

1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) (5%)  $\lim_{x \rightarrow 1^+} \frac{2x}{x-1} - \frac{2}{\ln x}$

(b) (5%)  $\lim_{x \rightarrow 0} \cos(x)^{\frac{1}{x^2}}$

(c) (5%)  $\lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(d) (5%)  $\lim_{x \rightarrow \infty} \frac{e^x}{x}$

**Ans:**

(a)  $\lim_{x \rightarrow 1^+} \frac{2x}{x-1} - \frac{2}{\ln x} = \lim_{x \rightarrow 1^+} \frac{2x \ln x - 2(x-1)}{(x-1) \ln x}$  (L' Hôpital' s rule)  $= 2 \lim_{x \rightarrow 1^+} \frac{\ln x}{\ln x + \frac{x-1}{x}}$  (L'

Hôpital' s rule)  $= 2 \lim_{x \rightarrow 1^+} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = 1$

(b)  $y = \lim_{x \rightarrow 0} \cos(x)^{\frac{1}{x^2}}$

$\ln y = \ln \lim_{x \rightarrow 0} \cos(x)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} \ln \cos(x)^{\frac{1}{x^2}} =$

$\lim_{x \rightarrow 0} \frac{1}{x^2} \ln \cos(x)$  (L' Hôpital' s rule)  $= \lim_{x \rightarrow 0} \frac{\frac{-\sin(x)}{\cos(x)}}{2x}$  (L' Hôpital' s rule)  $=$

$\lim_{x \rightarrow 0} \frac{-\sec^2(x)}{2} = \frac{-1}{2}$

Since  $\ln y = \frac{-1}{2}$  Therefore,  $y = e^{\frac{-1}{2}}$

(c)  $\lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \rightarrow \infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = 1$

Since  $\lim_{x \rightarrow \infty} e^{-2x} = 0$

(d)  $\lim_{x \rightarrow \infty} \frac{e^x}{x}$  (L'Hôpital's rule)  $= \lim_{x \rightarrow \infty} e^x = \infty$

2. (15%) Solve the following problems

(a) Evaluate  $\int_{-2}^2 x e^{-x^4} dx$

(b) Find  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left[ \frac{1}{1 + \frac{i^2}{n^2}} \right]$

(c)  $\int_0^5 (5 - |x - 5|) dx$

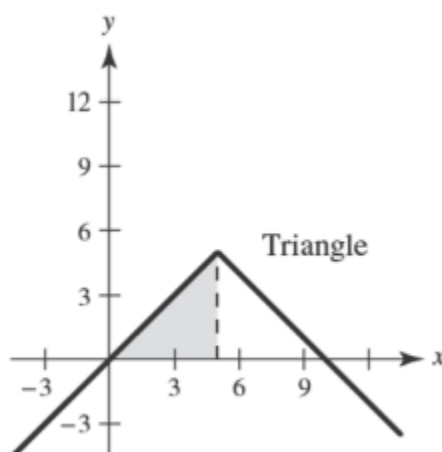
**Ans:**

(a) Note that let  $f(x) = x e^{-x^4}$ , we have  $f(-x) = -f(x)$ . It is an odd function,

therefore,  $\int_{-2}^2 x e^{-x^4} dx = 0$

(b)  $\lim_{n \rightarrow \infty} \left[ \frac{1}{1 + \frac{1}{n^2}} + \frac{1}{1 + \frac{2^2}{n^2}} + \frac{1}{1 + \frac{3^2}{n^2}} + \cdots + \frac{1}{1 + \frac{n^2}{n^2}} \right] \frac{1}{n} = \int_0^1 \frac{1}{1+x^2} dx = \tan^{-1}(x) \Big|_0^1 = \frac{\pi}{4}$

(c) It is the region below, thus  $\int_0^5 (5 - |x - 5|) dx = \frac{1}{2} 5 \times 5 = \frac{25}{2}$



3. (9%) Let  $f(x) = \int_1^x \sqrt{1+t^4} dt$

(a) (3%) Show that  $f(x)$  has an inverse function

(b) (3%) What is the value of  $f^{-1}(x)$  when  $x = 0$

(c) (3%) What is the value of  $(f^{-1})'(x)$  when  $x = 0$

**Ans:**

(a) Note that by the fundamental theorem of calculus  $f'(x) = \sqrt{1+x^4} > 0$  for all  $x \rightarrow f$  is strictly increasing therefore is one to one and has an inverse function.

(b) Because  $f(1) = 0$ , we know that  $f^{-1}(0) = 1$

(c)  $f'(x) = \sqrt{1+x^4}$

Because  $f$  is differentiable and has an inverse function, we have

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{\sqrt{2}}$$

4. (10%) Compute the derivative.

(a) (5%)  $f(x) = \log_7 \frac{x\sqrt{x-1}}{2}$

(b) (5%)  $f(x) = e^{\sin(x)} + \cos(e^{2x}) - 2^{\sqrt{x}}$

**Ans:**

(a)  $f(x) = \log_7 x + \frac{1}{2} \log_7(x-1)$

$$f'(x) = \frac{1}{x \ln(7)} + \frac{1}{2(x-1) \ln(7)}$$

(b)  $f'(x) = \cos(x) e^{\sin(x)} - 2e^{2x} \sin(e^{2x}) - 2^{\sqrt{x}} \ln(2) \frac{1}{2\sqrt{x}}$

5. (25%) Evaluate the following integral.

(a) (5%)  $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

(b) (5%)  $\int \frac{\cos(1-\ln \theta) d\theta}{\theta}$

(c) (5%)  $\int_0^1 3^{x^2+2x} (x+1) dx$

(d) (5%)  $\int \frac{4 \tan^{-1}\left(\frac{x}{2}\right)}{4+x^2} dx$

(e) (5%)  $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x) \sqrt{\sin^2(x) - \frac{1}{4}}} dx$

**Ans:**

(a)  $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}} = \int_1^{\sqrt{2}} \frac{2udu}{2\sqrt{4-u^2}u}$  (Let  $u = \sqrt{x+1} \rightarrow u^2 = x+1, 2udu = dx,$

$\sqrt{3-x} = \sqrt{4-u^2}$ )

$$= \int_1^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}} = \sin^{-1} \frac{u}{2} \Big|_1^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

(b)  $u = (1 - \ln \theta), du = \frac{-1}{\theta} d\theta$

$$\int \frac{\cos(1 - \ln \theta) d\theta}{\theta} = \int -\cos(u) du = -\sin(u) + C = -\sin(1 - \ln \theta) + C$$

(c) Let  $u = x^2 + 2x, du = 2(x + 1)dx$

$$\int_0^1 3^{x^2+2x}(x+1)dx = \int_0^3 3^u \frac{1}{2} du = \frac{1}{2 \ln 3} 3^u \Big|_0^3 = \frac{26}{2 \ln 3} = \frac{13}{\ln 3}$$

(d) Let  $u = \tan^{-1}\left(\frac{x}{2}\right), du = \frac{2}{4+x^2} dx$

$$\int \frac{4 \tan^{-1}\left(\frac{x}{2}\right)}{4+x^2} dx = 2 \int u du = u^2 + C = \left(\tan^{-1}\left(\frac{x}{2}\right)\right)^2 + C$$

(e) Let  $u = \sin(x), du = \cos(x)dx$

$$\begin{aligned} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x) \sqrt{\sin^2(x) - \frac{1}{4}}} dx &= \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{u \sqrt{u^2 - \frac{1}{4}}} = 2 \sec^{-1} 2|u| \Big|_{\frac{\sqrt{3}}{2}}^1 \\ &= 2 \sec^{-1} 2 - 2 \sec^{-1} \sqrt{3} \end{aligned}$$

6. (7%) Find the area of the given region bounded by the graphs of  $x = \cos(y)$  and  $x = \frac{1}{2}$  on the interval  $\frac{\pi}{3} \leq y \leq \frac{7\pi}{3}$ .

**Ans:**

$$\begin{aligned} A &= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{1}{2} - \cos(y) dy + \int_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} \cos(y) - \frac{1}{2} dy = \left[ \frac{y}{2} - \sin(y) \right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} + \left[ \sin(y) - \frac{y}{2} \right]_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} = \\ &\frac{\pi}{3} + 2\sqrt{3} \end{aligned}$$

7. (7%) Find the volume of the solid formed by revolving the region bounded by the graphs of the equations  $y = \frac{1}{x^2}, y = 0, x = 2, x = 5$  about the y-axis.

**Ans:**

$$V = 2\pi \int_2^5 x \frac{1}{x^2} dx = 2\pi \left[ \int_2^5 \frac{1}{x} dx \right] = 2\pi \ln|x| \Big|_2^5 = 2\pi \ln \frac{5}{2}$$

8. (7%) Find the area of the surface generated by revolving the curve  $y = \frac{x^3}{18}$  on the interval  $3 \leq x \leq 6$  about the  $x$ -axis.

**Ans:**

$$y' = \frac{x^2}{6}, 1 + (y')^2 = \frac{(36+x^4)}{36}$$

$$S = 2\pi \int_3^6 \frac{x^3}{18} \sqrt{\frac{(36+x^4)}{36}} dx = \frac{\pi}{54} \int_3^6 \sqrt{36+x^4} x^3 dx$$

Let  $u = 36 + x^4, du = 4x^3 dx$

$$\begin{aligned} S &= \frac{\pi}{54} \int_3^6 \sqrt{36+x^4} x^3 dx = \frac{\pi}{216} \int_{117}^{1332} \sqrt{u} du = \frac{\pi}{324} u^{\frac{3}{2}} \Big|_{117}^{1332} \\ &= \frac{\pi}{324} (1332^{\frac{3}{2}} - 117^{\frac{3}{2}}) \end{aligned}$$