $= \frac{\tan\theta}{2}$ $\geq$ Area of sector $= \frac{\theta}{2}$ $\geq$ Area of triangle $= \frac{\sin\theta}{2}$

- Multiplying each expression by $2/\sin\theta$ produces

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

- Because $\cos\theta = \cos(-\theta)$ and $(\sin\theta)/\theta = [\sin(-\theta)]/(-\theta)$, you can conclude that this inequality is valid for all nonzero θ in the open interval $(-\pi/2, \pi/2)$.

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- Finally, because $\lim_{\theta\to 0}\cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$.

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2.

Example 9 (A limit involving a trigonometric function)

Find the limit: $\lim_{x\to 0} \frac{\tan x}{x}$.

Example 10 (A limit involving a trigonometric function)

Find the limit: $\lim_{x\to 0} \frac{\sin 4x}{x}$.

Table of Contents

Continuity at a point and on an open interval

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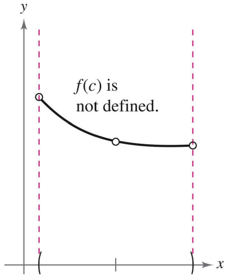

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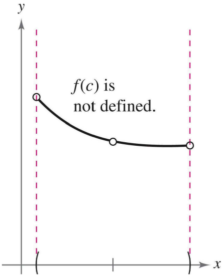

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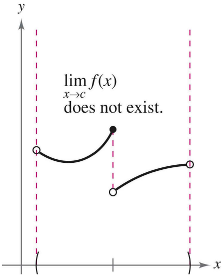

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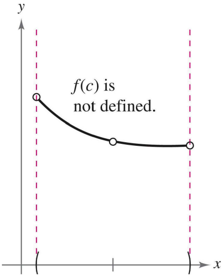

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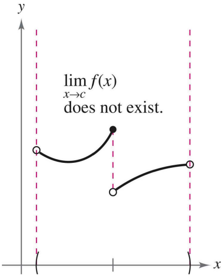

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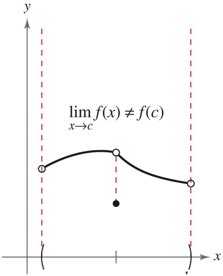

(c)

Figure 14: Three conditions the graph of f is not continuous at $x = c$.

- It appears that continuity at $x = c$ can be destroyed by any one of the following conditions.
  1. The function is not defined at $x = c$.
  2. The limit of $f(x)$ does not exist at $x = c$.
  3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

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  3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.
- If none of the three conditions above is true, the function f is called continuous at c, as indicated in the following important definition.

Definition 1.2 (Continuity)

Continuity at a point: A function f is continuous at c if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$

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Continuity on an open interval: A function is continuous on an open interval (a, b) if it is continuous at each point in the interval.
Continuity on $\mathbb{R}$: A function that is continuous on the entire real line $(-\infty, \infty)$ is everywhere continuous.

- Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a discontinuity at c.

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- Discontinuities fall into two categories: removable and nonremovable.
  - A discontinuity at c is called **removable** if f can be made continuous by appropriately defining (or redefining $f(c)$).
  - For instance, the functions shown in Figures 14(a) and 14(c) have removable discontinuities at c and the function shown in Figure 14(b) has a **nonremovable** discontinuity at c.

Example 1 (Continuity of a function)

Discuss the continuity of each function. **a.** $f(x) = \frac{1}{x}$ **b.** $g(x) = \frac{x^2-1}{x-1}$

c. $h(x) = \begin{cases} x+1, & x \le 0 \\ x^2+1, & x > 0 \end{cases}$ **d.** $y = \sin x$

a.

b.

c.

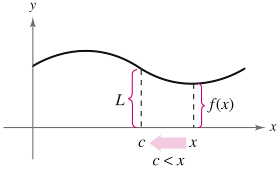
d.

One-sided limits and continuity on a closed interval

- To understand continuity on a closed interval, we first need to look at a different type of limit called a <u>one-sided limit</u>.

One-sided limits and continuity on a closed interval

- To understand continuity on a closed interval, we first need to look at a different type of limit called a one-sided limit.
- For example, the limit from the right (or right-hand limit) means that x approaches c from values greater than c [see Figure 15(a)].



(a) Limit from right.

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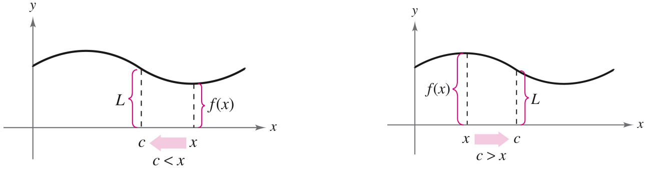

(a) Limit from right.

(b) Limit from left.

Figure 15: One-sided limits.

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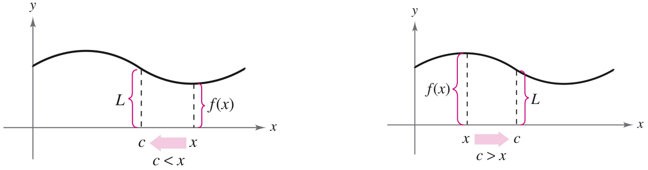

(a) Limit from right.

(b) Limit from left.

Figure 15: One-sided limits.

- This limit is denoted as

$$\lim_{x \to c^+} f(x) = L. \qquad \text{Limit from the right}$$

- Similarly, the limit from the left (or left-hand limit) means that x approaches c from values less than c [see Figure 15(b)].
- This limit is denoted as

$$\lim_{x \to c^-} f(x) = L. \qquad \text{Limit from the left}$$

- Similarly, the limit from the left (or left-hand limit) means that x approaches c from values less than c [see Figure 15(b)].
- This limit is denoted as

$$\lim_{x \to c^-} f(x) = L. \qquad \text{Limit from the left}$$

- One-sided limits are useful in taking limits of functions involving radicals. For instance, if n is an even integer,

$$\lim_{x \to 0^+} \sqrt[n]{x} = 0.$$

Definition 1.3 (One-sided limit, c.f. definition 1.1)

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c^+} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$c < x < c + \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

To define the limit from the left, replace $c < x < c + \delta$ with $c - \delta < x < c$.

Example 2 (A one-sided limit)

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

- One-sided limits can be used to investigate the behavior of step functions. One common type of step function is the greatest integer function $\lfloor x \rfloor$, defined by

$$\lfloor x \rfloor = \text{greatest integer } n \text{ such that } n \leq x.$$

- For instance, $\lfloor 2.5 \rfloor = 2$ and $\lfloor -2.5 \rfloor = -3$.

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Example 3 (The greatest integer function)

Find the limit of $f(x) = \lfloor x \rfloor$ as x approaches 0 from the left and from the right.

Theorem 1.10 (The existence of a limit)

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \textit{and} \quad \lim_{x \to c^+} f(x) = L.$$

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Definition 1.4 (Continuity on a closed interval)

A function f is continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

The function f is continuous from the right at a and continuous from the left at b (see Figure 16).

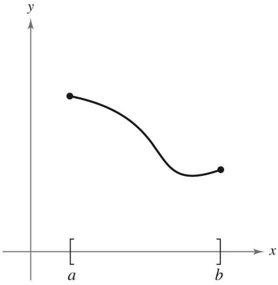

Figure 16: Continuous function on a closed interval.

Example 4 (Continuity on a closed interval)

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

Properties of continuity

Theorem 1.11 (Properties of continuity, c.f. Theorem 1.2)

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. *Scalar multiple:* bf
2. *Sum or difference:* $f \pm g$
3. *Product:* fg
4. *Quotient:* $\frac{f}{g}$, *if* $g(c) \neq 0$

- The following types of functions are continuous at every point in their domains.
    1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
    2. Rational: $r(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$
    3. Radical: $f(x) = \sqrt[n]{x}$
    4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

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- By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains!

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- By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains!

Example 6 (Applying properties of continuity)

By Theorem 1.11, it follows that each of the functions below is continuous at every point in its domain.

$$f(x) = x + \sin x, \quad f(x) = 3\tan x, \quad f(x) = \frac{x^2+1}{\cos x}. \qquad \blacksquare$$

- The next theorem, which is a consequence of Theorem 1.5, allows you to determine the continuity of composite functions such as

$$f(x) = \sin 3x, \quad f(x) = \sqrt{x^2 + 1}, \quad f(x) = \tan \frac{1}{x}.$$

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Theorem 1.12 (Continuity of a composite function)

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Example 7 (Testing for continuity)

Describe the interval(s) on which each function is continuous.

a. $f(x) = \tan x$ **b.** $g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c. $h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

a.

b.

c.

The Intermediate Value Theorem

A theorem verifying that the graph of a continuous function is connected.

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Theorem 1.13 (The Intermediate Value Theorem)

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

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Theorem 1.13 (The Intermediate Value Theorem)

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

- The Intermediate Value Theorem tells us that at least one number c exists, but it does not provide a method for finding c. Such theorems are called existence theorems. A proof of this theorem is based on a property of real numbers called *completeness*.

- The Intermediate Value Theorem states that for a continuous function f, if x takes on all values between a and b, $f(x)$ must take on all values between $f(a)$ and $f(b)$.

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- Suppose that a girl is 160 centimeters tall on her thirteenth birthday and 165 centimeters tall on her fourteenth birthday. Then, for any height h between 160 centimeters and 165 centimeters, there must have been a time t when her height was exactly h. This seems reasonable because human growth is continuous and a person's height does not abruptly change from one value to another.

- The Intermediate Value Theorem guarantees the existence of at least one number c in the closed interval $[a, b]$. There may, of course, be more than one number c such that $f(c) = k$, as shown in Figure 17.

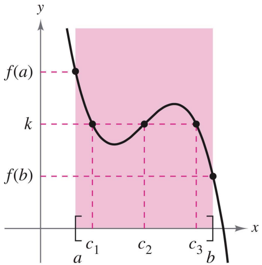

Figure 17: Intermediate Value Theorem: f is continuous on $[a, b]$. (There exists three c's such that $f(c) = k$.)

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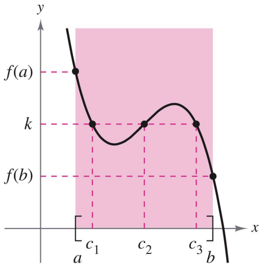

Figure 17: Intermediate Value Theorem: f is continuous on $[a, b]$. (There exists three c's such that $f(c) = k$.)

- A function that is not continuous does not necessarily exhibit the intermediate value property.

- For example, the graph of the function shown in Figure 18 jumps over the horizontal line given by $y = k$, and for this function there is no value of c in $[a, b]$ such that $f(c) = k$.

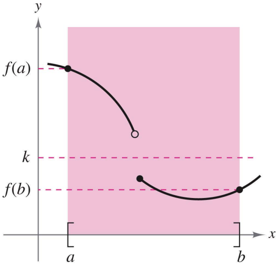

Figure 18: f is not continuous on $[a, b]$.

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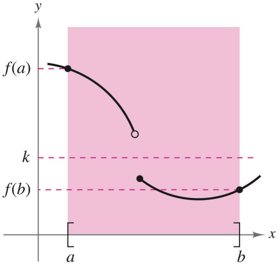

Figure 18: f is not continuous on $[a, b]$.

- The Intermediate Value Theorem can often be used to locate the zeros of a function that is continuous on a closed interval: if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, the Intermediate Value Theorem guarantees the existence of at least one zero of f in the closed interval $[a, b]$.

Example 8 (An application of the Intermediate Value Theorem)

Use the Intermediate Value Theorem to show that the polynomial function $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

- The *Bisection Method* for approximating the real zeros of a continuous function are similar to the method used in Example 8.

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- If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a+b)/2]$ or $[(a+b)/2, b]$.

- The *Bisection Method* for approximating the real zeros of a continuous function are similar to the method used in Example 8.
- If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval $[a, (a + b)/2]$ or $[(a + b)/2, b]$.
- From the sign of $f([a + b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

Table of Contents

Infinite limits

- Let f be the function given by $3/(x-2)$. From Figure 19 and the table, you can see that $f(x)$ decreases without bound as x approaches 2 from the left, and $f(x)$ increases without bound as x approaches 2 from the right.

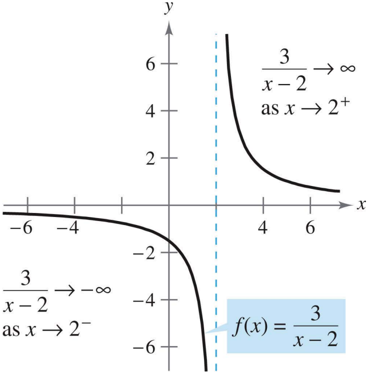

Figure 19: $f(x) = \frac{3}{x-2}$ increases and decreases without bound as x approaches 2.

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

- This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty$$

$f(x)$ decreases without bound as x approaches 2 from the left

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty$$

$f(x)$ increases without bound as x approaches 2 from the right

Definition 1.5 (Infinite limit, c.f. definition 1.1)

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 20).

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means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.
To define the infinite limit from the left, replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the infinite limit from the right, replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

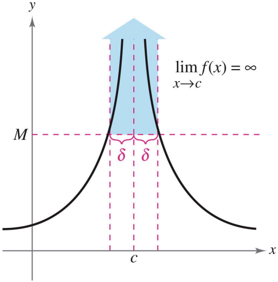
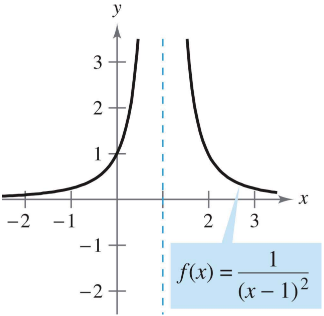


Figure 20: Infinite limits.

Example 1 (Determining infinite limits from a graph)

Determine the limit of each function shown in Figure 21 as x approaches 1 from the left and from the right.



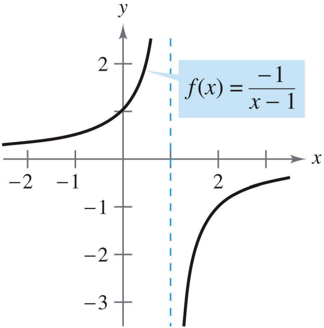


Figure 21: $f(x) = \frac{1}{(x-1)^2}$ and $f(x) = \frac{-1}{x-1}$ have an asymptote at $x = 1$.

a.

b.

Vertical asymptotes

Definition 1.6 (Vertical asymptote)

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a vertical asymptote of the graph of f.

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If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a vertical asymptote of the graph of f.

Theorem 1.14 (Vertical asymptotes)

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

Example 2 (Find vertical asymptotes)

Determine all vertical asymptotes of the graph of each function. **a.** $f(x) = \frac{1}{2(x+1)}$ **b.** $f(x) = \frac{x^2+1}{x^2-1}$ **c.** $f(x) = \cot x$

a.

b.

c.

- Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. If both the numerator and the denominator are 0 at $x = c$, you obtain the indeterminate form $0/0$, and you cannot determine the limit at $x = c$ without further investigation, as illustrated next.

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Example 3 (A rational function with common factors)

Determine all vertical asymptotes of the graph of

$$h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Example 4 (Determining infinite limits)

Find each limit

$$\lim_{x \to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x - 1}.$$

Theorem 1.15 (Properties of infinite limits)

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \textit{and} \quad \lim_{x \to c} g(x) = L.$$

1. *Sum or difference:* $\lim_{x \to c} [f(x) \pm g(x)] = \infty$
2. *Product:*

$$\lim_{x \to c} [f(x)g(x)] = \infty, \quad \underline{L > 0}$$
$$\lim_{x \to c} [f(x)g(x)] = -\infty, \quad \underline{L < 0}$$

3. *Quotient:* $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$

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3. *Quotient:* $\lim_{x\to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Example 5 (Determining limits)

a. $\lim_{x\to 0}\left(1+\frac{1}{x^2}\right)$ **b.** $\lim_{x\to 1^-}\frac{x^2+1}{\cot \pi x}$

c. $\lim_{x\to 0^+} 3\cot x$ **d.** $\lim_{x\to 0^-}\left(x^2+\frac{1}{x}\right)$