Calculus (I), Final Exam 2025/1/3

Note that e is euler constant in all the following questions

1. (15%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

   (a) $\lim_{x\to 0^+} \sin(x)^{\frac{1}{\ln(x)}}$

   (b) $\lim_{x\to 2^+} \frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4}$

   (c) $\lim_{x\to\infty} 2x \cdot tan(\frac{1}{x})$

2. (4%) Suppose f is a function such that $f(1) = 1, f'(1) = 3, f''(1) = 5,$ $f(2) = -2, f'(4) = -4, f''(6) = -6,$ f' and f'' are both continuous everywhere. Evaluate $\int_1^2 f'(x)dx$ and $\int_1^2 f''(x)dx$.

3. (9%) Let $f(x) = x^3 + 3x + 1$
   (a) (3%) Show that $f(x)$ has an inverse function
   (b) (3%) What is the value of $f^{-1}(x)$ when $x = 5$
   (c) (3%) What is the value of $(f^{-1})'(x)$ when $x = 5$

4. (20%) Evaluate the following integral.
   (a) $\int \cos(x) \times \sin(\sin(x))$ dx

   (b) $\int_0^1 (5^x - 3^x)dx$

   (c) $\int_1^{e^2} \ln(x)\, dx$

   (d) $\int tan^3(\theta)sec^4(\theta)\, d\theta$

   (e) $\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2} dx$

5. (4%) Sketch the region enclosed by the given curves and find its area: $y = \frac{1}{x}, y = x,$ and $y = 4x$ for $x \geq 0$

6. (12%) Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2, y = 1$ and the y-axis ($x = 0$) about the following lines:
   (a) About $x -$ axis (by the disk/washer method)
   (b) About $y -$axis (by the shell method)

7. (12%)
   (a) (3%) Write down the formula for finding the arc length of a curve defined by $y = \frac{x^2}{4} - 2x$ in terms of the variable x.
   (b) (3%) Use trigonometric substitution to express the integral for the arc length obtained in part (a) in terms of the variable θ.
   (c) (6%) Solve the integral you obtained in part (b) and calculate the total arc length of the curve over the interval $[4,8]$.

8. (4%) If we have an arc which is part of the circules $x^2 + y^2 = 4$ between the points $(-\sqrt{3}, 1)$ and $(\sqrt{3}, 1)$. Find the area of the surface generated by revolving the arc about the x-axis.

9. (20%) Evaluate the following integral. (If the integral is diverge, you should point it out)
   (a) $\int x \cdot arcsin(x^2)\, dx$
   (b) $\int \frac{x^2+2x}{x^3-x^2+x-1} dx$
   (c) $\int_0^\infty \frac{1}{e^x+e^{-x}} dx$

Derivative	Integrals
$\frac{d\sin^{-1}u}{dx} = \frac{u'}{\sqrt{1-u^2}}$	$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$
$\frac{d\,cos^{-1}u}{dx} = \frac{-u'}{\sqrt{1-u^2}}$	$\int \frac{du}{a^2+u^2} = \frac{1}{a}tan^{-1}\frac{u}{a} + C$
$\frac{d\tan^{-1}u}{dx} = \frac{u'}{1+u^2}$	$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}sec^{-1}\frac{\lvert u\rvert}{a} + C$
$\frac{d\,cot^{-1}u}{dx} = \frac{-u'}{1+u^2}$	
$\frac{d\sec^{-1}u}{dx} = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	
$\frac{d\,csc^{-1}u}{dx} = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$	