Chapter 10 Conics, Parametric Equations, and Polar Coordinates

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1. Plane curves and parametric equations
2. Parametric equations and calculus
3. Polar coordinates and polar graphs
4. Area and arc length in polar coordinates

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Plane curves and parametric equations

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Plane curves and parametric equations

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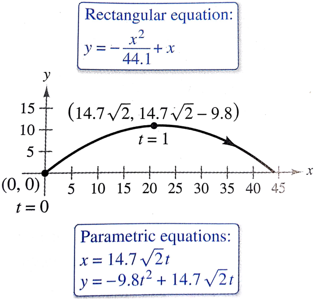
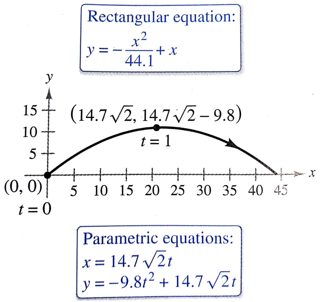
- Consider the path followed by an object that is propelled into the air at an angle of 45°.
- If the initial velocity of the object is 29.4 meters per second, the object travels the parabolic path given by

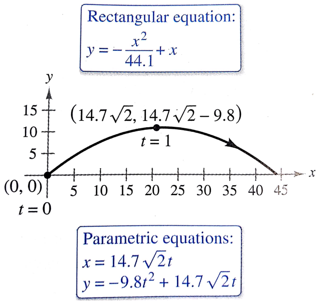
$$y = -\frac{x^2}{44.1} + x \qquad \text{Rectangular equation}$$

  as shown in Figur below.


Rectangular equation:
$y = -\frac{x^2}{44.1} + x$
y
15
10
5
$(14.7\sqrt{2}, 14.7\sqrt{2} - 9.8)$
$t = 1$
x
(0, 0)
$t = 0$
5 10 15 20 25 30 35 40 45
Parametric equations:
$x = 14.7\sqrt{2}t$
$y = -9.8t^2 + 14.7\sqrt{2}t$



- To get more information about the time, you can introduce a third variable t, called a **parameter**.



- To get more information about the time, you can introduce a third variable t, called a **parameter**.
- By writing both x and y as functions of t, you obtain the **parametric equations**

$$x = 14.7\sqrt{2}t \quad \text{and} \quad y = -9.8t^2 + 14.7\sqrt{2}t.$$

- From this set of equations, you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point $(14.7\sqrt{2}, 14.7\sqrt{2} - 9.8)$ and so on.

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- For this particular motion problem, x and y are continuous functions of t, and the resulting path is called a **plane curve**.

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Definition 10.1 (Plane curve)

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

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are called **parametric equations** and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve**, denoted by C.

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Example 1 (Sketching a curve)

Sketch the curve described by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \quad -2 \le t \le 3.$$

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t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

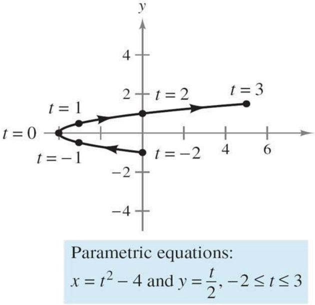

Figure 1: Parametric equations: $x = t^2 - 4$, $y = \frac{t}{2}$, $-2 \leq t \leq 3$.

- By plotting these points in order of increasing t and using the continuity of f and g, you obtain the curve C shown in Figure 1. Note that the arrows on the curve indicate its orientation as t increases from -2 to 3. ■

Eliminating the parameter

- Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter**.

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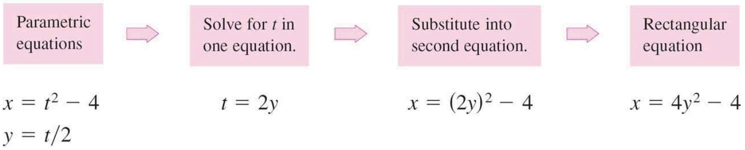
- For instance, you can eliminate the parameter from the set of parametric equations in Example 1 as follows.



- Once you have eliminated the parameter, you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$, as shown in Example 1.

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- The range of x and y implied by the parametric equations may be altered by the change to rectangular form.
- In such instances the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations.

Example 2 (Adjusting the domain after eliminating the parameter)

Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.

Example 3 (Using trigonometry to eliminate a parameter)

Sketch the curve represented by

$$x = 3\cos\theta \quad \text{and} \quad y = 4\sin\theta, \quad 0 \le \theta \le 2\pi$$

by eliminating the parameter and finding the corresponding rectangular equation.

Finding parametric equations

Example 4 (Finding parametric equation for a given graph)

Find a set of parametric equations that represents the graph of $y = 1 - x^2$, using each of the following parameters.

a. $t = x$ **b.** The slope $m = \frac{\mathrm{d}y}{\mathrm{d}x}$ at the point (x, y).

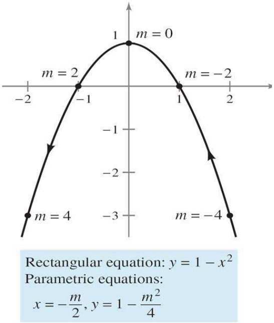

Figure 2: Parametric equation: $x = \frac{m}{2}$, $y = 1 - \frac{m^2}{4}$. Rectangular equation: $y = 1 - x^2$.

Example 5 (Parametric equations for a cycloid)

Determine the curve traced by a point P on the circumference of a circle of radius a rolling along a straight line in a plane. Such a curve is called a **cycloid**.

- Let the parameter θ be the measure of the circle's rotation, and let the point $P = (x, y)$ begin at the origin.
- When $\theta = 0$, P is at the origin. When $\theta = \pi$, P is at a maximum point $(\pi a, 2a)$.
- When $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$. From Figure 3, you can see that $\angle APC = 180^\circ - \theta$.

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- When $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$. From Figure 3, you can see that $\angle APC = 180^\circ - \theta$.
- So,

$$\sin\theta = \sin(180^\circ - \theta) = \sin(\angle APC) = \frac{\overline{AC}}{a} = \frac{\overline{BD}}{a}$$

$$\cos\theta = -\cos(180^\circ - \theta) = -\cos(\angle APC) = \frac{\overline{AP}}{-a}$$

which implies that $\overline{AP} = -a\cos\theta$ and $\overline{BD} = a\sin\theta$.

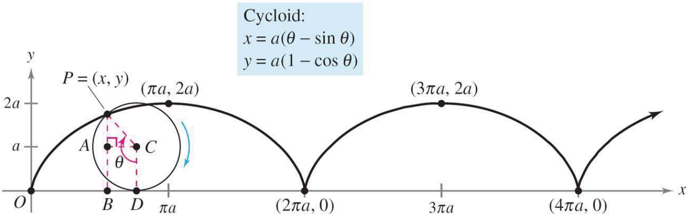
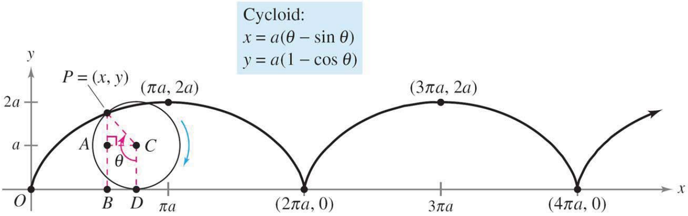


Figure 3: Cycloid: $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.



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- Because the circle rolls along the x-axis, you know that

$$\overline{OD} = \overset{\frown}{PD} = a\theta.$$

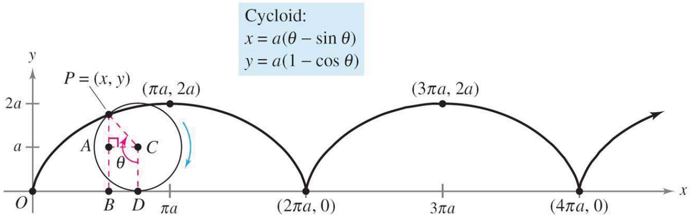

Figure 3: Cycloid: $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

- Because the circle rolls along the x-axis, you know that

$$\overline{OD} = \overset{\frown}{PD} = a\theta.$$

- Furthermore, because $\overline{BA} = \overline{DC} = a$, you have

$$x = \overline{OD} - \overline{BD} = a\theta - a\sin\theta, \qquad y = \overline{BA} + \overline{AP} = a - a\cos\theta.$$

- So, the parametric equations are

$$x = a(\theta - \sin\theta) \quad \text{and} \quad y = a(1 - \cos\theta). \blacksquare$$

- The cycloid in Figure 3 has sharp corners (**cusps**) at the values $x = 2n\pi a$. Notice that the derivatives $x'(\theta)$ and $y'(\theta)$ are both zero at the points for which $\theta = 2n\pi$.

$$\begin{aligned} x(\theta) &= a(\theta - \sin\theta) & y(\theta) &= a(1 - \cos\theta) \\ x'(\theta) &= a - a\cos\theta & y'(\theta) &= a\sin\theta \\ x'(2n\pi) &= 0 & y'(2n\pi) &= 0 \end{aligned}$$

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Definition 10.2 (Smooth curve)

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called **piecewise smooth** if it is smooth on each subinterval of some partition of I.

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Slope and tangent lines

- The projectile is represented by the parametric equations

$$x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t$$

  You know that these equations enable you to locate the position of the projectile at a given time.

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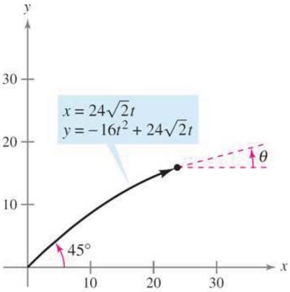

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Theorem 10.7 (Parametric form of the derivative)

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}, \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0.$$

Example 1 (Differentiation and parametric form)

Find $\mathrm{d}y/\mathrm{d}x$ for the curve given by $x = \sin t$ and $y = \cos t$.

- Because $\mathrm{d}y/\mathrm{d}x$ is a function of t, you can use Theorem 10.7 repeatedly to find higher-order derivatives.

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- For instance,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x/\mathrm{d}t} = \frac{-\sec^2 t}{\cos t} = -\sec^3 t$$

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Example 2 (Finding slope and concavity)

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point $(2, 3)$.

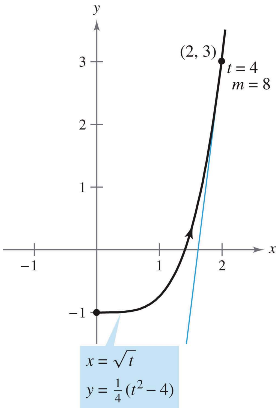

Figure 4: The graph is concave upward at $(2, 3)$, when $t = 4$.

Example 3 (A curve with two tangent lines at a point)

The prolate cycloid given by

$$x = 2t - \pi \sin t \quad \text{and} \quad y = 2 - \pi \cos t$$

crosses itself at the point $(0, 2)$, as shown in Figure 5. Find the equations of both tangent lines at this point.

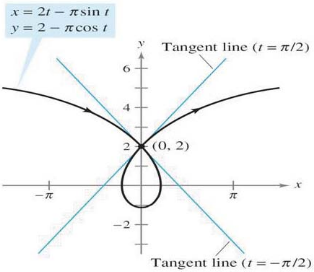

Figure 5: This prolate cycloid has two tangent lines at the point $(0, 2)$.

Remark

If $\mathrm{d}y/\mathrm{d}t = 0$ and $\mathrm{d}x/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$. Similarly, if $\mathrm{d}x/\mathrm{d}t = 0$ and $\mathrm{d}y/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Arc length

Theorem 10.8 (Arc length in parametric form)

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t.$$

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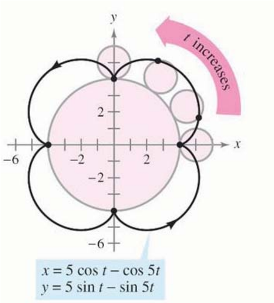
- If a circle rolls along a line, a point on its circumference will trace a path called a **cycloid**.
- If the circle rolls around the circumference of another circle, the path of the point is an **epicycloid**.

Example 4 (Finding arc length)

A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The epicycloid traced by a point on the circumference of the smaller circle is given by

$$x = 5\cos t - \cos 5t \quad \text{and} \quad y = 5\sin t - \sin 5t.$$

Find the distance traveled by point in one trip about the larger circle.



Theorem 10.9 (Area of a surface of revolution)

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the x-axis: $g(t) \ge 0$
2. $S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the y-axis: $f(t) \ge 0$

- These formulas are easy to remember if you think of the differential of arc length as

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

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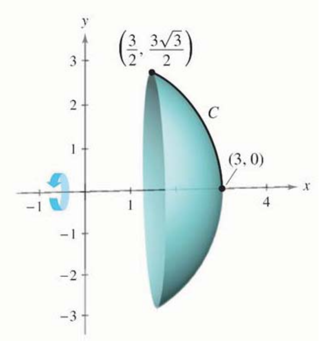
- Then the formulas are written as follows.
  1. $S = 2\pi \int_a^b g(t)\,\mathrm{d}s$ **2.** $S = 2\pi \int_a^b f(t)\,\mathrm{d}s$

Example 5 (Finding the area of a surface of revolution)

Let C be the arc of the circle

$$x^2 + y^2 = 9$$

from $(3, 0)$ to $(3/2, 3\sqrt{3}/2)$, as shown below. Find the area of the surface formed by revolving C about the x-axis.



Area between the Parametric equation and axis

We know that the area under a curve $y = F(x)$ from a to b is

$$A = \int_a^b F(x)\,dx,$$

where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations

$$x = f(t), \quad y = g(t), \quad t_1 \leq t \leq t_2,$$

then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_{t_1}^{t_2} g(t) f'(t)\,dt$$

- As t varies from t_1 to t_2, the corresponding values of x vary from $a = f(t_1)$ to $b = f(t_2)$, and we have

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- The area under the curve in terms of x and y is

$$A = \int_a^b y\,dx.$$

- By substituting $y = g(t)$ and $dx = f'(t)dt$, the area becomes

$$A = \int_{t_1}^{t_2} g(t)f'(t)\,dt.$$

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Polar coordinates

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- In this section you will study a coordinate system called the **polar coordinate system**.

Polar coordinates

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- In this section you will study a coordinate system called the **polar coordinate system**.
- To form the polar coordinate system in the plane, fix a point O, called the **pole** (or origin), and construct from O an initial ray called the **polar axis**, as shown in Figure 6.

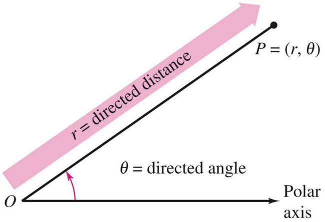

Figure 6: The definition of polar coordinates.

- Then each point P in the plane can be assigned **polar coordinates** (r, θ), as follows

$$r = \text{directed distance from } O \text{ to } P$$

$$\theta = \text{directed angle, counterclockwise from polar axis to segment } \overline{OP}$$

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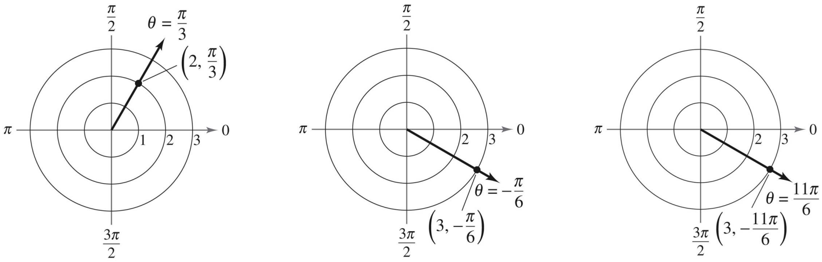
- Figure 7 shows three points on the polar coordinate system.



(a) (b) (c)

Figure 7: Points $\left(2, \frac{\pi}{3}\right)$, $\left(3, -\frac{\pi}{6}\right)$, $\left(3, \frac{11}{6}\pi\right)$ on the polar coordinates system.

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- In general, the point (r, θ) can be written as

$$(r, \theta) = (r, \theta + 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta + (2n+1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

Coordinate conversion

- To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown below:

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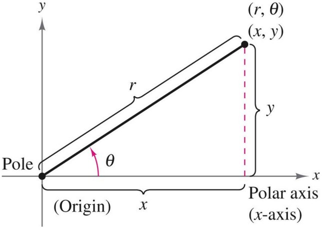

- Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$ the definitions of the trigonometric functions imply that $\tan\theta = \frac{y}{x}$, $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

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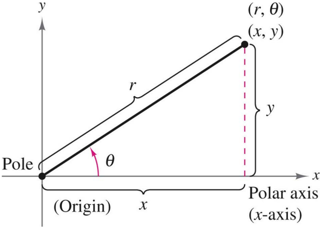

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- If $r < 0$, you can show that the same relationships hold.

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Theorem 10.10 (Polar-to-rectangular conversion)

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.
1. $x = r\cos\theta$ *and* $y = r\sin\theta$. **2.** $\tan\theta = \frac{y}{x}$ *and* $r^2 = x^2 + y^2$.

Example 1 (Polar-to-rectangular conversion)

a. Convert the point $(r, \theta) = (2, \pi)$ to rectangular coordinates

b. Convert the point $(r, \theta) = (\sqrt{3}, \pi/6)$ to rectangular coordinates

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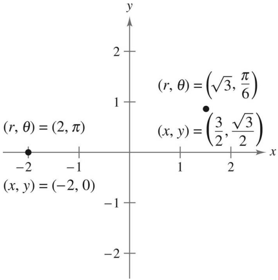

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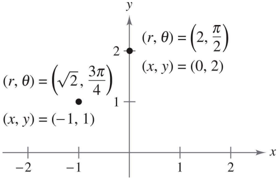


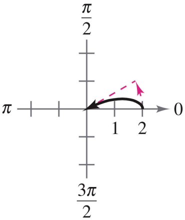
Polar graphs

Example 3 (Graphing polar equations)

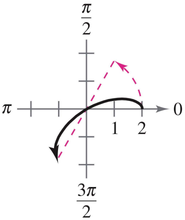
Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.
a. $r = 2$ **b.** $\theta = \frac{\pi}{3}$ **c.** $r = \sec\theta$

Example 4 (Sketching a polar graph)

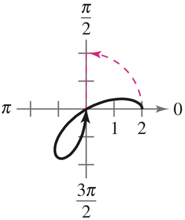
Sketch the graph of $r = 2\cos 3\theta$.



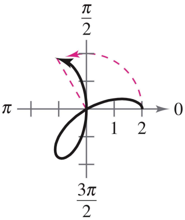
(a) $0 \le \theta \le \pi/6$.



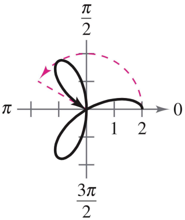
(b) $0 \le \theta \le \pi/3$.



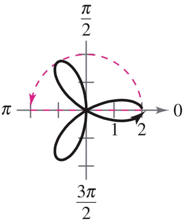
(c) $0 \le \theta \le \pi/2$.



(d) $0 \le \theta \le 2\pi/3$.



(e) $0 \le \theta \le 5\pi/6$.



(f) $0 \le \theta \le \pi$.

Figure 8: Sketching a polar graph.

Slope and tangent lines

- To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To find the slope in polar form, use the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

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- Using the parametric form of $\mathrm{d}y/\mathrm{d}x$ given in Theorem 10.7, you have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

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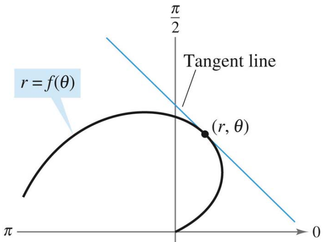

Theorem 10.11 (Slope in polar form)

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

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provided that $\mathrm{d}x/\mathrm{d}\theta \neq 0$ at (r, θ).

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- From Theorem 10.11, you can make the following observations.
  1. Solution to $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ yield horizontal tangents, provided that $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$.
  2. Solution to $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$ yield vertical tangents, provided that $\frac{\mathrm{d}y}{\mathrm{d}\theta} \neq 0$.

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- If $\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$ are simultaneously 0, no conclusion can be drawn about tangent lines.

Example 5 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \leq \theta \leq \pi$.

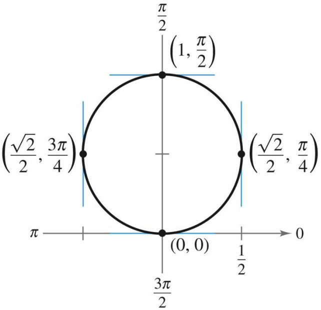

Figure 9: Horizontal and vertical tangent lines of $r = \sin\theta$.

Example 6 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$.

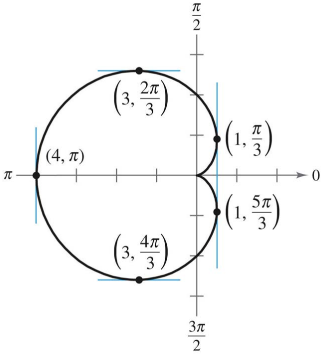

Figure 10: Horizontal and vertical tangent lines of $r = 2(1 - \cos\theta)$.

- This graph is called a **cardioid**. Note that both derivatives $(\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta)$ are 0 when $\theta = 0$.

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- Using this information alone, you don't know whether the graph has a horizontal or vertical tangent line at the pole. From Figure 10, however, you can see that the graph has a **cusp** at the pole.

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Theorem 10.12 (Tangent lines at the pole)

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Special polar graphs

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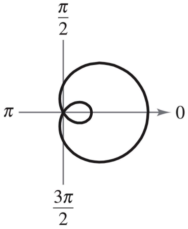

(a) $\frac{a}{b} < 1$.
Limacon with inner loop.

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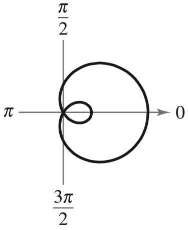

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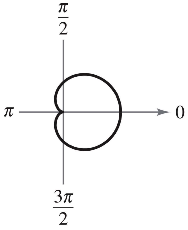

(b) $\frac{a}{b} = 1$.
Cardioid (heart-shaped).

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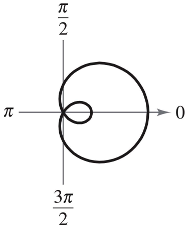

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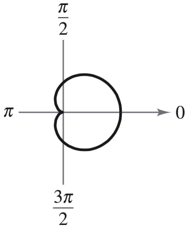

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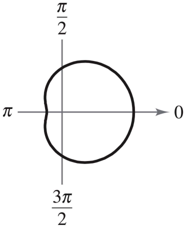

(c) $1 < \frac{a}{b} < 2$.
Dimpled limaçon.

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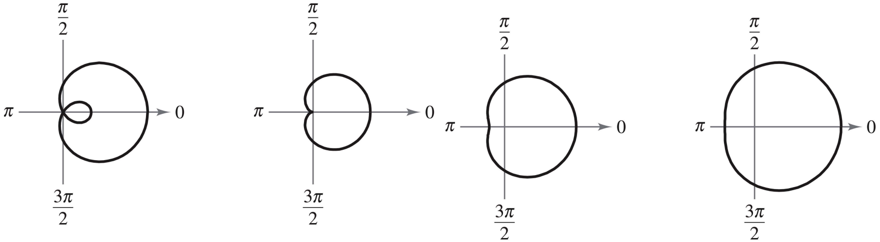


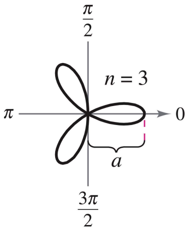
(a) $\frac{a}{b} < 1$. Limacon with inner loop.

(b) $\frac{a}{b} = 1$. Cardioid (heart-shaped).

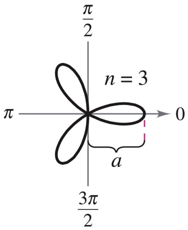
(c) $1 < \frac{a}{b} < 2$. Dimpled limaçon.

(d) $\frac{a}{b} \geq 2$. Convex limaçon.

Figure 11: Limacon: $r = a \pm b \cos\theta$, $r = a \pm b \sin\theta$ $(a > 0, b > 0)$.



(a) $r = a \cos n\theta$.
Rose curve.



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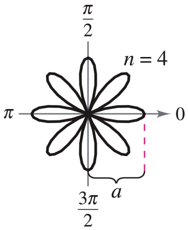
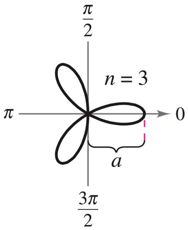


(b) $r = a\cos n\theta$.
Rose curve.



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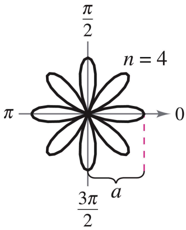

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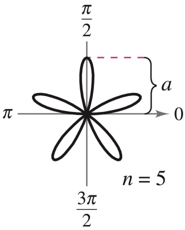
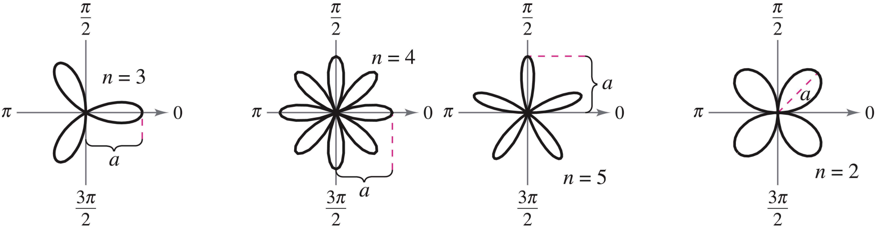


(c) $r = a \sin n\theta$.
Rose curve.



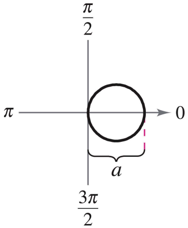
(a) $r = a\cos n\theta$.
Rose curve.

(b) $r = a\cos n\theta$.
Rose curve.

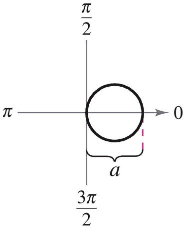
(c) $r = a\sin n\theta$.
Rose curve.

(d) $r = a\sin n\theta$.
Rose curve.

Figure 12: Rose curves: n petals if n is odd, $2n$ petals if n is even ($n \geq 2$).



(a) $r = a\cos\theta$.
Circle.



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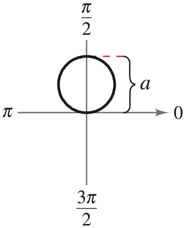

(b) $r = a\sin\theta$.
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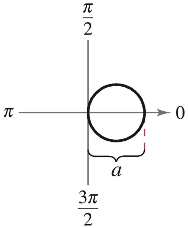

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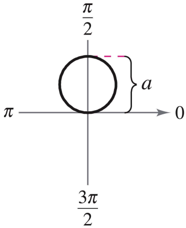

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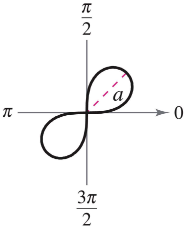
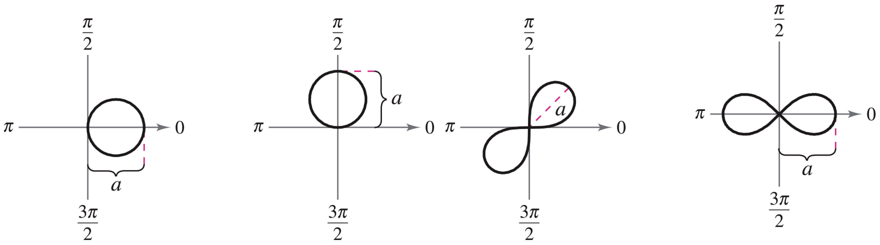


(c) $r^2 = a^2\sin 2\theta$.
Lemniscate.



(a) $r = a\cos\theta$. Circle.

(b) $r = a\sin\theta$. Circle.

(c) $r^2 = a^2 \sin 2\theta$. Lemniscate.

(d) $r^2 = a^2 \cos 2\theta$. Lemniscate.

Figure 13: Circles and Lemniscate.

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Area of a polar region

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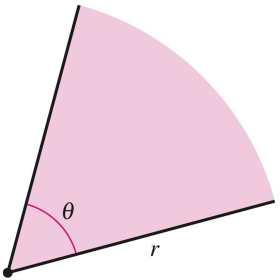

Figure 14: The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.

- Consider the function given by $r = f(\theta)$, where f is continuous and nonnegative on the interval given by $\alpha \leq \theta \leq \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 15.

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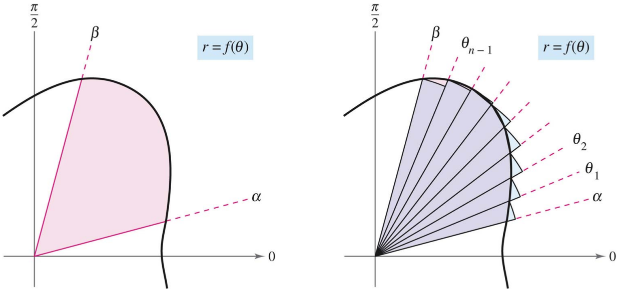

Figure 15: Area in polar coordinates.

- To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = \beta$$

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- Radius of ith sector $= f(\theta_i)$. Central angle of ith sector $= \frac{\beta - \alpha}{n} = \triangle\theta$

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$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right) \triangle\theta [f(\theta_i)]^2.$$

- Taking the limit as $n \to \infty$ produces

$$A = \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \triangle\theta = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta.$$

Theorem 10.13 (Area in polar coordinates)

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta, \quad 0 < \beta - \alpha \leq 2\pi.$$

Example 1 (Finding the area of a polar region)

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.

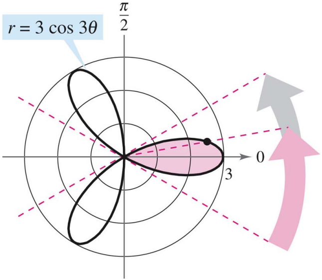

Figure 16: The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$.

Example 2 (Finding the area bounded by a single curve)

Find the area of the region lying between the inner and outer loops of the limacon $r = 1 - 2\sin\theta$.

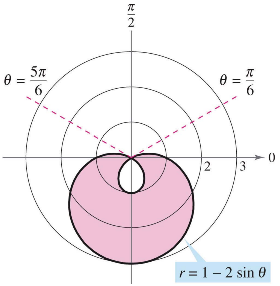

Figure 17: The area between the inner and outer loops is approximately 8.34.

Points of intersection of polar graphs

- Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs.

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- Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs.
- For example, consider the points of intersection of the graphs of $r = 1 - 2\cos\theta$ and $r = 1$ as shown in Figure 18.

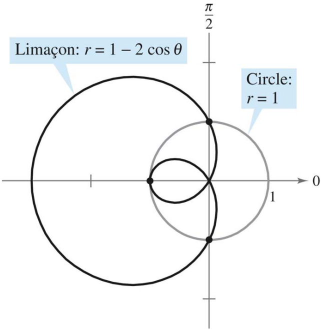

Figure 18: Limacon: $r = 1 - 2\cos\theta$ and three points intersection: $(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$.

- If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

$$r = 1 - 2\cos\theta \qquad 1 = 1 - 2\cos\theta \qquad \cos\theta = 0 \qquad \theta = \frac{\pi}{2},\ \frac{3\pi}{2}$$

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- However, from Figure 18 you can see that there is a third point of intersection that did not show up when the two polar equations were solved simultaneously.
- The reason the third point was not found is that it does not occur with the same coordinates in the two graphs!

- On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of $r = 1 - 2\cos\theta$, the point occurs with coordinates $(-1, 0)$.

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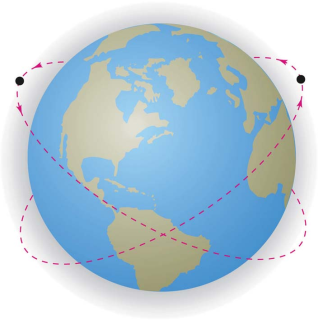

Figure 19: The paths of satellites can cross without causing a collision.

Example 3 (Finding the area of a region between two curves)

Find the area of the region common to the two regions bounded by the following curves.

$$r = -6\cos\theta \qquad \text{Circle} \qquad\qquad r = 2 - 2\cos\theta \qquad \text{Cardioid}$$

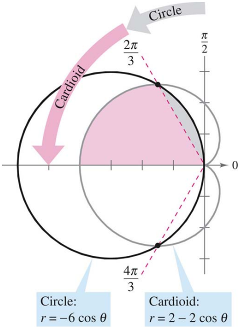

Figure 20: Find the area between circle $r = -6\cos\theta$ and cardioid $r = 2 - 2\cos\theta$.

Arc length in polar form

- The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations.

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Theorem 10.14 (Arc length of a polar curve)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

Example 4 (Finding the length of a polar curve)

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$ as shown in Figure 21.

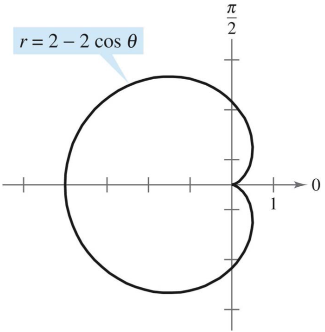

Figure 21: The cardioid $r = 2 - 2\cos\theta$.

Area of a surface of revolution

- The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

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Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

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1. $S = 2\pi \int_\alpha^\beta y\,\mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta$
   About the polar axis

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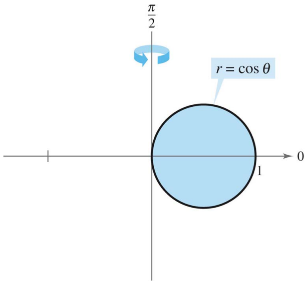
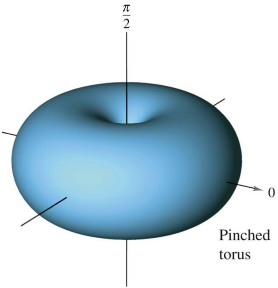
1. $S = 2\pi \int_\alpha^\beta y\,\mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta$
   About the polar axis
2. $S = 2\pi \int_\alpha^\beta x\,\mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta$
   About the line $\theta = \frac{\pi}{2}$

Example 5 (Finding the area of a surface of revolution)

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 22.



(a) Circle.



(b) Pinched torus.

Figure 22: Revolving a circle $r = \cos\theta$ around $x = \frac{\pi}{2}$.