If the limit does not exist or has an infinite limit, you should point it out. In addition, do not use the L'Hôpital's rule to solve the limit problem.

1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)
   (a) $\frac{x^2-2x-8}{x^2+3x+2}$
   (b) $\frac{2sin(x^2)}{1-cos(x)}$
   (c) $\frac{1}{x}\left(x\ )+\frac{2}{x})\right)$
   (d) $\frac{\sqrt{1+x}-1}{|x^2+x|}$

Ans:

(a) $\frac{x^2-2x-8}{x^2+3x+2} = \frac{(x+2)(x-4)}{(x+2)(x+1)} = \frac{(x-4)}{(x+1)} = 6$

(b) $\frac{2sinsin\ (x^2)}{1-coscos\ (x)} = \frac{2sinsin\ (x^2)\ (1+coscos\ (x)\ )}{(1-coscos\ (x)\ )(1+coscos\ (x)\ )} = \frac{(x^2)\ (1+coscos\ (x)\ )}{sin^2x} =$
$\frac{(x^2)\ x^2(1+coscos\ (x)\ )}{x^2sin^2x} = \frac{(x^2)}{x^2}\frac{x}{sinsin\ (x)}\frac{x}{sinsin\ (x)}\ (1+cos\ cos\ (x)\ ) =$
$2\frac{sinsin\ (t)}{t}\ \frac{x}{sinsin\ (x)}\ \frac{x}{sinsin\ (x)}\ (1+cos\ cos\ (x)\ )$ (Let $t = x^2$)
$= 4$

(c) For any $x > 0, -2 \leq x\ )+\frac{2}{x}) \leq 2 \Rightarrow -\frac{2}{x} \leq \frac{1}{x}\left(x\ )+\frac{2}{x})\right) \leq \frac{2}{x}$,
   In addition, $-\frac{2}{x} = 0$ and $\frac{2}{x} = 0$
   According to Squeeze theorem, $\frac{1}{x}\left(x\ )+\frac{2}{x})\right) = 0$

(d) $\frac{\sqrt{1+x}-1}{|x^2+x|} = \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{(x^2+x)(\sqrt{1+x}+1)} = \frac{x}{(x^2+x)(\sqrt{1+x}+1)} = \frac{1}{(x+1)(\sqrt{1+x}+1)} = \frac{1}{2}$

$$\frac{\sqrt{1+x}-1}{|x^2+x|} = \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{-(x^2+x)(\sqrt{1+x}+1)} = \frac{x}{-(x^2+x)(\sqrt{1+x}+1)}$$
$$= \frac{1}{-(x+1)(\sqrt{1+x}+1)} = -\frac{1}{2}$$

Therefore, the limit does not exist!

2. (8%)
   Suppose $f(x) = \{-ax^2 - x - a \quad if\ x < -1\ ax^2 + bx + 6\ \ if - 1 \leq x < 2\ 3x^2 - bx - b \quad if\ x \geq 2$ is a continuous function on $(-\infty, \infty)$. What are the values of a and b?

Ans:

(a)

Since f is continuous at -1, we know $f(x)\ = f(x)$. Therefore, $-ax^2 - x - a\ = ax^2 + bx + 6\ \rightarrow -a + 1 - a = a - b + 6 \rightarrow 3a - b = -5$.

On the other hand, f is continuous at 2, we know $f(x)\ = f(x)$. Therefore, $ax^2 + bx + 6\ = 3x^2 - bx - b\ \rightarrow 4a + 2b + 6 = 12 - 2b - b \rightarrow 4a + 5b = 6$.

Solving the two equations we get $a = -1, b = 2$

3. (8%) Proof that $f(x) = 3x^3 + 2x - sin(x)\ has\ exactly\ one\ real\ root.$ (Hint:

use the mean value theorem)

Ans:

$f(1) > 0, f(-1) < 0$ by the intermediate value theorem, it has at least one real root between -1 and 0.

Assume the real root is a and there is a second real root b. Then, by the mean value theorem, there is a c such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. However, $f'(x) = 9x^2 + 2 - cos(x) > 0$. Contradict, therefore, there is only one real root.

4. (15%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Find the derivative of $f(x) = \sqrt{1 + cot(x^2)}$

(b) Given $f(x) = \frac{x^2}{(0-x)(1-x)(2-x)...(2023-x)}$, what is the value of $f'(0)$?

(c) Let $f(x) = \{cos\ cos\ (2x)\ \ if\ x \leq 0\ ax\ \ \ \ if\ x > 0$, where a is a constant. Find the value of a makes $f(x)$ differentiable at 0..

Ans:

(a) $f(x) = \sqrt{1 + cot(x^2)} = (1 + cot(x^2))^{\frac{1}{2}} \to f'(x) = \frac{1}{2}(1 + cot\ cot\ (x^2)\ )^{\frac{-1}{2}}(-csc^2(x^2))2x = -\frac{csc^2(x^2)\cdot x}{\sqrt{1+cot(x^2)}}$

(b) $f'(0) = \frac{f(0+\Delta x)-f(0)}{\Delta x} = \frac{\frac{(\Delta x)^2}{-\Delta x(1-\Delta x)(2-\Delta x)...(2023-\Delta x)}}{\Delta x} = \frac{-1}{(1-\Delta x)(2-\Delta x)...(2023-\Delta x)} = \frac{-1}{2023!}$

(c) Since $f(x)$ is not continuous at 0, there is no value of a that can make it differentiable.

5. (8%) Given the graph $x^2 + xy + y^2 = 12$.
   (a) Express y' in terms of x and y
   (b) Find the extrema of the graph by checking the critical number

   Ans:

   (a) $\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(12)$

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{(x + 2y)}$$

   (b) The critical number occurs at $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist

   When $\frac{dy}{dx} = 0 \to y = -2x$, substitute back to the original equation we get $x = \pm 2,\ y = \mp 4$

   When $\frac{dy}{dx}$ does not exist, $x = -2y$, substitute back to the original equation we get $x = \mp 4,\ y = \pm 2$

   Therefore, the graph has maximum at (-2,4) at minimum at (2,-4)

6. (20%) Let $f(x) = \frac{-x^2-4x-7}{x+3}$

(a) Find the open intervals on which f is increasing or decreasing. Indicates the extreme values
(b) Find the open intervals on which f is concave upward or concave downward. Indicates the points of inflection
(c) Find all the asymptotes (Vertical/horizontal/Slant)
(d) Sketch the graph of $f(x)$
(e) What is the domain and range of $f(x)$?

Ans: Note that the original function is undefined at $x = -3$, therefore we should include it in the following table.
(a)
(b) $f(x) = \frac{-x^2-4x-7}{x+3}, f'(x) = \frac{-(x+1)(x+5)}{(x+3)^2}, f''(x) = \frac{-8}{(x+3)^3}$

	$(-\infty, -5)$	$(-5, -3)$	$(-3, -1)$	$(-1, \infty)$
測試值	-6	-4	-2	0
f'的正負號	-	+	+	-
f''的正負號	+	+	-	-
結論	遞減/向上凹	遞增/向上凹	遞增/向下凹	遞減/向下凹

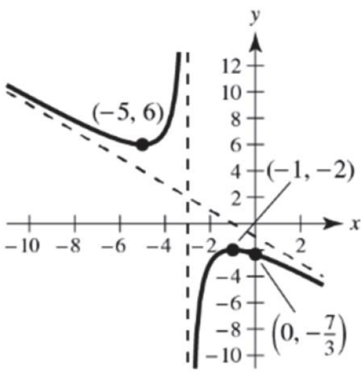
The critical numbers are $x = -1, -5$. f is increasing on $(-5, -3)$ and $(-3, -1)$ since $f'(x) > 0$, f is decreasing on $(-\infty, -5)$ and $(-1, \infty)$ since $f'(x) < 0$.
Local (global) maxima is $(-5, 6)$ and local (global) minima is $(-1, -2)$.
There are no possible points of inflection. f is concave downward on $(-3, \infty)$ since $f''(x) < 0$, f is concave upward on $(-\infty, -3)$ since $f''(x) > 0$.

(c) Since $f(x) = \pm\infty$ → No horizontal asymptote
Since $f(x) = -\infty$ and $f(x) = \infty$ vertical asymptote at $x = -3$

$$\frac{-x^2-4x-7}{x+3} = -x - 1 - \frac{4}{x+3} \text{ (Using long division)}$$

$$f(x) - (-x - 1 - \frac{4}{x+3}) = 0 \rightarrow y = -x - 1 \text{ is a slant asymptote}$$

(d)



(e) Domain is entire real line except -3. Range is $(-\infty, -2] \cup [6, \infty)$.

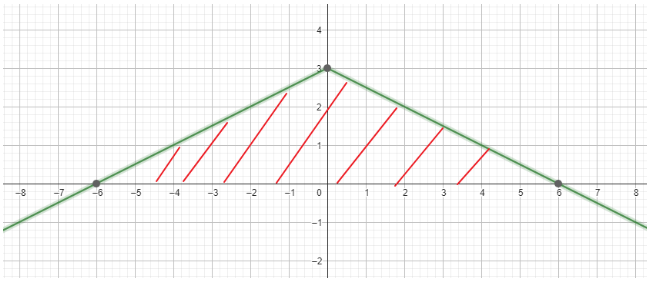
7. (15%) Evaluate the following expression. Remember the meaning and the definition of definite integral when solving the following question

(a) $\int \quad 3x - \frac{6}{x^3} + 5\, sec\, sec\, (x)\, tan(x)\, dx$
(b) $\int_{-6}^{6} 3 - |\frac{x}{2}| dx$
(c) $\frac{2^5}{n^5}(1^4 + 2^4 + 3^4 + \cdots + (2n)^4)$

Ans:

(a) $\frac{3x^2}{2} + \frac{3}{x^2} + 5\, sec\, sec\, (x)\ + C$

(b) $\int_{-6}^{6} 3 - |\frac{x}{2}|dx$ can be considered as the area in the following graph colored with red slash



Therefore, $\int_{-6}^{6} 3 - |\frac{x}{2}|dx = \frac{1}{2} 12 \times 3 = 18$

(c) $\frac{1}{n}(\frac{1^4+2^4+3^4\ldots+(2n)^4}{n^4})\ = 2^5 \sum_{i=1}^{n}(\frac{i}{n})^4 + \frac{1}{n}\sum_{i=n+1}^{2n}(\frac{i}{n})^4\ = 2^5(\int_0^1 x^4dx + \int_1^2 x^4dx) = 2^5 \frac{1}{5}x^5]2\ 0\ = \frac{2^{10}}{5}$

8. (8%) Find $\frac{d}{dx}\int_x^{x^2} \sqrt{1+t^2}dt$. (Hint: Let $F(x) = \int_1^x \sqrt{1+t^2}dt$ and use the fundamental theorem of calculus)

Ans: Let $F(x) = \int_1^x \sqrt{1+t^2}dt$, since $\sqrt{1+t^2}$ is continuous, by the fundamental theorem of calculus, $F'(x) = \sqrt{1+x^2}$. Also $F(b) - F(a) = \int_a^b \quad \sqrt{1+t^2}dt, a, b \in R$, therefore

$$\frac{d}{dx}\int_x^{x^2} \sqrt{1+t^2}dt = \frac{d}{dx}\left[\int_x^1 \sqrt{1+t^2}dt + \int_1^{x^2} \sqrt{1+t^2}dt\right]$$
$$= -\sqrt{1+x^2} + 2x\sqrt{1+x^4}$$

9. (8%) Find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\frac{\sqrt{1+tan\,(t)}}{cos^2(t)} + t^3sin^2(t))dt$.

Ans:

Note that $t^3sin^2(t)$ is an odd function, so we only need to deal with the first term.

Let $u = 1 +\ tan\,(t)\ \ \rightarrow du = sec^2(t)dt$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\frac{\sqrt{1+tan\,(t)}}{cos^2(t)} + t^3sin^2(t))dt = \int_0^2 \sqrt{u}du\ \ = \frac{4\ \sqrt{2}}{3}$$