1. (5%) Determine whether the sequence is convergent or divergent. If convergent, find its limit.

   (a) $a_n = n^{\frac{1}{n}}$

   (b) $a_n = \frac{\ln(n^3)}{n}$

Ans:

(a) Let $y = \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{\frac{1}{n}} \to \ln y = \lim_{n\to\infty} \frac{\ln n}{n}$. As n becomes large, $\ln n$ grows much more slowly than n. In fact, using standard limits or L'Hôpital's rule, we get $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. Therefore, $y = e^0 = 1 \to a_n = n^{\frac{1}{n}}$ converges to 1

(b) Since $a_n = \frac{\ln(n^3)}{n} = \frac{3\ln(n)}{n}$. Again As n becomes large, $\ln n$ grows much more slowly than n, we get $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3\ln n}{n} = 0$. Therefore, $a_n = \frac{\ln(n^3)}{n}$ converges to 0.

2. (5%) Determine whether the series is convergent or divergent. In addition, please indicate the test you use.

   (a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

   (b) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n-1}\right)^{n+1}$

Ans:

(a) Using the ratio test, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!}| = \lim_{n\to\infty} |\frac{(n+1)}{(n+1)^{n+1}} n^n| =$

$$\lim_{n\to\infty} |\frac{1}{(\frac{n+1}{n})^n}| = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

by the Ratio Test the series converges.

(b) Using the root test, $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n-1}\right)^{n+1}\right|} = \lim_{n\to\infty} \left(\frac{2n+3}{3n-1}\right)^{\frac{n+1}{n}} =$

$$\lim_{n\to\infty} \left(\frac{2n+3}{3n-1}\right)^{\frac{1}{n}} \left(\frac{2n+3}{3n-1}\right) = \lim_{n\to\infty} \left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right)^{\frac{1}{n}} \left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right) = \frac{2}{3} \lim_{n\to\infty} \left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right)^{\frac{1}{n}}$$

Let $y = \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right)^{\frac{1}{n}} \to \ln y = \lim_{n\to\infty}\frac{1}{n}\left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right) = 0 \to y = 1$

Therefore, $\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+3}{3n-1}\right)^{n+1}\right|} = \frac{2}{3}\lim_{n\to\infty}\left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right)^{\frac{1}{n}} = \frac{2}{3} < 1$

by the Root Test the series converges.

3. (5%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.

   (a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n-3}$

   (b) $\sum_{n=1}^{\infty}(-1)^n\frac{\arctan(n)}{n^2+1}$

Ans:

(a) Firstly, we check whether $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{5n-3}\right| = \sum_{n=1}^{\infty}\frac{1}{5n-3}$. By limit comparison test to the divergent series $\sum_{n=1}^{\infty}\frac{1}{n}$, $\lim_{n\to\infty}\frac{\frac{1}{5n-3}}{\frac{1}{n}} = \frac{1}{5}$ which means $\sum_{n=1}^{\infty}\frac{1}{5n-3}$ is also divergent.

$\sum_{n=1}^{\infty}\frac{(-1)^n}{5n-3}$ is converge by the alternating series test, since $\lim_{n\to\infty}\frac{1}{5n-3} = 0$ and $\frac{1}{5(n+1)-3} < \frac{1}{5n-3}$ for $n > 1$. Therefore, the original series is <u>conditionally converge</u>.

(b) Firstly, we check whether $\sum_{n=1}^{\infty}\left|(-1)^n\frac{\arctan(n)}{n^2+1}\right| = \sum_{n=1}^{\infty}\frac{\arctan(n)}{n^2+1}$ is converge or not. Let $f(x) = \frac{\arctan(x)}{x^2+1}$. Since $f'(x) = \frac{1-2x arctan(x)}{(x+1)^2} < 0$ for $x \geq 1$, f is positive, continuous and decreasing for $x \geq 1$. $\int_1^{\infty}\frac{\arctan(x)}{x^2+1}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}u\,du = \frac{1}{2}u^2\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{3\pi^2}{32}$. Therefore, the original series is <u>absolute converge</u> by the integral test.

4. (8%) Find the interval of convergence of the power series (Be sure to check the for the convergence at the endpoints of the intervals)

(a) $\sum_{n=1}^{\infty} \frac{x^n}{2^n(n^2+1)}$

(b) $\sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^n}$

Ans:

(a) $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{\frac{x^{n+1}}{2^{n+1}((n+1)^2+1)}}{\frac{x^n}{2^n(n^2+1)}}| = \lim_{n\to\infty} \left|\frac{n^2+1}{n^2+2n+2}\frac{1}{2}\right| |x| = \frac{1}{2}|x|$. By the ratio test, the series converges for $|x| < 2$

When $x = 2$: $\sum_{n=1}^{\infty} \frac{x^n}{2^n(n^2+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$. It is converging by limit comparison test to the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

When $x = -2$: $\sum_{n=1}^{\infty} \frac{x^n}{2^n(n^2+1)} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1}$. The series is converge by the alternating series test, since $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ and $\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$ for $n > 1$.

So $R = 2$ and the interval of convergence is $[-2,2]$

(b) $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{(n+1)(x+1)^{n+1}}{3^{n+1}} \times \frac{3^n}{n(x+1)^n}| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| \left|\frac{x+1}{3}\right| = \left|\frac{x+1}{3}\right|$. By the ratio test, the series converges for $|x+1| < 3$

When $x = 2$: $\sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^n} = \sum_{n=1}^{\infty} n$. Which is clearly diverge by the n-th term Test.

When $x = -4$: $\sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n$. Which is clearly diverge by the n-th term Test.

So $R = 3$ and the interval of convergence is $(-4,2)$

5. (8%) Use a power series to approximate $\int_0^1 \frac{1-e^{-x^2}}{x^2} dx$ with an error of less than 0.01

Ans: $\int_0^1 \frac{1-e^{-x^2}}{x^2} dx = \int_0^1 \frac{1-(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}\dots)}{x^2} dx = \int_0^1 \left(1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{4!} \dots\right) dx =$
$\left[x - \frac{x^3}{3\times 2!} + \frac{x^5}{5\times 3!} - \frac{x^7}{7\times 4!} \dots\right]_0^1 = 1 - \frac{1}{3\times 2!} + \frac{1}{5\times 3!} - \frac{x^7}{7\times 4!} + \cdots$

$$\int_0^1 \frac{1-e^{-x^2}}{x^2}dx = 1 - \frac{1}{3\times 2!} + \frac{1}{5\times 3!} - \frac{1}{7\times 4!} + \cdots + \frac{(-1)^n}{(2n+1)(n+1)!} + \cdots$$

It is an alternating series and since $|R_N| \le a_{N+1} = \frac{1}{(2\times3+1)(3+1)!} < 0.01$ when $\mathrm{N} =$

3. Therefore we know that $\int_0^1 \frac{1-e^{-x^2}}{x}dx \approx 1 - \frac{1}{3\times2!} + \frac{1}{5\times3!} \approx \frac{13}{15}$

6. (10%) Evaluate the following expression:

(a) (2.5%) $\frac{1}{1\times2} - \frac{1}{3\times2^3} + \frac{1}{5\times2^5} - \frac{1}{7\times2^7} - \cdots$

(b) (2.5%) $\sum_{n=1}^{\infty}\left(\frac{1}{e^n} + \frac{1}{n(n+2)}\right)$

(c) (5%) If $\lim_{x\to0}\frac{\sin(3x)+a+bx+cx^3}{x^3} = 0$, what is the value of a, b, c?

Ans:

(a) $\frac{1}{1\times2} - \frac{1}{3\times2^3} + \frac{1}{5\times2^5} - \frac{1}{7\times2^7} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)2^{2n+1}} = \arctan(\frac{1}{2})$

(b) $\sum_{n=1}^{\infty}\left(\frac{1}{e^n} + \frac{1}{n(n+2)}\right) = \sum_{n=1}^{\infty}\frac{1}{e^n} + \sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \frac{\frac{1}{e}}{1-\frac{1}{e}} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{e-1} +$

$\frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots - \frac{1}{n+1} - \frac{1}{n+2} + \cdots\right] = \frac{1}{e-1} + \frac{1}{2}\left[1 + \frac{1}{2}\right] = \frac{1}{e-1} + \frac{3}{4}$

(c) $\lim_{x\to0}\frac{\sin(3x)+a+bx+cx^3}{x^3} = \lim_{x\to0}\frac{\left(3x - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5\ldots\right) + (a+bx+cx^3)}{x^3} =$

$\lim_{x\to0}\frac{a+(3+b)x+\left(c-\frac{3^3}{3!}\right)x^3 + x^3\left(\frac{3^5x^2}{5!} - \cdots\right)}{x^3} = \lim_{x\to0}\frac{a}{x^3} + \frac{3+b}{x^2} + \left(c - \frac{3^3}{3!}\right) + \left(\frac{3^5x^2}{5!} - \cdots\right) =$

$\lim_{x\to0}\frac{a}{x^3} + \frac{3+b}{x^2} + \left(c - \frac{3^3}{3!}\right) + \lim_{x\to0}\frac{3^5x^2}{5!} - \cdots$

Since the second term is 0, For the limit to exist (and equal 0) as $x \to 0$, the numerator must have no constant term or x term (since these would lead to terms which blow up) the first term should also be 0, and so $a = 0, b = -3, c = \frac{27}{6} = \frac{9}{2}$.

7. (12%) Find the first three nonzero terms in the Maclaurin Series for each function
   (a) $f(x) = e^{-x}\,ln(1+2x)$
   (b) $f(x) = \sqrt{1+2x}cos(x)$
   (c) $f(x) = \frac{1}{1+x-6x^2}$

Ans:

(a) $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 + \cdots$$

$$e^{-x}\, ln(1+2x) \approx 2x - 4x^2 + \frac{17}{3}x^3$$

(b) $\sqrt{1+2x} = (1+2x)^{\frac{1}{2}} = 1 + \frac{1}{2}2x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(2x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(2x)^3 + \cdots$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$$\sqrt{1+2x}co\,s(x) \approx 1 + x - x^2$$

(c) $\frac{1}{1+x-6x^2} = \frac{1}{1-(-x+6x^2)} = 1 + (-\mathrm{x} + 6x^2) + (x^2 - 12x^3 + 36x^4) + \cdots \approx$

$1 - \mathrm{x} + 7x^2$

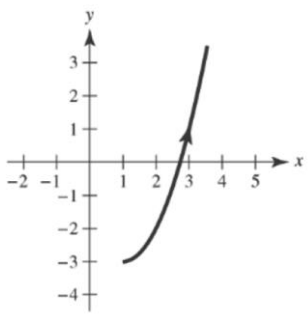
Or $\frac{1}{1+x-6x^2} = \frac{1}{(1+3x)(1-2x)} = \frac{1}{(1+3x)}\frac{1}{(1-2x)} = (1 - 3x + \frac{-1(-2)}{2!}9x^2 + \cdots)(1 -$

$2x + \frac{-1(-2)}{2!}4x^2 + \cdots) \approx 1 - \mathrm{x} + 7x^2$

8. (6%) Sketch the curve represented by the parametric equations (indicate the orientation of the curve) $x = \sqrt{t} + 1, y = t - 3,\ t \geq 0$ and write the corresponding rectangular equation by eliminating the parameter.

Ans:

$$x - 1 = \sqrt{t} \rightarrow (x-1)^2 = t \rightarrow y = (x-1)^2 - 3, x \geq 1$$



9. (9%) Find the area of the surface generated by revolving the curve $x = 4t, y = 3t + 1,\ 0 \leq \mathrm{t} \leq 2$ about the x-axis

Ans:

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = 3, \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = 5$$

$$S = 2\pi \int_0^2 (3t+1)\, 5dt = 10\pi \left[\frac{3t^2}{2} + t\right]_0^2 = 80\pi$$

10. (10%) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the parametric equation $x = 3 + cos(\theta)$ and $y = 2 + 4sin(\theta)$ and find the slope and concavity at $\theta = \frac{\pi}{6}$.

Ans:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\cos(\theta)}{-\sin(\theta)} = -4\cot(\theta),\ \frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-4\cot(\theta)]}{-\sin(\theta)} = \frac{4csc^2(\theta)}{-\sin(\theta)}$$

$$= -4csc^3(\theta)$$

At $\theta = \frac{\pi}{6}$, the slope is $\frac{dy}{dx} = -4\sqrt{3}$ and $\frac{d^2y}{dx^2} = -32$ which is concave downward.

11. (10%) Sketch the polar graph and find the arc length and area of the polar graph $r = 8(1 + \cos(\theta))$

Ans:

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(8(1+\cos(\theta)))^2 + (-8\sin(\theta))^2}$$

$$S = \int_0^{2\pi} \sqrt{\left(8(1+\cos(\theta))\right)^2 + (-8\sin(\theta))^2} = 8\sqrt{2}\int_0^{2\pi} \sqrt{1+\cos(\theta)}\, d\theta$$

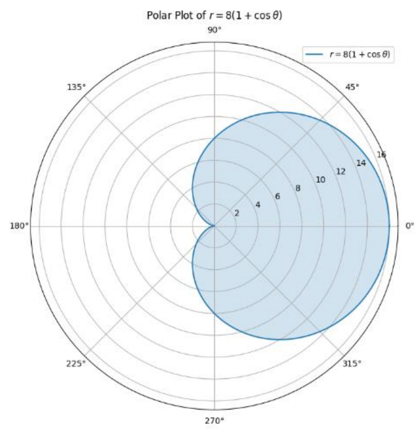
$$= 8\sqrt{2}\int_0^{2\pi} \sqrt{2\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 16\int_0^{2\pi} \left|cos\left(\frac{\theta}{2}\right)\right| d\theta$$

$$= 16\left[\int_0^{\pi} cos\left(\frac{\theta}{2}\right) d\theta + \int_{\pi}^{2\pi} -cos\left(\frac{\theta}{2}\right) d\theta\right]$$

$$= 32\left[sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} - 32\left[sin\left(\frac{\theta}{2}\right)\right]_{\pi}^{2\pi} = 64$$

$$A = \frac{1}{2}\int_0^{2\pi} [8(1+\cos(\theta))]^2 d\theta = 32\int_0^{2\pi} \left(1 + 2\cos(\theta) + cos^2(\theta)\right) d\theta$$

$$= 32\left[\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{sin2(\theta)}{4}\right]_0^{2\pi} = 96\pi$$



12. (12%)

(a) Find an equation in rectangular coordinates for the surface represented by cylindrical coordinates $z = r^2 sin^2\theta + 3rcos(\theta) + 1$.

(b) Find an equation in rectangular coordinates for the surface represented by spherical coordinates $\rho = 9\sec(\theta)$.

(c) Convert the rectangular equation $x^2 + y^2 + z^2 = 9$ to equation in cylindrical coordinates.

(d) Convert the rectangular equation $x^2 - y^2 = 2z$ to equation in spherical coordinates.

Ans:

(a) $z = y^2 + 3x + 1$

(b) $\rho\cos(\theta) = 9 \rightarrow z = 9$

(c) $r^2 + z^2 = 9$

(d) $\rho^2 sin^2\Phi cos^2(\theta) - \rho^2 sin^2\Phi sin^2\theta = 2\rho\cos(\Phi) \rightarrow \rho sin^2\Phi\cos(2\theta) - 2\cos(\Phi) = 0 \rightarrow \rho = 2\sec(2\theta)\cos(\Phi)csc^2(\Phi)$

Function	Taylor series	Interval of convergence
$\frac{1}{x}$	$1-(x-1)+(x-1)^2-(x-1)^3+\cdots.+(-1)^n(x-1)^n+\cdots.$	$0<x<2$
$\frac{1}{1+x}$	$1-x+x^2-x^3+\cdots.+(-1)^n x^n+\cdots.$	$-1<x<1$
$\ln x$	$(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots.+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots.$	$0<x\le 2$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots.+\frac{x^n}{n!}+\cdots.$	$-\infty<x<\infty$
$sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots.+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots.$	$-\infty<x<\infty$
$cos(x)$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots.+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots.$	$-\infty<x<\infty$
$arctan(x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots.+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots.$	$-1\le \mathrm{x}\le 1$
$arcsin(x)$	$x+\frac{x^3}{2\times 3}+\frac{1\times 3x^5}{2\times 4\times 5}+\cdots.+\frac{(2n)!\,x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots.$	$-1\le \mathrm{x}\le 1$
$(1+x)^k$	$1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}$ $+\cdots.+\frac{k(k-1)\ldots(k-n+1)x^n}{n!}+\cdots.$	$-1<x<1$

Derivative	Integrals
$\frac{d\sin^{-1}u}{dx}=\frac{u'}{\sqrt{1-u^2}}$	$\int\frac{du}{\sqrt{a^2-u^2}}=\sin^{-1}\frac{u}{a}+C$
$\frac{d\,cos^{-1}u}{dx}=\frac{-u'}{\sqrt{1-u^2}}$	$\int\frac{du}{a^2+u^2}=\frac{1}{a}tan^{-1}\frac{u}{a}+C$
$\frac{d\tan^{-1}u}{dx}=\frac{u'}{1+u^2}$	$\int\frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}sec^{-1}\frac{\|u\|}{a}+C$
$\frac{d\,cot^{-1}u}{dx}=\frac{-u'}{1+u^2}$	
$\frac{d\sec^{-1}u}{dx}=\frac{u'}{\|u\|\sqrt{u^2-1}}$	
$\frac{d\,csc^{-1}u}{dx}=\frac{-u'}{\|u\|\sqrt{u^2-1}}$	