Chapter 11 Vectors and the Geometry of Space

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Cylindrical surfaces

- You have already known two special types of surfaces.
  1. Spheres: $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$
  2. Planes: $ax+by+cz+d=0$
- A third type of surface in space is called a **cylindrical surface**, or simply a **cylinder**.
- To define a cylinder, consider the familiar right circular cylinder shown in Figure 1.

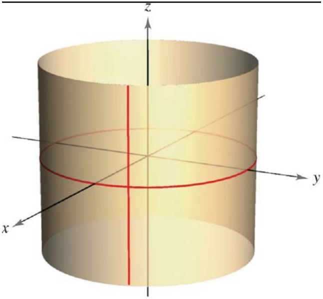


Figure 1: Right circular cylinder: $x^2+y^2=a^2$. Rulings are parallel to the z-axis.

- You can imagine that this cylinder is generated by a vertical line moving around the circle $x^2 + y^2 = a^2$ in the xy-plane.
- This circle is called a **generating curve** for the cylinder, as indicated in the following definition.

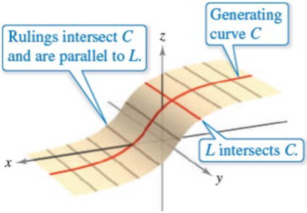


Figure 2: Right cylinder. Rulings are perpendicular to the plane containing C.

Definition 11.1 (Cylinder)

Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a cylinder. C is called the generating curve (or **directrix**) of the cylinder, and the parallel lines are called **rulings**.

- For the right circular cylinder shown in Figure 1, the equation of the generating curve is

$$x^2 + y^2 = a^2. \qquad \text{Equation of generating curve in } xy\text{-plane}$$

- To find an equation of the cylinder, note that you can generate any one of the rulings by fixing the values of x and y and then allowing z to take on all real values.
- In this sense, the value of z is arbitrary and is, therefore, not included in the equation. In other words, the equation of this cylinder is simply the equation of its generating curve.

$$x^2 + y^2 = a^2 \qquad \text{Equation of cylinder in space}$$

Definition 11.2 (Equation of cylinders)

The equation of a cylinder whose ruling are parallel to one of the coordinate axes contain only the variables corresponding to the other two axes.

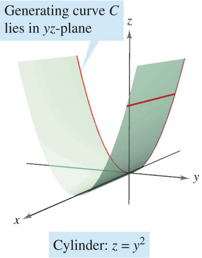
Example 1 (Sketching a cylinder)

Sketch the surface represented by each equation.

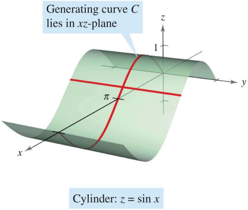
a. $z = y^2$ **b.** $z = \sin x,\ 0 \le x \le 2\pi$.

a.
- The graph is a cylinder whose generating curve, $z = y^2$, is a parabola in the yz-plane.
- The rulings of the cylinder are parallel to the x-axis.

b.
- The graph is a cylinder generated by the sine curve in the xz-plane.
- The rulings are parallel to the y-axis. ∎



(a) Rulings are parallel to the x-axis.



(b) Rulings are parallel to the y-axis.

Quadric surfaces

- The fourth basic type of surface in space is a **quadric surface**. Quadric surfaces are the three-dimensional analogs of conic sections.

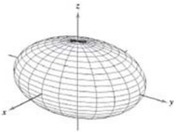
Definition 11.3 (Quadric surface)

The equation of a quadric surface in space is a second-degree equation in three variables. The **general form** of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

There are six basic types of quadric surfaces: **ellipsoid**, **hyperboloid of one sheet**, **hyperboloid of two sheets**, **elliptic cone**, **elliptic paraboloid**, and **hyperbolic paraboloid**.

- The intersection of a surface with a plane is called the **trace of the surface** in the plane.
- To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes. The traces of quadric surfaces are conics.
- These traces, together with the **standard form** of the equation of each quadric surface, are shown in the following tables.

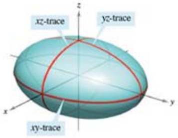


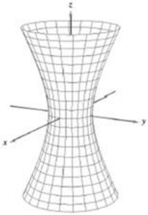
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.



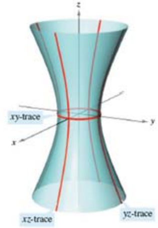


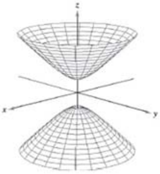
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



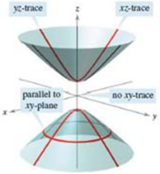


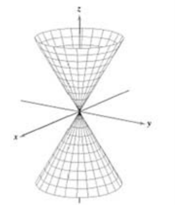
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



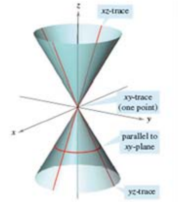


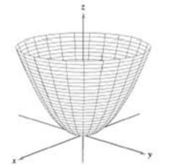
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



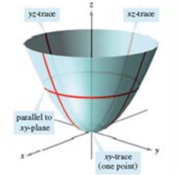


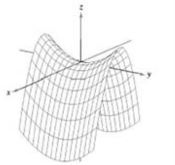
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



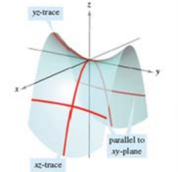


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



Example 2 (Sketching a quadric surface)

Classify and sketch the surface given by

$$4x^2 - 3y^2 + 12z^2 + 12 = 0.$$

- To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

$$xy\text{-trace } (z=0): \quad \frac{y^2}{4}-\frac{x^2}{3}=1 \qquad \text{Hyperbola}$$

$$xz\text{-trace } (y=0): \quad \frac{x^2}{3}+\frac{z^2}{1}=-1 \qquad \text{No trace}$$

$$yz\text{-trace } (x=0): \quad \frac{y^2}{4}-\frac{z^2}{1}=1 \qquad \text{Hyperbola}$$

■

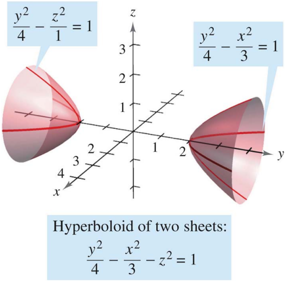


Figure 4: Hyperboloid of two sheets: $\frac{y^2}{4}-\frac{x^2}{3}-z^2=1.$

Example 3 (Sketching a quadric surface)

Classify and sketch the surface given by $x - y^2 - 4z^2 = 0$.

- Some convenient traces are as follows.

$$
\begin{array}{rll}
xy\text{-trace } (z=0): & x = y^2 & \text{Parabola} \\
xz\text{-trace } (y=0): & x = 4z^2 & \text{Parabola} \\
\text{parallel to } yz\text{-plane } (x=4): & \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1 & \text{Ellipse}
\end{array}
$$

- The surface is an elliptic paraboloid, as shown in Figure 5. ■

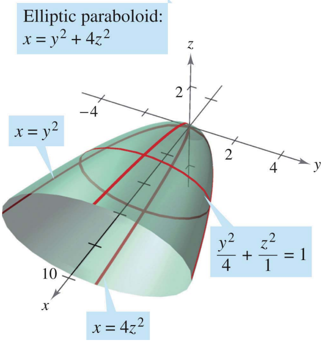
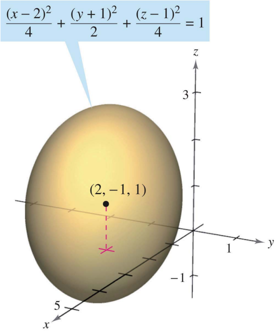


Figure 5: Elliptic paraboloid.

Example 4 (A quadric surface not centered at the origin)

Classify and sketch the surface given by
$x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0.$



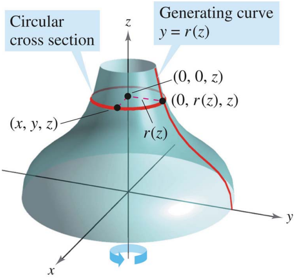
$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{2} + \frac{(z-1)^2}{4} = 1$
z
3
(2, −1, 1)
y
1
−1
5
x

Surfaces of revolution

- The fifth special type of surface you will study is called a **surface of revolution**. We now look at how to find its equation.
- Consider the graph of the **radius function**

$$y = r(z) \qquad \text{Generating curve}$$

in the yz-plane.
- If this graph is revolved around the z-axis, it forms a surface of revolution.



- The trace of the surface in the plane $z = z_0$ is a circle whose radius is $r(z_0)$ and whose equation is

$$x^2 + y^2 = [r(z_0)]^2 . \qquad \text{Circular trace in plane: } z = z_0$$

- Replacing z_0 with z produces an equation that is valid for all values of z.
- You can obtain equations for surfaces of revolution for the other two axes, and the results are summarized as follows.

Definition 11.4 (Surface of revolution)

If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

1. Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$

Example 5 (Finding an equation for a surface of revolution)

Find an equation for the surface of revolution formed by revolving (a) the graph of $y = 1/z$ about the z-axis and (b) the graph of $9x^2 = y^3$ about the y-axis.

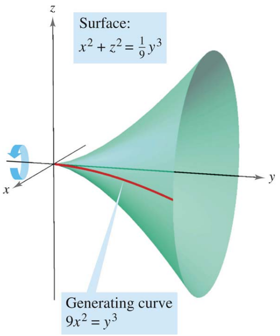


Figure 6: Surface of revolution: $x^2 + z^2 = \frac{1}{9}y^3$ with generating curve $9x^2 = y^3$ about the y-axis.

Example 6 (Finding a generating curve for a surface of revolution)

Find a generating curve and the axis of revolution for the surface given by

$$x^2 + 3y^2 + z^2 = 9.$$

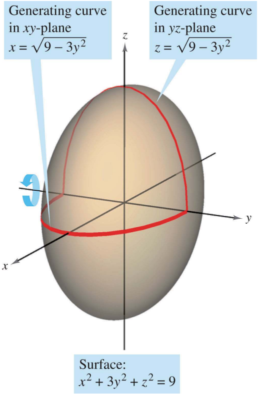


Figure 7: Finding a generating curve for a surface of revolution: not unique.

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Cylindrical coordinates

- The **cylindrical coordinate system**, is an extension of the polar coordinates in the plane to three-dimensional space.

Definition 11.5 (The cylindrical coordinate system)

In a cylindrical coordinate system, a point P in space is represented by an ordered triple (r, θ, z).

1. (r, θ) is a polar representation of the projection of P in the xy-plane.
2. z is the directed distance from (r, θ) to P.

- To convert from rectangular to cylindrical coordinates (or vice versa), use the following conversion guidelines for polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

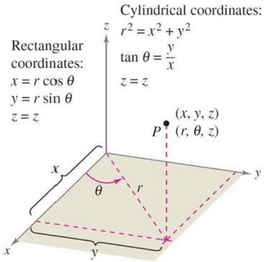


Figure 8: The relationship between cylindrical and rectangular coordinates.

- The point $(0,0,0)$ is called the **pole**.

- Moreover, because the representation of a point in the polar coordinate system is not unique, it follows that the representation in the cylindrical coordinate system is also not unique!

Example 1 (Converting from cylindrical to rectangular coordinates)

Convert the point $(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$ to rectangular coordinates.

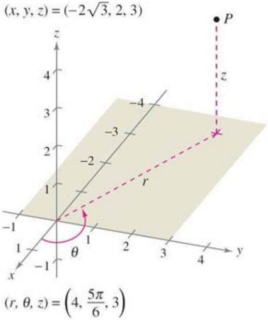


Figure 9: Converting $(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$ to $(x, y, z) = \left(-2\sqrt{3}, 2, 3\right)$.

Example 2 (Converting from rectangular to cylindrical coordinate)

Convert the point $(x, y, z) = (1, \sqrt{3}, 2)$ to cylindrical coordinates.

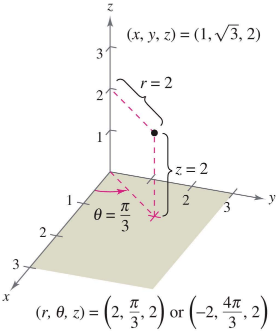


Figure 10: Converting from rectangular to cylindrical coordinates.

- Cylindrical coordinates are especially convenient for representing cylindrical surfaces and surfaces of revolution with the z-axis as the axis of symmetry:

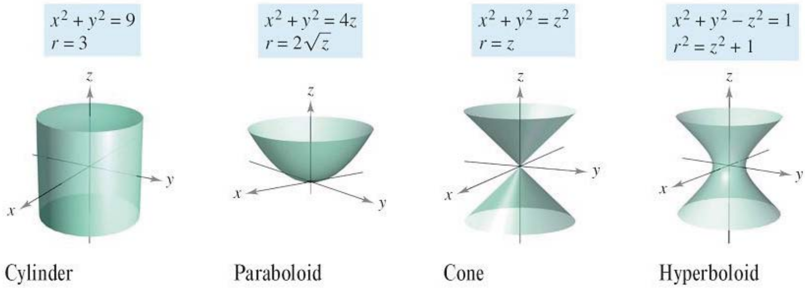
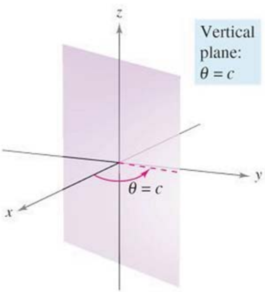


Figure 11: Different cylindrical equations.

- Vertical planes containing the z-axis and horizontal planes also have simple cylindrical coordinate equations:



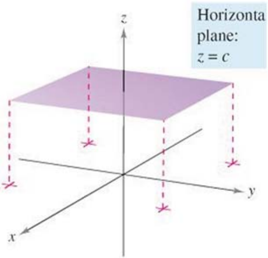
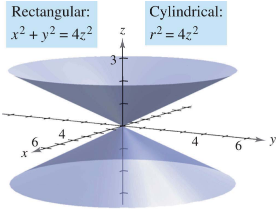


Figure 12: Vertical plane: $\theta = c$ and horizontal plane: $z = c$.

Example 3 (Rectangular-to-cylindrical conversion)

Find an equation in cylindrical coordinates for the surface represented by each rectangular equation.
a. $x^2 + y^2 = 4z^2$ **b.** $y^2 = x$



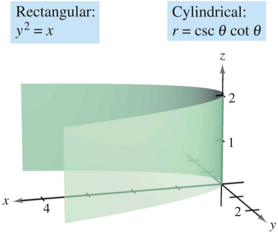


Figure 13: Rectangular-to-cylindrical conversion.

Example 4 (Cylindrical-to-rectangular conversion)

Find an equation in rectangular coordinates for the surface represented by the cylindrical equation

$$r^2 \cos 2\theta + z^2 + 1 = 0.$$

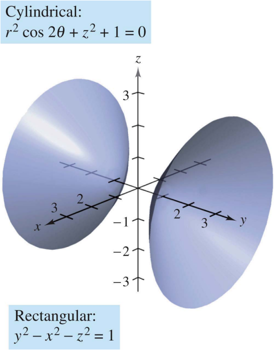


Figure 14: Cylindrical-to-rectangular conversion.

- In the **spherical coordinate system**, each point is represented by an ordered triple: the first coordinate is a distance, and the second and third coordinates are angles.
- This system is similar to the latitude-longitude system used to identify points on the surface of Earth.
- For example, the point on the surface of Earth whose latitude is 40° North (of the equator) and whose longitude is 80° West (of the prime meridian) is shown in Figure 15. Assuming that the Earth is spherical and has a radius of 6371 kilometers, you would label this point as

$$(4000, -80^\circ, 50^\circ).$$

Radius

80° clockwise from prime meridian

50° down from North Pole

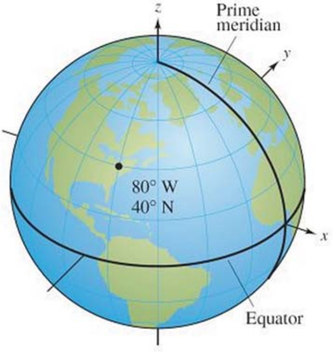


Figure 15: Spherical coordinate of 80° W 40° N is $(4000, -80^\circ, 50^\circ)$.

Definition 11.6 (The spherical coordinate system)

In a spherical coordinate system, a point P in space is represented by an ordered triple (ρ, θ, ϕ).

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the same angle used in cylindrical coordinates for $r \geq 0$.
3. ϕ is the angle between the positive z-axis and the line segment $\overrightarrow{OP}$, $0 \leq \phi \leq \pi$.

Note that the first and third coordinates, ρ and ϕ, are nonnegative. ρ is the lowercase Greek letter rho, and ϕ is the lowercase Greek letter phi.

- The relationship between rectangular and spherical coordinates is illustrated in Figure 16.

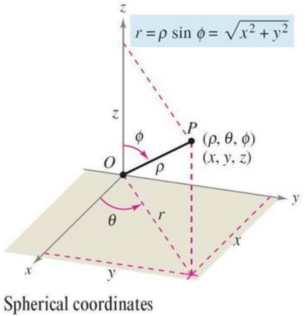


Figure 16: The relationship between rectangular coordinate (x, y, z) and spherical coordinates (ρ, θ, ϕ) where $r = \rho \sin \phi = \sqrt{x^2 + y^2}$.

- To convert from one system to the other, use the following.
- Spherical to rectangular:

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$

- Rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2, \quad \tan\theta = \frac{y}{x}, \quad \phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

- To change coordinates between the cylindrical and spherical systems, use the following.
- Spherical to cylindrical ($r \geq 0$):

$$r^2 = \rho^2 \sin^2\phi, \quad \theta = \theta, \quad z = \rho\cos\phi.$$

- Cylindrical to spherical ($r \geq 0$):

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \phi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right).$$

- The spherical coordinate system is useful primarily for surfaces in space that have a point or center of symmetry.
- For example, Figure 17 shows three surfaces with simple spherical equations.

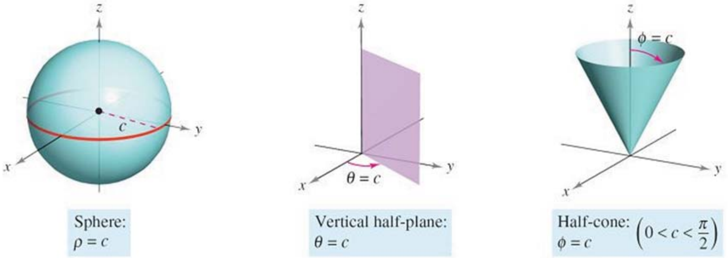


Figure 17: Three surfaces with simple spherical equations.

Example 5 (Rectangular-to-spherical conversion)

Find an equation in spherical coordinates for the surface represented by each rectangular equation.

a. Cone: $x^2 + y^2 = z^2$ **b.** Sphere: $x^2 + y^2 + z^2 - 4z = 0$

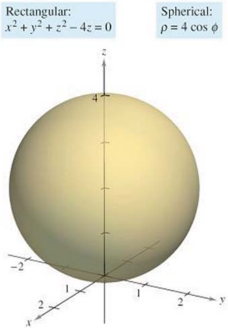


Figure 18: $x^2 + y^2 + z^2 - 4z = 0$ in rectangular coordinate is equivalent to $\rho = 4\cos\phi$ in spherical coordinate.