1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) (5%) $\lim\limits_{x \to 1^+} \dfrac{2x}{x-1} - \dfrac{2}{\ln x}$

(b) (5%) $\lim\limits_{x \to 0} \cos(x)^{\frac{1}{x^2}}$

(c) (5%) $\lim\limits_{x \to \infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

(d) (5%) $\lim\limits_{x \to \infty} \dfrac{e^x}{x}$

Ans:

(a) $\lim\limits_{x \to 1^+} \dfrac{2x}{x-1} - \dfrac{2}{\ln x} = \lim\limits_{x \to 1^+} \dfrac{2x \ln x - 2(x-1)}{(x-1)\ln x}$(L' Hôpital' s rule)$= 2\lim\limits_{x \to 1^+} \dfrac{\ln x}{\ln x + \frac{x-1}{x}}$ (L'

Hôpital' s rule)$= 2\lim\limits_{x \to 1^+} \dfrac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = 1$

(b) $y = \lim\limits_{x \to 0} \cos(x)^{\frac{1}{x^2}}$

$\ln y = \ln \lim\limits_{x \to 0} \cos(x)^{\frac{1}{x^2}} = \lim\limits_{x \to 0} \ln \cos(x)^{\frac{1}{x^2}} =$

$\lim\limits_{x \to 0} \dfrac{1}{x^2} \ln \cos(x)$ (L' Hôpital' s rule) $= \lim\limits_{x \to 0} \dfrac{\frac{-\sin(x)}{\cos(x)}}{2x}$(L' Hôpital' s rule)$=$

$\lim\limits_{x \to 0} \dfrac{-\sec^2(x)}{2} = \dfrac{-1}{2}$

Since $\ln y = \dfrac{-1}{2}$ Therefore, $y = e^{\frac{-1}{2}}$

(c) $\lim\limits_{x \to \infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x \to \infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = 1$

Since $\lim\limits_{x \to \infty} e^{-2x} = 0$

(d) $\lim\limits_{x \to \infty} \dfrac{e^x}{x}$ (L'Hôpital's rule) $= \lim\limits_{x \to \infty} e^x = \infty$

2. (15%) Solve the following problems

(a) Evaluate $\int_{-2}^{2} xe^{-x^4} dx$

(b) Find $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\frac{1}{1+\frac{i^2}{n^2}}\right]$

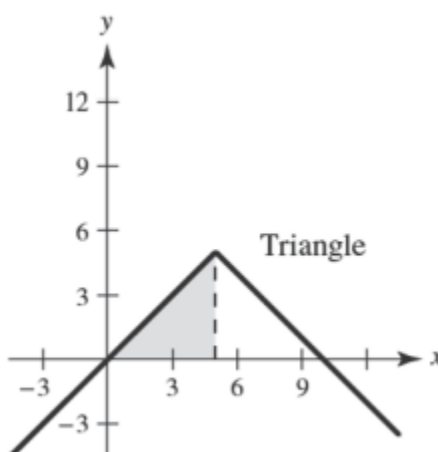(c) $\int_{0}^{5}(5 - |x - 5|)\, dx$

Ans:

(a) Note that let $f(x) = xe^{-x^4}$, we have $f(-x) = -f(-x)$. It is an odd function,

   therefore, $\int_{-2}^{2} xe^{-x^4}\, dx = 0$

(b) $\lim\limits_{n\to\infty}\left[\frac{1}{1+\frac{1}{n^2}} + \frac{1}{1+\frac{2^2}{n^2}} + \frac{1}{1+\frac{3^2}{n^2}} + \cdots + \frac{1}{1+\frac{n^2}{n^2}}\right]\frac{1}{n} = \int_{0}^{1}\frac{1}{1+x^2}dx = tan^{-1}(x)|_{0}^{1} = \frac{\pi}{4}$

(c) It is the region below, thus $\int_{0}^{5}(5 - |x - 5|)\, dx = \frac{1}{2}5 \times 5 = \frac{25}{2}$

3. (9%) Let $f(x) = \int_{1}^{x} \sqrt{1 + t^4}\, dt$

    (a) (3%) Show that $f(x)$ has an inverse function
    (b) (3%) What is the value of $f^{-1}(x)$ when $x = 0$
    (c) (3%) What is the value of $(f^{-1})'(x)$ when $x = 0$

    Ans:

(a) Note that $f'(x) = \sqrt{1 + x^4} > 0$ for all $x \to f$ is strictly increasing therefore is
    one to one and has an inverse function.

(b) Because $f(1) = 0$, we know that $f^{-1}(0) = 1$

(c) $f'(x) = \sqrt{1 + x^4}$

    Because f is differentiable and has an inverse function, we have

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{\sqrt{2}}$$

4. (10%) Compute the derivative.

(a) (5%) $f(x) = \log_7 \frac{x\sqrt{x-1}}{2}$

(b) (5%) $f(x) = e^{\sin(x)} + \cos(e^{2x}) - 2^{\sqrt{x}}$

Ans:

(a) $f(x) = \log_7 x + \frac{1}{2}\log_7(x-1)$

$$f'(x) = \frac{1}{x\ln(7)} + \frac{1}{2(x-1)\ln(7)}$$

(b) $f'(x) = \cos(x) e^{\sin(x)} - 2e^{2x}\sin(e^{2x}) - 2^{\sqrt{x}}\ln(2)\frac{1}{2\sqrt{x}}$

5. (25%) Evaluate the following integral.

(a) (5%) $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

(b) (5%) $\int \frac{\cos(1-\ln\theta)d\theta}{\theta}$

(c) (5%) $\int_0^1 3^{x^2+2x}(x+1)dx$

(d) (5%) $\int \frac{4\tan^{-1}\left(\frac{x}{2}\right)}{4+x^2}dx$

(e) (5%) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)\sqrt{\sin^2(x)-\frac{1}{4}}}dx$

Ans:

(a) $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}} = \int_1^{\sqrt{2}} \frac{2udu}{2\sqrt{4-u^2}u}$ (Let $u = \sqrt{x+1} \to u^2 = x+1, 2udu = dx,$

$\sqrt{3-x} = \sqrt{4-u^2})$

$$= \int_1^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}} = \sin^{-1}\frac{u}{2}\Big]_1^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

(b) $u = (1-\ln\theta),\ du = \frac{-1}{\theta}d\theta$

$$\int \frac{\cos(1-\ln\theta)d\theta}{\theta} = \int -\cos(u)du = -\sin(u) + C = -\sin(1-\ln\theta) + C$$

(c) Let $u = x^2 + 2x, du = 2(x+1)dx$

$$\int_0^1 3^{x^2+2x}(x+1)dx = \int_0^3 3^u \frac{1}{2}\,du = \frac{1}{2\ln 3}3^u\Big|_0^3 = \frac{26}{2\ln 3} = \frac{13}{\ln 3}$$

(d) Let $u = tan^{-1}\left(\frac{x}{2}\right), du = \frac{2}{4+x^2}dx$

$$\int \frac{4tan^{-1}(\frac{x}{2})}{4+x^2}dx = 2\int u\,du = u^2 + C = \left(tan^{-1}(\frac{x}{2})\right)^2 + C$$

(e) Let $u = \sin(x), du = \cos(x)dx$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)\sqrt{\sin^2(x) - \frac{1}{4}}}dx = \int_{\frac{\sqrt{3}}{2}}^{1} \frac{du}{u\sqrt{u^2 - \frac{1}{4}}} = 2\sec^{-1}2|u|\,\Big|_{\frac{\sqrt{3}}{2}}^{1}$$

$$= 2\sec^{-1}2 - 2\sec^{-1}\sqrt{3}$$

6. (7%) Find the area of the given region bounded by the graphs of $x = \cos(y)$ and $x = \frac{1}{2}$ on the interval $\frac{\pi}{3} \le y \le \frac{7\pi}{3}$.

Ans:

$$A = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{1}{2} - \cos(y)\,dy + \int_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} \cos(y) - \frac{1}{2}dy = \frac{y}{2} - \sin(y)\Big]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} + \sin(y) - \frac{y}{2}\Big]_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} =$$

$$\frac{\pi}{3} + 2\sqrt{3}$$

7. (7%) Find the volume of the solid formed by revolving the region bounded by the graphs of the equations $y = \frac{1}{x^2}, y = 0, x = 2, x = 5$ about the y-axis.

Ans:

$$V = 2\pi \int_2^5 x\frac{1}{x^2}dx = 2\pi\left[\int_2^5 \frac{1}{x}dx\right] = 2\pi\ln|x|\,|_2^5 = 2\pi\ln\frac{5}{2}$$

8. (7%) Find the area of the surface generated by revolving the curve $y = \dfrac{x^3}{18}$ on

   the interval $3 \le x \le 6$ abouty the x-axis.

   Ans:

   $y' = \dfrac{x^2}{6}, 1 + (y')^2 = \dfrac{(36+x^4)}{36}$

   $$S = 2\pi \int_3^6 \frac{x^3}{18}\sqrt{\frac{(36+x^4)}{36}}\, dx = \frac{\pi}{54}\int_3^6 \sqrt{36+x^4}\, x^3\, dx$$

   Let $u = 36 + x^4, du = 4x^3 dx$

   $$S = \frac{\pi}{54}\int_3^6 \sqrt{36+x^4}\, x^3\, dx = \frac{\pi}{216}\int_{117}^{1332} \sqrt{u}\, du = \frac{\pi}{324} u^{\frac{3}{2}}\Big|_{117}^{1332}$$

   $$= \frac{\pi}{324}\left(1332^{\frac{3}{2}} - 117^{\frac{3}{2}}\right)$$