

# Chapter 11 Vectors and the Geometry of Space

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1 Surfaces in space

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# Cylindrical surfaces

- You have already known two special types of surfaces.
  - ① Spheres:  $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$
  - ② Planes:  $ax + by + cz + d = 0$
- A third type of surface in space is called a **cylindrical surface**, or simply a **cylinder**.
- To define a cylinder, consider the familiar right circular cylinder shown in Figure 1.

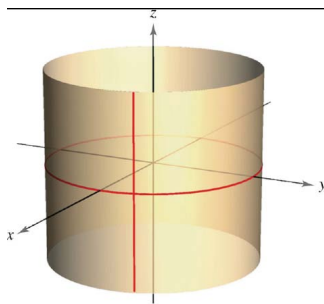


Figure 1: Right circular cylinder:  $x^2 + y^2 = a^2$ . Rulings are parallel to the  $z$ -axis.

- You can imagine that this cylinder is generated by a vertical line moving around the circle  $x^2 + y^2 = a^2$  in the  $xy$ -plane.
- This circle is called a **generating curve** for the cylinder, as indicated in the following definition.

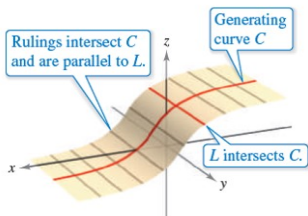


Figure 2: Right cylinder. Rulings are perpendicular to the plane containing  $C$ .

## Definition 11.1 (Cylinder)

Let  $C$  be a curve in a plane and let  $L$  be a line not in a parallel plane. The set of all lines parallel to  $L$  and intersecting  $C$  is called a cylinder.  $C$  is called the generating curve (or **directrix**) of the cylinder, and the parallel lines are called **rulings**.

- For the right circular cylinder shown in Figure 1, the equation of the generating curve is

$$x^2 + y^2 = a^2. \quad \text{Equation of generating curve in } xy\text{-plane}$$

- To find an equation of the cylinder, note that you can generate any one of the rulings by fixing the values of  $x$  and  $y$  and then allowing  $z$  to take on all real values.
- In this sense, the value of  $z$  is arbitrary and is, therefore, not included in the equation. In other words, the equation of this cylinder is simply the equation of its generating curve.

$$x^2 + y^2 = a^2 \quad \text{Equation of cylinder in space}$$

### Definition 11.2 (Equation of cylinders)

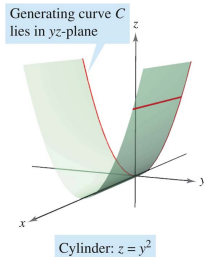
The equation of a cylinder whose ruling are parallel to one of the coordinate axes contain only the variables corresponding to the other two axes.

## Example 1 (Sketching a cylinder)

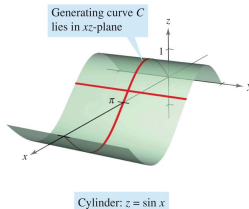
Sketch the surface represented by each equation.

**a.**  $z = y^2$     **b.**  $z = \sin x$ ,  $0 \leq x \leq 2\pi$ .

- a.**
- The graph is a cylinder whose generating curve,  $z = y^2$ , is a parabola in the  $yz$ -plane.
  - The rulings of the cylinder are parallel to the  $x$ -axis.
- b.**
- The graph is a cylinder generated by the sine curve in the  $xz$ -plane.
  - The rulings are parallel to the  $y$ -axis.



(a) Rulings are parallel to the  $x$ -axis.



(b) Rulings are parallel to the  $y$ -axis.



# Quadric surfaces

- The fourth basic type of surface in space is a **quadric surface**.  
Quadric surfaces are the three-dimensional analogs of conic sections.

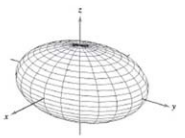
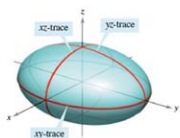
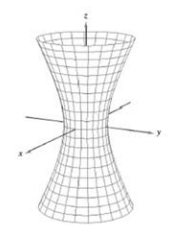
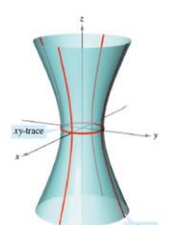
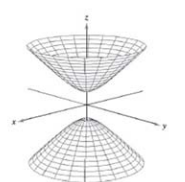
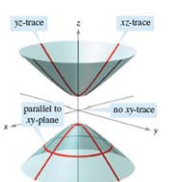
## Definition 11.3 (Quadric surface)

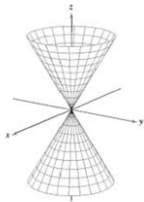
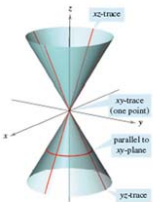

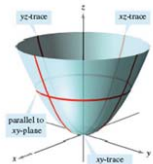
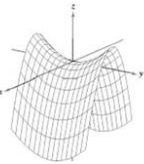
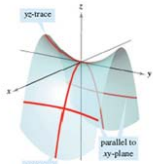
The equation of a quadric surface in space is a second-degree equation in three variables. The **general form** of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

There are six basic types of quadric surfaces: **ellipsoid**, **hyperboloid of one sheet**, **hyperboloid of two sheets**, **elliptic cone**, **elliptic paraboloid**, and **hyperbolic paraboloid**.

- The intersection of a surface with a plane is called the **trace of the surface** in the plane.
- To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes. The traces of quadric surfaces are conics.
- These traces, together with the **standard form** of the equation of each quadric surface, are shown in the following tables.

	<p style="text-align: center;"><b>Ellipsoid</b></p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <table><tr><td><u>Trace</u></td><td><u>Plane</u></td></tr><tr><td>Ellipse</td><td>Parallel to <math>xy</math>-plane</td></tr><tr><td>Ellipse</td><td>Parallel to <math>xz</math>-plane</td></tr><tr><td>Ellipse</td><td>Parallel to <math>yz</math>-plane</td></tr></table> <p>The surface is a sphere if <math>a = b = c \neq 0</math>.</p>	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to $xy$ -plane	Ellipse	Parallel to $xz$ -plane	Ellipse	Parallel to $yz$ -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to $xy$ -plane									
Ellipse	Parallel to $xz$ -plane									
Ellipse	Parallel to $yz$ -plane									
	<p style="text-align: center;"><b>Hyperboloid of One Sheet</b></p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <table><tr><td><u>Trace</u></td><td><u>Plane</u></td></tr><tr><td>Ellipse</td><td>Parallel to <math>xy</math>-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to <math>xz</math>-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to <math>yz</math>-plane</td></tr></table> <p>The axis of the hyperboloid corresponds to the variable whose coefficient is negative.</p>	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to $xy$ -plane	Hyperbola	Parallel to $xz$ -plane	Hyperbola	Parallel to $yz$ -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to $xy$ -plane									
Hyperbola	Parallel to $xz$ -plane									
Hyperbola	Parallel to $yz$ -plane									
	<p style="text-align: center;"><b>Hyperboloid of Two Sheets</b></p> $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <table><tr><td><u>Trace</u></td><td><u>Plane</u></td></tr><tr><td>Ellipse</td><td>Parallel to <math>xy</math>-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to <math>xz</math>-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to <math>yz</math>-plane</td></tr></table> <p>The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.</p>	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to $xy$ -plane	Hyperbola	Parallel to $xz$ -plane	Hyperbola	Parallel to $yz$ -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to $xy$ -plane									
Hyperbola	Parallel to $xz$ -plane									
Hyperbola	Parallel to $yz$ -plane									

	<p style="text-align: center;"><b>Elliptic Cone</b></p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <table><tr><th>Trace</th><th>Plane</th></tr><tr><td>Ellipse</td><td>Parallel to <math>xy</math>-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to <math>xz</math>-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to <math>yz</math>-plane</td></tr></table> <p>The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.</p>	Trace	Plane	Ellipse	Parallel to $xy$ -plane	Hyperbola	Parallel to $xz$ -plane	Hyperbola	Parallel to $yz$ -plane	
Trace	Plane									
Ellipse	Parallel to $xy$ -plane									
Hyperbola	Parallel to $xz$ -plane									
Hyperbola	Parallel to $yz$ -plane									
	<p style="text-align: center;"><b>Elliptic Paraboloid</b></p> $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <table><tr><th>Trace</th><th>Plane</th></tr><tr><td>Ellipse</td><td>Parallel to <math>xy</math>-plane</td></tr><tr><td>Parabola</td><td>Parallel to <math>xz</math>-plane</td></tr><tr><td>Parabola</td><td>Parallel to <math>yz</math>-plane</td></tr></table> <p>The axis of the paraboloid corresponds to the variable raised to the first power.</p>	Trace	Plane	Ellipse	Parallel to $xy$ -plane	Parabola	Parallel to $xz$ -plane	Parabola	Parallel to $yz$ -plane	
Trace	Plane									
Ellipse	Parallel to $xy$ -plane									
Parabola	Parallel to $xz$ -plane									
Parabola	Parallel to $yz$ -plane									
	<p style="text-align: center;"><b>Hyperbolic Paraboloid</b></p> $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ <table><tr><th>Trace</th><th>Plane</th></tr><tr><td>Hyperbola</td><td>Parallel to <math>xy</math>-plane</td></tr><tr><td>Parabola</td><td>Parallel to <math>xz</math>-plane</td></tr><tr><td>Parabola</td><td>Parallel to <math>yz</math>-plane</td></tr></table> <p>The axis of the paraboloid corresponds to the variable raised to the first power.</p>	Trace	Plane	Hyperbola	Parallel to $xy$ -plane	Parabola	Parallel to $xz$ -plane	Parabola	Parallel to $yz$ -plane	
Trace	Plane									
Hyperbola	Parallel to $xy$ -plane									
Parabola	Parallel to $xz$ -plane									
Parabola	Parallel to $yz$ -plane									

## Example 2 (Sketching a quadric surface)

Classify and sketch the surface given by

$$4x^2 - 3y^2 + 12z^2 + 12 = 0.$$

- Begin by writing the equation in standard form.

$$4x^2 - 3y^2 + 12z^2 + 12 = 0$$

$$\frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 = 0$$

$$\frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} = 1$$

- You can conclude that the surface is a hyperboloid of two sheets with the  $y$ -axis as its axis.

- To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

$$\text{xy-trace } (z = 0) : \frac{y^2}{4} - \frac{x^2}{3} = 1 \quad \text{Hyperbola}$$

$$\text{xz-trace } (y = 0) : \frac{x^2}{3} + \frac{z^2}{1} = -1 \quad \text{No trace}$$

$$\text{yz-trace } (x = 0) : \frac{y^2}{4} - \frac{z^2}{1} = 1 \quad \text{Hyperbola}$$

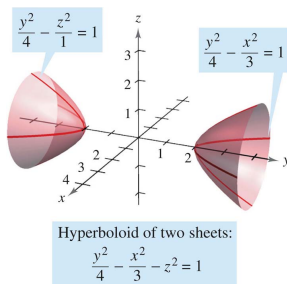


Figure 4: Hyperboloid of two sheets:  $\frac{y^2}{4} - \frac{x^2}{3} - z^2 = 1$ .

### Example 3 (Sketching a quadric surface)

Classify and sketch the surface given by  $x - y^2 - 4z^2 = 0$ .

- Because  $x$  is raised only to the first power, the surface is a paraboloid.
- The axis of the paraboloid is the  $x$ -axis. In the standard form, the equation is

$$x = y^2 + 4z^2.$$

- Some convenient traces are as follows.

$$\text{xy-trace } (z = 0) : x = y^2 \quad \text{Parabola}$$

$$\text{xz-trace } (y = 0) : x = 4z^2 \quad \text{Parabola}$$

$$\text{parallel to yz-plane } (x = 4) : \frac{y^2}{4} + \frac{z^2}{1} = 1 \quad \text{Ellipse}$$

- The surface is an elliptic paraboloid, as shown in Figure 5. ■

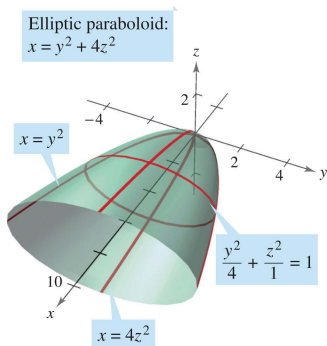


Figure 5: Elliptic paraboloid.



### Example 4 (A quadric surface not centered at the origin)

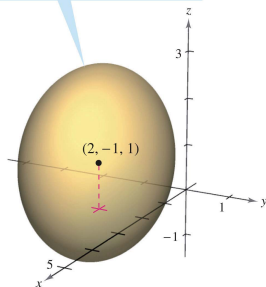
Classify and sketch the surface given by  
 $x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0$ .

- Completing the square for each variable produces the following.

$$\begin{aligned}(x^2 - 4x + \quad) + 2(y^2 + 2y + \quad) + (z^2 - 2z + \quad) &= -3 \\(x^2 - 4x + 4) + 2(y^2 + 2y + 1) + (z^2 - 2z + 1) &= -3 + 4 + 2 + 1 \\(x - 2)^2 + 2(y + 1)^2 + (z - 1)^2 &= 4 \\\frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{2} + \frac{(z - 1)^2}{4} &= 1\end{aligned}$$

- From this equation, you can see that the quadric surface is an ellipsoid that is centered at  $(2, -1, 1)$ . ■

$$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{2} + \frac{(z-1)^2}{4} = 1$$



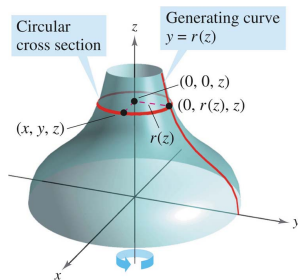
# Surfaces of revolution

- The fifth special type of surface you will study is called a **surface of revolution**. We now look at how to find its equation.
- Consider the graph of the **radius function**

$$y = r(z) \quad \text{Generating curve}$$

in the  $yz$ -plane.

- If this graph is revolved around the  $z$ -axis, it forms a surface of revolution.



- The trace of the surface in the plane  $z = z_0$  is a circle whose radius is  $r(z_0)$  and whose equation is

$$x^2 + y^2 = [r(z_0)]^2. \quad \text{Circular trace in plane: } z = z_0$$

- Replacing  $z_0$  with  $z$  produces an equation that is valid for all values of  $z$ .
- You can obtain equations for surfaces of revolution for the other two axes, and the results are summarized as follows.

## Definition 11.4 (Surface of revolution)

If the graph of a radius function  $r$  is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

- 1 Revolved about the  $x$ -axis:  $y^2 + z^2 = [r(x)]^2$
- 2 Revolved about the  $y$ -axis:  $x^2 + z^2 = [r(y)]^2$
- 3 Revolved about the  $z$ -axis:  $x^2 + y^2 = [r(z)]^2$

### Example 5 (Finding an equation for a surface of revolution)

Find an equation for the surface of revolution formed by revolving (a) the graph of  $y = 1/z$  about the  $z$ -axis and (b) the graph of  $9x^2 = y^3$  about the  $y$ -axis.

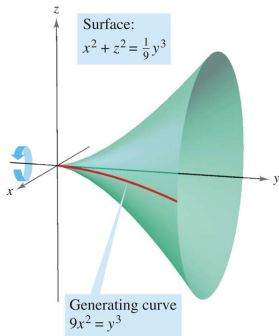
- a. An equation for the surface of revolution formed by revolving the graph of  $y = \frac{1}{z}$  (radius function) about the  $z$ -axis is

$$x^2 + y^2 = [r(z)]^2 \qquad x^2 + y^2 = \left(\frac{1}{z}\right)^2.$$

- b. To find an equation for the surface formed by revolving the graph of  $9x^2 = y^3$  about the  $y$ -axis, solve for  $x$  in terms of  $y$  to obtain  $x = \frac{1}{3}y^{3/2} = r(y)$  (radius function). So, the equation for surface is

$$x^2 + z^2 = [r(y)]^2 \qquad x^2 + z^2 = \left(\frac{1}{3}y^{3/2}\right)^2 \qquad x^2 + z^2 = \frac{1}{9}y^3.$$

The graph is shown in Figure 6.



**Figure 6:** Surface of revolution:  $x^2 + z^2 = \frac{1}{9}y^3$  with generating curve  $9x^2 = y^3$  about the  $y$ -axis.

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# Cylindrical coordinates

- The **cylindrical coordinate system**, is an extension of the polar coordinates in the plane to three-dimensional space.

## Definition 11.5 (The cylindrical coordinate system)

In a cylindrical coordinate system, a point  $P$  in space is represented by an ordered triple  $(r, \theta, z)$ .

- 1  $(r, \theta)$  is a polar representation of the projection of  $P$  in the  $xy$ -plane.
- 2  $z$  is the directed distance from  $(r, \theta)$  to  $P$ .

- To convert from rectangular to cylindrical coordinates (or vice versa), use the following conversion guidelines for polar coordinates:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \quad z = z$$

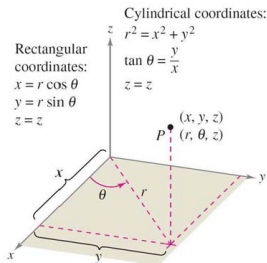


Figure 7: The relationship between cylindrical and rectangular coordinates.

- The point  $(0, 0, 0)$  is called the **pole**.

- Moreover, because the representation of a point in the polar coordinate system is not unique, it follows that the representation in the cylindrical coordinate system is also not unique!

### Example 1 (Converting from cylindrical to rectangular coordinates)

Convert the point  $(r, \theta, z) = (4, \frac{5\pi}{6}, 3)$  to rectangular coordinates.

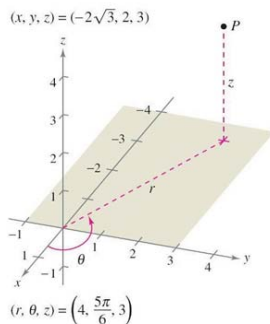


Figure 8: Converting  $(r, \theta, z) = \left(4, \frac{5\pi}{6}, 3\right)$  to  $(x, y, z) = (-2\sqrt{3}, 2, 3)$ .

## Example 2 (Converting from rectangular to cylindrical coordinate)

Convert the point  $(x, y, z) = (1, \sqrt{3}, 2)$  to cylindrical coordinates.

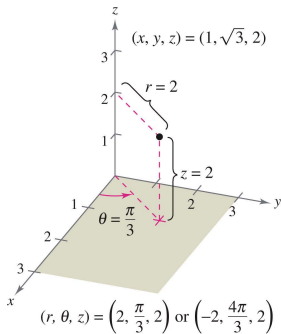


Figure 9: Converting from rectangular to cylindrical coordinates.

- Cylindrical coordinates are especially convenient for representing cylindrical surfaces and surfaces of revolution with the z-axis as the axis of symmetry:

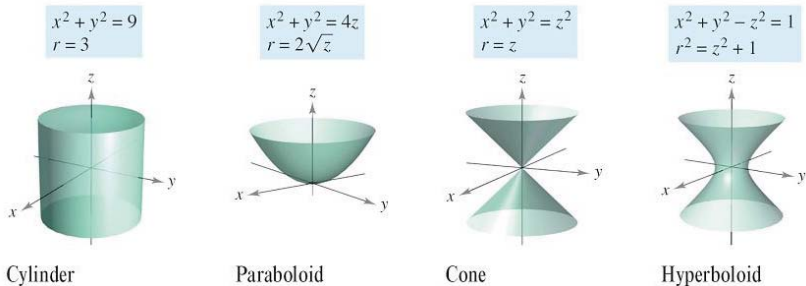


Figure 10: Different cylindrical equations.

- Vertical planes containing the  $z$ -axis and horizontal planes also have simple cylindrical coordinate equations:

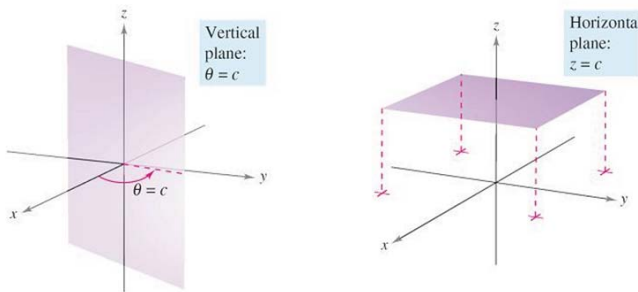


Figure 11: Vertical plane:  $\theta = c$  and horizontal plane:  $z = c$ .



### Example 3 (Rectangular-to-cylindrical conversion)

Find an equation in cylindrical coordinates for the surface represented by each rectangular equation.

**a.**  $x^2 + y^2 = 4z^2$      **b.**  $y^2 = x$

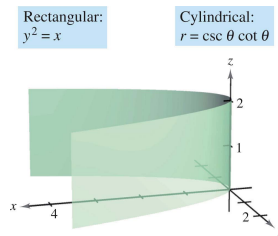
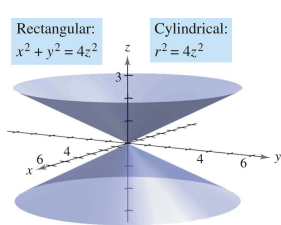


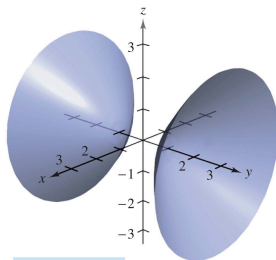
Figure 12: Rectangular-to-cylindrical conversion.

### Example 4 (Cylindrical-to-rectangular conversion)

Find an equation in rectangular coordinates for the surface represented by the cylindrical equation

$$r^2 \cos 2\theta + z^2 + 1 = 0.$$

Cylindrical:  
 $r^2 \cos 2\theta + z^2 + 1 = 0$



Rectangular:  
 $y^2 - x^2 - z^2 = 1$

Figure 13: Cylindrical-to-rectangular conversion.

# Spherical coordinates

- In the **spherical coordinate system**, each point is represented by an ordered triple: the first coordinate is a distance, and the second and third coordinates are angles.
- This system is similar to the latitude-longitude system used to identify points on the surface of Earth.
- For example, the point on the surface of Earth whose latitude is  $40^\circ$  North (of the equator) and whose longitude is  $80^\circ$  West (of the prime meridian) is shown in Figure 14. Assuming that the Earth is spherical and has a radius of 6371 kilometers, you would label this point as

$$(4000, -80^\circ, 50^\circ).$$

Radius       $80^\circ$  clockwise from  
prime meridian       $50^\circ$  down from  
North Pole

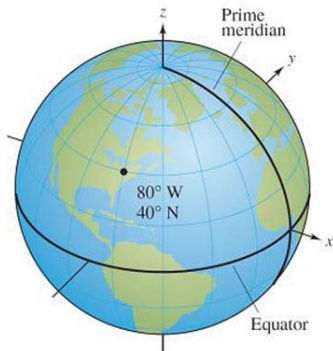


Figure 14: Spherical coordinate of  $80^\circ$  W  $40^\circ$  N is  $(4000, -80^\circ, 50^\circ)$ .

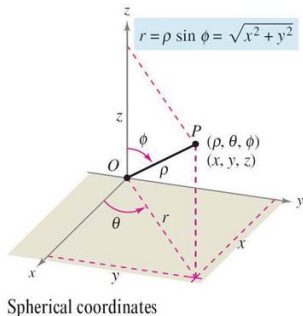
## Definition 11.6 (The spherical coordinate system)

In a spherical coordinate system, a point  $P$  in space is represented by an ordered triple  $(\rho, \theta, \phi)$ .

1.  $\rho$  is the distance between  $P$  and the origin,  $\rho \geq 0$ .
2.  $\theta$  is the same angle used in cylindrical coordinates for  $r \geq 0$ .
3.  $\phi$  is the angle between the positive  $z$ -axis and the line segment  $\overrightarrow{OP}$ ,  $0 \leq \phi \leq \pi$ .

Note that the first and third coordinates,  $\rho$  and  $\phi$ , are nonnegative.  $\rho$  is the lowercase Greek letter rho, and  $\phi$  is the lowercase Greek letter phi.

- The relationship between rectangular and spherical coordinates is illustrated in Figure 15.



**Figure 15:** The relationship between rectangular coordinate  $(x, y, z)$  and spherical coordinates  $(\rho, \theta, \phi)$  where  $r = \rho \sin \phi = \sqrt{x^2 + y^2}$ .

- To convert from one system to the other, use the following.
- Spherical to rectangular:

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$



- Rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2, \quad \tan \theta = \frac{y}{x}, \quad \phi = \arccos \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right).$$

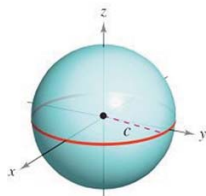
- To change coordinates between the cylindrical and spherical systems, use the following.
- Spherical to cylindrical ( $r \geq 0$ ):

$$r^2 = \rho^2 \sin^2 \phi, \quad \theta = \theta, \quad z = \rho \cos \phi.$$

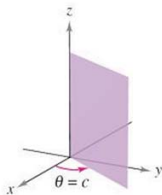
- Cylindrical to spherical ( $r \geq 0$ ):

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \phi = \arccos \left( \frac{z}{\sqrt{r^2 + z^2}} \right).$$

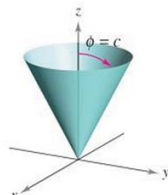
- The spherical coordinate system is useful primarily for surfaces in space that have a point or center of symmetry.
- For example, Figure 16 shows three surfaces with simple spherical equations.



Sphere:  
 $\rho = c$



Vertical half-plane:  
 $\theta = c$



Half-cone:  $\left(0 < c < \frac{\pi}{2}\right)$   
 $\phi = c$

Figure 16: Three surfaces with simple spherical equations.

### Example 5 (Rectangular-to-spherical conversion)

Find an equation in spherical coordinates for the surface represented by each rectangular equation.

- a.** Cone:  $x^2 + y^2 = z^2$      **b.** Sphere:  $x^2 + y^2 + z^2 - 4z = 0$

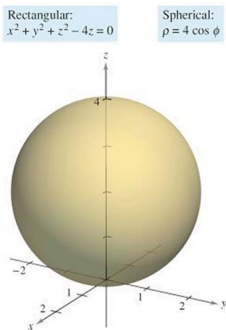


Figure 17:  $x^2 + y^2 + z^2 - 4z = 0$  in rectangular coordinate is equivalent to  $\rho = 4 \cos \phi$  in spherical coordinate.