Calculus (I), Midterm Exam 2024/11/08

1. (12%) Determine the following limit.

**(a) (3%)** $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1}$

**(b) (3%)** $\lim_{x\to 2}\frac{x^2-4}{|x-2|}$

**(c) (3%)** $\lim_{x\to 0}\frac{3}{1+\frac{2}{x}}$

**(d) (3%)** $\lim_{x\to\infty}\frac{\cos(\pi x)}{x+1}$

**Ans:**

(a) $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1}\frac{(x+3)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1}\frac{(x+3)}{(x+1)} = 2$

(b) $\lim_{x\to 2^+}\frac{x^2-4}{x-2} = \lim_{x\to 2^+} x+2 = 4$ and $\lim_{x\to 2^-}\frac{x^2-4}{-(x-2)} = \lim_{x\to 2^-} -(x+2) = -4$

Therefore, the limit does not exist.

(c) $\lim_{x\to 0}\frac{3}{1+\frac{2}{x}} = \lim_{x\to 0}\frac{3}{\frac{x+2}{x}} = \lim_{x\to 0}\frac{3x}{x+2} = 0$

(d) Since $\frac{-1}{x+1} \le \frac{cos(\pi x)}{x+1} \le \frac{1}{x+1}$ and $\lim_{x\to\infty}\frac{1}{x+1} = 0 = \lim_{x\to\infty}\frac{-1}{x+1}$. By the squeeze theorem, $\lim_{x\to\infty}\frac{cos(\pi x)}{x+1} = 0.$

2. (6%)

If $f(x)$ and $g(x)$ are both continuous function with $\lim_{x\to 2}[3f(x)+g(x)] = 4$ and $\lim_{x\to 2}[f(x)-2g(x)] = 6$ . Find

**(a)** (2%) $\lim_{x\to 2} f(x)$ **(b)** (2%) $g(2)$ **(c)** (2%) $\lim_{x\to 2^-} f(x)g(x)$

**Ans:**

Since $f(x)$ and $g(x)$ are continuous at $x=2$, we have:

$\lim_{x\to 2} f(x) = f(2)$ and $\lim_{x\to 2} g(x) = g(2)$

From the given limits, let $L = f(2)$ and $M = g(2)$

$$3L + M = 4, L - 2M = 6 \to L = 2, M = -2$$

(a) $\lim_{x\to 2} f(x) = L = 2$

(b) $g(2) = M = -2$

(c) $\lim_{x\to 2^-} f(x)g(x) = -4$

3. (10%)

Let $f(\text{x}) = \begin{cases} \sin(3x) & \text{for } x \le 0 \\ mx & \text{for } x > 0 \end{cases}$

**(a)** (5%) Find all values of $m$ that make $f$ continuous at 0

**(b)** (5%) Find all the values of $m$ that make $f$ differentiable at 0

**Ans:**

(a)

$\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x)$ → m can be any real number.

(b) Considering the alternative form of derivative:

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{sin(3x)}{x} = \lim_{x\to 0^-}\frac{3sin(3x)}{3x} = 3\lim_{t\to 0^-}\frac{\sin(t)}{t}\,(Let\ t=3x)$$
$$= 3$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{mx}{x}=m$$

Since it is differentiable, we have $m=3$

4. (5%) If $f(x)=x^2+2x-3$, use the definition of the derivative of a function to compute $f'(x)$

**Ans:**

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)\ -f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2+2(x+\Delta x)-3-(x^2+2x-3)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2+2\Delta x}{\Delta x}=\lim_{\Delta x\to 0}2x+\Delta x+2=2x+2$$

5. (5%) Verify that $f(x)=x^3+2x+4$ satisfies the hypotheses of the Mean Value Theorem on $[-1,1]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

**Ans:**

The function $f(x)=x^3+2x+4$ is a polynomial function. Polynomial functions are continuous everywhere on $R$, including the closed interval $[-1,1]$. Again, since $f(x)$ is a polynomial function, it is differentiable everywhere on $R$, including the open interval $(-1,1)$. Therefore, the hypotheses of the Mean Value Theorem are satisfied.

$$f(1)=7, f(-1)=1$$

In addition, $f'(x)=3x^2+2$

By MVT, we have $f'(c)=\frac{f(1)-f(-1)}{1-(-1)}=3\to 3c^2+2=3\to 3c^2=1\to c=\pm\frac{\sqrt{3}}{3}$

Both of them lies in $(-1,1)$. Therefore, $c=\pm\frac{\sqrt{3}}{3}$.

6. (12%)
   **(a)** (5%) Find the equation of the tangent line to the graph of $f(x)=\frac{x+8}{\sqrt{3x+1}}$ at the point (0,8)
   **(b)** (3%) Use chain rule to find the derivative of $g(x)=sin(2x^2+3cos(x))$
   **(c)** (4%) Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the expression $2xy-1=3x+y^2$

   **Ans:**

(a) $f'(x)=\frac{(3x+1)^{\frac{1}{2}}(1)-(x+8)\frac{1}{2}(3x+1)^{\frac{-1}{2}}(3)}{3x+1}$

$f'(0)=\frac{1-4(3)}{1}=-11$. The tangent line is $y-8=-11(x-0)\to y=-11x+8$

(b) $g'(x)=cos(2x^2+3cos(x))\times(4x-3\ sin\ sin\ (x)\ )$

(c) Differentiate both side with respect to $x$, we have

$$2y+2x\frac{dy}{dx}=3+2y\frac{dy}{dx}\to\frac{dy}{dx}=\frac{3-2y}{2x-2y}$$

$$\frac{d^2y}{dx^2}=\frac{-2\frac{dy}{dx}(2x-2y)-2(3-2y)(1-\frac{dy}{dx})}{4(x-y)^2}=\frac{(-4x+6)\frac{dy}{dx}+(4y-6)}{4(x-y)^2}$$

$$=\frac{(-4x+6)\left(\frac{3-2y}{2x-2y}\right)+(4y-6)}{4(x-y)^2}=\frac{-12x+8xy+9-4y^2}{4(x-y)^3}$$

7. (20%) Let $f(x) = \frac{x^4+x^2+4x}{x}$

**(a)** (4%) Find the critical points and possible points of inflection for $f(x)$

**(b)** (3%) Find the open intervals on which $f(x)$ is increasing or decreasing

**(c)** (3%) Find the open intervals of concavity for $f(x)$

**(d)** (4%) Find all asymptotes of $f(x)$

**(e)** (6%) Sketch the graph of $f(x)$, labeling intercepts, relative extrema, points of inflection, and asymptotes.

**Ans:** Note that the original function is undefined at $x = 0$, therefore we should include it in the following table.

$f(x) = x^3 + x + 4,\ x \neq 0,\ f'(x) = 3x^2 + 1 > 0$

$f''(x) = 6x$

| | $(-\infty, \mathbf{0})$ | $(\mathbf{0}, \infty)$ |
|---|---|---|
| 測試值 | -1 | 1 |
| $\boldsymbol{f'}$的正負號 | + | + |
| $\boldsymbol{f''}$的正負號 | - | + |
| 結論 | 遞增/向下凹 | 遞減/上下凹 |

(a) Note that $x$ is not define at $x = 0$, we should not include it in the critical numbers or possible points of inflection

There is no critical numbers **(**$f' = 0$**)**

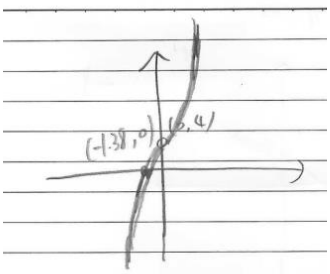
There is no possible points of inflection **(**$f'' = 0$**)**

(b) Increasing $(-\infty, 0), (0, \infty)$.

(c) Upward: $(0, \infty)$. Downward $(-\infty, 0)$

(d) Since $\lim_{x\to\pm\infty} f(x) = \pm\infty$ → No horizontal asymptote. No vertical asymptote since $f(x)$ is undefined at $x = 0$

(e) Graph




There is no relative extrema or point of inflection. No $y$ intercept

Using Newton's method or bisection method, we have $x$ intercept roughly equals -1.38. (Other approximation methods are also acceptable, like trial and error)

8. (8%)

**(a)** (4%) Find the point on the graph y= $\sqrt{x-8}$ that is closest to the point (12,0).

**(b)** (4%) Use Newton's method with the initial approximation $x_1 = -1$ to find $x_3$, the third approximation to the solution of the equation $2x^3 - 3x^2 + 2 = 0$

**Ans:**

(a) The distance between (12,0) and a point $(x, y)$ on the graph of $\sqrt{x-8}$ is

$$d = \sqrt{(x-12)^2 + y^2} = \sqrt{(x-12)^2 + x - 8}$$

Minimize $d^2 = f(x) = (x-12)^2 + x - 8$. Note that $f'(x) = 2(x-12) + 1 = 2x - 23$. The only critical point is $x = \frac{23}{2}$. When $x = \frac{23}{2}, y = \sqrt{\frac{7}{2}}, d = \frac{\sqrt{15}}{2}$. Therefore, $x = \frac{23}{2}$ is relative minimum (Testing the critical number using First Derivative Test). Therefore, the closest point is $(\frac{23}{2}, \sqrt{\frac{7}{2}})$.

(b) Newton's Method iteratively improves an estimate $x_n$ of a root of a function $f(x)$ using the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = 6x^2 - 6x$$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $\frac{f(x_n)}{f'(x_n)}$ | $x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|---|
| 1 | -1 | -3 | 12 | -0.25 | -0.75 |
| 2 | -0.75 | −0.53125 | 7.875 | 0.06746031746 | −0.68254 |
| 3 | −0.68254 | | | | |

9. (8%) Use differentials to approximate $\sqrt{1 + sin(0.01)}$

**Ans:** Let $f(x) = \sqrt{1 + sin(x)}, f'(x) = \frac{cos(x)}{2\sqrt{1+sin(x)}}$

$$f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt{1 + sin(x)} + \frac{cos(x)}{2\sqrt{1 + sin(x)}}dx$$

Choosing $x = 0$ and $dx = 0.01$.

$$f(x + \Delta x) = \sqrt{1 + sin(0.01)} \approx \sqrt{1 + sin\ sin\ (0)} + \frac{cos\ cos\ (0)}{2\sqrt{1 + sin\ sin\ (0)}}0.01$$

$$= 1 + \frac{0.01}{2} = 1.005$$

10. (14%) Solve the following problems

**(a)** (7%) Evaluate $\int \frac{3}{\sqrt[3]{x}} + x^2 + 2dx$

**(b)** (7%) Find $\lim_{n\to\infty} \frac{1}{n}\left[\sqrt{\frac{n^2-1^2}{n^2}} + \sqrt{\frac{n^2-2^2}{n^2}} + \sqrt{\frac{n^2-3^2}{n^2}} + \cdots + \sqrt{\frac{n^2-n^2}{n^2}}\right]$

**Ans:**

$(a)\int \frac{3}{\sqrt[3]{x}} + x^2 + 2dx = \int 3x^{\frac{-1}{3}} + x^2 + 2\,dx = \frac{9}{2}x^{\frac{2}{3}} + \frac{1}{3}x^3 + 2x + C$

(b) $\lim_{n\to\infty} \frac{1}{n}\left[\sqrt{\frac{n^2-1^2}{n^2}} + \sqrt{\frac{n^2-2^2}{n^2}} + \sqrt{\frac{n^2-3^2}{n^2}} + \cdots + \sqrt{\frac{n^2-n^2}{n^2}}\right] = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{1 - \left(\frac{k}{n}\right)^2} = \int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$