Chapter 8 Integration Techniques and Improper Integrals

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Fitting Integrands to Basic Integration Rules

Table 1: Review of basic integration rules ($a > 0$)

1. $\int kf(u)\,\mathrm{d}u = k\int f(u)\,\mathrm{d}u$	2. $\int [f(u) \pm g(u)]\,\mathrm{d}u$ $= \int f(u)\,\mathrm{d}u \pm \int g(u)\,\mathrm{d}u$
3. $\int \mathrm{d}u = u + C$	4. $\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$
5. $\int \frac{\mathrm{d}u}{u} = \ln\|u\| + C$	6. $\int e^u\,\mathrm{d}u = e^u + C$
7. $\int a^u\,\mathrm{d}u = \left(\frac{1}{\ln a}\right) a^u + C$	8. $\int \sin u\,\mathrm{d}u = -\cos u + C$
9. $\int \cos u\,\mathrm{d}u = \sin u + C$	10. $\int \tan u\,\mathrm{d}u = -\ln\|\cos u\| + C$
11. $\int \cot u\,\mathrm{d}u = \ln\|\sin u\| + C$	12. $\int \sec u\,\mathrm{d}u = \ln\|\sec u + \tan u\| + C$
13. $\int \csc u\,\mathrm{d}u = -\ln\|\csc u + \cot u\| + C$	14. $\int \sec^2 u\,\mathrm{d}u = \tan u + C$
15. $\int \csc^2 u\,\mathrm{d}u = -\cot u + C$	16. $\int \sec u \tan u\,\mathrm{d}u = \sec u + C$
17. $\int \csc u \cot u\,\mathrm{d}u = -\csc u + C$	18. $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$	20. $\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{\|u\|}{a} + C$

Example 1 (A comparison of three similar integrals)

Find each integral.

a. $\int \frac{4}{x^2+9}\,\mathrm{d}x$ **b.** $\int \frac{4x}{x^2+9}\,\mathrm{d}x$ **c.** $\int \frac{4x^2}{x^2+9}\,\mathrm{d}x$

Example 2 (Using two basic rules to solve a single integral)

Evaluate $\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,\mathrm{d}x$.

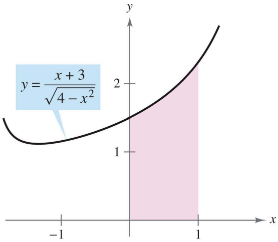

Figure 1: The area of the region is approximately 1.839.

Example 3 (A substitution involving $a^2 - u^2$)

Find $\int \frac{x^2}{\sqrt{16-x^6}}\,\mathrm{d}x$.

Example 4 (A disguised form of the Log Rule)

Find $\int \frac{1}{1+e^x}\,\mathrm{d}x$.

Example 5 (A disguised form of the Power Rule)

Find $\int (\cot x) \ln(\sin x) \, \mathrm{d}x$.

Example 6 (Using trigonometric identities)

Find $\int \tan^2 2x \, \mathrm{d}x$.

Procedures for fitting integrands to basic integration

Technique	Example
Expand (numerator).	$(1+e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$
Complete the square.	$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$
Divide improper rational function.	$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$
Add and subtract terms in numerator.	$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate	$\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x}$ $= \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$

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Integration by parts

- In this section you will study an important integration technique called integration by parts. This technique can be applied to a wide variety of functions and is particularly useful for integrands involving products of algebraic and transcendental functions.
- For instance, integration by parts works well with integrals such as

$$\int x \ln x \, \mathrm{d}x, \quad \int x^2 e^x \, \mathrm{d}x, \quad \text{and} \quad \int e^x \sin x \, \mathrm{d}x.$$

- Integration by parts is based on the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = uv' + vu'.$$

- If u' and v' are continuous, you can integrate both sides of this equation to obtain

$$uv = \int uv' \, \mathrm{d}x + \int vu' \, \mathrm{d}x = \int u \, \mathrm{d}v + \int v \, \mathrm{d}u.$$

Theorem 8.1 (Integration by Parts)

If u and v are functions of x and have continuous derivatives, then

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u = uv - \int vu'\,\mathrm{d}x.$$

Guidelines for integration by parts

1. Try letting $\mathrm{d}v$ be the most complicated portion of the integration rule. Then u will be remaining factor(s) of the integrand.
2. Trying letting u be the portion of the integrated whose derivative is a function simpler than u (LIATE). Then $\mathrm{d}v$ will be the remaining factor(s) of the integrand.

Note that $\mathrm{d}v$ always includes the $\mathrm{d}x$ of the original integrand.

Example 1 (Integration by parts)

Find $\int xe^x \, \mathrm{d}x$.

Example 2 (Integration by parts)

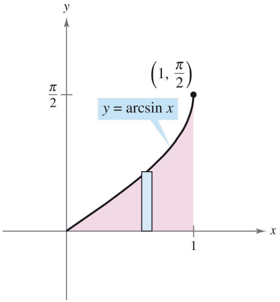
Find $\int x^2 \ln x \, \mathrm{d}x$.

Example 3 (An integrand with a single term)

Evaluate $\int \ln x \, \mathrm{d}x$.

Example 4 (An integrand with a single term)

Evaluate $\int_0^1 \sin^{-1} x \, \mathrm{d}x$.



y
$\left(1, \frac{\pi}{2}\right)$
$\frac{\pi}{2}$
$y = \arcsin x$
x
1

Example 5 (Repeated use of integration by parts)

Find $\int x^2 \sin x \, \mathrm{d}x$.

Example 6 (Integration by parts)

Find $\int \sec^3 x \, \mathrm{d}x$.

Summary of common integrals using integration by parts (LIATE)

1. For integrals of the form

$$\int x^n e^{ax}\,\mathrm{d}x, \quad \int x^n \sin ax\,\mathrm{d}x, \quad \text{or} \quad \int x^n \cos ax\,\mathrm{d}x$$

   let $u = x^n$ and let $\mathrm{d}v = e^{ax}\,\mathrm{d}x$, $\sin ax\,\mathrm{d}x$, $\cos ax\,\mathrm{d}x$.

2. For integrals of the form

$$\int x^n \ln x\,\mathrm{d}x, \quad \int x^n \arcsin ax\,\mathrm{d}x, \quad \text{or} \quad \int x^n \arctan ax\,\mathrm{d}x$$

   let $u = \ln x$, $\arcsin ax$, or $\arctan x$ and let $\mathrm{d}v = x^n\,\mathrm{d}x$.

3. For integrals of the form

$$\int e^{ax} \sin bx\,\mathrm{d}x, \quad \text{or} \quad \int e^{ax} \cos bx\,\mathrm{d}x$$

   let $u = \sin bx$ or $\cos bx$ and let $\mathrm{d}v = e^{ax}\,\mathrm{d}x$.

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Integrals involving powers of sine and cosine

- In this section you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x \, \mathrm{d}x \quad \text{and} \quad \int \sec^m x \tan^n x \, \mathrm{d}x$$

where either m or n is a positive integer.

- Break them into combinations of trigonometric integrals so you can apply the Power Rule. For instance, you can evaluate $\int \sin^5 x \cos x \, \mathrm{d}x$ by letting $u = \sin x$. Then, $\mathrm{d}u = \cos x \, \mathrm{d}x$ and you have

$$\int \sin^5 x \cos x \, \mathrm{d}x = \int u^5 \, \mathrm{d}u = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

- To break up $\int \sin^m x \cos^n x \, \mathrm{d}x$ into forms to which you can apply the Power Rule, use the following identities.

$$\sin^2 x + \cos^2 x = 1 \quad \sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Guidelines for evaluating integrals involving powers of sine and cosine

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x \,\mathrm{d}x = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x \,\mathrm{d}x}^{\text{Save for } \mathrm{d}u}$$
$$= \int (1-\cos^2 x)^k \cos^n x \sin x \,\mathrm{d}x$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x \,\mathrm{d}x = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to cosines}} \overbrace{\cos x \,\mathrm{d}x}^{\text{Save for } \mathrm{d}u}$$
$$= \int \sin^m x (1-\sin^2 x)^k \cos x \,\mathrm{d}x$$

3. If the power of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

Example 1 (Power of sine is odd and positive)

Find $\int \sin^3 x \cos^4 x \, \mathrm{d}x$.

Example 2 (Power of cosine is odd and positive)

Find $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,\mathrm{d}x$, as shown in Figure 2.

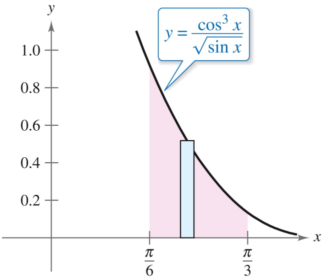

Figure 2: The area of the region is approximately 0.239.

Example 3 (Power of cosine is even and nonnegative)

Find $\int \cos^4 x \, \mathrm{d}x$.

Wallis's Formulas

a. If n is odd ($n \geq 3$), then

$$\int_0^{\pi/2} \cos^n x \, \mathrm{d}x = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right).$$

b. If n is even ($n \geq 2$), then

$$\int_0^{\pi/2} \cos^n x \, \mathrm{d}x = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

These formulas are also valid if $\cos^n x$ is replaced by $\sin^n x$.

- The following guidelines can help you evaluate integrals of the form $\int \sec^m x \tan^n x \,\mathrm{d}x$

Guidelines for evaluating integrals involving powers of secant and tangent

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{even}}} x \tan^n x \,\mathrm{d}x = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x \,\mathrm{d}x}^{\text{Save for } \mathrm{d}u}$$

$$= \int (1+\tan^2 x)^{k-1} \tan^n x \sec^2 x \,\mathrm{d}x$$

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x \, \mathrm{d}x = \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x \, \mathrm{d}x}^{\text{Save for } \mathrm{d}u}$$
$$= \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x \, \mathrm{d}x$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x \, \mathrm{d}x = \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} \, \mathrm{d}x$$
$$= \int \tan^{n-2} x (\sec^2 x - 1) \, \mathrm{d}x$$

4. If the integral is of the form $\int \sec^m x \,\mathrm{d}x$, where m is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.
5. If none of the first four guidelines applies, try converting to sines and cosines.

Example 4 (Power of tangent is odd and positive)

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,\mathrm{d}x$.

Example 5 (Power of secant is even and positive)

Find $\int \sec^4 3x \tan^3 3x \, \mathrm{d}x$.

Example 6 (Power of tangent is even)

Evaluate $\int_0^{\pi/4} \tan^4 x \, \mathrm{d}x$.

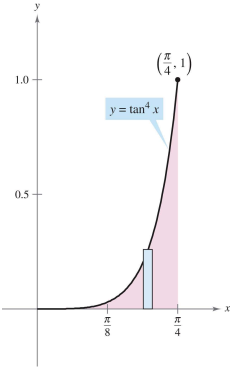

Figure 3: The area of the region is approximately 0.119.

Example 7 (Converting to sines and cosines)

Find $\int \frac{\sec x}{\tan^2 x}\,\mathrm{d}x$.

Integrals involving sine-cosine products with different angles

- Integrals involving the products of sines and cosines of two different angles occur in many applications.
- In such instances you can use the following product-to-sum identities.

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$
$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$
$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$

Example 8 (Using Product-to-Sum Identities)

Find $\int \sin 5x \cos 4x \, \mathrm{d}x$.

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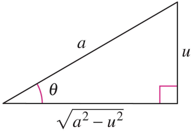
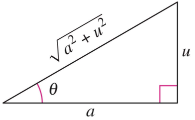
Trigonometric substitution

- Use trigonometric substitution to evaluate integrals involving the radicals
$$\sqrt{a^2-u^2}, \quad \sqrt{a^2+u^2}, \quad \text{and} \quad \sqrt{u^2-a^2}.$$
- The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities
$$\cos^2\theta = 1-\sin^2\theta, \quad \sec^2\theta = 1+\tan^2\theta, \quad \text{and} \quad \tan^2\theta = \sec^2\theta - 1$$
- For example, if $a>0$, let $u = a\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then
$$\sqrt{a^2-u^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$
- Note that $\cos\theta \ge 0$, because $-\pi/2 \le \theta \le \pi/2$.

1. For integrals involving $\sqrt{a^2-u^2}$, let $u = a\sin\theta$. Then $\sqrt{a^2-u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.

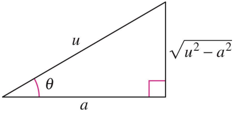


2. For integrals involving $\sqrt{a^2+u^2}$, let $u = a\tan\theta$. Then $\sqrt{a^2+u^2} = a\sec\theta$, where $-\pi/2 \le \theta \le \pi/2$.



3. For integrals involving $\sqrt{u^2-a^2}$, let $u = a\sec\theta$.

Then $\sqrt{u^2-a^2} = \begin{cases} a\tan\theta & \text{if } u > a, \text{ where } 0 \le \theta < \pi/2 \\ -a\tan\theta, & \text{if } u < -a, \text{ where } \pi/2 < \theta \le \pi. \end{cases}$



Example 1 (Trigonometric substitution: $u = a\sin\theta$)

Find $\int \frac{\mathrm{d}x}{x^2\sqrt{9-x^2}}$.

Example 2 (Trigonometric substitution: $u = a\tan\theta$)

Find $\int \frac{\mathrm{d}x}{\sqrt{4x^2+1}}$.

Example 3 (Trigonometric substitution: rational powers)

Find $\int \frac{\mathrm{d}x}{(x^2+1)^{3/2}}$.

Example 4 (Converting the limits of integration)

Evaluate $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,\mathrm{d}x$.

Theorem 8.2 (Special integration formulas ($a > 0$) (Exercise 65))

1. $\int \sqrt{a^2 - u^2}\,\mathrm{d}u = \frac{1}{2}\left(a^2 \arcsin \frac{u}{a} + u\sqrt{a^2 - u^2}\right) + C$
2. $\int \sqrt{u^2 - a^2}\,\mathrm{d}u = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right|\right) + C, \quad u > a$
3. $\int \sqrt{u^2 + a^2}\,\mathrm{d}u = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right) + C$

Example 5 (Finding arc length)

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 4).

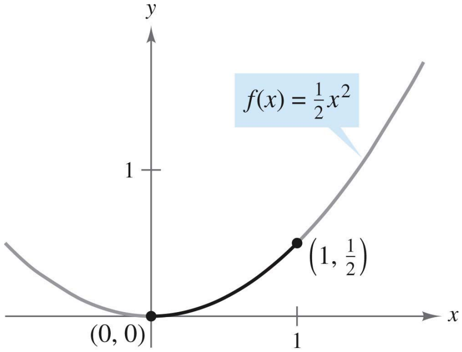


Figure 4: The arc length of the curve of $f(x) = \frac{1}{2}x^2$.

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Partial fractions

- The Method of Partial Fractions is a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas.
- To see the benefit of the Method of Partial Fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6}\, \mathrm{d}x.$$

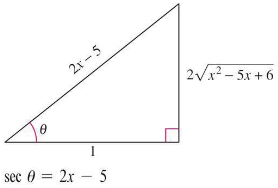

Figure 5: Trigonometric substitution.

- To evaluate this integral without partial fractions, you can complete the square and use trigonometric substitution (see Figure 5) to obtain

$$
\begin{aligned}
\int \frac{1}{x^2-5x+6}\,\mathrm{d}x &= \int \frac{\mathrm{d}x}{(x-5/2)^2-(1/2)^2} \qquad a=\frac{1}{2},\ x-\frac{5}{2}=\frac{1}{2}\sec\theta \\
&= \int \frac{(1/2)\sec\theta\tan\theta\,\mathrm{d}\theta}{(1/4)\tan^2\theta} \qquad \mathrm{d}x=\frac{1}{2}\sec\theta\tan\theta\,\mathrm{d}\theta \\
&= 2\int \csc\theta\,\mathrm{d}\theta = -2\ln|\csc\theta+\cot\theta|+C \\
&= 2\ln|\csc\theta-\cot\theta|+C \\
&= 2\ln\left|\frac{2x-5}{2\sqrt{x^2-5x+6}}-\frac{1}{2\sqrt{x^2-5x+6}}\right|+C \\
&= 2\ln\left|\frac{x-3}{\sqrt{x^2-5x+6}}\right|+C \\
&= 2\ln\left|\frac{\sqrt{x-3}}{\sqrt{x-2}}\right|+C = \ln\left|\frac{x-3}{x-2}\right|+C \\
&= \ln|x-3|-\ln|x-2|+C.
\end{aligned}
$$

- Now, suppose you had observed that

$$\frac{1}{x^2-5x+6}=\frac{1}{x-3}-\frac{1}{x-2}. \qquad \text{Partial fraction decomposition}$$

- Then you could evaluate the integral easily, as follows.

$$\begin{aligned}\int \frac{1}{x^2-5x+6}\,\mathrm{d}x &= \int \left(\frac{1}{x-3}-\frac{1}{x-2}\right)\mathrm{d}x \\ &= \ln|x-3|-\ln|x-2|+C\end{aligned}$$

- This method is clearly preferable to trigonometric substitution. However, its use depends on the <u>ability to factor the denominator</u>, x^2-5x+6, and to find the partial fractions

$$\frac{1}{x-3} \quad \text{and} \quad -\frac{1}{x-2}.$$

1. Divide if improper: If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain
$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$
where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.
2. Factor denominator: Completely factor the denominator into factors of the form
$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$
where ax^2+bx+c is irreducible.

3. Linear factors: For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. Quadratic factors: For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Linear factors

Example 1 (Distinct linear factors)

Write the partial fraction decomposition for $\frac{1}{x^2-5x+6}$.

Example 2 (Repeated linear factors)

Find $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,\mathrm{d}x$.

Quadratic factors

Example 3 (Distinct linear and quadratic factors)

Find $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,\mathrm{d}x$.

Example 4 (Repeated quadratic factors)

Find $\int \frac{8x^3+13x}{(x^2+2)^2}\,\mathrm{d}x$.

Guidelines for solving the basic equation

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.

1. It is not necessary to use the partial fractions technique on all rational functions.

$$\int \frac{x^2+1}{x^3+3x-4}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x-4}\,\mathrm{d}x = \frac{1}{3}\ln\left|x^3+3x-4\right| + C$$

2. If the integrand is not in reduced form, reducing it may eliminate the need for partial fractions.

$$\begin{aligned}\int \frac{x^2-x-2}{x^3-2x-4}\,\mathrm{d}x &= \int \frac{(x+1)(x-2)}{(x-2)(x^2+2x+2)}\,\mathrm{d}x \\ &= \int \frac{x+1}{x^2+2x+2}\,\mathrm{d}x = \frac{1}{2}\ln\left|x^2+2x+2\right| + C\end{aligned}$$

3. Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution $u = \sin x$ allows you to write

$$\int \frac{\cos x}{\sin x(\sin x - 1)}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u(u-1)}. \qquad u = \sin x,\ \mathrm{d}u = \cos x\,\mathrm{d}x$$

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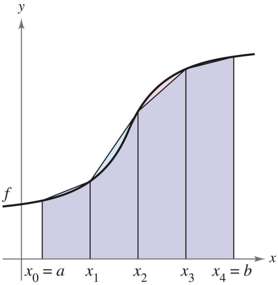
The Trapezoidal Rule

- One way to approximate a definite integral is to use n trapezoids.
- In the development of this method, assume that f is continuous and positive on the interval $[a, b]$.
- So, the definite integral
$$\int_a^b f(x)\,\mathrm{d}x$$
represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$.



- First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- Then form a trapezoid for each subinterval (see Figure 6).

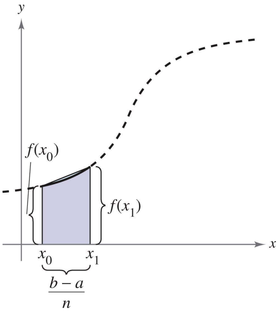


Figure 6: The area of the first trapezoid is $\left[\frac{f(x_0)+f(x_1)}{2}\right]\left(\frac{b-a}{n}\right)$.

- The area of the ith trapezoid is

$$\text{Area of } i\text{th trapezoid} = \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right).$$

- This implies that the sum of the areas of the n trapezoids

$$\begin{aligned}\text{Area} &= \left(\frac{b-a}{n}\right)\left[\frac{f(x_0)+f(x_1)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right] \\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + f(x_1) + f(x_1) + \cdots + f(x_{n-1}) + f(x_n)] \\ &= \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)].\end{aligned}$$

- Letting $\Delta x = (b-a)/n$, you can take the limits as $n \to \infty$ to obtain

$$
\begin{aligned}
& \lim_{n\to\infty} \left(\frac{b-a}{2n}\right) [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \\
= & \lim_{n\to\infty} \left[\frac{[f(a) - f(b)]\Delta x}{2} + \sum_{i=1}^{n} f(x_i)\Delta x \right] \\
= & \lim_{n\to\infty} \frac{[f(a) - f(b)](b-a)}{2n} + \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,\mathrm{d}x.
\end{aligned}
$$

The result is summarized in the following theorem.

Theorem 8.3 (The Trapezoidal Rule)

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,\mathrm{d}x$ is given by

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Moreover, as $n \to \infty$, the right hand side approaches $\int_a^b f(x)\,\mathrm{d}x$.

Example 1 (Approximation with the Trapezoidal Rule)

Use the Trapezoidal Rule to approximate

$$\int_0^{\pi} \sin x \, \mathrm{d}x.$$

Compare the results for $n = 4$ and $n = 8$, as shown in Figure 7.

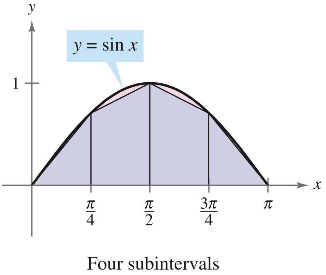


Figure 7: Trapezoidal approximations for $\sin x$, $0 \le x \le \pi$.

- Compare with the Midpoint Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$

- One way to view the trapezoidal approximation of a definite integral is to say that on each subinterval you approximate f by a first-degree polynomial.
- In Simpson's Rule, you take this procedure one step further and approximate f by second-degree polynomials.
- Before presenting Simpson's Rule, we list a theorem for evaluating integrals of polynomials of degree 2 (or less).

Theorem 8.4 (Integral of $p(x) = Ax^2 + Bx + C$)

If $p(x) = Ax^2 + Bx + C$, *then*

$$\int_a^b p(x)\,\mathrm{d}x = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$

Simpson's Rule

- To develop Simpson's Rule for approximating a definite integral, you again partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b-a)/n$.
- This time, however, n required to be even, and the subintervals are grouped in pairs such that

$$a = \underbrace{x_0 < x_1 < x_2}_{[x_0, x_2]} \underbrace{< x_3 < x_4}_{[x_2, x_4]} < \cdots < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2}, x_n]} = b.$$

- On each (double) subinterval $[x_{i-2}, x_i]$, you can approximate f by a polynomial p of degree less than or equal to 2.

- For example, on the subinterval $[x_0, x_2]$, choose the polynomial of least degree passing through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2), as shown in Figure 8.

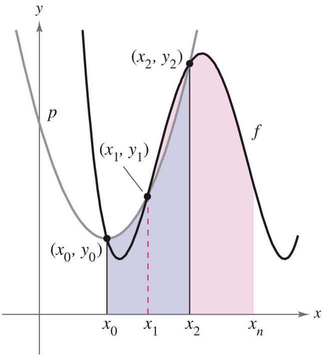


Figure 8: Simpson Rule: $\int_{x_0}^{x_2} p(x)\,\mathrm{d}x \approx \int_{x_0}^{x_2} f(x)\,\mathrm{d}x$.

- Now, using p as an approximation of f on this subinterval, you have, by Theorem 8.4,

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,\mathrm{d}x &\approx \int_{x_0}^{x_2} p(x)\,\mathrm{d}x \\ &= \frac{x_2 - x_0}{6}\left[p(x_0) + 4p\left(\frac{x_0 + x_2}{2}\right) + p(x_2)\right] \\ &= \frac{2[(b-a)/n]}{6}\left[p(x_0) + 4p(x_1) + p(x_2)\right] \\ &= \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + f(x_2)\right].\end{aligned}$$

- Repeating this procedure on the entire interval $[a, b]$ produces the following theorem.

Theorem 8.5 (Simpson's Rule)

Let f be continuous on $[a, b]$ and let n be an even integer. The Simpson's Rule for approximating $\int_a^b f(x)\,\mathrm{d}x$ is

$$\int_a^b f(x)\,\mathrm{d}x \approx$$
$$\frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,\mathrm{d}x$.

Example 2 (Approximation with Simpson's Rule)

Use Simpson's Rule to approximate

$$\int_0^{\pi} \sin x \, \mathrm{d}x.$$

Compare the results for $n = 4$ and $n = 8$.

Error analysis

- If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be.
- The following theorem, gives the formulas for estimating the errors involved in the use of Simpson's Rule and the Trapezoidal Rule.
- In general, when using an approximation, you can think of the error E as the difference between $\int_a^b f(x)\,\mathrm{d}x$ and the approximation.

Theorem 8.6 (Errors in the Trapezoidal Rule and Simpson's Rule)

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\,\mathrm{d}x$ by the Trapezoidal Rule is

$$|E| \leq \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right], \quad a \leq x \leq b.$$

Moreover, if f has a continuous fourth derivative on $[a, b]$, then the error E is approximating $\int_a^b f(x)\,\mathrm{d}x$ by Simpson's Rule is

$$|E| \leq \frac{(b-a)^5}{180n^4}\left[\max|f^{(4)}(x)|\right], \quad a \leq x \leq b.$$

Example 3 (The approximate error in the Trapezoidal Rule)

Determine a value of n such that the Trapezoidal Rule will approximate the value of $\int_0^1 \sqrt{1+x^2}\,\mathrm{d}x$ with an error that is less than or equal to 0.01.

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Improper integrals with infinite limits of integration

- The definition of a definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

requires that the interval $[a, b]$ be finite.

- A procedure for evaluating integrals that do not satisfy these requirements - usually because either one or both of the limits of integration are <u>infinite</u>, or f has a finite number of <u>infinite discontinuities</u> in the interval $[a, b]$.
- Integrals that possess either property are improper integrals. A function f is said to have an infinite discontinuity at c if, from the right or left,

$$\lim_{x \to c} f(x) = \infty \qquad \text{or} \qquad \lim_{x \to c} f(x) = -\infty.$$

Definition 8.1 (Improper integrals with infinite integration limits)

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^b f(x)\,\mathrm{d}x.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

where c is any real number.

In the above first two cases, the improper integral converges if the limit exists-otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integral on the right diverges.

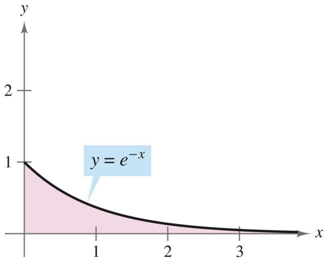
Example 1 (An improper integral that diverges)

Evaluate $\int_1^{\infty} \frac{\mathrm{d}x}{x}$.

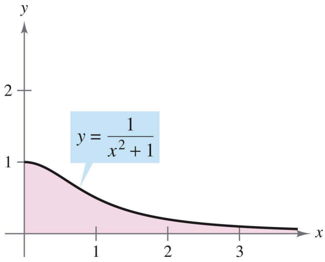
Example 2 (Improper integrals that converge)

Evaluate each improper integral.

a. $\int_0^\infty e^{-x}\,\mathrm{d}x$ **b.** $\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$



(a) The area of the unbounded region is 1.



(b) The area of the unbounded region is $\pi/2$.

Example 3 (Using L'Hôpital's Rule with an improper integral)

Evaluate $\int_1^\infty (1-x)e^{-x}\,\mathrm{d}x$.

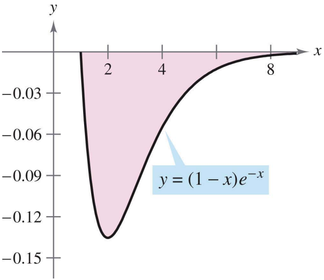


Figure 10: The area of the unbounded region is $1/e$.

Example 4 (Infinite upper and lower limits of integration)

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$.

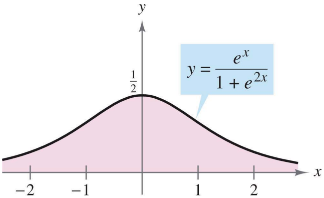


Figure 11: The area of the unbounded region is $\pi/2$.

Definition 8.2 (Improper integrals with infinite discontinuities)

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c \to b^-} \int_a^c f(x)\,\mathrm{d}x.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c \to a^+} \int_c^b f(x)\,\mathrm{d}x.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

In the above first two cases, the improper integral converges if the limit exists-otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integral on the right diverges.

Example 6 (An improper integral with an infinite discontinuity)

Evaluate $\int_0^1 \frac{\mathrm{d}x}{\sqrt[3]{x}}$.

Example 7 (An improper integrals that diverges)

Evaluate $\int_0^2 \frac{\mathrm{d}x}{x^3}$.

Example 8 (An improper integrals with an interior discontinuity)

Evaluate $\int_{-1}^{2} \frac{\mathrm{d}x}{x^3}$.

Example 9 (A doubly improper integral)

Evaluate $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x$.

Example 10 (An application involving arc length)

Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

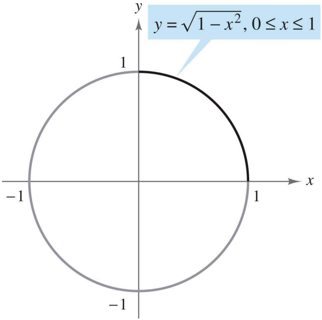


Figure 12: The circumference of the circle is 2π.

Theorem 8.7 (A special type of improper integral)

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \begin{cases} \frac{1}{p-1}, & \textit{if } p > 1 \\ \textit{diverges}, & \textit{if } p \le 1 \end{cases}$$

Example 11 (An application involving a solid of revolution)

The solid formed by revolving (about the x-axis) the unbounded region lying between the graph of $f(x) = 1/x$ and the x-axis ($x \geq 1$) is called Gabriel's Horn. (See Figure 13.) Show that this solid has a finite volume and an infinite surface area.

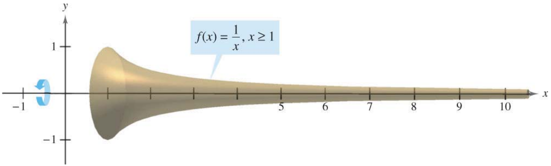


Figure 13: Gabriel's Horn has a finite volume and an infinite surface area.

- In some cases, it is impossible or hard to find the exact value of an improper integral, but it is important to determine whether the integral converges or diverges.

Theorem 8.8 (Comparison Test for Improper Integrals)

Suppose the function f and g are continuous and $0 \leq g(x) \leq f(x)$ on the interval $[a, \infty)$. It can be shown that if $\int_a^\infty f(x)\,\mathrm{d}x$ converges, then $\int_a^\infty g(x)\,\mathrm{d}x$ also converges, and if $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then $\int_a^\infty f(x)\,\mathrm{d}x$ also diverges.

Theorem 8.9 (Limit Comparison Test for Improper Integrals)

Suppose the function f and g are continuous and $0 < g(x)$ and $0 \leq f(x)$ on the interval $[a, \infty)$. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ for some finite $L > 0$, then $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent if and only if $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.

Example 12 (Comparison Test for Improper Integrals)

Determine whether $\int_1^\infty e^{-x^2}\,\mathrm{d}x$ converges of diverges.