

Chapter 7 Applications of Integration

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

November 13, 2023

Table of Contents

- 1 Area of a region between two curves
- 2 Volume: The Disk Method
- 3 Arc length and surfaces of revolution

Table of Contents

1 Area of a region between two curves

2 Volume: The Disk Method

3 Arc length and surfaces of revolution

Area of a region between two curves

- With a few modifications, you can extend the application of definite integrals from the area under a curve to the area between two curves.
- Consider two functions f and g that are continuous on the interval $[a, b]$.

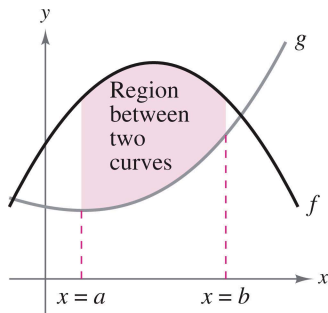


Figure 1: Area of a region between two curves.

- If, the graphs of both f and g lie above the x -axis, and the graph of g lies below the graph of f , you can geometrically interpret the area between the graphs as the area under the graph of g subtracted from the area under the graph of f , as shown in Figure 2.

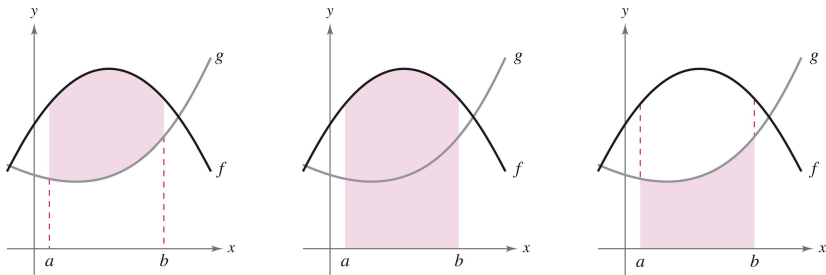


Figure 2: $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$

- To verify the reasonableness of the result shown in Figure 2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx .
- Then, as shown in Figure 3, sketch a representative rectangle of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the i th subinterval.

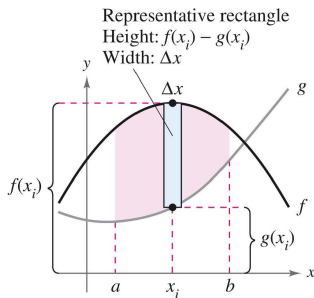


Figure 3: Representative rectangle. Height: $f(x_i) - g(x_i)$; Width: Δx .

- The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

- By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \rightarrow 0$ ($n \rightarrow \infty$), we obtain

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n [f(x_i) - g(x_i)]\Delta x.$$

- Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n [f(x_i) - g(x_i)]\Delta x = \int_a^b [f(x) - g(x)] dx.$$

Area of a region between two curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx.$$

- In Figure 1, the graphs of f and g are shown above the x -axis. This, however, is not necessary.
- The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ for all x in the interval $[a, b]$.
- Notice in Figure 4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x -axis.

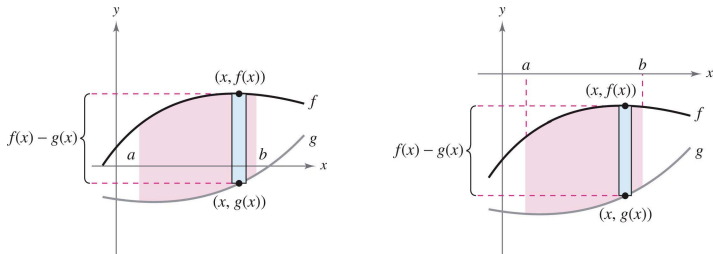


Figure 4: The height of a representative rectangle.

- Representative rectangles are used throughout this chapter in various applications of integration.
- A vertical rectangle (of width Δx) implies integration with respect to x , whereas a horizontal rectangle (of width Δy) implies integration with respect to y .

Example 1 (Finding the area of a region between two curves)

Find the area of the region bounded by the graphs of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

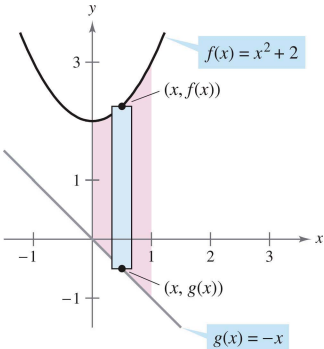


Figure 5: Region bounded by the graph of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

- In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly.
- A more common problem involves the area of a region bounded by two intersecting graphs, where the values of a and b must be calculated.

Example 2 (A region lying between two intersecting graphs)

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

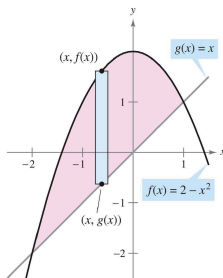


Figure 6: Region bounded by the graph of $f(x) = 2 - x^2$ and the graph of $g(x) = x$.

Example 3 (A region lying between two intersecting graphs)

The sine and cosine curves intersect infinitely many times, bounding regions of equal areas, as shown in Figure 7. Find the area of one of these regions.

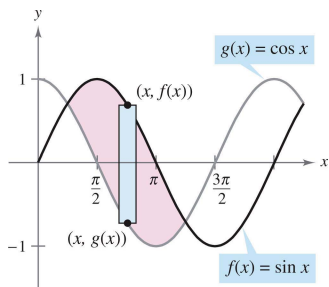


Figure 7: One of the regions bounded by the graphs of the sine and cosine functions.

Example 4 (Curves that intersect at more than two points)

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

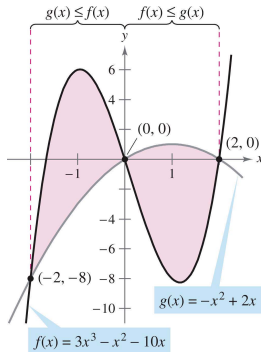
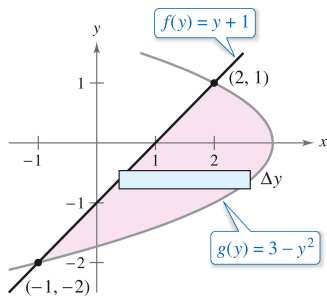


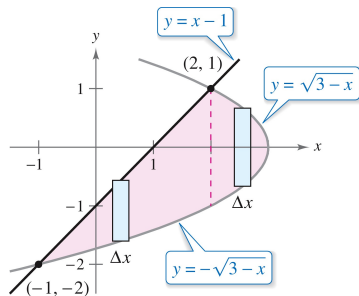
Figure 8: On $[-2, 0]$, $g(x) \leq f(x)$, and on $[0, 2]$, $f(x) \leq g(x)$.

Example 5 (Horizontal representative rectangles)

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.



(a) Horizontal rectangles
(integration with respect to y)



(b) Vertical rectangles
(integration with respect to x)

Figure 9: Horizontal rectangles v.s. vertical rectangles.

Table of Contents

- 1 Area of a region between two curves
- 2 Volume: The Disk Method
- 3 Arc length and surfaces of revolution

The Disk Method

- If a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution.
- The simplest such solid is a right circular cylinder or disk, which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, as shown in Figure 10.

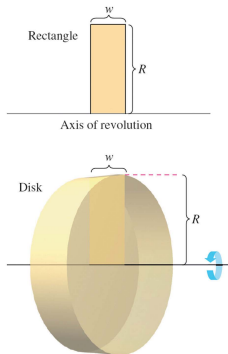


Figure 10: Volume of a disk: $\pi R^2 w$.

- The volume of such a disk is

$$\text{Volume of disk} = (\text{area of disk})(\text{width of disk}) = \pi R^2 w$$

where R is the radius of the disk and w is the width.

- To see how to use the volume of a disk to find the volume of a general solid of revolution, consider a solid of revolution formed by revolving the plane region in Figure 11 about the indicated axis.

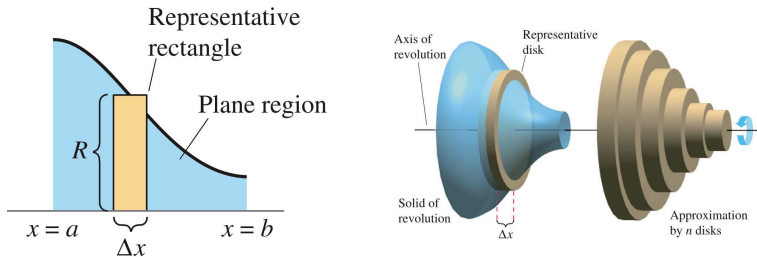


Figure 11: Disk Method.

- To determine the volume, consider a representative rectangle in the plane region. When this rectangle is revolved about the axis of revolution, it generates a representative disk whose volume is

$$\Delta V = \pi R^2 \Delta x.$$

- Approximating the volume of the solid by n such disks of width Δx and radius $R(x_i)$ produces

$$\text{Volume of solid} \approx \sum_{i=1}^n \pi [R(x_i)]^2 \Delta x = \pi \sum_{i=1}^n [R(x_i)]^2 \Delta x.$$

- This approximation appears to become better and better as $\|\Delta\| \rightarrow 0$ ($n \rightarrow \infty$). So, you can define the volume of the solid as

$$\text{Volume of solid} = \lim_{\|\Delta\| \rightarrow 0} \pi \sum_{i=1}^n [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2 dx.$$

- Schematically, the Disk Method looks like this.

Known precalculus formula		Representative element		New integration formula
Volume of disk $V = \pi R^2 w$	\Rightarrow	$\Delta V = \pi [R(x_i)]^2 \Delta x$	\Rightarrow	Solid of revolution $V = \pi \int_a^b [R(x)]^2 dx$

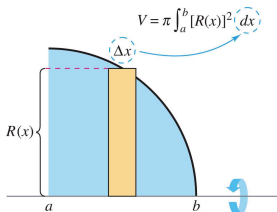
- A similar formula can be derived when the axis of revolution is vertical!

The Disk Method

To find the volume of a solid of revolution with the Disk Method, use one of the following, as shown below.

Horizontal axis of revolution

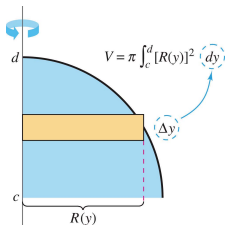
$$\frac{\text{Volume}}{\pi \int_a^b [R(x)]^2 dx} = \frac{V}{=}$$



(a) Horizontal axis of revolution.

Vertical axis of revolution

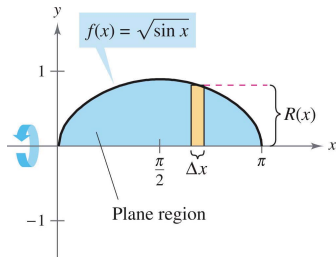
$$\frac{\text{Volume}}{\pi \int_c^d [R(y)]^2 dy} = \frac{V}{=}$$



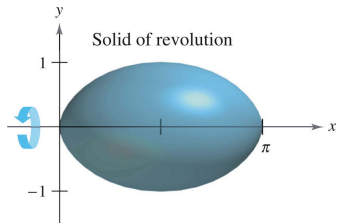
(b) Vertical axis of revolution.

Example 1 (Using the Disk Method)

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x -axis ($0 \leq x \leq \pi$) about the x -axis.



(a) Plane region.



(b) Solid of revolution.

Figure 13: Disk Method: $f(x) = \sqrt{\sin x}$.

Example 2 (Revolving about a line that is not a coordinate axis)

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$, as shown in Figure 14.

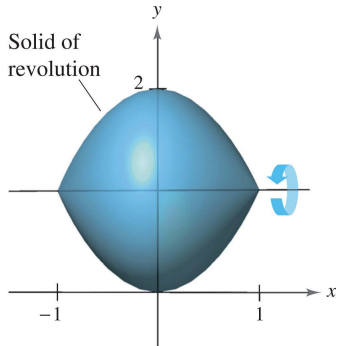
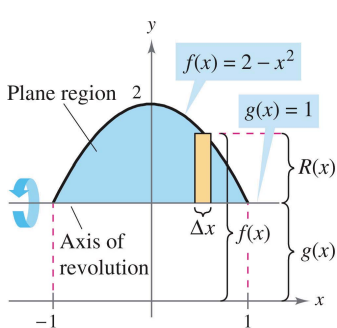


Figure 14: Revolving about a line that is not a coordinate axis.

The Washer Method

- The Disk Method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative washer.
- The washer is formed by revolving a rectangle about an axis. If r and R are the inner and outer radii of the washer and w is the width of the washer, the volume is given by $\text{Volume of washer} = \pi(R^2 - r^2)w$.
- To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an outer radius $R(x)$ and an inner radius $r(x)$, as shown in Figure 16.

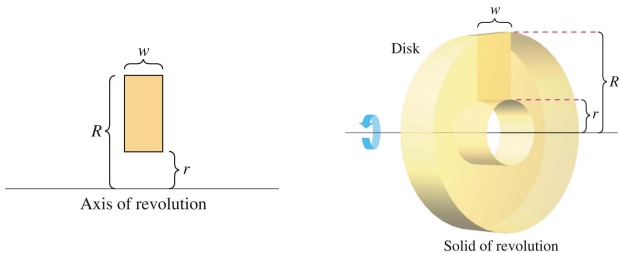


Figure 15: Washer Method.

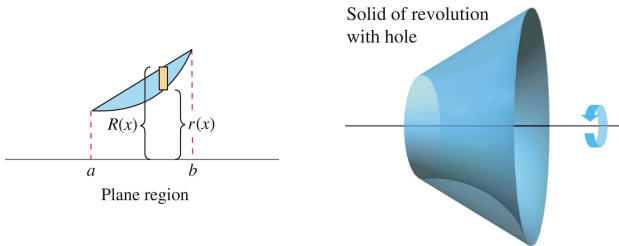


Figure 16: Solid of revolution with hole.

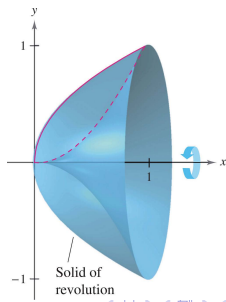
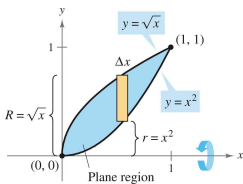
- If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx. \quad \text{Washer Method}$$

- Note that the integral involving the inner radius represents the volume of the hole and is subtracted from the integral involving the outer radius.

Example 3 (Using the Washer Method)

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x -axis, as shown below.



Example 4 (Integrating with respect to y , two-integral case)

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about y -axis, as shown in Figure 18.

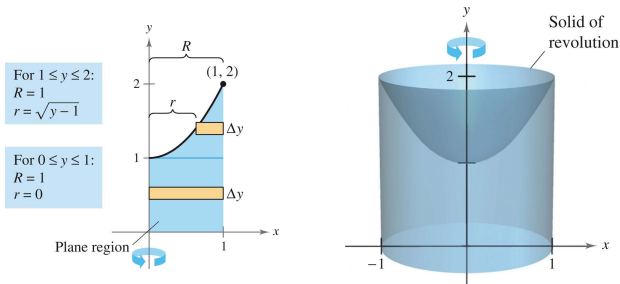


Figure 18: The volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about y -axis.

Solids with known cross sections

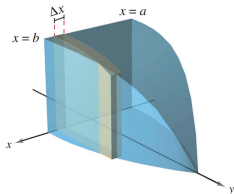
- With the Disk Method, you can find the volume of a solid having a circular cross section whose area is $A = \pi R^2$.
- This method can be generalized to solids of any shape, as long as you know a formula for the area of an arbitrary cross section. Some common cross sections are squares, rectangles, triangles, semicircles, and trapezoids.
- Volumes of solids with known cross sections are shown below

- ① For cross sections of area $A(x)$ taken perpendicular to the x -axis,

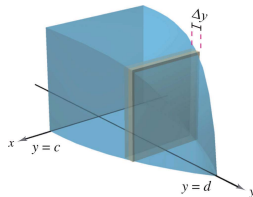
$$\text{Volume} = \int_a^b A(x) dx.$$

- ② For cross sections of area $A(y)$ taken perpendicular to the y -axis,

$$\text{Volume} = \int_c^d A(y) dy.$$



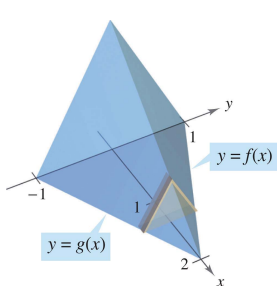
(a) Cross sections perpendicular to x -axis.



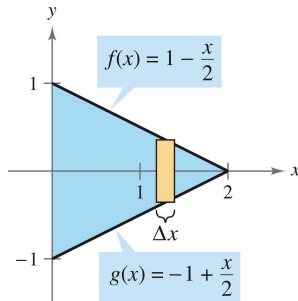
(b) Cross sections perpendicular to y -axis.

Example 6 (Triangular cross sections)

Find the volume of the solid shown in Figure 20. The base of the solid is the region bounded by the lines $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, and $x = 0$. The cross sections perpendicular to the x -axis are equilateral triangles.



(a) Cross sections are equilateral triangles.



(b) Triangular base in xy -plane.

Figure 20: Equilateral triangle cross section.

Table of Contents

- 1 Area of a region between two curves
- 2 Volume: The Disk Method
- 3 Arc length and surfaces of revolution

Arc length

- Definite integrals can also be use to find the arc length of curves and the areas of surfaces of revolution.
- In either case, an arc (a segment of a curve) is approximated by straight line segments whose lengths are given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- A rectifiable curve is one that has a finite arc length. You will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$.
- Such a function is continuously differentiable on $[a, b]$, and its graph on the interval $[a, b]$ is a smooth curve.

- Consider a function $y = f(x)$ that is continuously differentiable on the interval $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$:

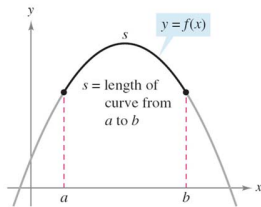
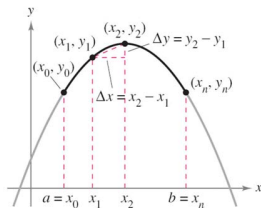


Figure 21: Arc length.

Definition 7.1 (Arc length)

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The arc length of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for a smooth curve given by $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Example 1 (The length of a line segment)

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of $f(x) = mx + b$

Example 2 (Finding arc length)

Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $[\frac{1}{2}, 2]$, as shown in Figure 22.

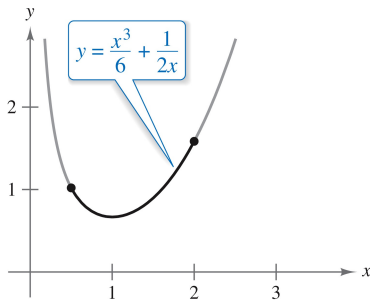


Figure 22: The arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on $[\frac{1}{2}, 2]$.

Example 4 (Finding arc length)

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$, as shown in Figure 23.

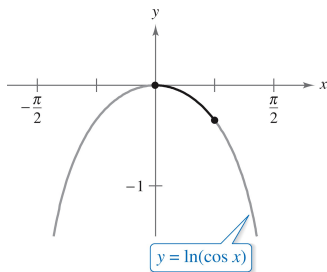


Figure 23: The arc length of the graph of y on $[0, \frac{\pi}{4}]$.

Area of a surface of revolution

Definition 7.2 (Surface of revolution)

If the graph of a continuous function is revolved about a line, the resulting surface is a surface of revolution.

- The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone.
- Consider the line segment in Figure 24, where L is the length of the line segment, r_1 is the radius at the left end of the line segment, and r_2 is the radius at the right end of the line segment.

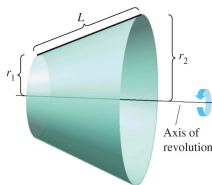


Figure 24: Surface of revolution.

- When the line segment is revolved about its axis of revolution, it forms a frustum of a right circular cone, with (Exercise 56)

$$S = 2\pi rL \quad \text{Lateral surface area of frustum}$$

where

$$r = \frac{1}{2}(r_1 + r_2). \quad \text{Average radius of frustum}$$

- Suppose the graph of a function f , having a continuous derivative on the interval $[a, b]$, is revolved about the x -axis to form a surface of revolution, as shown in Figure 25.

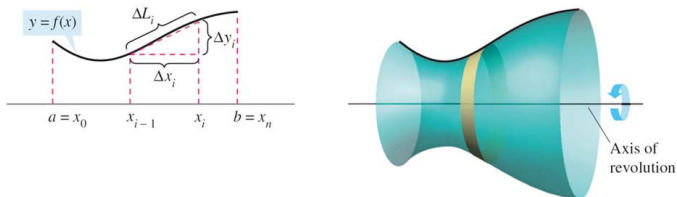


Figure 25: Surface of revolution.

- Let Δ be a partition of $[a, b]$, with width Δx_i . Then the line segment of length $\Delta L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$ generates a frustum of a cone.
- Let r_i be the average radius of this frustum. By the Intermediate Value Theorem, a point d_i exists (in the i th subinterval) such that $r_i = f(d_i)$. The lateral surface area ΔS_i of the frustum is

$$\Delta S_i = 2\pi r_i \Delta L_i = 2\pi f(d_i) \sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i.$$

- By Mean Value Theorem, a number c_i exists in (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}.$$

- So, $\Delta S_i = 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\Delta x_i$, and the total surface area can be approximated by

$$S \approx 2\pi \sum_{i=1}^n f(d_i)\sqrt{1 + [f'(c_i)]^2}\Delta x_i.$$

- It can be shown that limit of the right side as $\|\Delta\| \rightarrow 0$ ($n \rightarrow \infty$) is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} dx.$$

- In a similar manner, if the graph of f is revolved about the y -axis, then S is

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2} dx.$$

- In these two formulas for S , you can regard the products $2\pi f(x)$ and $2\pi x$ as the circumferences of the circles traced by a point (x, y) on the graph of f as it is revolved about the x -axis and the y -axis. The radius is $r = f(x)$, and $r = x$, respectively.

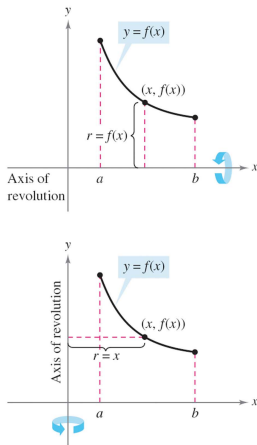


Figure 26: Revolve about x -axis and y -axis.

Definition 7.3 (Area of a surface of revolution)

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2} dx \quad y \text{ is a function of } x$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2} dy \quad x \text{ is a function of } y$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Example 5 (The area of a surface of revolution)

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x -axis, as shown in Figure 27.

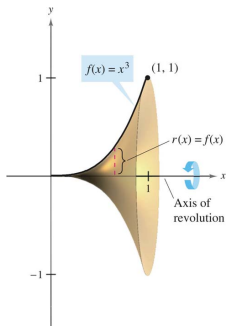


Figure 27: Area of a surface of revolution: $f(x) = x^3$ about x -axis.

Example 6 (The area of a surface of revolution)

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y -axis, as shown in Figure 28.

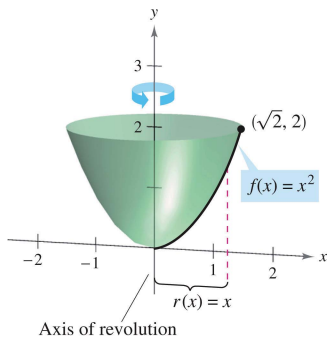


Figure 28: The area of a surface revolution: $f(x) = x^2$ about y -axis.