Chapter 10 Conics, Parametric Equations, and Polar Coordinates

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Plane curves and parametric equations

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Definition 10.1 (Plane curve)

If f and g are continuous functions of t on an interval I, then the equations

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are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. Taken together, the parametric equations and the graph are called a plane curve, denoted by C.

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Definition 10.2 (Smooth curve)

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called <u>smooth</u> if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called <u>piecewise smooth</u> if it is smooth on each subinterval of some partition of I.

Slope and tangent lines

- The projectile is represented by the parametric equations

$$x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t$$

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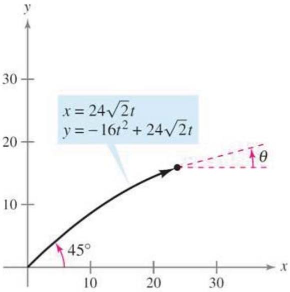

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Theorem 10.7 (Parametric form of the derivative)

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}, \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0.$$

Example 1 (Differentiation and parametric form)

Find $\mathrm{d}y/\mathrm{d}x$ for the curve given by $x = \sin t$ and $y = \cos t$.

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$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x/\mathrm{d}t} = \frac{-\sec^2 t}{\cos t} = -\sec^3 t$$

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$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]}{\mathrm{d}x/\mathrm{d}t} \\ &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(-\sec^3 t)}{\cos t} = \frac{-3\sec^2 t(\sec t\tan t)}{\cos t} = -3\sec^4 t\tan t.\end{aligned}$$

Example 2 (Finding slope and concavity)

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point $(2, 3)$.

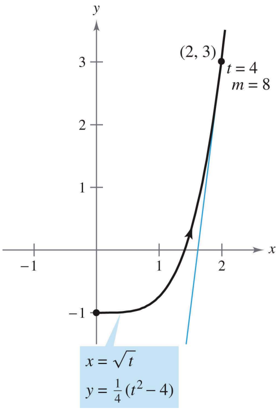

Figure 1: The graph is concave upward at $(2, 3)$, when $t = 4$.

Example 3 (A curve with two tangent lines at a point)

The prolate cycloid given by

$$x = 2t - \pi \sin t \quad \text{and} \quad y = 2 - \pi \cos t$$

crosses itself at the point $(0, 2)$, as shown in Figure 2. Find the equations of both tangent lines at this point.

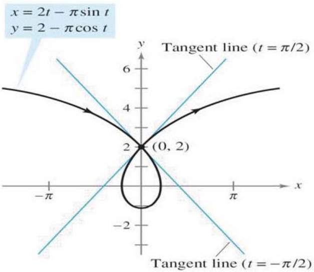

Figure 2: This prolate cycloid has two tangent lines at the point $(0, 2)$.

Remark

If $\mathrm{d}y/\mathrm{d}t = 0$ and $\mathrm{d}x/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$.
Similarly, if $\mathrm{d}x/\mathrm{d}t = 0$ and $\mathrm{d}y/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Arc length

Theorem 10.8 (Arc length in parametric form)

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t.$$

- If a circle rolls along a line, a point on its circumference will trace a path called a cycloid.

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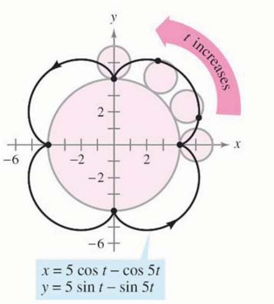
- If the circle rolls around the circumference of another circle, the path of the point is an <u>epicycloid</u>.

Example 4 (Finding arc length)

A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The epicycloid traced by a point on the circumference of the smaller circle is given by

$$x = 5\cos t - \cos 5t \quad \text{and} \quad y = 5\sin t - \sin 5t.$$

Find the distance traveled by point in one trip about the larger circle.



Area of a surface of revolution

Theorem 10.9 (Area of a surface of revolution)

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \leq t \leq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the x-axis: $g(t) \geq 0$
2. $S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the y-axis: $f(t) \geq 0$

- These formulas are easy to remember if you think of the differential of arc length as

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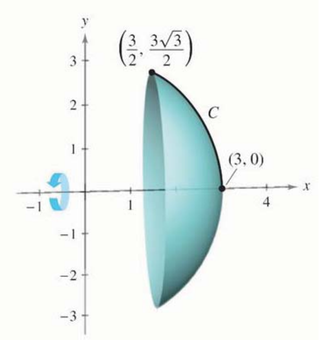
- Then the formulas are written as follows.
  1. $S = 2\pi \int_a^b g(t)\,\mathrm{d}s$ **2.** $S = 2\pi \int_a^b f(t)\,\mathrm{d}s$

Example 5 (Finding the area of a surface of revolution)

Let C be the arc of the circle

$$x^2 + y^2 = 9$$

from $(3, 0)$ to $(3/2, 3\sqrt{3}/2)$, as shown below. Find the area of the surface formed by revolving C about the x-axis.



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Polar coordinates

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- In this section you will study a coordinate system called the polar coordinate system.
- To form the polar coordinate system in the plane, fix a point O, called the pole (or origin), and construct from O an initial ray called the polar axis, as shown in Figure 3.

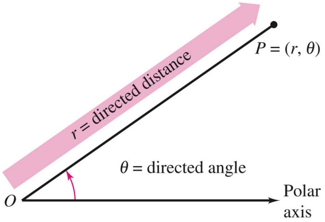

Figure 3: The definition of polar coordinates.

- Then each point P in the plane can be assigned polar coordinates (r, θ), as follows

$$r = \text{directed distance from } O \text{ to } P$$

$$\theta = \text{directed angle, counterclockwise from polar axis to segment } \overline{OP}$$

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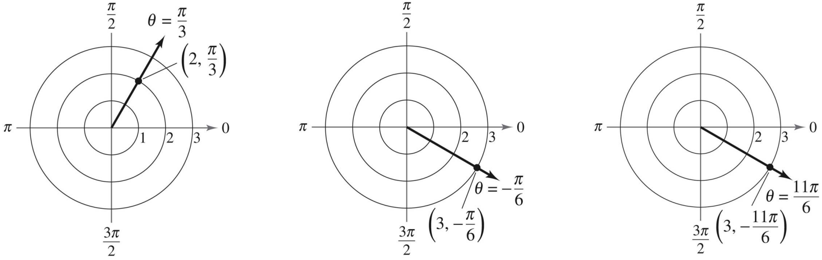
- Figure 4 shows three points on the polar coordinate system.



(a) (b) (c)

Figure 4: Points $\left(2, \frac{\pi}{3}\right)$, $\left(3, -\frac{\pi}{6}\right)$, $\left(3, \frac{11}{6}\pi\right)$ on the polar coordinates system.

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- For instance, coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point [parts (b) and (c) in Figure 4]. Also, because r is a directed distance, coordinates (r, θ) and $(-r, \pi + \theta)$ represent the same point.

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- In general, the point (r, θ) can be written as

$$(r, \theta) = (r, \theta + 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta + (2n+1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

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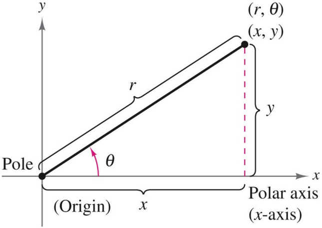

- Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$ the definitions of the trigonometric functions imply that $\tan\theta = \frac{y}{x}$, $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

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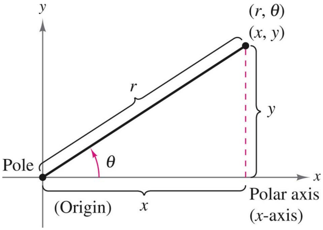

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- If $r < 0$, you can show that the same relationships hold.

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Theorem 10.10 (Polar-to-rectangular conversion)

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.
1. $x = r\cos\theta$ *and* $y = r\sin\theta$. **2.** $\tan\theta = \frac{y}{x}$ *and* $r^2 = x^2 + y^2$.

Example 1 (Polar-to-rectangular conversion)

a. Convert the point $(r, \theta) = (2, \pi)$ to rectangular coordinates

b. Convert the point $(r, \theta) = (\sqrt{3}, \pi/6)$ to rectangular coordinates

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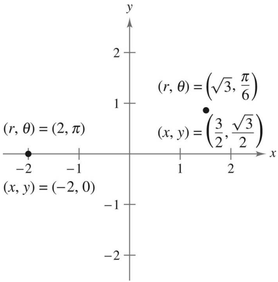

Example 2 (Rectangular-to-polar conversion)

a. Convert the point $(x, y) = (-1, 1)$ to polar coordinates

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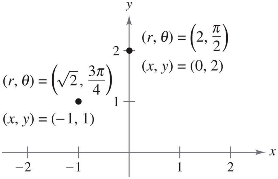


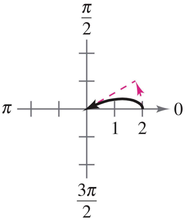
Example 3 (Graphing polar equations)

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

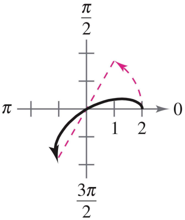
a. $r = 2$ **b.** $\theta = \frac{\pi}{3}$ **c.** $r = \sec\theta$

Example 4 (Sketching a polar graph)

Sketch the graph of $r = 2\cos 3\theta$.



(a) $0 \le \theta \le \pi/6$.



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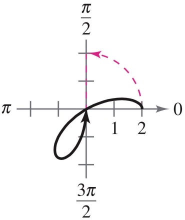
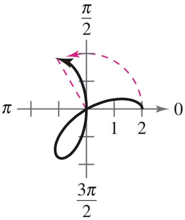
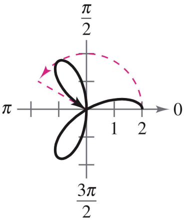


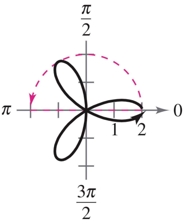
(c) $0 \le \theta \le \pi/2$.



(d)
$0 \le \theta \le 2\pi/3$.



(e)
$0 \le \theta \le 5\pi/6$.



(f) $0 \le \theta \le \pi$.

Figure 5: Sketching a polar graph.

Slope and tangent lines

- To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To find the slope in polar form, use the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

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- Using the parametric form of $\mathrm{d}y/\mathrm{d}x$ given in Theorem 10.7, you have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

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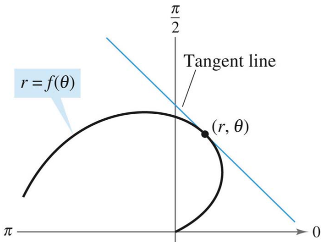

Theorem 10.11 (Slope in polar form)

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

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provided that $\mathrm{d}x/\mathrm{d}\theta \neq 0$ at (r, θ).

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- From Theorem 10.11, you can make the following observations.
  1. Solution to $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ yield horizontal tangents, provided that $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$.
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- If $\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$ are simultaneously 0, no conclusion can be drawn about tangent lines.

Example 5 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \leq \theta \leq \pi$.

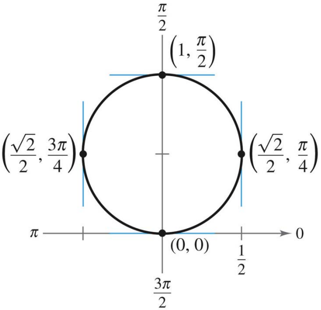

Figure 6: Horizontal and vertical tangent lines of $r = \sin\theta$.

Example 6 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$.

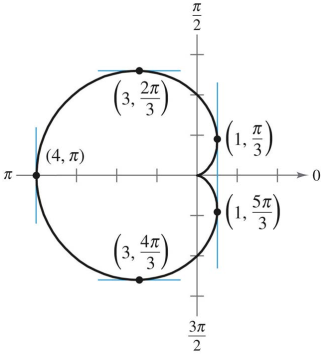

Figure 7: Horizontal and vertical tangent lines of $r = 2(1 - \cos\theta)$.

- This graph is called a <u>cardioid</u>. Note that both derivatives ($\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$) are 0 when $\theta = 0$.

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Theorem 10.12 (Tangent lines at the pole)

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Special polar graphs

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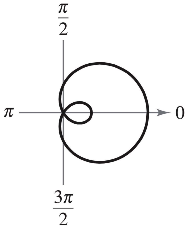

(a) $\frac{a}{b} < 1$.
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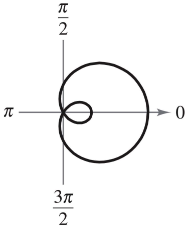

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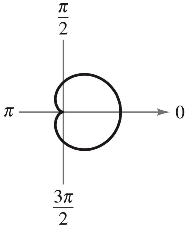

(b) $\frac{a}{b} = 1$.
Cardioid (heart-shaped).

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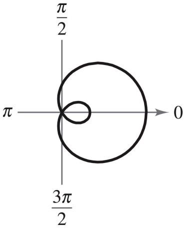

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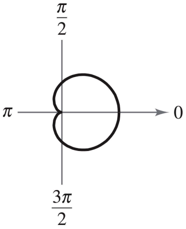

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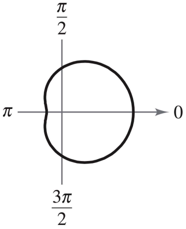

(c) $1 < \frac{a}{b} < 2$.
Dimpled limaçon.

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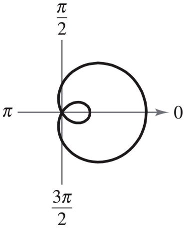

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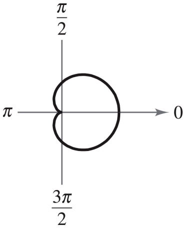

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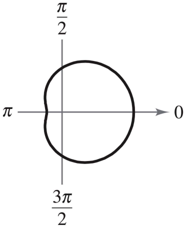

(c) $1 < \frac{a}{b} < 2$.
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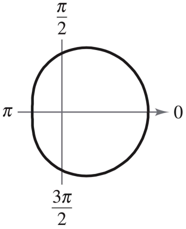
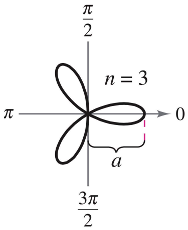
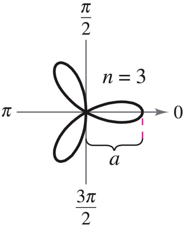


(d) $\frac{a}{b} \geq 2$.
Convex limaçon.

Figure 8: Limacon: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$ $(a > 0, b > 0)$.



(a) $r = a \cos n\theta$.
Rose curve.



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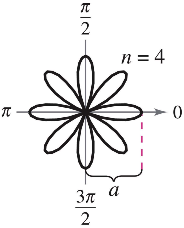

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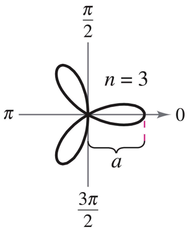

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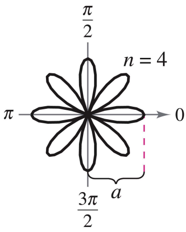

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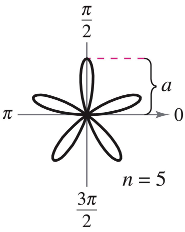
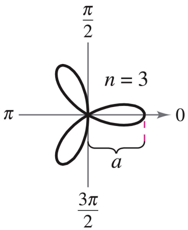


(c) $r = a \sin n\theta$. Rose curve.



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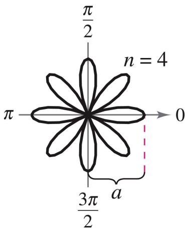

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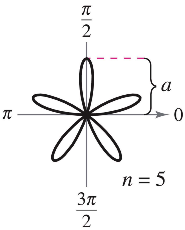
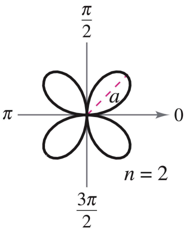
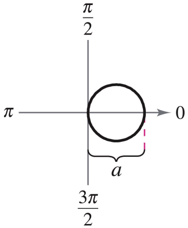


(c) $r = a\sin n\theta$. Rose curve.

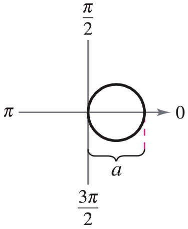


(d) $r = a\sin n\theta$. Rose curve.

Figure 9: Rose curves: n petals if n is odd, $2n$ petals if n is even $(n \geq 2)$.



(a) $r = a\cos\theta$.
Circle.



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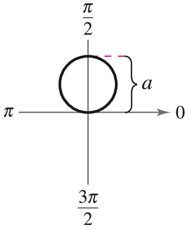
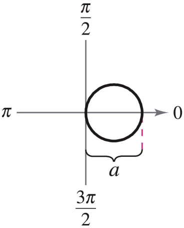
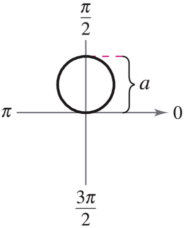


(b) $r = a\sin\theta$.
Circle.



(a) $r = a\cos\theta$.
Circle.



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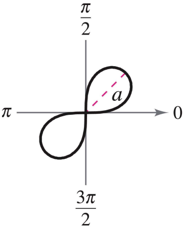
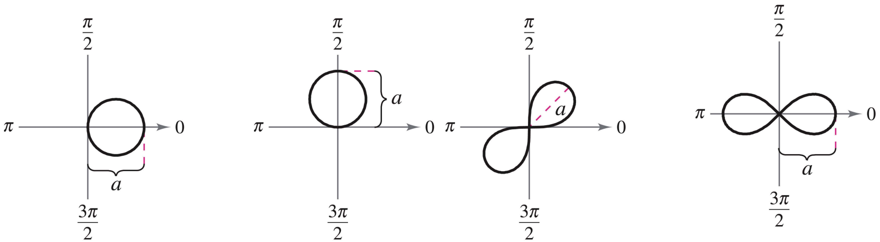


(c) $r^2 = a^2\sin 2\theta$.
Lemniscate.



(a) $r = a\cos\theta$. Circle.

(b) $r = a\sin\theta$. Circle.

(c) $r^2 = a^2\sin 2\theta$. Lemniscate.

(d) $r^2 = a^2\cos 2\theta$. Lemniscate.

Figure 10: Circles and Lemniscate.

Table of Contents

Area of a polar region

- The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area.

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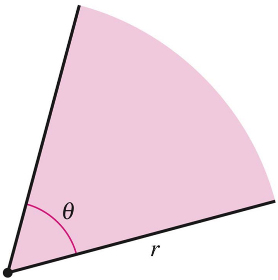

Figure 11: The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.

- Consider the function given by $r = f(\theta)$, where f is continuous and nonnegative on the interval given by $\alpha \leq \theta \leq \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 12.

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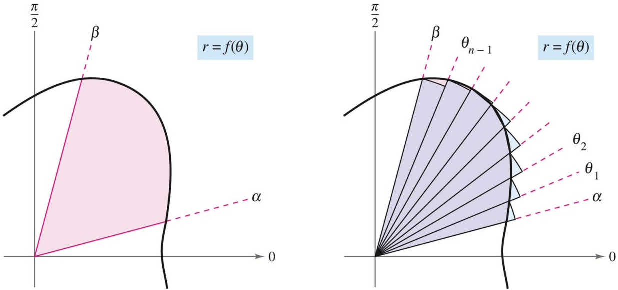

Figure 12: Area in polar coordinates.

- To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = \beta$$

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- Radius of ith sector $= f(\theta_i)$. Central angle of ith sector $= \frac{\beta-\alpha}{n} = \triangle\theta$

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$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right) \triangle\theta [f(\theta_i)]^2.$$

- Taking the limit as $n \to \infty$ produces

$$A = \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \triangle\theta = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta.$$

Theorem 10.13 (Area in polar coordinates)

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta, \quad 0 < \beta - \alpha \leq 2\pi.$$

Example 1 (Finding the area of a polar region)

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.

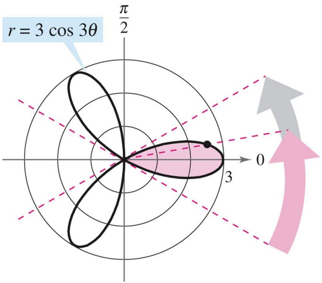

Figure 13: The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$.

Example 2 (Finding the area bounded by a single curve)

Find the area of the region lying between the inner and outer loops of the limacon $r = 1 - 2\sin\theta$.

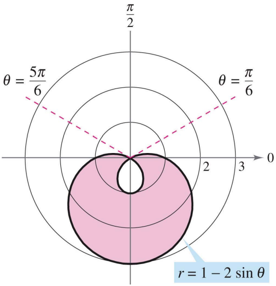

Figure 14: The area between the inner and outer loops is approximately 8.34.

Points of intersection of polar graphs

- Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs.

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- Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs.
- For example, consider the points of intersection of the graphs of $r = 1 - 2\cos\theta$ and $r = 1$ as shown in Figure 15.

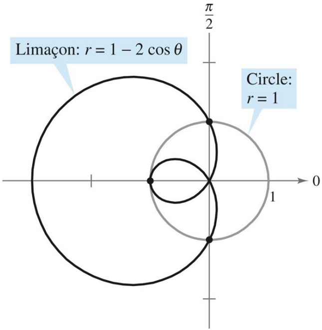

Figure 15: Limacon: $r = 1 - 2\cos\theta$ and three points intersection: $(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$.

- If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

$$r = 1 - 2\cos\theta \qquad 1 = 1 - 2\cos\theta \qquad \cos\theta = 0 \qquad \theta = \frac{\pi}{2},\ \frac{3\pi}{2}$$

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- The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$.
- However, from Figure 15 you can see that there is a third point of intersection that did not show up when the two polar equations were solved simultaneously.
- The reason the third point was not found is that it does not occur with the same coordinates in the two graphs!

- On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of $r = 1 - 2\cos\theta$, the point occurs with coordinates $(-1, 0)$.

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- You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 16.

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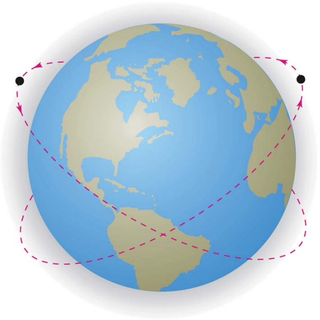

Figure 16: The paths of satellites can cross without causing a collision.

Example 3 (Finding the area of a region between two curves)

Find the area of the region common to the two regions bounded by the following curves.

$$r = -6\cos\theta \qquad \text{Circle} \qquad\qquad r = 2 - 2\cos\theta \qquad \text{Cardioid}$$

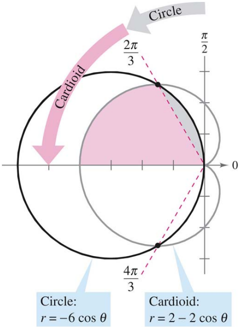

Figure 17: Find the area between circle $r = -6\cos\theta$ and cardioid $r = 2 - 2\cos\theta$.

Arc length in polar form

- The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations.

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Theorem 10.14 (Arc length of a polar curve)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta.$$

Example 4 (Finding the length of a polar curve)

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$ as shown in Figure 18.

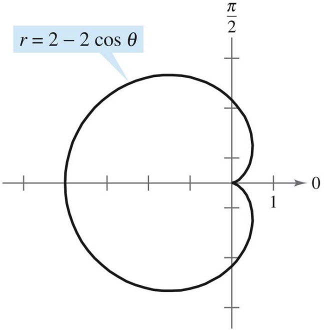

Figure 18: The cardioid $r = 2 - 2\cos\theta$.

Area of a surface of revolution

- The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

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Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

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Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

1. $S = 2\pi \int_\alpha^\beta y\,\mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta$
   About the polar axis

Area of a surface of revolution

- The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

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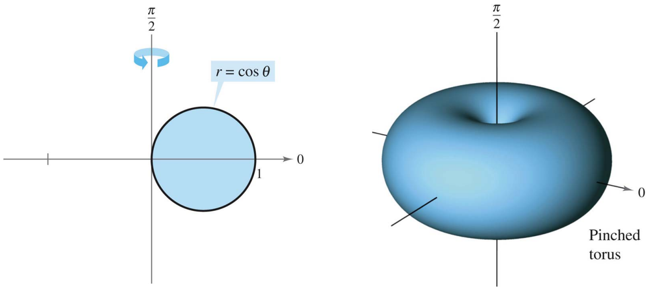
   About the polar axis
2. $S = 2\pi \int_{\alpha}^{\beta} x \,\mathrm{d}s = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta$
   About the line $\theta = \frac{\pi}{2}$

Example 5 (Finding the area of a surface of revolution)

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 19.



(a) Circle. (b) Pinched torus.

Figure 19: Revolving a circle $r = \cos\theta$ around $x = \frac{\pi}{2}$.