

Chapter 12 Vector-Valued Functions

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Space curves and vector-valued functions

- A plane curve is defined as the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where f and g are continuous functions of t on an interval I .

- A **space curve** C is the set of all ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations

$$x = f(t), \quad y = g(t), \quad \text{and} \quad z = h(t)$$

where f , g , and h are continuous functions of t on an interval I .

- A new type of function, called a **vector-valued function**, that maps real numbers to vectors is first introduced.

Definition 12.1 (Vector-valued function)

A function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad (\text{Plane})$$

or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad (\text{Space})$$

is a **vector-valued function**, where the **component functions** f , g , and h are real-valued functions of the parameter t . Vector-valued functions are sometimes denoted as $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

- Technically, a curve in the plane or in space consists of a collection of points and the defining parametric equations. Two different curves can have the same graph.
- For instance, each of the curves given by

$$\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} \quad \text{and} \quad \mathbf{r}(t) = \sin t^2 \mathbf{i} + \cos t^2 \mathbf{j}$$

has the unit circle as its graph, but these equations do not represent the same curve — because the circle is traced out in different ways!

- Be sure you see the distinction between the vector-valued function \mathbf{r} and the real-valued functions f , g , and h . They are functions of the real variable t , but $\mathbf{r}(t)$ is a vector, whereas $f(t)$, $g(t)$, and $h(t)$ are real numbers (for each specific value of t).

- Vector-valued functions serve dual roles in the representation of curves.
 - By letting the parameter t represent time, you can use a vector-valued function to represent motion along a curve.
 - Or, in the more general case, you can use a vector-valued function to trace the graph of a curve.

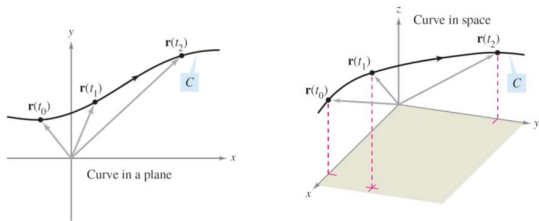


Figure 1: Curve C is traced out by the terminal point of position vector $\mathbf{r}(t)$.

- In either case, the terminal point of the position vector $\mathbf{r}(t)$ coincides with the point (x, y) or (x, y, z) on the curve given by the parametric equations, as shown in Figure 1.

- The arrowhead on the curve indicates the curve's orientation by pointing in the direction of increasing values of t .
- Unless stated otherwise, the domain of a vector-valued function \mathbf{r} is considered to be the intersection of the domains of the component functions f , g , and h .
- For instance, the domain of $\mathbf{r}(t) = \ln t \mathbf{i} + \sqrt{1-t} \mathbf{j} + t \mathbf{k}$ is the interval $(0, 1]$.

Example 1 (Sketching a plane curve)

Sketch the plane curve represented by the vector-valued function

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} - 3 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

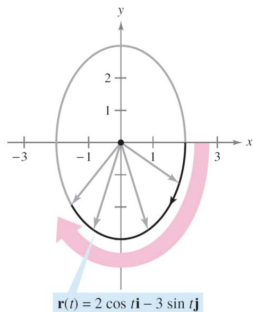


Figure 2: The ellipse $\mathbf{r}(t) = 2 \cos t \mathbf{i} - 3 \sin t \mathbf{j}$ is traced clockwise as t increases from 0 to 2π .

Example 2 (Sketching a space curve)

Sketch the space curve represented by the vector-valued function

$$\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + t \mathbf{k}, \quad 0 \leq t \leq 4\pi.$$

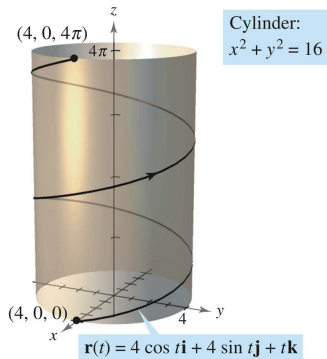
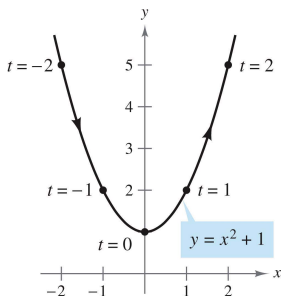


Figure 3: As t increases from 0 to 4π , two spirals on the helix are traced out.

Example 3 (Representing a graph by a vector-valued function)

Represent the parabola given by $y = x^2 + 1$ by a vector-valued function.



Example 4 (Representing a graph by a vector-valued function)

Sketch the space curve C represented by the intersection of the semiellipsoid

$$\frac{x^2}{12} + \frac{y^2}{24} + \frac{z^2}{4} = 1, \quad z \geq 0$$

and the parabolic cylinder $y = x^2$. Then, find a vector-valued function to represent the graph.

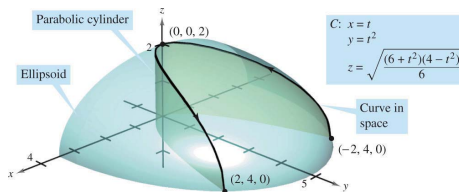


Figure 4: The curve C is the intersection of the semiellipsoid and the parabolic cylinder.

Limits and continuity

- To add or subtract two vector-valued functions (in the plane), you can write

$$\begin{aligned}\mathbf{r}_1 + \mathbf{r}_2 &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j} \\ \mathbf{r}_1 - \mathbf{r}_2 &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] - [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) - f_2(t)]\mathbf{i} + [g_1(t) - g_2(t)]\mathbf{j}.\end{aligned}$$

- To multiply and divide a vector-valued function by a scalar, you can write

$$\begin{aligned}c\mathbf{r}(t) &= c[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] = cf_1(t)\mathbf{i} + cg_1(t)\mathbf{j} \\ \frac{\mathbf{r}(t)}{c} &= \frac{[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}]}{c} = \frac{f_1(t)}{c}\mathbf{i} + \frac{g_1(t)}{c}\mathbf{j}, \quad c \neq 0.\end{aligned}$$

Definition 12.2 (The limit of a vector-valued function)

1. If \mathbf{r} is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left[\lim_{t \rightarrow a} f(t) \right] \mathbf{i} + \left[\lim_{t \rightarrow a} g(t) \right] \mathbf{j} \quad \text{Plane}$$

provided f and g have limits as $t \rightarrow a$.

2. If \mathbf{r} is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left[\lim_{t \rightarrow a} f(t) \right] \mathbf{i} + \left[\lim_{t \rightarrow a} g(t) \right] \mathbf{j} + \left[\lim_{t \rightarrow a} h(t) \right] \mathbf{k} \quad \text{Space}$$

provided f , g , and h have limits as $t \rightarrow a$.

- If $\mathbf{r}(t)$ approaches the vector \mathbf{L} as $t \rightarrow a$, the length of the vector $\mathbf{r}(t) - \mathbf{L}$ approaches 0.
- That is, $\|\mathbf{r}(t) - \mathbf{L}\| \rightarrow 0$ as $t \rightarrow a$. This is illustrated graphically in Figure 5.

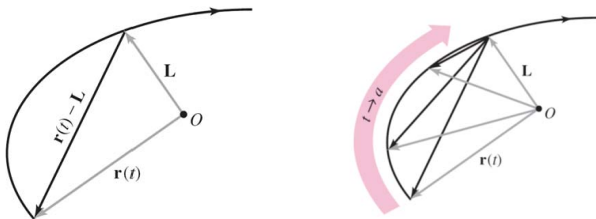


Figure 5: As t approaches a , $\mathbf{r}(t)$ approaches the limit \mathbf{L} . For the limit \mathbf{L} to exist, it is not necessary that $\mathbf{r}(a)$ be defined or that $\mathbf{r}(a)$ be equal to \mathbf{L} .

Definition 12.3 (Continuity of a vector-valued function)

A vector-valued function \mathbf{r} is **continuous at a point** given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \rightarrow a$ and

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function \mathbf{r} is continuous on an interval I if it is continuous at every point in the interval.

- A vector-valued function is continuous at $t = a$ if and only if each of its component function is continuous at $t = a$.

Example 5 (Continuity of vector-valued functions)

Discuss the continuity of the vector-valued function given by

$$\mathbf{r}(t) = t \mathbf{i} + a \mathbf{j} + (a^2 - t^2) \mathbf{k} \quad a \text{ is a constant}$$

at $t = 0$.

Example 6 (Continuity of vector-valued functions)

Determine the interval(s) on which the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t+1}\mathbf{j} + (t^2 + 1)\mathbf{k}$ is continuous.

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Differentiation of vector-valued functions

- The definition of the derivative of a vector-valued function parallels the definition given for real-valued functions.

Definition 12.4 (The derivative of a vector-valued function)

The derivative of a vector-valued function \mathbf{r} is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then \mathbf{r} is **differentiable at t** . If $\mathbf{r}'(t)$ exists for all t in an open interval I , then \mathbf{r} is differentiable on the interval I . Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

- Differentiation of vector-valued functions can be done on a **component-by-component basis**.
- To see why this is true, consider the function given by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}.$$

- Applying the definition of the derivative produces the following.

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\&= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t)\mathbf{i} + g(t + \Delta t)\mathbf{j} - f(t)\mathbf{i} - g(t)\mathbf{j}}{\Delta t} \\&= \lim_{\Delta t \rightarrow 0} \left\{ \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \mathbf{i} + \left[\frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \mathbf{j} \right\} \\&= \left\{ \lim_{\Delta t \rightarrow 0} \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \right\} \mathbf{i} + \left\{ \lim_{\Delta t \rightarrow 0} \left[\frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \right\} \mathbf{j} \\&= f'(t)\mathbf{i} + g'(t)\mathbf{j}\end{aligned}$$

- Note that the derivative of the vector-valued function \mathbf{r} is itself a vector-valued function.
- You can see from Figure 6 that $\mathbf{r}'(t)$ is a vector tangent to the curve given by $\mathbf{r}(t)$ and pointing in the direction of increasing t -values.

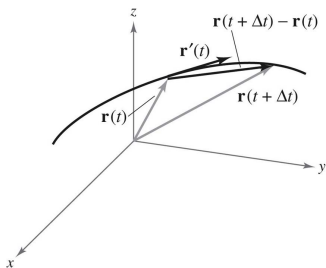


Figure 6: Definition of the derivative of a vector-valued functions.

Theorem 12.1 (Differentiation of vector-valued functions)

- ① If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t , then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}. \quad \text{Plane}$$

- ② If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are differentiable functions of t , then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \quad \text{Space}$$

Example 1 (Differentiation of vector-valued functions)

For the vector-valued function given by $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2)\mathbf{j}$, find $\mathbf{r}'(t)$. Then sketch the plane curve represented by $\mathbf{r}(t)$, and the graphs of $\mathbf{r}(1)$ and $\mathbf{r}'(1)$.

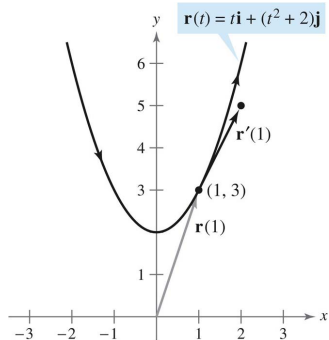


Figure 7: $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2)\mathbf{j}$

Example 2 (Higher-order differentiation)

For the vector-valued function given by $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + 2t \mathbf{k}$, find each of the following.

a. $\mathbf{r}'(t)$ **b.** $\mathbf{r}''(t)$ **c.** $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ **d.** $\mathbf{r}'(t) \times \mathbf{r}''(t)$

- The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is **smooth on an open interval** if f' , g' , and h' are continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t in the interval I .

Theorem 12.2 (Properties of the derivative)

Let \mathbf{r} and \mathbf{u} be differentiable vector-valued functions of t , let w be a differentiable real-valued function of t , and let c be scalar.

- ① $D_t [c \mathbf{r}(t)] = c \mathbf{r}'(t)$
- ② $D_t [\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
- ③ $D_t [w(t) \mathbf{r}(t)] = w(t) \mathbf{r}'(t) + w'(t) \mathbf{r}(t)$
- ④ $D_t [\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
- ⑤ $D_t [\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
- ⑥ $D_t [\mathbf{r}(w(t))] = \mathbf{r}'(w(t)) w'(t)$
- ⑦ If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Example 4 (Using properties of the derivative)

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t} \mathbf{i} - \mathbf{j} + \ln t \mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2 \mathbf{i} - 2t \mathbf{j} + \mathbf{k}$$

find **a.** $D_t [\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and **b.** $D_t [\mathbf{u}(t) \times \mathbf{u}'(t)]$.

Integration of vector-valued functions

- The following definition is a rational consequence of the definition of the derivative of a vector-valued function.

Definition 12.5 (Integration of vector-valued functions)

- If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, then the indefinite integral(antiderivative) of \mathbf{r} is

$$\int \mathbf{r}(t) dt = \left[\int f(t) dt \right] \mathbf{i} + \left[\int g(t) dt \right] \mathbf{j} \quad \text{Plane}$$

and its definite integral over the interval $a \leq t \leq b$ is

$$\int_a^b \mathbf{r}(t) dt = \left[\int_a^b f(t) dt \right] \mathbf{i} + \left[\int_a^b g(t) dt \right] \mathbf{j}.$$

Definition 12.5 (continue)

- If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are continuous on $[a, b]$, then the indefinite integral (antiderivative) of \mathbf{r} is

$$\int \mathbf{r}(t) dt = \left[\int f(t) dt \right] \mathbf{i} + \left[\int g(t) dt \right] \mathbf{j} + \left[\int h(t) dt \right] \mathbf{k} \quad \text{Space}$$

and its definite integral over the interval $a \leq t \leq b$ is

$$\int_a^b \mathbf{r}(t) dt = \left[\int_a^b f(t) dt \right] \mathbf{i} + \left[\int_a^b g(t) dt \right] \mathbf{j} + \left[\int_a^b h(t) dt \right] \mathbf{k}.$$

- The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector \mathbf{C} .

- For instance, if $\mathbf{r}(t)$ is a three-dimensional vector-valued function, then for the indefinite integral $\int \mathbf{r}(t) dt$, you obtain three constants of integration

$$\int f(t) dt = F(t) + C_1, \int g(t) dt = G(t) + C_2, \int h(t) dt = H(t) + C_3$$

where $F'(t) = f(t)$, $G'(t) = g(t)$, and $H'(t) = h(t)$.

- These three scalar constants produce one vector constant of integration,

$$\begin{aligned} \int \mathbf{r}(t) dt &= [F(t) + C_1] \mathbf{i} + [G(t) + C_2] \mathbf{j} + [H(t) + C_3] \mathbf{k} \\ &= [F(t) \mathbf{i} + G(t) \mathbf{j} + H(t) \mathbf{k}] + [C_1 \mathbf{i} + C_2 \mathbf{j} + C_3 \mathbf{k}] \\ &= \mathbf{R}(t) + \mathbf{C} \end{aligned}$$

where $\mathbf{R}'(t) = \mathbf{r}(t)$.

Example 5 (Integrating a vector-valued function)

Find the indefinite integral $\int (t \mathbf{i} + 3 \mathbf{j}) dt$.

Example 6 (Definite Integral of a vector-valued function)

Evaluate the integral

$$\int_0^1 \mathbf{r}(t) \, dt = \int_0^1 \left(\sqrt[3]{t} \mathbf{i} + \frac{1}{t+1} \mathbf{j} + e^{-t} \mathbf{k} \right) dt.$$