1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out.)

(a) $\lim\limits_{(x,y)\to(1,2)} \frac{(x-1)^2-(y-2)^2}{(x-1)^2+(y-2)^2}$

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{xycos(y)}{x^2+y^2}$

(c) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

(d) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

Ans:

(a) Let $u = x - 1, v = y - 2$ and $u = rcos(\theta), v = rsin(\theta)$

$$\lim_{(x,y)\to(1,2)} \frac{(x-1)^2-(y-2)^2}{(x-1)^2+(y-2)^2} = \lim_{(u,v)\to(0,0)} \frac{u^2-v^2}{u^2+v^2} = \lim_{r\to 0}\left(\frac{r^2(cos^2\theta - sin^2\theta)}{r^2}\right)$$
$$= cos^2\theta - sin^2\theta$$

which means that if we follow the trajectory of different line $u = rcos(\theta), v = rsin(\theta)$ to approach (0,0) we will get different value for different θ, therefore, the limit does not exist.

(b) Let $y = mx$, $\lim\limits_{(x,y)\to(0,0)} \frac{xycos(y)}{x^2+y^2} = \lim\limits_{x\to 0} \frac{xmxcos(mx)}{x^2(1+m^2)} = \frac{m}{1+m^2}$. which means that if we follow the trajectory of different line $y = mx$ to approach (0,0) we will get different value for different m, therefore, the limit does not exist.

(c) The limit does not exist, because along the path $y = z = 0$

We have $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim\limits_{(x,0,0)\to(0,0,0)} \frac{0}{x^2} = 0$

However, along the path $z = 0, x = y$

We have $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim\limits_{(x,x,0)\to(0,0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$

(d) Let $x = rcos(\theta), y = rsin(\theta)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{r\to 0} \frac{r^2}{\sqrt{r^2+1}-1} = \lim_{r\to 0} \frac{r^2(\sqrt{r^2+1}+1)}{r^2+1-1} = 2$$

2. (15%)

(a) Let $f(x,y) = \begin{cases} \frac{1}{x} sin(xy) & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$, find f_x when $(x,y) \neq (0,0)$ and when $(x,y) = (0,0)$, respectively

(b) Given the equation $\mathrm{w} = \sin(2x + 3y), x = s + t, y = s - t$, find $\frac{\partial \mathrm{w}}{\partial s}$ and $\frac{\partial \mathrm{w}}{\partial \mathrm{t}}$

(c) Considering the level surface defined by $z^4 + (\sin(x))z^2 + yz = 3$. Find an equation of the tangent plane at (0,2,1)

Ans:

(a) For $(x,y) \neq (0,0)$:

$$f_x(x,y) = \frac{-1}{x^2}\sin(xy) + \frac{y}{x}\cos(xy),$$

For $(x,y) = (0,0)$:

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

(b) Using the chain rule

$$\frac{\partial w}{\partial s} = 2\cos(2x + 3y) + 3\cos(2x + 3y) = 5\cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2\cos(2x + 3y) - 3\cos(2x + 3y) = -\cos(5s - t)$$

(c) $F(x,y,z) = z^4 + (sin(x))z^2 + yz = 4$

$$\nabla F = cos(x)\, z^2\boldsymbol{i} + z\boldsymbol{j} + (4z^3 + 2(sin(x))z + y)\boldsymbol{k}$$

$$\nabla F(0,2,1) = 1\boldsymbol{i} + 1\boldsymbol{j} + 6\boldsymbol{k}$$

$$(x - 0) + (y - 2) + 6(z - 1) = 0$$

3. (10%) Let $f(x,y) = \int_1^{2y-x^2} e^t\, dt$

Find the direction in which $f(x,y)$ increase most. What is the rate of increase?

Ans:

$$\nabla f(x,y) = \left(-2x \cdot e^{2y-x^2}, 2e^{2y-x^2}\right) = 2e^{2y-x^2}(-x\boldsymbol{i} + \boldsymbol{j})$$

Increasing most rapidly is the direction of the gradient. That is $\frac{\nabla \mathrm{f}}{|\nabla \mathrm{f}|} = (\frac{-x}{\sqrt{x^2+1}}\boldsymbol{i} + \frac{1}{\sqrt{x^2+1}}\boldsymbol{j})$

$$|\nabla f| = 2e^{2y-x^2}\sqrt{x^2 + 1}$$

4. (15%) Let $f(x,y) = e^{-\frac{xy}{2}}$

Use lagrange multiplier to find any extrema of the function subject to $x^2+y^2\le 1$

Ans:

Case 1: On the circle $x^2+y^2=1$

$$\begin{cases} \frac{-y}{2}e^{\frac{-xy}{2}} = 2x\lambda \\ \frac{-x}{2}e^{\frac{-xy}{2}} = 2y\lambda \end{cases} \rightarrow x^2=y^2$$

Combine with $x^2+y^2=1$, we have $\mathrm{x}=\pm\frac{\sqrt{2}}{2}$

Case 2: Inside the circle:

$$f_x=\frac{-y}{2}e^{\frac{-xy}{2}}=0, f_y=\frac{-x}{2}e^{\frac{-xy}{2}}=0$$

We get $(\mathrm{x},\mathrm{y})=(0,0)$ as the critical point.

$f_{xx}=\frac{-y^2}{4}e^{\frac{-xy}{2}}$, $f_{yy}=\frac{-x^2}{4}e^{\frac{-xy}{2}}$, $f_{xy}=\frac{-1}{2}e^{\frac{-xy}{2}}+\frac{xy}{4}e^{\frac{-xy}{2}}$

At $(0,0)$ $\mathrm{D}=f_{xx}f_{yy}-f_{xy}f_{yx}<0$ therefore $(0,0)$ is saddle point

Combining the two cases, we have maximum of $e^{\frac{1}{4}}$ at $(\pm\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2})$ and

minimum $e^{\frac{-1}{4}}$ at $(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2})$

5. (10%) Evaluate the following expression

(a) $\int_0^{\ln 5}\int_{e^x}^{5}\frac{1}{y}dydx$

(b) $\int_0^{\frac{\pi}{4}}\int_0^{\frac{y}{2}}\int_0^{\frac{1}{y}}\sin(y)\ \ dzdxdy$

Ans:

(a) $\int_0^{\ln 5}\int_{e^x}^{5}\frac{1}{y}dydx=\int_1^5\int_0^{\ln y}\frac{1}{y}dxdy=\int_1^5\left[\frac{x}{y}\right]_0^{\ln y}dy=\int_1^5\frac{\ln y}{y}dy=\int_0^{\ln 5}u\,du=$

$\frac{1}{2}u^2|_0^{\ln 5}=\frac{1}{2}(\ln 5)^2$

(b) $\int_0^{\frac{\pi}{4}}\int_0^{\frac{y}{2}}\int_0^{\frac{1}{y}}\sin(y)\ \ dzdxdy=\int_0^{\frac{\pi}{4}}\int_0^{\frac{y}{2}}\frac{\sin(y)}{y}dxdy=\frac{1}{2}\int_0^{\frac{\pi}{4}}\sin(y)\,dy=\frac{-1}{2}\cos(y)\,|_0^{\frac{\pi}{4}}=$

$\frac{-\sqrt{2}}{4}+\frac{1}{2}$

6. (13%) Evaluate $\int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dydx$

Ans:

$$R = \left\{(x,y)\middle|0 \le y \le \frac{\sqrt{2}}{2}, x \le y \le \sqrt{1-x^2}\right\} = \left\{(r,\theta)\middle|0 \le r \le 1, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\right\}$$

$$\int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dydx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 \sqrt{1-r^2}\, rdrd\theta \ \ (Let\ \ u = 1-r^2, du$$

$$= -2rdr) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{3}\frac{-1}{2} u^{\frac{3}{2}}|_1^0 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3} d\theta = \frac{1}{3}\theta|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{12}$$

7. (15%) Find the area of the surface given by $z = f(x,y) = 25 - x^2 - y^2$ that lies above the region R where $R = \{(x,y): x^2 + y^2 \le 25\}$

Ans:

$$f_x = -2x, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$S = 4\int_0^{\frac{\pi}{2}} \int_0^5 \sqrt{1+4r^2}\, rdrd\theta \ \ (Let\ \ u = 1 + 4r^2, du = 8rdr)$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_1^{101} \sqrt{u}\, dud\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}} (101\sqrt{101} - 1) d\theta = \frac{\pi}{6}(101\sqrt{101} - 1)$$

8. (15%) Evaluate the triple integral $\int\int\int_Q yzcos(x^3 - 1)dV$ where $Q = \{\frac{y}{2} \le$ $x \le 1, 0 \le y \le 2, 0 \le z \le 1\}$

Ans:

$$\iiint_Q x^2 + y^2 \ dV$$

$$= \int_0^1 \int_0^2 \int_{\frac{y}{2}}^{1} yzcos(x^3 - 1) \ \ dxdydz$$

$$= \int_0^1 \int_0^1 \int_0^{2x} yzcos(x^3 - 1) \ \ dydxdz$$

$$= \int_0^1 \int_0^1 2x^2 zcos(x^3 - 1)dxdz = \int_0^1 zdz \int_0^1 2x^2 cos(x^3 - 1)dx$$

$$= \frac{1}{2}\frac{2}{3}\sin(x^3 - 1)\,|_0^1 = \frac{1}{3}\sin(1)$$