Chapter 14 Multiple Integration

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- We can also integrate functions of several variables, for instance considering $f_x(x, y) = 2xy$:

$$\begin{aligned} f(x, y) &= \int f_x(x, y)\,\mathrm{d}x \\ &= \int 2xy\,\mathrm{d}x \\ &= y\int 2x\,\mathrm{d}x \\ &= x^2y + C(y) \end{aligned}$$

- The constant of integration "$C(y)$" is a function of y! We can only recover $f(x, y)$ partially.
- For now, we will focus on extending definite integrals to functions of several variables.

Iterated integrals

(1) $\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,\mathrm{d}x = f(x,y)\Big|_{h_1(y)}^{h_2(y)} =$
$f(h_2(y),y) - f(h_1(y),y)$ With respect to x

(2) $\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,\mathrm{d}y = f(x,y)\Big|_{g_1(x)}^{g_2(x)} =$
$f(x,g_2(x)) - f(x,g_1(x))$ With respect to y

- Note that the variable of integration cannot appear in either limit of integration.
- For instance, it makes no sense to write

$$\int_0^x y\,\mathrm{d}x.$$

Example 2 (The integral of an integral)

Evaluate

$$\int_1^2 \left[\int_1^x (2xy + 3y^2)\,\mathrm{d}y \right] \mathrm{d}x.$$

- The integral in Example 2 is an **iterated integral**. The brackets used in Example 2 are normally not written.
- Instead, iterated integrals are usually written simply as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \text{and} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

- The inside limits of integration can be variable with respect to the outer variable of integration. However, the outside limits of integration must be constant with respect to both variables of integration!

- For instance, in Example 2, the outside limits indicate that x lies in the interval $1 \le x \le 2$ and the inside limits indicate that y lies in the interval $1 \le y \le x$.
- Together, these two intervals determine the **region of integration** R of the iterated integral, as shown in Figure 1.

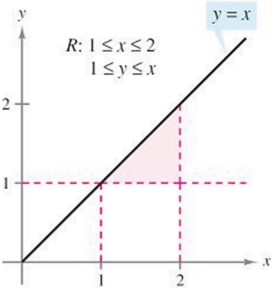
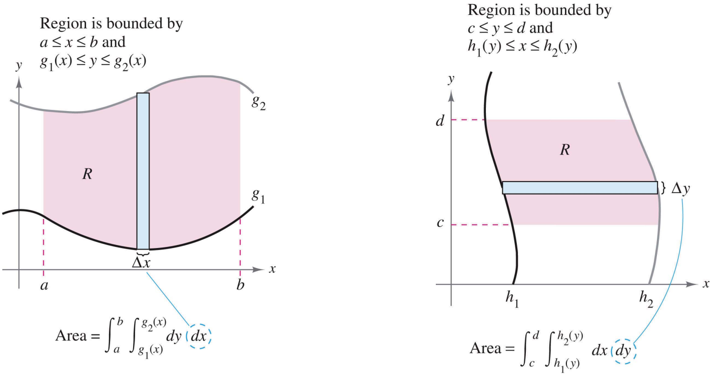


Figure 1: The region of integration for $\int_1^2 \int_1^x f(x, y)\,\mathrm{d}y\,\mathrm{d}x$.

Area of a plane region

- Consider the plane region (**vertically simple**) R bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, as shown in Figure 2(a).



(a) Vertically simple region.

(b) Horizontally simple region.

Figure 2: Vertically and horizontally simple regions.

- The area of R is given by the definite integral

$$\int_a^b [g_2(x) - g_1(x)]\,\mathrm{d}x. \qquad \text{Area of } R$$

- Using the Fundamental Theorem of Calculus, you can rewrite the integrand $g_2(x) - g_1(x)$ as a definite integral. If you consider x to be fixed and let y vary from $g_1(x)$ to $g_2(x)$, we have

$$\int_{g_1(x)}^{g_2(x)} \mathrm{d}y = y\,\Big|_{g_1(x)}^{g_2(x)} = g_2(x) - g_1(x).$$

- Combining these two integrals, you can write the area of the region R as an iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \mathrm{d}y\,\mathrm{d}x = \int_a^b y\,\Big|_{g_1(x)}^{g_2(x)} \mathrm{d}x = \int_a^b [g_2(x) - g_1(x)]\,\mathrm{d}x.$$

- A vertical rectangle implies the order $\mathrm{d}y\,\mathrm{d}x$, with the inside limits corresponding to the upper and lower bounds of the rectangle, as shown in Figure 2(a).
- This type of region is called **vertically simple**, because the outside limits of integration represent the vertical lines $x = a$ and $x = b$.
- Similarly, a horizontal rectangle implies the order $\mathrm{d}x\,\mathrm{d}y$, with the inside limits determined by the left and right bounds of the rectangle, as shown in Figure 2(b).
- This type of region is called **horizontally simple**, because the outside limits represent the horizontal lines $y = c$ and $y = d$.

Definition 14.1 (Area of a region in the plane)

1. If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} \mathrm{d}y \, \mathrm{d}x. \qquad \text{Figure 2}(a) \text{ (vertically simple)}$$

2. If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} \mathrm{d}x \, \mathrm{d}y. \qquad \text{Figure 2}(b) \text{ (horizontally simple)}$$

Example 4 (Finding area by an iterated integral)

Use an iterated integral to find the area of the region bounded by the graphs of

$$f(x) = \sin x \qquad\qquad g(x) = \cos x$$

between $x = \pi/4$ and $x = 5\pi/4$.

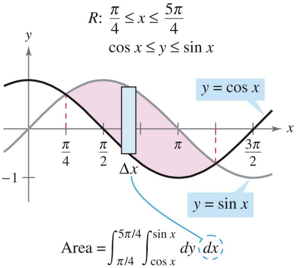


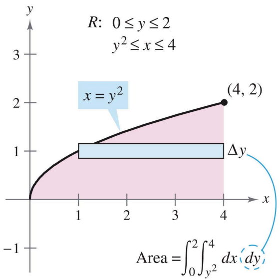
Figure 3: Area of $R = \int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} \mathrm{d}y\, \mathrm{d}x$.

Example 5 (Comparing different orders of integration)

Sketch the region whose area is represented by the integral

$$\int_0^2 \int_{y^2}^4 \mathrm{d}x \, \mathrm{d}y.$$

Then find another iterated integral using the order $\mathrm{d}y \, \mathrm{d}x$ to represent the same area and show that both integrals yield the same value.



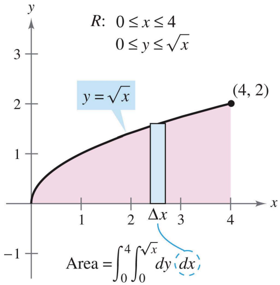


Figure 4: Comparing different orders of integration.

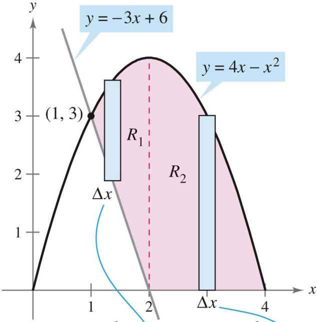
Example 6 (An area represented by two iterated integrals)

Find the area of the region R that lies below the parabola

$$y = 4x - x^2 \qquad \text{Parabola forms upper boundary}$$

above the x-axis, and above the line

$$y = -3x + 6. \qquad \text{Line and } x\text{-axis form lower boundary}$$



$$\text{Area} = \int_1^2 \int_{-3x+6}^{4x-x^2} dy\, dx + \int_2^4 \int_0^{4x-x^2} dy\, dx$$

Figure 5: An area represented by two iterated integrals.

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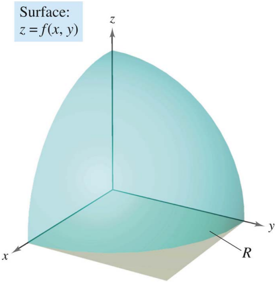
Double integrals and volume of a solid region

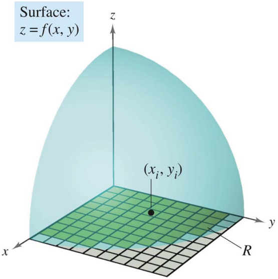
- You know that a definite integral over an interval uses a limit process to measures quantities such as area, volume and arc length.
- In this section, you will use a similar process to define the **double integral** of a function of two variables over a region in the plane.
- Consider a continuous function f such that $f(x, y) \geq 0$ for all (x, y) in a region R in the xy-plane. The goal is to find the volume of the solid region lying between the surface given by

$$z = f(x, y) \qquad \text{Surface lying above the } xy\text{-plane}$$

and the xy-plane, as shown in Figure 6(a).



(a) Volume of the solid region lying between the surface given by $z = f(x, y)$ and the xy-plane.



(b) The rectangles lying within R form as inner partition of R.

Figure 6: Volume of the solid region and rectangles.

- You can begin by superimposing a rectangular grid over the region, as shown in Figure 6(b).

- The rectangles within R form an inner partition Δ, whose norm $\|\Delta\|$ is defined as the length of the longest diagonal of the n rectangles. Next, choose a point (x_i, y_i) in each rectangle and form the rectangular prism whose height is $f(x_i, y_i)$, as shown in Figure 7(a).
- Because the area of the ith rectangle is

$$\Delta A_i \qquad \text{Area of } i\text{th rectangle}$$

it follows that the volume of the ith prism is

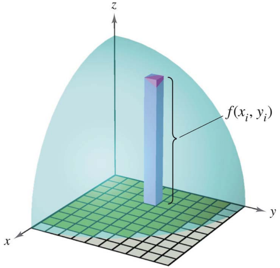
$$f(x_i, y_i)\Delta A_i. \qquad \text{Volume of } i\text{th prism}$$

- You can approximate the volume of the solid region by the Riemann sum of the volumes of all n prisms,

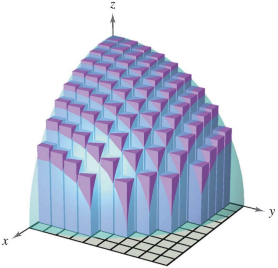
$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i \qquad \text{Riemann sum}$$

as shown in Figure 7(b).

- This approximation can be improved by tightening the mesh of the grid to form smaller and smaller rectangles!



(a) Rectangular prism whose base has an area ΔA_i, and whose height is $f(x_i, y_i)$.



(b) Volume approximated by rectangular prisms.

Figure 7: Rectangular prism and volume approximated by rectangular prisms.

Example 1 (Approximating the volume of a solid)

Approximate the volume of the solid lying between the paraboloid

$$f(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

and the square region R given by $0 \le x \le 1$, $0 \le y \le 1$. Use a partition made up of squares whose sides have a length of $\frac{1}{4}$.

- For this partition, it is convenient to choose the centers of the subregions as the points at which to evaluate $f(x, y)$.

$$\begin{array}{cccc}
\left(\frac{1}{8}, \frac{1}{8}\right) & \left(\frac{1}{8}, \frac{3}{8}\right) & \left(\frac{1}{8}, \frac{5}{8}\right) & \left(\frac{1}{8}, \frac{7}{8}\right) \\
\left(\frac{3}{8}, \frac{1}{8}\right) & \left(\frac{3}{8}, \frac{3}{8}\right) & \left(\frac{3}{8}, \frac{5}{8}\right) & \left(\frac{3}{8}, \frac{7}{8}\right) \\
\left(\frac{5}{8}, \frac{1}{8}\right) & \left(\frac{5}{8}, \frac{3}{8}\right) & \left(\frac{5}{8}, \frac{5}{8}\right) & \left(\frac{5}{8}, \frac{7}{8}\right) \\
\left(\frac{7}{8}, \frac{1}{8}\right) & \left(\frac{7}{8}, \frac{3}{8}\right) & \left(\frac{7}{8}, \frac{5}{8}\right) & \left(\frac{7}{8}, \frac{7}{8}\right)
\end{array}$$

- Because the area of each square is $\Delta A_i = 1/16$, you can approximate the volume by the sum

$$\sum_{i=1}^{16} f(x_i, y_i)\Delta A_i = \sum_{i=1}^{16} \left[1 - \frac{1}{2}x_i^2 - \frac{1}{2}y_i^2\right]\left[\frac{1}{16}\right] \approx 0.672.$$

- This approximation is shown graphically in Figure 8.
- The exact volume of the solid is $2/3$.

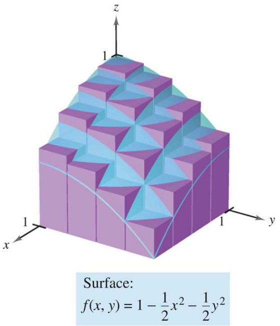


Figure 8: Volume approximated by rectangular prisms: $f(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$.

- In Example 1, note that by using finer partitions, you obtain better approximations of the volume. This observation suggests that you could obtain the exact volume by taking a limit. That is,

$$\text{Volume} = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i.$$

- The precise meaning of this limit is that the limit is equal to L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| L - \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i \right| < \varepsilon$$

for all partitions Δ of the plane region R (that satisfy $\|\Delta\| < \delta$) and for all possible choices of x_i and y_i in the ith region.

- Using the limit of a Riemann sum to define volume is a special case of using the limit to define a double integral. The general case, however, does not require that the function be positive or continuous.

Definition 14.2 (Double integral)

If f defined on a closed, bounded region R in the xy-plane, then the **double integral** of f over R is given by

$$\iint_R f(x,y)\,\mathrm{d}A = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i,y_i)\Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

- Having defined a double integral, you will see that a definite integral is occasionally referred to as a **single integral**.
- Sufficient conditions for the double integral of f on the region R to exist are that R can be written as a union of a finite number of nonoverlapping subregions (see Figure 9) that are vertically or horizontally simple and that f is continuous on the region R.

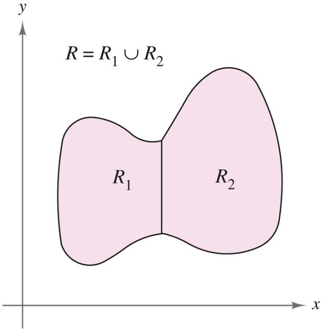


Figure 9: The regions are nonoverlapping if their intersection is a set that has an area of 0. In this figure, the area of the line segment that is common to R_1 and R_2 is 0.

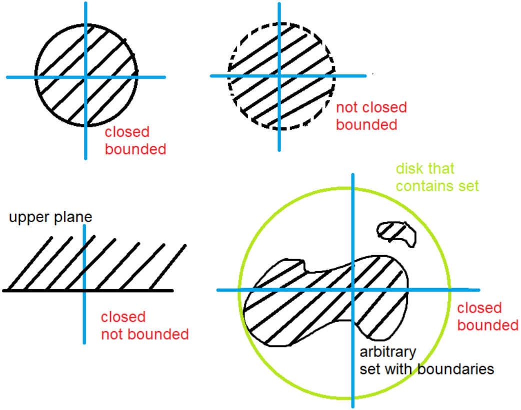


Figure 10: source: https://math.stackexchange.com/questions/1190640/what-is-the-difference-between-closed-and-bounded-in-terms-of-domains

- A double integral can be used to find the volume of a solid region that lies between the xy-plane and the surface given by $z = f(x, y)$.

Volume of a solid region: If f is integrable over a plane region R and $f(x, y) \geq 0$ for all (x, y) in R, then the volume of the solid region that lies above R and below the graph of f defined as

$$V = \iint_R f(x, y)\, \mathrm{d}A.$$

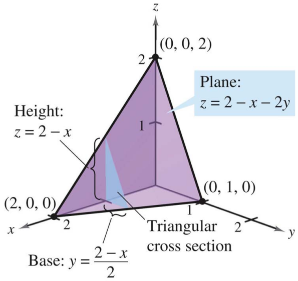
Theorem 14.1 (Properties of double integrals)

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\iint_R cf(x,y)\,\mathrm{d}A = c\iint_R f(x,y)\,\mathrm{d}A$
2. $\iint_R [f(x,y) \pm g(x,y)]\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}A \pm \iint_R g(x,y)\,\mathrm{d}A$
3. $\iint_R f(x,y)\,\mathrm{d}A \geq 0, \quad \text{if } f(x,y) \geq 0$
4. $\iint_R f(x,y)\,\mathrm{d}A \geq \iint_R g(x,y)\,\mathrm{d}A, \quad \text{if } f(x,y) \geq g(x,y)$
5. $\iint_R f(x,y)\,\mathrm{d}A = \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A$, *where R is the union of two nonoverlapping subregions R_1 and R_2.*

Evaluation of double integrals

- Consider the solid region bounded by the plane $z = f(x, y) = 2 - x - 2y$ and the three coordinate planes, as shown below:



- Each vertical cross section taken parallel to the yz-plane is a triangular region whose base has a length of $y = (2 - x)/2$ and whose height is $z = 2 - x$.

- This implies that for a fixed value of x, the area of the triangular cross section is

$$A(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\left(\frac{2-x}{2}\right)(2-x) = \frac{(2-x)^2}{4}.$$

- By the formula for the volume of a solid with known cross sections, the volume of the solid is

$$\text{Volume} = \int_a^b A(x)\,\mathrm{d}x = \int_0^2 \frac{(2-x)^2}{4}\,\mathrm{d}x = -\frac{(2-x)^3}{12}\bigg|_0^2 = \frac{2}{3}.$$

- This procedure works no matter how $A(x)$ is obtained as below!

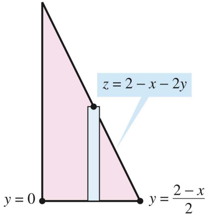


Figure 11: Triangular cross section $z = 2 - x - 2y$ where $0 \le y \le \frac{2-x}{2}$.

- That is, you consider x to be constant, and integrate $z = 2 - x - 2y$ from 0 to $(2-x)/2$ to obtain

$$A(x) = \int_0^{(2-x)/2} (2 - x - 2y)\,\mathrm{d}y = \left[(2-x)y - y^2\right]_0^{(2-x)/2} = \frac{(2-x)^2}{4}.$$

- Combining these results, you have the iterated integral

$$\text{Volume} = \iint_R f(x,y)\,\mathrm{d}A = \int_0^2 \int_0^{(2-x)/2} (2 - x - 2y)\,\mathrm{d}y\,\mathrm{d}x.$$

- To understand this procedure better, it helps to imagine the integration as two sweeping motions. For the inner integration, a vertical line sweeps out the area of a cross section. For the outer integration, the triangular cross section sweeps out the volume, as shown in Figure 12.

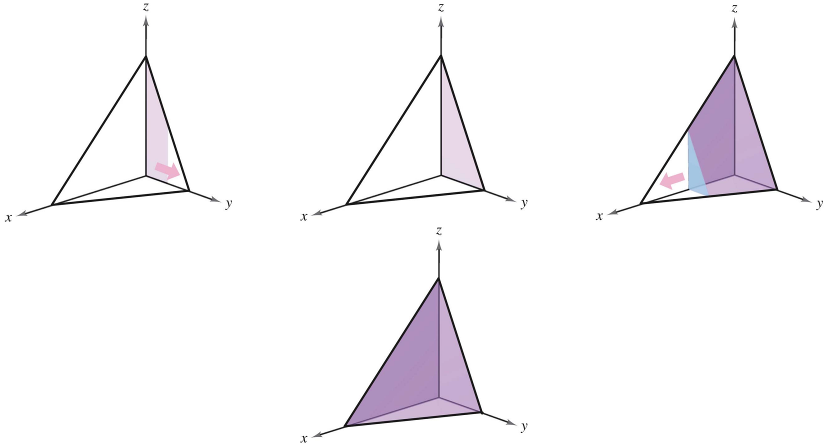


Figure 12: Integrate with respect to y to obtain the area of the cross section; integrate with respect to x to obtain the volume of the cross section.

Theorem 14.2 (Fubini's Theorem)

Let f be continuous on a plane region R.

1. *If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

2. *If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Example 2 (Evaluating a double integral as an iterated integral)

Evaluate $\iint_R \left(1 - \frac{1}{2}\,x^2 - \frac{1}{2}\,y^2\right)\,\mathrm{d}A$ where R is the region given by $0 \le x \le 1$, $0 \le y \le 1$.

Example 3 (Finding volume by a double integral)

Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane, as shown in Figure 13(a).

- This plane region is both vertically and horizontally simple, so the order $\mathrm{d}y\,\mathrm{d}x$ is appropriate.

$$\text{Variable bounds for } y\text{:} \quad -\sqrt{\frac{(4-x^2)}{2}} \le y \le \sqrt{\frac{(4-x^2)}{2}}$$

$$\text{Constant bounds for } x\text{:} \quad -2 \le x \le 2$$

- The volume is given by

$$\begin{aligned}
V &= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4-x^2-2y^2)\,\mathrm{d}y\,\mathrm{d}x \qquad \text{See Figure 13(b)} \\
&= \int_{-2}^{2}\left[(4-x^2)y-\frac{2y^3}{3}\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \mathrm{d}x = \frac{4}{3\sqrt{2}}\int_{-2}^{2}(4-x^2)^{3/2}\,\mathrm{d}x \\
&= \frac{4}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} 16\cos^4\theta\,\mathrm{d}\theta = \frac{64}{3\sqrt{2}}(2)\int_{0}^{\pi/2}\cos^4\theta\,\mathrm{d}\theta \\
&= \frac{128}{3\sqrt{2}}\left(\frac{3\pi}{16}\right) = 4\sqrt{2}\pi. \qquad \blacksquare
\end{aligned}$$

- Note that to evaluate the integral

$$\int_{-2}^{2}\left(4-x^{2}\right)^{3/2}\,\mathrm{d}x,$$

set $x=2\sin\theta$. Because $x\in[-2,2]$, the parameter θ ranges over $[-\frac{\pi}{2},\,\frac{\pi}{2}]$. With this choice

$$\mathrm{d}x=2\cos\theta\;\mathrm{d}\theta,\qquad 4-x^{2}=4\cos^{2}\theta,$$

so that

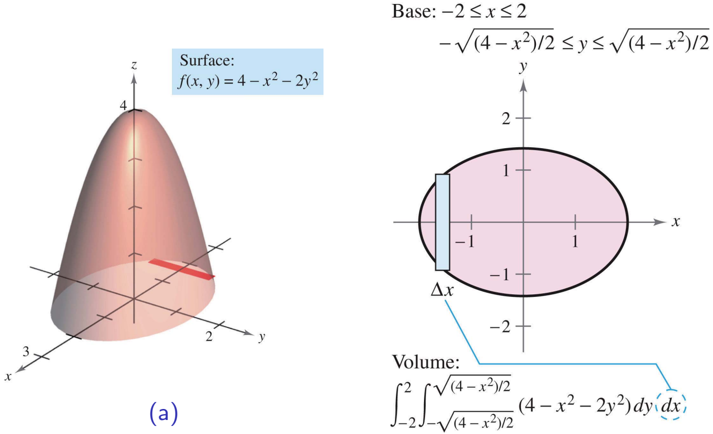
$$\left(4-x^{2}\right)^{3/2}\,\mathrm{d}x=\left(4\cos^{2}\theta\right)^{3/2}\left(2\cos\theta\;\mathrm{d}\theta\right)=16\cos^{4}\theta\;\mathrm{d}\theta.$$

Consequently,

$$\int_{-2}^{2}\left(4-x^{2}\right)^{3/2}\,\mathrm{d}x=\int_{-\pi/2}^{\pi/2}16\cos^{4}\theta\;\mathrm{d}\theta,$$

which is precisely the form used in the third line of the computation.

$$
\begin{aligned}
I &= \int_0^{\pi/2} \cos^4\theta\, d\theta = \int_0^{\pi/2} \left(\cos^2\theta\right)^2 d\theta \\
&= \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta \\
&= \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta \\
&= \frac{1}{4}\int_0^{\pi/2} \left(\frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\right) d\theta \\
&= \frac{1}{8}\int_0^{\pi/2} \left(3 + 4\cos 2\theta + \cos 4\theta\right) d\theta \\
&= \frac{1}{8}\Big[3\theta + 2\sin 2\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/2} = \frac{1}{8}\left(3\cdot\tfrac{\pi}{2}\right) = \frac{3\pi}{16}
\end{aligned}
$$



(a)

(b)

Figure 13: Finding volume by a double integral.

Example 4 (Comparing different orders of integration)

Find the volume of the solid region R bounded by the surface

$$f(x, y) = e^{-x^2} \qquad \text{Surface}$$

and the planes $z = 0$, $y = 0$, $y = x$, and $x = 1$, as shown in Figure 14.

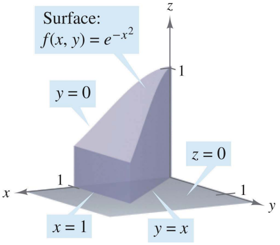
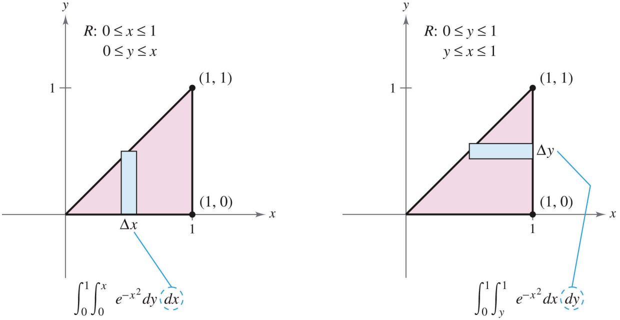


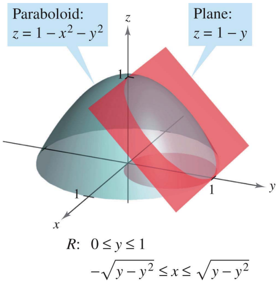
Figure 14: Base is bounded by $y = 0$, $y = x$, and $x = 1$.



(a) $dy\, dx$ (b) $dx\, dy$

Example 5 (Volume of a region bounded by two surfaces)

Find the volume of the solid region R bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$, as shown in Figure 16.



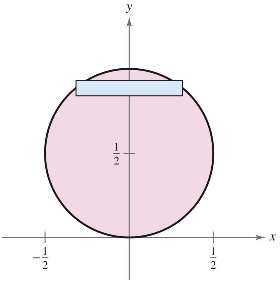


Figure 16: Volume of a region bounded by two surfaces.

- Because the volume of R is the difference between the volume under the paraboloid and the volume under the plane, you have

$$\begin{aligned}
\text{Volume} &= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y - \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (y-y^2-x^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left[(y-y^2)x - \frac{x^3}{3}\right]_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \mathrm{d}y = \frac{4}{3}\int_0^1 (y-y^2)^{3/2}\,\mathrm{d}y \\
&= \left(\frac{4}{3}\right)\left(\frac{1}{8}\right)\int_0^1 [1-(2y-1)^2]^{3/2}\,\mathrm{d}y \\
&= \frac{1}{6}\int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{2}\,\mathrm{d}\theta = \frac{1}{6}\int_0^{\pi/2} \cos^4\theta\,\mathrm{d}\theta \\
&= \left(\frac{1}{6}\right)\left(\frac{3\pi}{16}\right) = \frac{\pi}{32}.
\end{aligned}$$

■

- Set $2y - 1 = \sin\theta$. Then

$$y = \frac{1+\sin\theta}{2}, \qquad dy = \frac{\cos\theta}{2}\, d\theta,$$

and the limits $y = 0 \to 1$ correspond to $\theta = -\frac{\pi}{2} \to \frac{\pi}{2}$. Because

$$1 - (2y-1)^2 = 1 - \sin^2\theta = \cos^2\theta, \qquad \left[1 - (2y-1)^2\right]^{3/2} = \cos^3\theta,$$

we have

$$\left[1 - (2y-1)^2\right]^{3/2} dy = \cos^3\theta \left(\frac{\cos\theta}{2}\, d\theta\right) = \frac{\cos^4\theta}{2}\, d\theta.$$

Hence

$$\frac{4}{3}\cdot\frac{1}{8}\int_0^1 [1-(2y-1)^2]^{3/2} dy = \frac{1}{6}\int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{2}\, d\theta,$$

which is precisely the form used in the next step of the calculation.

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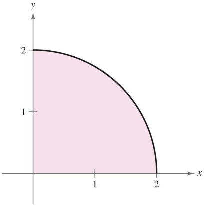
- The polar coordinates (r, θ) of a point can be converted as.

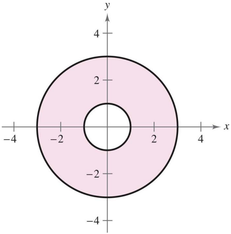
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \qquad r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

Example 1 (Using polar coordinates to describe a region)

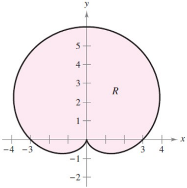
Use polar coordinates to describe each region



(a) A quarter circle with radius 2.



(b) Region R consists of all points between concentric circles of radii 1 and 3.

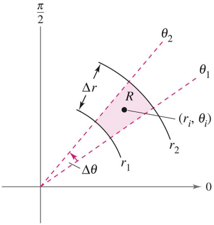


(c) Region R is a cardioid

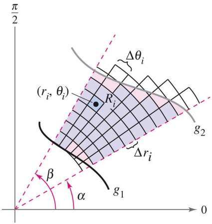
- The regions in Example 1 are special cases of **polar sectors**

$$R = \{(r, \theta) : r_1 \le r \le r_2,\ \theta_1 \le \theta \le \theta_2\} \qquad \text{Polar sector}$$

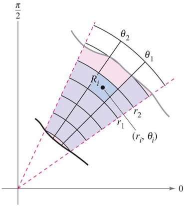
as shown in Figure 19(a).



(a) Polar sector $R = \{(r, \theta) : r_1 \le r \le r_2,\ \theta_1 \le \theta \le \theta_2\}$.



(b) Polar grid superimposed over region R.



(c) The polar sector R_i is the set of all points (r, θ) such that $r_1 \le r \le r_2$ and $\theta_1 \le \theta \le \theta_2$.

Figure 19: Polar sector.

- To define a double integral of a continuous function $z = f(x, y)$ in polar coordinates, consider a region R bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$.
- Instead of partitioning R into small rectangles, use a partition of small polar sectors. On R, superimpose a polar grid made of rays and circular arcs, as shown in Figure 19(b).
- The polar sectors R_i lying entirely within R form an inner polar partition Δ, whose norm $\|\Delta\|$ is the length of the longest diagonal of the n polar sectors. Now, consider a specific polar sector R_i, as shown in Figure 19(c).

- It can be shown that the area of R_i is

$$\begin{aligned}\Delta A_i &= \left(\frac{\Delta\theta_i}{2\pi}\right)\left(\pi r_2^2 - \pi r_1^2\right) \\ &= \frac{r_2 + r_1}{2}(r_2 - r_1)\Delta\theta_i = r_i \Delta r_i \Delta\theta_i \qquad \text{Area of } R_i\end{aligned}$$

where $r_i = \frac{r_2+r_1}{2}$, $\Delta r_i = r_2 - r_1$ and $\Delta\theta_i = \theta_2 - \theta_1$.

- This implies that the volume of the solid of height $f(r_i \cos\theta_i, r_i \sin\theta_i)$ above R_i is approximately

$$f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i$$

and you have

$$\iint_R f(x, y)\,\mathrm{d}A \approx \sum_{i=1}^{n} f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i.$$

- The sum on the right can be interpreted as a Riemann sum for $f(r\cos\theta, r\sin\theta)r$.

- The region R corresponds to a horizontally simple region S in the $r\theta$-plane, as shown in Figure 20.

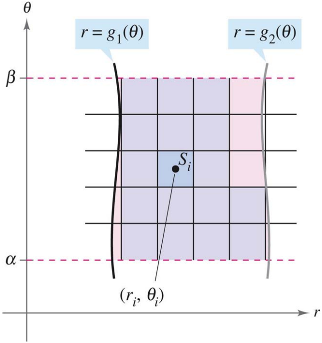


Figure 20: $S = \{(r, \theta) \mid g_1(\theta) \leq r \leq g_2(\theta),\ \alpha \leq \theta \leq \beta\}$.

- The polar sectors R_i correspond to rectangles S_i, and the area ΔA_i of S_i is $\Delta r_i \Delta\theta_i$.

- So, the right-hand side of the equation corresponds to the double integral

$$\iint_S f(r\cos\theta, r\sin\theta)r\,\mathrm{d}A.$$

- From this, you can write

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \iint_S f(r\cos\theta, r\sin\theta)r\,\mathrm{d}A \\ &= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\theta.\end{aligned}$$

Theorem 14.3 (Change of variables to polar form)

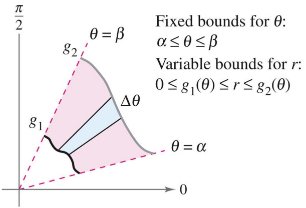
Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x, y)\, \mathrm{d}A = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, \mathrm{d}r\, \mathrm{d}\theta.$$

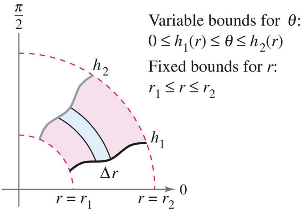
- The region R is restricted to two basic types,

$$\iint_R f(x, y)\, \mathrm{d}A = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, \mathrm{d}r\, \mathrm{d}\theta \quad r\textbf{-simple region}$$

$$\iint_R f(x, y)\, \mathrm{d}A = \int_{r_1}^{r_2} \int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta) r\, \mathrm{d}\theta\, \mathrm{d}r \quad \theta\textbf{-simple region}$$



(a) r-simple region.



(b) θ-simple region.

Figure 21: r-simple region and θ-simple region.

Example 2 (Evaluating a double polar integral)

Let R be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate the integral $\iint_R (x^2 + y)\,\mathrm{d}A$.

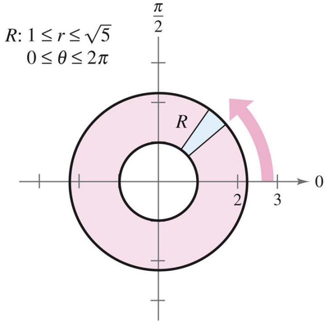


Figure 22: r-simple region, R: $1 \le r \le \sqrt{5}$, $0 \le \theta \le 2\pi$.

Example 3 (Change of variables to polar coordinates)

Use polar coordinates to find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2} \qquad \text{Hemisphere forms upper surface}$$

and below with the circular region R given by

$$x^2 + y^2 \leq 4 \qquad \text{Circular region forms lower surface}$$

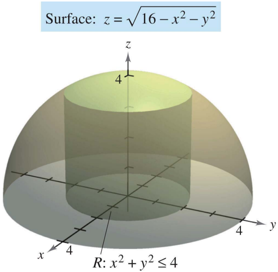


Figure 23: Change of variables to polar coordinates.

Example 4 (Finding areas of polar regions)

Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

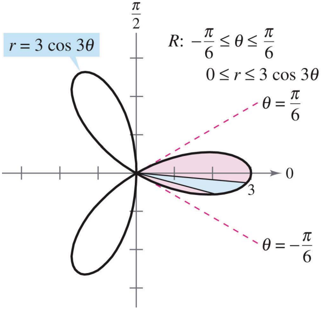


Figure 24: Find area of the polar region: rose curve with three petals.

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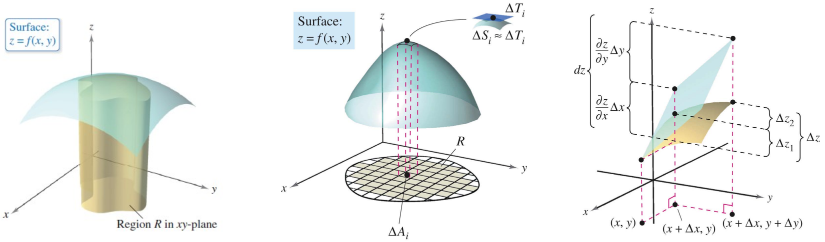
Surface area

- You know about the solid region lying between a surface and a closed and bounded region R in the xy-plane, as shown in Figure 25(a).



(a) (b) (c)

Figure 25: Region R in xy plane with surface: $z = f(x, y)$.

- For example, you know how to find the extrema of f on R.

- In this section, you will learn how to find the upper surface area of the solid. To begin, consider a surface S given by

$$z = f(x, y) \qquad \text{Surface defined over a region } R$$

defined over a region R.

- Assume that R is closed and bounded and that f has continuous first partial derivatives. To find the surface area, construct an inner partition of R consisting of n rectangles, where the area of the ith rectangle R_i is $\Delta A_i = \Delta x_i \Delta y_i$, as shown in Figure 25(b).
- In each R_i let (x_i, y_i) be the point that is closest to the origin. At the point $(x_i, y_i, z_i) = (x_i, y_i, f(x_i, y_i))$ on the surface S, construct a tangent plane T_i.

- The area of the portion of the tangent plane that lies directly above R_i is approximately equal to the area of the surface lying directly above R_i. That is, $\Delta T_i \approx \Delta S_i$. So, the surface area of S is given by

$$\sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \Delta T_i.$$

- To find the area of the parallelogram ΔT_i, note that its sides are given by the vectors

$$\mathbf{u} = \Delta x_i \,\mathbf{i} + f_x(x_i, y_i)\Delta x_i \,\mathbf{k} \quad \text{and} \quad \mathbf{v} = \Delta y_i \,\mathbf{j} + f_y(x_i, y_i)\Delta y_i \,\mathbf{k}.$$

- The area of ΔT_i is given by $\|\mathbf{u} \times \mathbf{v}\|$, where

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_i)\Delta x_i \\ 0 & \Delta y_i & f_y(x_i, y_i)\Delta y_i \end{vmatrix} \\ &= -f_x(x_i, y_i)\Delta x_i \Delta y_i \,\mathbf{i} - f_y(x_i, y_i)\Delta x_i \Delta y_i \,\mathbf{j} + \Delta x_i \Delta y_i \,\mathbf{k} \\ &= \left(-f_x(x_i, y_i)\,\mathbf{i} - f_y(x_i, y_i)\,\mathbf{j} + \mathbf{k}\right)\Delta A_i. \end{aligned}$$

- So, the area of ΔT_i is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{[f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2 + 1}\ \Delta A_i$ and

$$\text{Surface area of } S \approx \sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2}\ \Delta A_i.$$

Definition 14.3 (Surface area)

If f and its partial derivatives are continuous on the closed region R in the xy-plane, then the area of the surface S given by $z = f(x, y)$ over R is defined as

$$\text{Surface area} = \iint_R \mathrm{d}S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\,\mathrm{d}A.$$

- As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

Length on x-axis:	$\int_a^b \mathrm{d}x$
Arc length in xy-plane:	$\int_a^b \mathrm{d}s = \int_a^b \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x$
Area in xy-plane:	$\iint_R \mathrm{d}A$
Surface area in space:	$\iint_R \mathrm{d}S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\,\mathrm{d}A$

Example 1 (The surface area of a plane region)

Find the surface area of the portion of the plane

$$z = 2 - x - y$$

that lies above the circle $x^2 + y^2 \leq 1$ in the first quadrant, as shown in Figure 26.

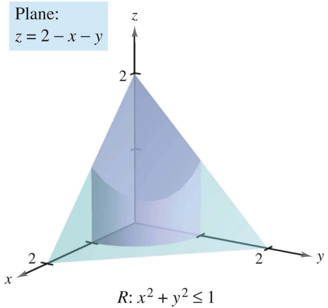


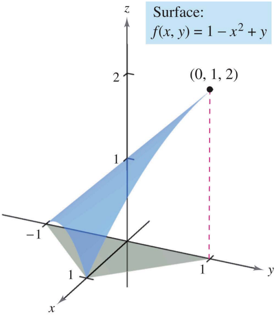
Figure 26: Plane: $z = 2 - x - y$ and R: $x^2 + y^2 \leq 1$.

Example 2 (Finding surface area)

Find the area of the portion of the surface

$$f(x, y) = 1 - x^2 + y$$

that lies above the triangular region with vertices $(1, 0, 0)$, $(0, -1, 0)$, and $(0, 1, 0)$, as shown in Figure 27(a).



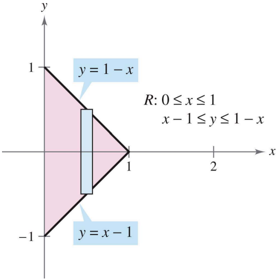


Figure 27: Finding surface area.

- In Figure 27(b), you can see that the bounds for R are $0 \leq x \leq 1$ and $x - 1 \leq y \leq 1 - x$. So, the integral becomes

$$
\begin{aligned}
S &= \int_0^1 \int_{x-1}^{1-x} \sqrt{2 + 4x^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 y\sqrt{2+4x^2}\Big]_{x-1}^{1-x}\,\mathrm{d}x \\
&= \int_0^1 \left[(1-x)\sqrt{2+4x^2} - (x-1)\sqrt{2+4x^2}\right]\,\mathrm{d}x \\
&= \int_0^1 \left(2\sqrt{2+4x^2} - 2x\sqrt{2+4x^2}\right)\,\mathrm{d}x \\
&= \left[x\sqrt{2+4x^2} + \ln(\sqrt{2}x + \sqrt{1+2x^2}) - \frac{(2+4x^2)^{3/2}}{6}\right]_0^1 \\
&= \sqrt{6} + \ln(\sqrt{2} + \sqrt{3}) - \sqrt{6} - \ln\sqrt{1} + \frac{1}{3}\sqrt{2} \approx 1.618. \qquad \blacksquare
\end{aligned}
$$

Note

$$\int \sqrt{2+4x^2}\,dx = \sqrt{2}\int \sqrt{1+2x^2}\,dx$$

$$\xrightarrow{x=\frac{\tan\theta}{\sqrt{2}},\ dx=\frac{\sec^2\theta}{\sqrt{2}}\,d\theta} \int \sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{\sqrt{1+2x^2}\sqrt{2}x}{2} + \tfrac{1}{2}\ln\bigl(\sqrt{1+2x^2}+\sqrt{2}x\bigr) + C.$$

$$\int \sec^3\theta\, d\theta = \int \sec\theta\ \sec^2\theta\, d\theta$$

$$\xrightarrow{(u=\sec\theta,\ dv=\sec^2\theta\, d\theta)} \sec\theta\tan\theta - \int \tan\theta\cdot\sec\theta\tan\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta\ \tan^2\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta\,(\sec^2\theta - 1)\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$\Longrightarrow\ 2\int \sec^3\theta\, d\theta\ =\ \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$$

$$\boxed{\int \sec^3\theta\, d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C}.$$

Example 3 (Change of variables to polar coordinate)

Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit circle, as shown in Figure 28.

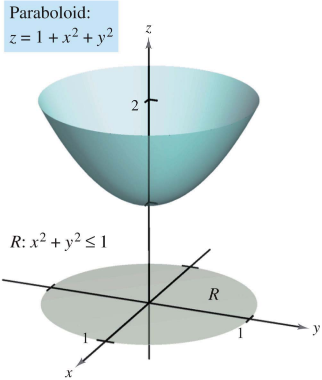


Figure 28: Surface area of the paraboloid $z = 1 + x^2 + y^2$.

Example 4 (Change of variables to polar coordinate)

Find the surface area of the portion of the hemisphere

$$f(x, y) = \sqrt{25 - x^2 - y^2}$$

that lies above the region R bounded by the circle $x^2 + y^2 \leq 9$, as shown in Figure 29.

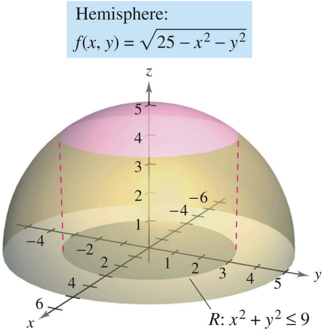


Figure 29: Surface area of the portion of a hemisphere.

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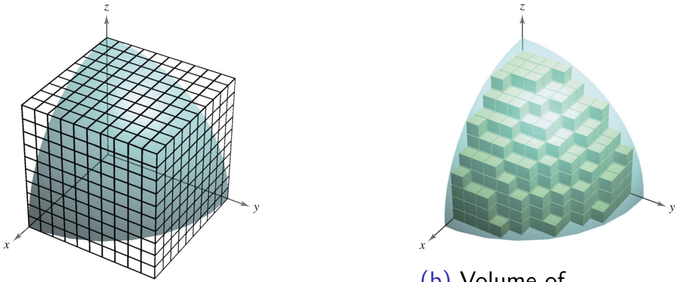
Triple integrals

- The procedure used to define a **triple integral** follows that used for double integrals.
- Consider a function f of three variables that is continuous over a bounded solid region Q. Then, encompass Q with a network of boxes and form the inner partition consisting of all boxes lying entirely within Q, as shown in Figure 30.



(a) Solid region Q.

(b) Volume of $Q \approx \sum_{i=1}^{n} \Delta V_i$.

Figure 30: Volume of solid region $Q \approx \sum_{i=1}^{n} \Delta V_i$.

- The volume of the ith box is

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i. \qquad \text{Volume of } i\text{th box}$$

- The norm $\|\Delta\|$ of the partition is the length of the longest diagonal of the n boxes in the partition. Now, choose a point (x_i, y_i, z_i) in each box and form the Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i.$$

- Taking the limit as $\|\Delta\| \to 0$ leads to the following definition.

Definition 14.4 (Triple integral)

If f is continuous over a bounded solid region Q, then the triple integral of f over Q is defined as

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta V_i$$

provided the limit exists. The volume of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q \mathrm{d}V.$$

1. $\iiint_Q cf(x,y,z)\,\mathrm{d}V = c\iiint_Q f(x,y,z)\,\mathrm{d}V$
2. $\iiint_Q [f(x,y,z) \pm g(x,y,z)]\,\mathrm{d}V =$
   $\iiint_Q f(x,y,z)\,\mathrm{d}V \pm \iiint_Q g(x,y,z)\,\mathrm{d}V$
3. $\iiint_Q f(x,y,z)\,\mathrm{d}V = \iiint_{Q_1} f(x,y,z)\,\mathrm{d}V + \iiint_{Q_2} f(x,y,z)\,\mathrm{d}V$

Theorem 14.4 (Evaluation by iterated integrals)

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x,y) \le z \le g_2(x,y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Example 1 (Evaluating a triple iterated integral)

Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x (y + 2z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x.$$

- For the first integration, hold x and y constant and integrate with respect to z.

$$\begin{aligned}
\int_0^2 \int_0^x \int_0^{x+y} e^x (y + 2z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x &= \int_0^2 \int_0^x e^x (yz + z^2)\Big|_0^{x+y}\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_0^2 \int_0^x e^x (x^2 + 3xy + 2y^2)\, \mathrm{d}y\, \mathrm{d}x
\end{aligned}$$

- For the second integration, hold x constant and integrate with respect to y.

$$\int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 e^x \left(x^2y + \frac{3xy^2}{2} + \frac{2y^3}{3} \right)\Bigg|_0^x \,\mathrm{d}x$$
$$= \frac{19}{6}\int_0^2 x^3 e^x \,\mathrm{d}x$$

- Finally, integrate with respect to x.

$$\frac{19}{6}\int_0^2 x^3 e^x \,\mathrm{d}x = \frac{19}{6}\, e^x(x^3 - 3x^2 + 6x - 6)\Big|_0^2 = 19\left(\frac{e^2}{3} + 1\right) \approx 65.797 \ \blacksquare$$

$$
\begin{aligned}
\int x^3 e^x \, dx &= x^3 e^x - \int 3x^2 e^x \, dx \qquad\qquad (u = x^3,\ dv = e^x dx) \\
&= x^3 e^x - 3\Big(x^2 e^x - \int 2x \, e^x \, dx\Big) \\
&= x^3 e^x - 3x^2 e^x + 6\Big(x e^x - \int e^x \, dx\Big) \\
&= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\
&= e^x\big(x^3 - 3x^2 + 6x - 6\big) + C.
\end{aligned}
$$

- To find the limits for a particular order of integration, it is generally advisable first to determine the innermost limits, which may be functions of the outer two variables.
- Then, by projecting the solid Q onto the coordinate plane of the outer two variables, you can determine their limits of integration by the methods used for double integrals. For instance, to evaluate

$$\iiint_Q f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

first determine the limits for z, and then the integral has the form

$$\iint \left[\int_{g_1(x,y)}^{g_2(x,y)} \mathrm{d}z\right] \mathrm{d}y\,\mathrm{d}x.$$

- By projecting the solid Q onto the xy-plane, you can determine the limits for x and y as you did for double integrals, as shown in Figure 31.

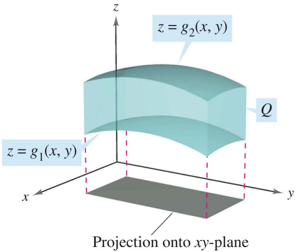


Figure 31: Solid region Q lies between $z = g_1(x, y)$ and $z = g_2(x, y)$.

Example 2 (Using a triple integral to find volume)

Find the volume of the ellipsoid given by $4x^2 + 4y^2 + z^2 = 16$.

- From the order $\mathrm{d}z \ \mathrm{d}y \ \mathrm{d}x$, you first determine the bounds for z.

$$0 \leq z \leq 2\sqrt{4 - x^2 - y^2}$$

- In Figure 32(b), you can see that the boundaries for x and y are $0 \leq x \leq 2$ and $0 \leq y \leq \sqrt{4 - x^2}$, so the volume of the ellipsoid is

$$\begin{aligned} V = \iiint_Q \mathrm{d}V &= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2\sqrt{4-x^2-y^2}} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\ &= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} z \Big|_0^{2\sqrt{4-x^2-y^2}} \mathrm{d}y \, \mathrm{d}x \\ &= 16 \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{(4 - x^2) - y^2} \, \mathrm{d}y \, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= 8\int_0^2 \left[y\sqrt{4-x^2-y^2} + (4-x^2)\sin^{-1}\left(\frac{y}{\sqrt{4-x^2}}\right)\right]_0^{\sqrt{4-x^2}} \, \mathrm{d}x \\
&= 8\int_0^2 \left[0 + (4-x^2)\sin^{-1}(1) - 0 - 0\right] \, \mathrm{d}x \\
&= 8\int_0^2 (4-x^2)\left(\frac{\pi}{2}\right) \, \mathrm{d}x = 4\pi\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{64\pi}{3}. \qquad \blacksquare
\end{aligned}
$$

$$I = 16\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{(4-x^2)-y^2}\,dy\,dx$$

$$= 16\int_0^2 \Big[\int_0^{\sqrt{A}} \sqrt{A-y^2}\,dy\Big]_{A=4-x^2} dx \qquad (A := 4-x^2)$$

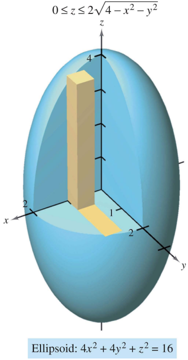
Set $y = \sqrt{A}\sin\theta$ so $dy = \sqrt{A}\cos\theta\,d\theta$:

$$\int_0^{\sqrt{A}} \sqrt{A-y^2}\,dy = A\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{A}{2}\int_0^{\pi/2} \big(1+\cos 2\theta\big)\,d\theta$$

$$= \frac{A}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = \frac{y}{2}\sqrt{A-y^2} + \frac{A}{2}\sin^{-1}\Big(\frac{y}{\sqrt{A}}\Big)\Bigg|_0^{\sqrt{A}}$$

$$I = 16\int_0^2 \Big[\tfrac{y}{2}\sqrt{A-y^2} + \tfrac{A}{2}\sin^{-1}\Big(\tfrac{y}{\sqrt{A}}\Big)\Big]_{y=0}^{y=\sqrt{A}}{}_{A=4-x^2}\,dx$$

$$= 8\int_0^2 \Big[y\sqrt{4-x^2-y^2} + (4-x^2)\sin^{-1}\Big(\tfrac{y}{\sqrt{4-x^2}}\Big)\Big]_0^{\sqrt{4-x^2}} dx.$$



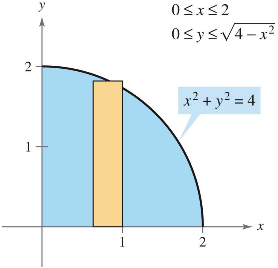


Figure 32: Using a triple integral to find volume.

Example 3 (Changing the order of integration)

Evaluate

$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

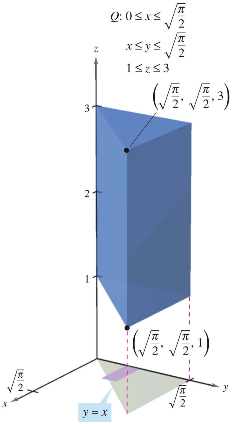


Figure 33: Changing the order of integration.

Example 4 (Determining the limits of integration)

Set up a triple integral for the volume of each solid region.

a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$ and lying between the vertical planes $x + y = 1$ and $x + y = 3$.

b. The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$.

c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$.

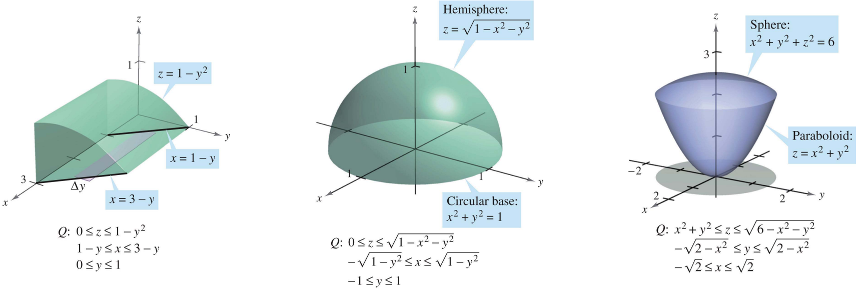


Figure 34: Determining the limits of integration.

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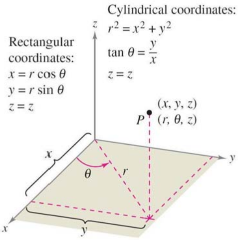
Triple integrals in cylindrical coordinates

- The rectangular conversion equations for cylindrical coordinates are

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

- In this coordinate system, the simplest solid region is a cylindrical block determined by

$$r_1 \le r \le r_2, \quad \theta_1 \le \theta \le \theta_2, \quad z_1 \le z \le z_2$$



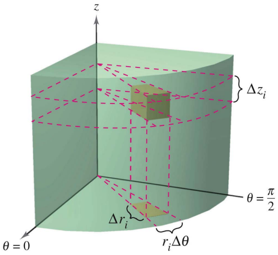


Figure 35: Volume of cylindrical block: $\Delta V_i = r_i \Delta r_i \Delta\theta_i \Delta z_i$.

- To obtain the cylindrical coordinate form of a triple integral, suppose that Q is a solid region whose projection R onto the xy-plane can be described in polar coordinates.
- That is,

$$Q = \{(x, y, z) : (x, y) \text{ is in } R,\ \ h_1(x, y) \le z \le h_2(x, y)\}$$

and

$$R = \{(r,\ \theta) : \theta_1 \le \theta \le \theta_2,\ \ g_1(\theta) \le r \le g_2(\theta)\}.$$

- If f is a continuous function on the solid Q, you can write the triple integral of f over Q as

$$\iiint_Q f(x, y, z)\,\mathrm{d}V = \iint_R \left[\int_{h_1(x,\ y)}^{h_2(x,\ y)} f(x, y, z)\,\mathrm{d}z\right] \mathrm{d}A$$

where the double integral over R is evaluated in polar coordinates.

- That is, R is a plane region that is either r-simple or θ-simple. If R is r-simple, the iterated form of the triple integral in cylindrical form is triple integral in cylindrical coordinates

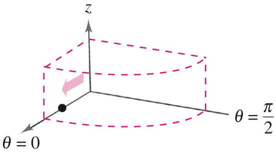
$$\iiint_Q f(x, y, z)\,\mathrm{d}V = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

- To visualize a particular order of integration, it helps to view the iterated integral in terms of three sweeping motions—each adding another dimension to the solid.
- For instance, in the order

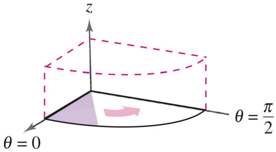
$$\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

  the first integration occurs in the r-direction as a point sweeps out a ray.

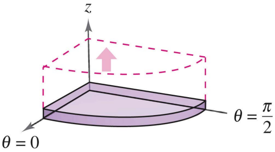
- Then, as θ increases, the line sweeps out a sector. Finally, as z increases, the sector sweeps out a solid wedge, as shown in Figure 36.



(a) Integrate with respect to r.



(b) Integrate with respect to θ.



(c) Integrate with respect to z.

Figure 36: Triple integrals in cylindrical coordinates.

Example 1 (Finding volume in cylindrical coordinates)

Find the volume of the solid region Q cut from the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $r = 2\sin\theta$, as shown in Figure 37.

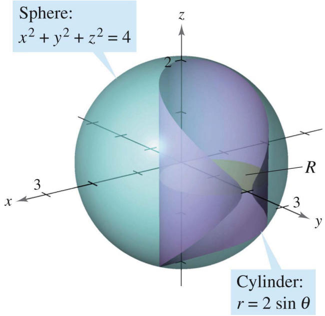


Figure 37: Sphere: $x^2 + y^2 + z^2 = 4$ and cylinder: $r = 2\sin\theta$.

Triple integrals in spherical coordinates

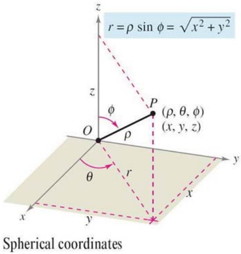
- The rectangular conversion equations for spherical coordinates are

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi.$$

- In this coordinate system, the simplest region is a spherical block

$$\{(\rho, \theta, \phi) : \rho_1 \le \rho \le \rho_2,\ \theta_1 \le \theta \le \theta_2,\ \phi_1 \le \phi \le \phi_2\}$$

where $\rho_1 \ge 0,\ \theta_2 - \theta_1 \le 2\pi$, and $0 \le \phi_1 \le \phi_2 \le \pi$, as below.



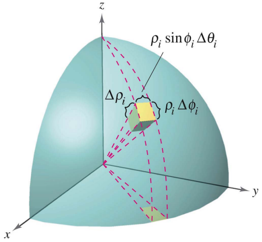


Figure 38: Spherical block: $\Delta V_i \approx \rho_i^2 \sin\phi_i \Delta\rho_i \Delta\phi_i \Delta\theta_i$.

- If (ρ, θ, ϕ) is a point in the interior of such a block, then the volume of the block can be determined as follows:
- One side is length $\Delta\rho$. Another side is $\rho\Delta\phi$. Finally, the third side is given by the length of an arc of angle $\Delta\theta$ in a circle of radius $\rho \sin\phi$. Thus:

$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\rho \sin\phi\Delta\theta) = \rho^2 \sin\phi\Delta\rho\Delta\phi\Delta\theta.$$

- Using the usual process involving an inner partition, summation, and a limit, you can develop the following version of a triple integral in spherical coordinates for a continuous function f defined on the solid region Q. triple integral in spherical coordinates

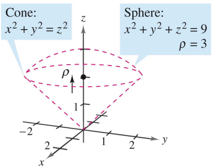
$$\begin{aligned}&\iiint_Q f(x,y,z)\,\mathrm{d}V\\&=\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta\end{aligned}$$

- Triple integrals in spherical coordinates are evaluated with iterated integrals. You can visualize a particular order of integration by viewing the iterated integral in terms of three sweeping motions—each adding another dimension to the solid.

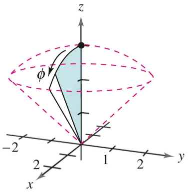
- For instance, the iterated integral

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

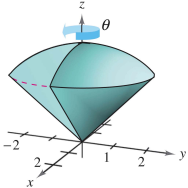
is illustrated in Figure 39.



(a) ρ varies from 0 to 3 with ϕ and θ held constant.



(b) ϕ varies from 0 to $\pi/4$ with θ held constant.



(c) θ varies from 0 to 2π.

Figure 39: Cone: $x^2 + y^2 = z^2$ and sphere: $x^2 + y^2 + z^2 = 9$, $\rho = 3$.

Example 2 (Finding volume in spherical coordinates)

Find the volume of the solid region Q bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$, as shown in Figure 40.

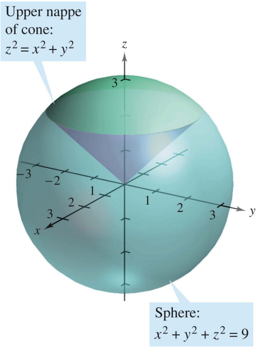


Figure 40: Cone: $x^2 + y^2 = z^2$ and sphere: $x^2 + y^2 + z^2 = 9$, $\rho = 3$.

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Jacobians

- For the single integral
$$\int_a^b f(x)\,\mathrm{d}x$$
you can change variables by letting $x = g(u)$, so that $\mathrm{d}x = g'(u)\,\mathrm{d}u$
$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(g(u))g'(u)\,\mathrm{d}u$$
where $a = g(c)$ and $b = g(d)$.
- The change of variables process introduces an additional factor $g'(u)$ into the integrand. This also occurs in the case of double integrals
$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(g(u,v),h(u,v))\underbrace{\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|}_{\text{Jacobian}}\,\mathrm{d}u\,\mathrm{d}v$$
where the change of variables $x = g(u,v)$ and $y = h(u,v)$ introduces a factor called the **Jacobian** of x and y with respect to u and v.

- In defining the Jacobian, it is convenient to use the following determinant notation.

Definition 14.5 (Jacobian)

If $x = g(u, v)$ and $y = h(u, v)$, then the Jacobian of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Example 1 (The Jacobian for rectangular-to-polar conversion)

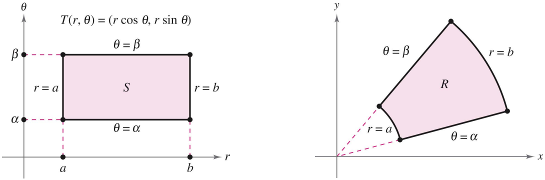
Find the Jacobian for the change of variables defined by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

- Example 1 points out that the change of variables from rectangular to polar coordinates for a double integral can be written as

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \iint_S f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\mathrm{d}r\,\mathrm{d}\theta\end{aligned}$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane, as shown in Figure 41.



(a) $r\theta$-plane (b) xy-plane

Figure 41: S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane where $\alpha \le \theta \le \beta$ and $a \le r \le b$.

- In general, a change of variables is given by a one-to-one transformation T from a region S in the uv-plane to a region R in the xy-plane, to be given by

$$T(u,v) = (x,y) = (g(u,v), h(u,v))$$

  where g and h have continuous first partial derivatives in the region S.
- Note that the point (u,v) lies in S and the point (x,y) lies in R.

Example 2 (Finding a change of variables to simplify a region)

Let R be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in Figure 42. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u- or v-axis).

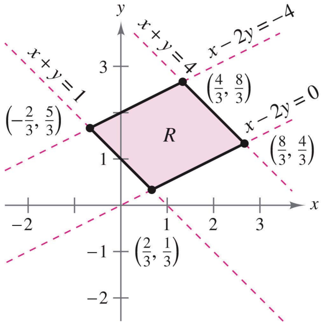


Figure 42: Region R in the xy-plane.

Change of variables for double integrals

Theorem 14.5 (Change of variables for double integrals)

Let R be a vertically or horizontally simple region in the xy-plane, and let S be a vertically or horizontally simple region in the uv-plane. Let T from S to R be given by $T(u,v) = (x,y) = (g(u,v), h(u,v))$, where g and h have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x,y)/\partial(u,v)$ is nonzero on S, then

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

Example 3 (Using a change of variables to simplify a region)

Let R be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in below. Evaluate the double integral

$$\iint_R 3xy \, \mathrm{d}A.$$

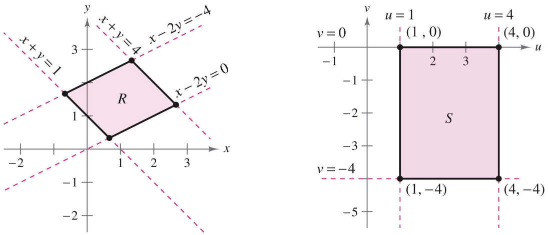


Figure 43: Using a change of variables to simplify a region bounded by $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$.

Example 4 (Using a change of variables to simplify an integrand)

Let R be the region bounded by the square with vertices $(0,1)$, $(1,2)$, $(2,1)$, and $(1,0)$. Evaluate the integral

$$\iint_R (x+y)^2 \sin^2(x-y)\,\mathrm{d}A.$$