

1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out.)

(a) $\lim_{(x,y) \rightarrow (1,2)} \frac{(x-1)^2 - (y-2)^2}{(x-1)^2 + (y-2)^2}$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy \cos(y)}{x^2 + y^2}$

(c) $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2}$

(d) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

Ans:

- (a) Let $u = x - 1, v = y - 2$ and $u = r \cos(\theta), v = r \sin(\theta)$

$$\lim_{(x,y) \rightarrow (1,2)} \frac{(x-1)^2 - (y-2)^2}{(x-1)^2 + (y-2)^2} = \lim_{(u,v) \rightarrow (0,0)} \frac{u^2 - v^2}{u^2 + v^2} = \lim_{r \rightarrow 0} \left(\frac{r^2(\cos^2 \theta - \sin^2 \theta)}{r^2} \right) = \cos^2 \theta - \sin^2 \theta$$

which means that if we follow the trajectory of different line $u = r \cos(\theta), v = r \sin(\theta)$ to approach (0,0) we will get different value for different θ , therefore, the limit does not exist.

- (b) Let $y = mx$, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy \cos(y)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{xm \cos(mx)}{x^2(1+m^2)} = \frac{m}{1+m^2}$. which means that if

we follow the trajectory of different line $y = mx$ to approach (0,0) we will get different value for different m , therefore, the limit does not exist.

- (c) The limit does not exist, because along the path $y = z = 0$

We have $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2} = \lim_{(x,0,0) \rightarrow (0,0,0)} \frac{0}{x^2} = 0$

However, along the path $z = 0, x = y$

We have $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2} = \lim_{(x,x,0) \rightarrow (0,0,0)} \frac{x^2}{x^2 + x^2} = \frac{1}{2}$

- (d) Let $x = r \cos(\theta), y = r \sin(\theta)$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \lim_{r \rightarrow 0} \frac{r^2}{\sqrt{r^2 + 1} - 1} = \lim_{r \rightarrow 0} \frac{r^2(\sqrt{r^2 + 1} + 1)}{r^2 + 1 - 1} = 2$$

2. (15%)

- (a) Let $f(x, y) = \begin{cases} \frac{1}{x} \sin(xy) & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0) \end{cases}$, find f_x when $(x, y) \neq (0, 0)$ and when $(x, y) = (0, 0)$, respectively
- (b) Given the equation $w = \sin(2x + 3y)$, $x = s + t$, $y = s - t$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$
- (c) Considering the level surface defined by $z^4 + (\sin(x))z^2 + yz = 3$. Find an equation of the tangent plane at $(0, 2, 1)$

Ans:

- (a) For $(x, y) \neq (0, 0)$:

$$f_x(x, y) = \frac{-1}{x^2} \sin(xy) + \frac{y}{x} \cos(xy),$$

For $(x, y) = (0, 0)$:

$$f_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$$

- (b) Using the chain rule

$$\frac{\partial w}{\partial s} = 2 \cos(2x + 3y) + 3 \cos(2x + 3y) = 5 \cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2 \cos(2x + 3y) - 3 \cos(2x + 3y) = -\cos(5s - t)$$

- (c) $F(x, y, z) = z^4 + (\sin(x))z^2 + yz - 3$

$$\nabla F = \cos(x) z^2 \mathbf{i} + z \mathbf{j} + (4z^3 + 2(\sin(x))z + y) \mathbf{k}$$

$$\nabla F(0, 2, 1) = 1 \mathbf{i} + 1 \mathbf{j} + 6 \mathbf{k}$$

$$(x - 0) + (y - 2) + 6(z - 1) = 0$$

3. (10%) Let $f(x, y) = \int_1^{2y-x^2} e^t dt$

Find the direction in which $f(x, y)$ increase most. What is the rate of increase?

Ans:

$$\nabla f(x, y) = (-2x \cdot e^{2y-x^2}, 2e^{2y-x^2}) = 2e^{2y-x^2}(-x\mathbf{i} + \mathbf{j})$$

Increasing most rapidly is the direction of the gradient. That is $\frac{\nabla f}{|\nabla f|} = \left(\frac{-x}{\sqrt{x^2+1}} \mathbf{i} + \right.$

$$\left. \frac{1}{\sqrt{x^2+1}} \mathbf{j} \right)$$

$$|\nabla f| = 2e^{2y-x^2} \sqrt{x^2+1}$$

4. (15%) Let $f(x, y) = e^{-\frac{xy}{2}}$

Use lagrange multiplier to find any extrema of the function subject to $x^2 + y^2 \leq 1$

Ans:

Case 1: On the circle $x^2 + y^2 = 1$

$$\begin{cases} \frac{-y}{2} e^{\frac{-xy}{2}} = 2x\lambda \\ \frac{-x}{2} e^{\frac{-xy}{2}} = 2y\lambda \end{cases} \rightarrow x^2 = y^2$$

Combine with $x^2 + y^2 = 1$, we have $x = \pm \frac{\sqrt{2}}{2}$

Case 2: Inside the circle:

$$f_x = \frac{-y}{2} e^{\frac{-xy}{2}} = 0, f_y = \frac{-x}{2} e^{\frac{-xy}{2}} = 0$$

We get $(x, y) = (0, 0)$ as the critical point.

$$f_{xx} = \frac{-y^2}{4} e^{\frac{-xy}{2}}, f_{yy} = \frac{-x^2}{4} e^{\frac{-xy}{2}}, f_{xy} = \frac{-1}{2} e^{\frac{-xy}{2}} + \frac{xy}{4} e^{\frac{-xy}{2}}$$

At $(0, 0)$ $D = f_{xx}f_{yy} - f_{xy}f_{yx} < 0$ therefore $(0, 0)$ is saddle point

Combining the two cases, we have maximum of $e^{\frac{1}{4}}$ at $(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2})$ and

minimum $e^{\frac{-1}{4}}$ at $(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2})$

5. (10%) Evaluate the following expression

$$(a) \int_0^{\ln 5} \int_{e^x}^5 \frac{1}{y} dy dx$$

$$(b) \int_0^{\frac{\pi}{4}} \int_0^{\frac{y}{2}} \int_0^{\frac{1}{y}} \sin(y) dz dx dy$$

Ans:

$$(a) \int_0^{\ln 5} \int_{e^x}^5 \frac{1}{y} dy dx = \int_1^5 \int_0^{\ln y} \frac{1}{y} dx dy = \int_1^5 \left[\frac{x}{y} \right]_0^{\ln y} dy = \int_1^5 \frac{\ln y}{y} dy = \int_0^{\ln 5} u du =$$

$$\frac{1}{2} u^2 \Big|_0^{\ln 5} = \frac{1}{2} (\ln 5)^2$$

$$(b) \int_0^{\frac{\pi}{4}} \int_0^{\frac{y}{2}} \int_0^{\frac{1}{y}} \sin(y) dz dx dy = \int_0^{\frac{\pi}{4}} \int_0^{\frac{y}{2}} \frac{\sin(y)}{y} dx dy = \frac{1}{2} \int_0^{\frac{\pi}{4}} \sin(y) dy = \frac{-1}{2} \cos(y) \Big|_0^{\frac{\pi}{4}} =$$

$$\frac{-\sqrt{2}}{4} + \frac{1}{2}$$

6. (13%) Evaluate $\int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} dy dx$

Ans:

$$R = \left\{ (x, y) \left| 0 \leq y \leq \frac{\sqrt{2}}{2}, x \leq y \leq \sqrt{1-x^2} \right. \right\} = \left\{ (r, \theta) \left| 0 \leq r \leq 1, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \right. \right\}$$

$$\int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} dy dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 \sqrt{1-r^2} r dr d\theta \quad (\text{Let } u = 1-r^2, du = -2r dr)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[-\frac{1}{3} u^{\frac{3}{2}} \right]_0^1 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[-\frac{1}{3} (1-r^2)^{\frac{3}{2}} \right]_0^1 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3} d\theta = \frac{1}{3} \theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{12}$$

7. (15%) Find the area of the surface given by $z = f(x, y) = 25 - x^2 - y^2$ that lies above the region R where $R = \{(x, y): x^2 + y^2 \leq 25\}$

Ans:

$$f_x = -2x, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = 4 \int_0^{\frac{\pi}{2}} \int_0^5 \sqrt{1 + 4r^2} r dr d\theta \quad (\text{Let } u = 1 + 4r^2, du = 8r dr)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_1^{101} \sqrt{u} du d\theta = \frac{1}{3} \int_0^{\frac{\pi}{2}} (101\sqrt{101} - 1) d\theta = \frac{\pi}{6} (101\sqrt{101} - 1)$$

8. (15%) Evaluate the triple integral $\int \int \int_Q yz \cos(x^3 - 1) dV$ where $Q = \{ \frac{y}{2} \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 1 \}$

Ans:

$$\begin{aligned}
& \int \int \int_Q x^2 + y^2 \, dV \\
&= \int_0^1 \int_0^2 \int_{\frac{y}{2}}^1 yz \cos(x^3 - 1) \, dx dy dz \\
&= \int_0^1 \int_0^1 \int_0^{2x} yz \cos(x^3 - 1) \, dy dx dz \\
&= \int_0^1 \int_0^1 2x^2 z \cos(x^3 - 1) dx dz = \int_0^1 z dz \int_0^1 2x^2 \cos(x^3 - 1) dx \\
&= \frac{1}{2} \frac{2}{3} \sin(x^3 - 1) \Big|_0^1 = \frac{1}{3} \sin(1)
\end{aligned}$$