

# Chapter 8 Integration Techniques and Improper Integrals

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

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# Fitting Integrands to Basic Integration Rules

Table 1: Review of basic integration rules ( $a > 0$ )

- |   |   |
|---|---|
| 1. $\int kf(u) du = k \int f(u) du$                                   | 2. $\int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$                                  |
| 3. $\int du = u + C$  | 4. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$   |
| 5. $\int \frac{du}{u} = \ln  u  + C$                                  | 6. $\int e^u du = e^u + C$  |
| 7. $\int a^u du = \left(\frac{1}{\ln a}\right) a^u + C$               | 8. $\int \sin u du = -\cos u + C$   |
| 9. $\int \cos u du = \sin u + C$                                      | 10. $\int \tan u du = -\ln  \cos u  + C$  |
| 11. $\int \cot u du = \ln  \sin u  + C$                               | 12. $\int \sec u du = \ln  \sec u + \tan u  + C$  |
| 13. $\int \csc u du = -\ln  \csc u + \cot u  + C$                     | 14. $\int \sec^2 u du = \tan u + C$   |
| 15. $\int \csc^2 u du = -\cot u + C$                                  | 16. $\int \sec u \tan u du = \sec u + C$  |
| 17. $\int \csc u \cot u du = -\csc u + C$                             | 18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$                              |
| 19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$ | 20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{ u }{a} + C$ |

## Example 1 (A comparison of three similar integrals)

Find each integral.

**a.**  $\int \frac{4}{x^2+9} dx$     **b.**  $\int \frac{4x}{x^2+9} dx$     **c.**  $\int \frac{4x^2}{x^2+9} dx$

## Example 2 (Using two basic rules to solve a single integral)

Evaluate  $\int_0^1 \frac{x+3}{\sqrt{4-x^2}} dx$ .

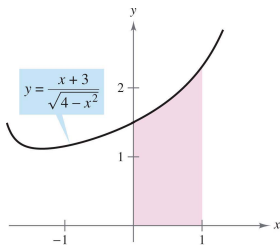


Figure 1: The area of the region is approximately 1.839.

### Example 3 (A substitution involving $a^2 - u^2$ )

Find  $\int \frac{x^2}{\sqrt{16-x^6}} dx$ .



## Example 4 (A disguised form of the Log Rule)

Find  $\int \frac{1}{1+e^x} dx$ .

### Example 5 (A disguised form of the Power Rule)

Find  $\int (\cot x) \ln(\sin x) dx$ .

## Example 6 (Using trigonometric identities)

Find  $\int \tan^2 2x \, dx$ .

## Procedures for fitting integrands to basic integration

Technique	Example
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$
Complete the square.	$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$
Divide improper rational function.	$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$
Add and subtract terms in numerator.	$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate	$\frac{1}{1+\sin x} = \left( \frac{1}{1+\sin x} \right) \left( \frac{1-\sin x}{1-\sin x} \right) =$ $\frac{1-\sin x}{1-\sin^2 x}$ $= \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$

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# Integration by parts

- In this section you will study an important integration technique called integration by parts. This technique can be applied to a wide variety of functions and is particularly useful for integrands involving products of algebraic and transcendental functions.
- For instance, integration by parts works well with integrals such as

$$\int x \ln x \, dx, \quad \int x^2 e^x \, dx, \quad \text{and} \quad \int e^x \sin x \, dx.$$

- Integration by parts is based on the formula

$$\frac{d}{dx} [uv] = uv' + vu'.$$

- If  $u'$  and  $v'$  are continuous, you can integrate both sides of this equation to obtain

$$uv = \int uv' \, dx + \int vu' \, dx = \int u \, dv + \int v \, du.$$

## Theorem 8.1 (Integration by Parts)

If  $u$  and  $v$  are functions of  $x$  and have continuous derivatives, then

$$\int u \, dv = uv - \int v \, du = uv - \int vu' \, dx.$$

### Guidelines for integration by parts

- 1 Try letting  $dv$  be the most complicated portion of the integration rule. Then  $u$  will be remaining factor(s) of the integrand.
- 2 Trying letting  $u$  be the portion of the integrated whose derivative is a function simpler than  $u$  (LIATE). Then  $dv$  will be the remaining factor(s) of the integrand.

Note that  $dv$  always includes the  $dx$  of the original integrand.

## Example 1 (Integration by parts)

Find  $\int x e^x dx$ .



## Example 2 (Integration by parts)

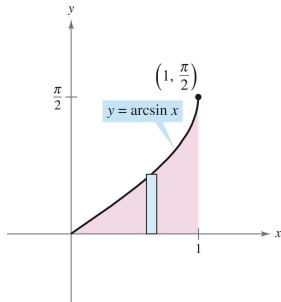
Find  $\int x^2 \ln x \, dx$ .

### Example 3 (An integrand with a single term)

Evaluate  $\int \ln x \, dx$ .

### Example 4 (An integrand with a single term)

Evaluate  $\int_0^1 \sin^{-1} x \, dx$ .



## Example 5 (Repeated use of integration by parts)

Find  $\int x^2 \sin x \, dx$ .



## Example 6 (Integration by parts)

Find  $\int \sec^3 x \, dx$ .





## Summary of common integrals using integration by parts (LIATE)

- ① For integrals of the form

$$\int x^n e^{ax} dx, \quad \int x^n \sin ax dx, \quad \text{or} \quad \int x^n \cos ax dx$$

let  $u = x^n$  and let  $dv = e^{ax} dx, \sin ax dx, \cos ax dx$ .

- ② For integrals of the form

$$\int x^n \ln x dx, \quad \int x^n \arcsin ax dx, \quad \text{or} \quad \int x^n \arctan ax dx$$

let  $u = \ln x, \arcsin ax, \text{ or } \arctan x$  and let  $dv = x^n dx$ .

- ③ For integrals of the form

$$\int e^{ax} \sin bx dx, \quad \text{or} \quad \int e^{ax} \cos bx dx$$

let  $u = \sin bx \text{ or } \cos bx$  and let  $dv = e^{ax} dx$ .

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# Integrals involving powers of sine and cosine

- In this section you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x \, dx \quad \text{and} \quad \int \sec^m x \tan^n x \, dx$$

where either  $m$  or  $n$  is a positive integer.

- Break them into combinations of trigonometric integrals so you can apply the Power Rule. For instance, you can evaluate  $\int \sin^5 x \cos x \, dx$  by letting  $u = \sin x$ . Then,  $du = \cos x \, dx$  and you have

$$\int \sin^5 x \cos x \, dx = \int u^5 \, du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

- To break up  $\int \sin^m x \cos^n x \, dx$  into forms to which you can apply the Power Rule, use the following identities.

$$\sin^2 x + \cos^2 x = 1 \quad \sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

## Guidelines for evaluating integrals involving powers of sine and cosine

- 1 If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines.

$$\begin{aligned}\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x \, dx &= \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x \, dx}^{\text{Save for } du} \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx\end{aligned}$$

- 2 If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines.

$$\begin{aligned}\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x \, dx &= \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x \, dx}^{\text{Save for } du} \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx\end{aligned}$$

- 3 If the power of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

### Example 1 (Power of sine is odd and positive)

Find  $\int \sin^3 x \cos^4 x \, dx$ .

## Example 2 (Power of cosine is odd and positive)

Find  $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$ , as shown in Figure 2.

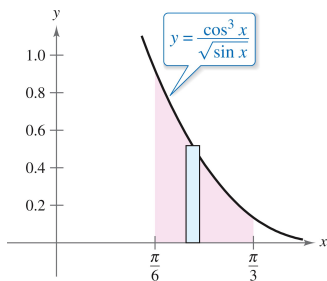


Figure 2: The area of the region is approximately 0.239.



### Example 3 (Power of cosine is even and nonnegative)

Find  $\int \cos^4 x \, dx$ .

## Wallis's Formulas

**a.** If  $n$  is odd ( $n \geq 3$ ), then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) \left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right).$$

**b.** If  $n$  is even ( $n \geq 2$ ), then

$$\int_0^{\pi/2} \cos^n x \, dx = \left(\frac{1}{2}\right) \left(\frac{3}{4}\right) \left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right) \left(\frac{\pi}{2}\right).$$

These formulas are also valid if  $\cos^n x$  is replaced by  $\sin^n x$ .

# Integrals involving powers of secant and tangent

- The following guidelines can help you evaluate integrals of the form  $\int \sec^m x \tan^n x \, dx$

## Guidelines for evaluating integrals involving powers of secant and tangent

- 1 If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\begin{aligned} \int \sec^{\overbrace{2k}^{\text{even}}} x \tan^n x \, dx &= \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x}^{\text{Save for } du} \, dx \\ &= \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx \end{aligned}$$

- ② If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\begin{aligned}\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x \, dx &= \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x \, dx}^{\text{Save for } du} \\ &= \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x \, dx\end{aligned}$$

- ③ If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\begin{aligned}\int \tan^n x \, dx &= \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} \, dx \\ &= \int \tan^{n-2} x (\sec^2 x - 1) \, dx\end{aligned}$$

- ④ If the integral is of the form  $\int \sec^m x \, dx$ , where  $m$  is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.
- ⑤ If none of the first four guidelines applies, try converting to sines and cosines.

### Example 4 (Power of tangent is odd and positive)

Find  $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$ .

### Example 5 (Power of secant is even and positive)

Find  $\int \sec^4 3x \tan^3 3x \, dx$ .

### Example 6 (Power of tangent is even)

Evaluate  $\int_0^{\pi/4} \tan^4 x \, dx$ .



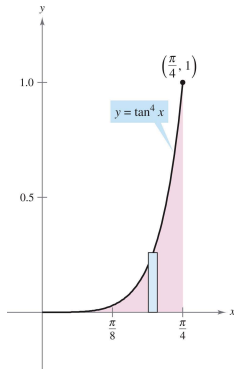


Figure 3: The area of the region is approximately 0.119.

## Example 7 (Converting to sines and cosines)

Find  $\int \frac{\sec x}{\tan^2 x} dx$ .

# Integrals involving sine-cosine products with different angles

- Integrals involving the products of sines and cosines of two different angles occur in many applications.
- In such instances you can use the following product-to-sum identities.

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m - n)x] - \cos[(m + n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m - n)x] + \sin[(m + n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m - n)x] + \cos[(m + n)x])$$

## Example 8 (Using Product-to-Sum Identities)

Find  $\int \sin 5x \cos 4x \, dx$ .

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# Trigonometric substitution

- Use trigonometric substitution to evaluate integrals involving the radicals

$$\sqrt{a^2 - u^2}, \quad \sqrt{a^2 + u^2}, \quad \text{and} \quad \sqrt{u^2 - a^2}.$$

- The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities

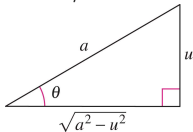
$$\cos^2 \theta = 1 - \sin^2 \theta, \quad \sec^2 \theta = 1 + \tan^2 \theta, \quad \text{and} \quad \tan^2 \theta = \sec^2 \theta - 1$$

- For example, if  $a > 0$ , let  $u = a \sin \theta$ , where  $-\pi/2 \leq \theta \leq \pi/2$ . Then

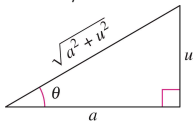
$$\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta.$$

- Note that  $\cos \theta \geq 0$ , because  $-\pi/2 \leq \theta \leq \pi/2$ .

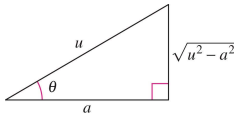
- ① For integrals involving  $\sqrt{a^2 - u^2}$ , let  $u = a \sin \theta$ . Then  $\sqrt{a^2 - u^2} = a \cos \theta$ , where  $-\pi/2 \leq \theta \leq \pi/2$ .



- ② For integrals involving  $\sqrt{a^2 + u^2}$ , let  $u = a \tan \theta$ . Then  $\sqrt{a^2 + u^2} = a \sec \theta$ , where  $-\pi/2 \leq \theta \leq \pi/2$ .



- ③ For integrals involving  $\sqrt{u^2 - a^2}$ , let  $u = a \sec \theta$ .  
Then  $\sqrt{u^2 - a^2} = \begin{cases} a \tan \theta & \text{if } u > a, \text{ where } 0 \leq \theta < \pi/2 \\ -a \tan \theta, & \text{if } u < -a, \text{ where } \pi/2 < \theta \leq \pi. \end{cases}$



## Example 1 (Trigonometric substitution: $u = a \sin \theta$ )

Find  $\int \frac{dx}{x^2 \sqrt{9-x^2}}$ .





## Example 2 (Trigonometric substitution: $u = a \tan \theta$ )

Find  $\int \frac{dx}{\sqrt{4x^2+1}}$ .

### Example 3 (Trigonometric substitution: rational powers)

Find  $\int \frac{dx}{(x^2+1)^{3/2}}$ .

## Example 4 (Converting the limits of integration)

Evaluate  $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x} dx$ .



## Theorem 8.2 (Special integration formulas ( $a > 0$ ) (Exercise 65))

$$\textcircled{1} \int \sqrt{a^2 - u^2} \, du = \frac{1}{2} \left( a^2 \arcsin \frac{u}{a} + u\sqrt{a^2 - u^2} \right) + C$$

$$\textcircled{2} \int \sqrt{u^2 - a^2} \, du = \frac{1}{2} \left( u\sqrt{u^2 - a^2} - a^2 \ln \left| u + \sqrt{u^2 - a^2} \right| \right) + C, \quad u > a$$

$$\textcircled{3} \int \sqrt{u^2 + a^2} \, du = \frac{1}{2} \left( u\sqrt{u^2 + a^2} + a^2 \ln \left| u + \sqrt{u^2 + a^2} \right| \right) + C$$

# Applications

## Example 5 (Finding arc length)

Find the arc length of the graph of  $f(x) = \frac{1}{2}x^2$  from  $x = 0$  to  $x = 1$  (see Figure 4).

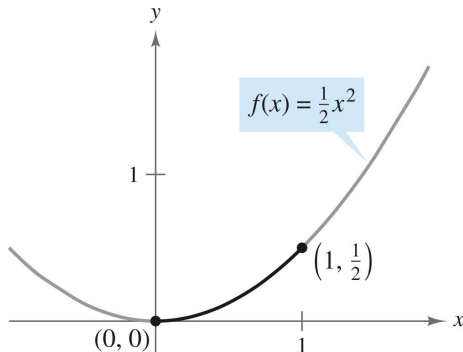


Figure 4: The arc length of the curve of  $f(x) = \frac{1}{2}x^2$ .





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# Partial fractions

- The Method of Partial Fractions is a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas.
- To see the benefit of the Method of Partial Fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6} dx.$$

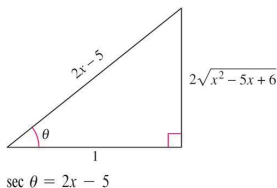


Figure 5: Trigonometric substitution.

- To evaluate this integral without partial fractions, you can complete the square and use trigonometric substitution (see Figure 5) to obtain

$$\begin{aligned}
 \int \frac{1}{x^2 - 5x + 6} dx &= \int \frac{dx}{(x - 5/2)^2 - (1/2)^2} \quad a = \frac{1}{2}, \quad x - \frac{5}{2} = \frac{1}{2} \sec \theta \\
 &= \int \frac{(1/2) \sec \theta \tan \theta d\theta}{(1/4) \tan^2 \theta} \quad dx = \frac{1}{2} \sec \theta \tan \theta d\theta \\
 &= 2 \int \csc \theta d\theta = -2 \ln |\csc \theta + \cot \theta| + C \\
 &= 2 \ln |\csc \theta - \cot \theta| + C \\
 &= 2 \ln \left| \frac{2x - 5}{2\sqrt{x^2 - 5x + 6}} - \frac{1}{2\sqrt{x^2 - 5x + 6}} \right| + C \\
 &= 2 \ln \left| \frac{x - 3}{\sqrt{x^2 - 5x + 6}} \right| + C \\
 &= 2 \ln \left| \frac{\sqrt{x - 3}}{\sqrt{x - 2}} \right| + C = \ln \left| \frac{x - 3}{x - 2} \right| + C \\
 &= \ln |x - 3| - \ln |x - 2| + C.
 \end{aligned}$$

- Now, suppose you had observed that

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}. \quad \text{Partial fraction decomposition}$$

- Then you could evaluate the integral easily, as follows.

$$\begin{aligned} \int \frac{1}{x^2 - 5x + 6} dx &= \int \left( \frac{1}{x - 3} - \frac{1}{x - 2} \right) dx \\ &= \ln |x - 3| - \ln |x - 2| + C \end{aligned}$$

- This method is clearly preferable to trigonometric substitution. However, its use depends on the ability to factor the denominator,  $x^2 - 5x + 6$ , and to find the partial fractions

$$\frac{1}{x - 3} \quad \text{and} \quad -\frac{1}{x - 2}.$$

- ① Divide if improper: If  $N(x)/D(x)$  is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of  $N_1(x)$  is less than the degree of  $D(x)$ . Then apply Steps 2, 3, and 4 to the proper rational expression  $N_1(x)/D(x)$ .

- ② Factor denominator: Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where  $ax^2 + bx + c$  is irreducible.

- ③ Linear factors: For each factor of the form  $(px + q)^m$ , the partial fraction decomposition must include the following sum of  $m$  fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

- ④ Quadratic factors: For each factor of the form  $(ax^2 + bx + c)^n$ , the partial fraction decomposition must include the following sum of  $n$  fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

[t]

## Example 1 (Distinct linear factors)

Write the partial fraction decomposition for  $\frac{1}{x^2-5x+6}$ .





## Example 2 (Repeated linear factors)

Find  $\int \frac{5x^2+20x+6}{x^3+2x^2+x} dx$ .



# Quadratic factors

## Example 3 (Distinct linear and quadratic factors)

Find  $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)} dx$ .





### Example 4 (t)

[Repeated quadratic factors] Find  $\int \frac{8x^3+13x}{(x^2+2)^2} dx$ .



## Guidelines for solving the basic equation

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### Linear Factors

- 1 Substitute the roots of the distinct linear factors in the basic equation.
  - 2 For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of  $x$  and solve for the remaining coefficients.
- 

### Quadratic Factors

- 1 Expand the basic equation.
- 2 Collect terms according to powers of  $x$ .
- 3 Equate the coefficients of like powers to obtain a system of linear equations involving  $A$ ,  $B$ ,  $C$ , and so on.
- 4 Solve the system of linear equations.



- ① It is not necessary to use the partial fractions technique on all rational functions.

$$\int \frac{x^2 + 1}{x^3 + 3x - 4} dx = \frac{1}{3} \int \frac{3x^2 + 3}{x^3 + 3x - 4} dx = \frac{1}{3} \ln |x^3 + 3x - 4| + C$$

- ② If the integrand is not in reduced form, reducing it may eliminate the need for partial fractions.

$$\begin{aligned} \int \frac{x^2 - x - 2}{x^3 - 2x - 4} dx &= \int \frac{(x+1)(x-2)}{(x-2)(x^2 + 2x + 2)} dx \\ &= \int \frac{x+1}{x^2 + 2x + 2} dx = \frac{1}{2} \ln |x^2 + 2x + 2| + C \end{aligned}$$

- ③ Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution  $u = \sin x$  allows you to write

$$\int \frac{\cos x}{\sin x(\sin x - 1)} dx = \int \frac{du}{u(u-1)}. \quad u = \sin x, du = \cos x dx$$

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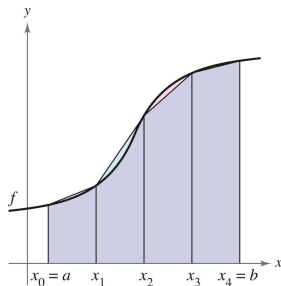
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# The Trapezoidal Rule

- One way to approximate a definite integral is to use  $n$  trapezoids.
- In the development of this method, assume that  $f$  is continuous and positive on the interval  $[a, b]$ .
- So, the definite integral

$$\int_a^b f(x) dx$$

represents the area of the region bounded by the graph of  $f$  and the  $x$ -axis, from  $x = a$  to  $x = b$ .



- First, partition the interval  $[a, b]$  into  $n$  subintervals, each of width  $\Delta x = (b - a)/n$ , such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- Then form a trapezoid for each subinterval (see Figure 6).

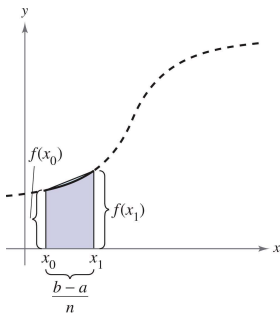


Figure 6: The area of the first trapezoid is  $\left[ \frac{f(x_0) + f(x_1)}{2} \right] \left( \frac{b-a}{n} \right)$ .

- The area of the  $i$ th trapezoid is

$$\text{Area of } i\text{th trapezoid} = \left[ \frac{f(x_{i-1}) + f(x_i)}{2} \right] \left( \frac{b-a}{n} \right).$$

- This implies that the sum of the areas of the  $n$  trapezoids

$$\begin{aligned} \text{Area} &= \left( \frac{b-a}{n} \right) \left[ \frac{f(x_0) + f(x_1)}{2} + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2} \right] \\ &= \left( \frac{b-a}{2n} \right) [f(x_0) + f(x_1) + f(x_1) + \cdots + f(x_{n-1}) + f(x_n)] \\ &= \left( \frac{b-a}{2n} \right) [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]. \end{aligned}$$

- Letting  $\Delta x = (b - a)/n$ , you can take the limits as  $n \rightarrow \infty$  to obtain

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \left( \frac{b - a}{2n} \right) [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \\
 &= \lim_{n \rightarrow \infty} \left[ \frac{[f(a) - f(b)]\Delta x}{2} + \sum_{i=1}^n f(x_i)\Delta x \right] \\
 &= \lim_{n \rightarrow \infty} \frac{[f(a) - f(b)](b - a)}{2n} + \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x) dx.
 \end{aligned}$$

The result is summarized in the following theorem.

## Theorem 8.3 (The Trapezoidal Rule)

Let  $f$  be continuous on  $[a, b]$ . The Trapezoidal Rule for approximating  $\int_a^b f(x) dx$  is given by

$$\int_a^b f(x) dx \approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

Moreover, as  $n \rightarrow \infty$ , the right hand side approaches  $\int_a^b f(x) dx$ .

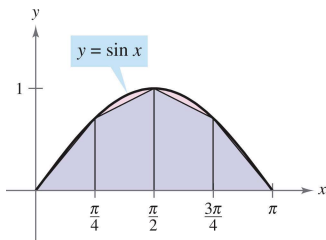
## Example 1 (Approximation with the Trapezoidal Rule)

Use the Trapezoidal Rule to approximate

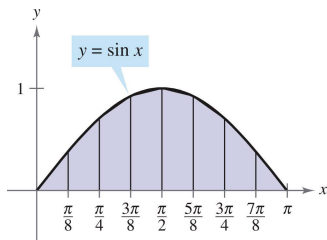
$$\int_0^{\pi} \sin x \, dx.$$

Compare the results for  $n = 4$  and  $n = 8$ , as shown in Figure 7.





Four subintervals



Eight subintervals

**Figure 7:** Trapezoidal approximations for  $\sin x$ ,  $0 \leq x \leq \pi$ .

- Compare with the Midpoint Rule

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f\left(\frac{x_i + x_{i-1}}{2}\right) \Delta x$$

- One way to view the trapezoidal approximation of a definite integral is to say that on each subinterval you approximate  $f$  by a first-degree polynomial.
- In Simpson's Rule, you take this procedure one step further and approximate  $f$  by second-degree polynomials.
- Before presenting Simpson's Rule, we list a theorem for evaluating integrals of polynomials of degree 2 (or less).

## Theorem 8.4 (Integral of $p(x) = Ax^2 + Bx + C$ )

If  $p(x) = Ax^2 + Bx + C$ , then

$$\int_a^b p(x) \, dx = \left( \frac{b-a}{6} \right) \left[ p(a) + 4p\left(\frac{a+b}{2}\right) + p(b) \right].$$

# Simpson's Rule

- To develop Simpson's Rule for approximating a definite integral, you again partition the interval  $[a, b]$  into  $n$  subintervals, each of width  $\Delta x = (b - a)/n$ .
- This time, however,  $n$  required to be even, and the subintervals are grouped in pairs such that

$$a = \underbrace{x_0 < x_1 < x_2}_{[x_0, x_2]} < \underbrace{x_3 < x_4}_{[x_2, x_4]} < \cdots < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2}, x_n]} = b.$$

- On each (double) subinterval  $[x_{i-2}, x_i]$ , you can approximate  $f$  by a polynomial  $p$  of degree less than or equal to 2.

- For example, on the subinterval  $[x_0, x_2]$ , choose the polynomial of least degree passing through the points  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , as shown in Figure 8.

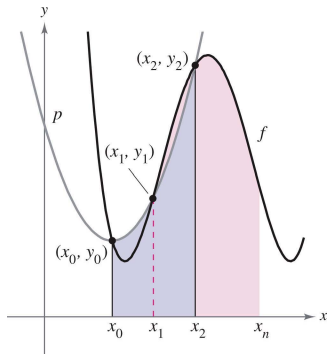


Figure 8: Simpson Rule:  $\int_{x_0}^{x_2} p(x) dx \approx \int_{x_0}^{x_2} f(x) dx$ .

- Now, using  $p$  as an approximation of  $f$  on this subinterval, you have, by Theorem 8.4,

$$\begin{aligned}
 \int_{x_0}^{x_2} f(x) \, dx &\approx \int_{x_0}^{x_2} p(x) \, dx \\
 &= \frac{x_2 - x_0}{6} \left[ p(x_0) + 4p\left(\frac{x_0 + x_2}{2}\right) + p(x_2) \right] \\
 &= \frac{2[(b-a)/n]}{6} [p(x_0) + 4p(x_1) + p(x_2)] \\
 &= \frac{b-a}{3n} [f(x_0) + 4f(x_1) + f(x_2)].
 \end{aligned}$$

- Repeating this procedure on the entire interval  $[a, b]$  produces the following theorem.

## Theorem 8.5 (Simpson's Rule)

Let  $f$  be continuous on  $[a, b]$  and let  $n$  be an even integer. The Simpson's Rule for approximating  $\int_a^b f(x) \, dx$  is

$$\int_a^b f(x) \, dx \approx \frac{b-a}{3n} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

Moreover, as  $n \rightarrow \infty$ , the right-hand side approaches  $\int_a^b f(x) \, dx$ .

## Example 2 (Approximation with Simpson's Rule)

Use Simpson's Rule to approximate

$$\int_0^{\pi} \sin x \, dx.$$

Compare the results for  $n = 4$  and  $n = 8$ .



# Error analysis

- If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be.
- The following theorem, gives the formulas for estimating the errors involved in the use of Simpson's Rule and the Trapezoidal Rule.
- In general, when using an approximation, you can think of the error  $E$  as the difference between  $\int_a^b f(x) \, dx$  and the approximation.

## Theorem 8.6 (Errors in the Trapezoidal Rule and Simpson's Rule)

If  $f$  has a continuous second derivative on  $[a, b]$ , then the error  $E$  in approximating  $\int_a^b f(x) dx$  by the Trapezoidal Rule is

$$|E| \leq \frac{(b-a)^3}{12n^2} \left[ \max |f''(x)| \right], \quad a \leq x \leq b.$$

Moreover, if  $f$  has a continuous fourth derivative on  $[a, b]$ , then the error  $E$  in approximating  $\int_a^b f(x) dx$  by Simpson's Rule is

$$|E| \leq \frac{(b-a)^5}{180n^4} \left[ \max |f^{(4)}(x)| \right], \quad a \leq x \leq b.$$

### Example 3 (The approximate error in the Trapezoidal Rule)

Determine a value of  $n$  such that the Trapezoidal Rule will approximate the value of  $\int_0^1 \sqrt{1+x^2} \, dx$  with an error that is less than or equal to 0.01.



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# Improper integrals with infinite limits of integration

- The definition of a definite integral

$$\int_a^b f(x) dx$$

requires that the interval  $[a, b]$  be finite.

- A procedure for evaluating integrals that do not satisfy these requirements - usually because either one or both of the limits of integration are infinite, or  $f$  has a finite number of infinite discontinuities in the interval  $[a, b]$ .
- Integrals that possess either property are improper integrals. A function  $f$  is said to have an infinite discontinuity at  $c$  if, from the right or left,

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow c} f(x) = -\infty.$$

## Definition 8.1 (Improper integrals with infinite integration limits)

- ① If  $f$  is continuous on the interval  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

- ② If  $f$  is continuous on the interval  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

- ③ If  $f$  is continuous on the interval  $(-\infty, \infty)$ , then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$

where  $c$  is any real number.

In the above first two cases, the improper integral converges if the limit exists-otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integral on the right diverges.

### Example 1 (An improper integral that diverges)

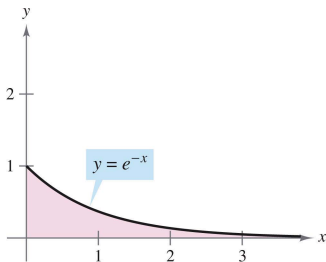
Evaluate  $\int_1^{\infty} \frac{dx}{x}$ .



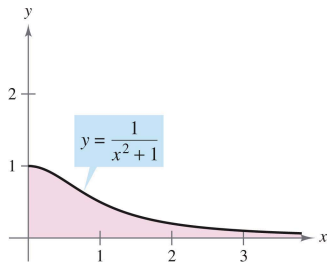
## Example 2 (Improper integrals that converge)

Evaluate each improper integral.

**a.**  $\int_0^{\infty} e^{-x} dx$     **b.**  $\int_0^{\infty} \frac{1}{1+x^2} dx$



(a) The area of the unbounded region is 1.



(b) The area of the unbounded region is  $\pi/2$ .

### Example 3 (Using L'Hôpital's Rule with an improper integral)

Evaluate  $\int_1^{\infty} (1 - x)e^{-x} dx$ .

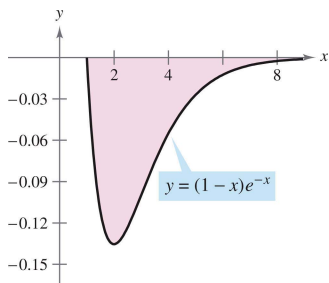


Figure 10: The area of the unbounded region is  $1/e$ .

## Example 4 (Infinite upper and lower limits of integration)

Evaluate  $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$ .

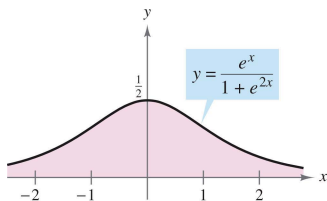


Figure 11: The area of the unbounded region is  $\pi/2$ .

## Definition 8.2 (Improper integrals with infinite discontinuities)

- ① If  $f$  is continuous on the interval  $[a, b)$  and has an infinite discontinuity at  $b$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

- ② If  $f$  is continuous on the interval  $(a, b]$  and has an infinite discontinuity at  $a$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

- ③ If  $f$  is continuous on the interval  $[a, b]$ , except for some  $c$  in  $(a, b)$  at which  $f$  has an infinite discontinuity, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

In the above first two cases, the improper integral converges if the limit exists-otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integral on the right diverges.

### Example 6 (An improper integral with an infinite discontinuity)

Evaluate  $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ .



## Example 7 (An improper integrals that diverges)

Evaluate  $\int_0^2 \frac{dx}{x^3}$ .

## Example 8 (An improper integrals with an interior discontinuity)

Evaluate  $\int_{-1}^2 \frac{dx}{x^3}$ .

## Example 9 (A doubly improper integral)

Evaluate  $\int_0^{\infty} \frac{1}{\sqrt{x(x+1)}} dx$ .

## Example 10 (An application involving arc length)

Use the formula for arc length to show that the circumference of the circle  $x^2 + y^2 = 1$  is  $2\pi$ .

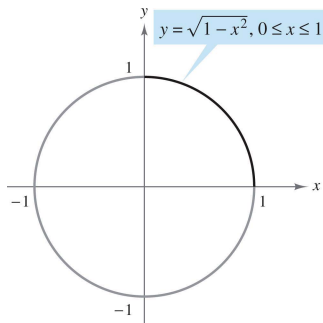


Figure 12: The circumference of the circle is  $2\pi$ .

## Theorem 8.7 (A special type of improper integral)

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \leq 1 \end{cases}$$

## Example 11 (An application involving a solid of revolution)

The solid formed by revolving (about the  $x$ -axis) the unbounded region lying between the graph of  $f(x) = 1/x$  and the  $x$ -axis ( $x \geq 1$ ) is called Gabriel's Horn. (See Figure 13.) Show that this solid has a finite volume and an infinite surface area.

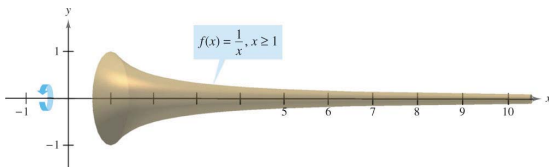


Figure 13: Gabriel's Horn has a finite volume and an infinite surface area.





- In some cases, it is impossible or hard to find the exact value of an improper integral, but it is important to determine whether the integral converges or diverges.

### Theorem 8.8 (Comparison Test for Improper Integrals)

*Suppose the function  $f$  and  $g$  are continuous and  $0 \leq g(x) \leq f(x)$  on the interval  $[a, \infty)$ . It can be shown that if  $\int_a^\infty f(x) dx$  converges, then  $\int_a^\infty g(x) dx$  also converges, and if  $\int_a^\infty g(x) dx$  diverges, then  $\int_a^\infty f(x) dx$  also diverges.*

## Example 12 (Comparison Test for Improper Integrals)

Determine whether  $\int_1^{\infty} e^{-x^2} dx$  converges or diverges.