Chapter 10 Conics, Parametric Equations, and Polar Coordinates

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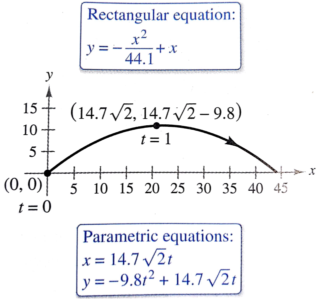
Plane curves and parametric equations

- We have been representing a graph by a single equation involving two variables. In this section, we will study case in which three variables are used!
- Consider the path followed by an object that is propelled into the air at an angle of 45°.
- If the initial velocity of the object is 29.4 meters per second, the object travels the parabolic path given by

$$y = -\frac{x^2}{44.1} + x \qquad \text{Rectangular equation}$$

as shown in Figur below.



- To get more information about the time, you can introduce a third variable t, called a **parameter**.
- By writing both x and y as functions of t, you obtain the **parametric equations**

$$x = 14.7\sqrt{2}t \quad \text{and} \quad y = -9.8t^2 + 14.7\sqrt{2}t.$$

- From this set of equations, you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point $(14.7\sqrt{2}, 14.7\sqrt{2} - 9.8)$ and so on.
- For this particular motion problem, x and y are continuous functions of t, and the resulting path is called a **plane curve**.

Definition 10.1 (Plane curve)

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve**, denoted by C.

- When sketching a curve represented by a set of parametric equations, you can plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t.
- By plotting the resulting points in order of increasing values of t, the curve is traced out in a specific direction. This is called the **orientation** of the curve.

Example 1 (Sketching a curve)

Sketch the curve described by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \quad -2 \le t \le 3.$$

- For values of t on the given interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

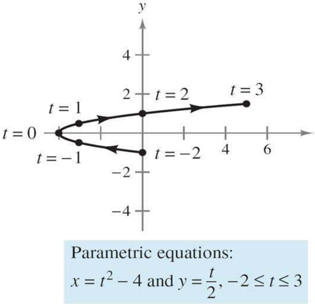


Figure 1: Parametric equations: $x = t^2 - 4$, $y = \frac{t}{2}$, $-2 \leq t \leq 3$.

- By plotting these points in order of increasing t and using the continuity of f and g, you obtain the curve C shown in Figure 1. Note that the arrows on the curve indicate its orientation as t increases from -2 to 3. ■

Eliminating the parameter

- Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter**.
- For instance, you can eliminate the parameter from the set of parametric equations in Example 1 as follows.

Parametric equations	$\Rightarrow$	Solve for t in one equation.	$\Rightarrow$	Substitute into second equation.	$\Rightarrow$	Rectangular equation
$x = t^2 - 4$ $y = t/2$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

- Once you have eliminated the parameter, you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$, as shown in Example 1.
- The range of x and y implied by the parametric equations may be altered by the change to rectangular form.
- In such instances the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations.

Example 2 (Adjusting the domain after eliminating the parameter)

Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.

Example 3 (Using trigonometry to eliminate a parameter)

Sketch the curve represented by

$$x = 3\cos\theta \quad \text{and} \quad y = 4\sin\theta, \quad 0 \le \theta \le 2\pi$$

by eliminating the parameter and finding the corresponding rectangular equation.

Finding parametric equations

Example 4 (Finding parametric equation for a given graph)

Find a set of parametric equations that represents the graph of $y = 1 - x^2$, using each of the following parameters.
a. $t = x$ **b.** The slope $m = \frac{\mathrm{d}y}{\mathrm{d}x}$ at the point (x, y).

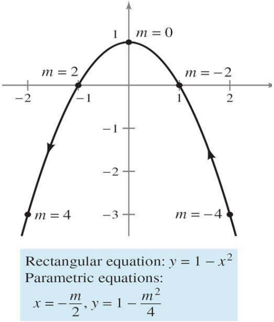


Figure 2: Parametric equation: $x = \frac{m}{2}$, $y = 1 - \frac{m^2}{4}$. Rectangular equation: $y = 1 - x^2$.

Example 5 (Parametric equations for a cycloid)

Determine the curve traced by a point P on the circumference of a circle of radius a rolling along a straight line in a plane. Such a curve is called a **cycloid**.

- Let the parameter θ be the measure of the circle's rotation, and let the point $P = (x, y)$ begin at the origin.
- When $\theta = 0$, P is at the origin. When $\theta = \pi$, P is at a maximum point $(\pi a, 2a)$.
- When $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$. From Figure 3, you can see that $\angle APC = 180^\circ - \theta$.
- So,

$$\sin\theta = \sin(180^\circ - \theta) = \sin(\angle APC) = \frac{\overline{AC}}{a} = \frac{\overline{BD}}{a}$$

$$\cos\theta = -\cos(180^\circ - \theta) = -\cos(\angle APC) = \frac{\overline{AP}}{-a}$$

which implies that $\overline{AP} = -a\cos\theta$ and $\overline{BD} = a\sin\theta$.

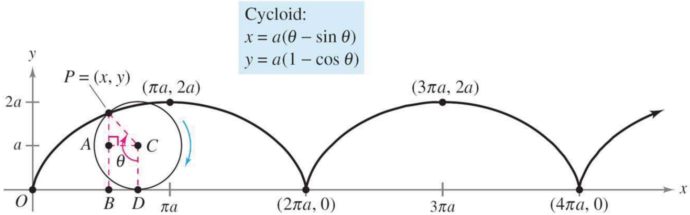


Figure 3: Cycloid: $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

- Because the circle rolls along the x-axis, you know that

$$\overline{OD} = \overset{\frown}{PD} = a\theta.$$

- Furthermore, because $\overline{BA} = \overline{DC} = a$, you have

$$x = \overline{OD} - \overline{BD} = a\theta - a\sin\theta, \qquad y = \overline{BA} + \overline{AP} = a - a\cos\theta.$$

- So, the parametric equations are

$$x = a(\theta - \sin\theta) \quad \text{and} \quad y = a(1 - \cos\theta).\blacksquare$$

- The cycloid in Figure 3 has sharp corners (**cusps**) at the values $x = 2n\pi a$. Notice that the derivatives $x'(\theta)$ and $y'(\theta)$ are both zero at the points for which $\theta = 2n\pi$.

$$\begin{aligned} x(\theta) &= a(\theta - \sin\theta) & \qquad y(\theta) &= a(1 - \cos\theta) \\ x'(\theta) &= a - a\cos\theta & \qquad y'(\theta) &= a\sin\theta \\ x'(2n\pi) &= 0 & \qquad y'(2n\pi) &= 0 \end{aligned}$$

- Between these points, the cycloid is called **smooth**.

Definition 10.2 (Smooth curve)

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called **piecewise smooth** if it is smooth on each subinterval of some partition of I.

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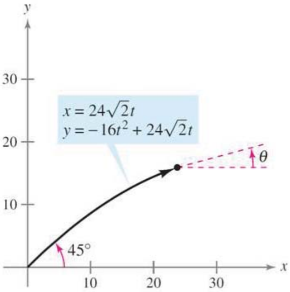
Slope and tangent lines

- The projectile is represented by the parametric equations

$$x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t$$

  You know that these equations enable you to locate the position of the projectile at a given time.
- You also know that the object is initially projected at an angle of 45°.



- But how can you find the angle θ representing the object's direction at some other time t?
- The following theorem answers this question by giving a formula for the slope of the tangent line as a function of t:

Theorem 10.7 (Parametric form of the derivative)

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}, \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0.$$

Example 1 (Differentiation and parametric form)

Find $\mathrm{d}y/\mathrm{d}x$ for the curve given by $x = \sin t$ and $y = \cos t$.

- Because $\mathrm{d}y/\mathrm{d}x$ is a function of t, you can use Theorem 10.7 repeatedly to find higher-order derivatives.
- For instance,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x/\mathrm{d}t} = \frac{-\sec^2 t}{\cos t} = -\sec^3 t$$

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]}{\mathrm{d}x/\mathrm{d}t} \\ &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(-\sec^3 t)}{\cos t} = \frac{-3\sec^2 t(\sec t\tan t)}{\cos t} = -3\sec^4 t\tan t.\end{aligned}$$

Example 2 (Finding slope and concavity)

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point $(2, 3)$.

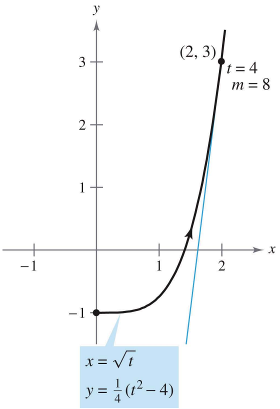


Figure 4: The graph is concave upward at $(2, 3)$, when $t = 4$.

Example 3 (A curve with two tangent lines at a point)

The prolate cycloid given by

$$x = 2t - \pi \sin t \quad \text{and} \quad y = 2 - \pi \cos t$$

crosses itself at the point $(0, 2)$, as shown in Figure 5. Find the equations of both tangent lines at this point.

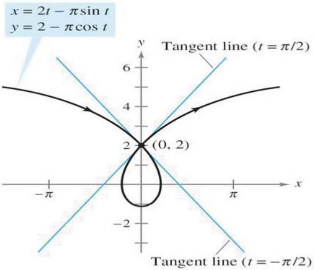


Figure 5: This prolate cycloid has two tangent lines at the point $(0, 2)$.

Remark

If $\mathrm{d}y/\mathrm{d}t = 0$ and $\mathrm{d}x/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$. Similarly, if $\mathrm{d}x/\mathrm{d}t = 0$ and $\mathrm{d}y/\mathrm{d}t \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Arc length

Theorem 10.8 (Arc length in parametric form)

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t.$$

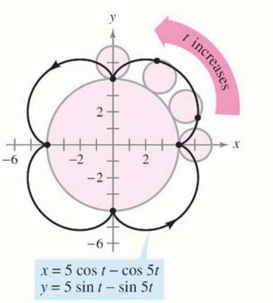
- If a circle rolls along a line, a point on its circumference will trace a path called a **cycloid**.
- If the circle rolls around the circumference of another circle, the path of the point is an **epicycloid**.

Example 4 (Finding arc length)

A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The epicycloid traced by a point on the circumference of the smaller circle is given by

$$x = 5\cos t - \cos 5t \quad \text{and} \quad y = 5\sin t - \sin 5t.$$

Find the distance traveled by point in one trip about the larger circle.



Area of a surface of revolution

Theorem 10.9 (Area of a surface of revolution)

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \leq t \leq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the x-axis: $g(t) \geq 0$
2. $S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$
   Revolution about the y-axis: $f(t) \geq 0$

- These formulas are easy to remember if you think of the differential of arc length as

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

- Then the formulas are written as follows.

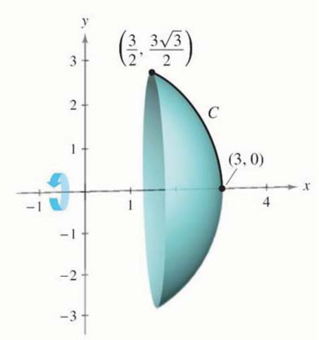
  1. $S = 2\pi \int_a^b g(t)\,\mathrm{d}s$ **2.** $S = 2\pi \int_a^b f(t)\,\mathrm{d}s$

Example 5 (Finding the area of a surface of revolution)

Let C be the arc of the circle

$$x^2 + y^2 = 9$$

from $(3, 0)$ to $(3/2, 3\sqrt{3}/2)$, as shown below. Find the area of the surface formed by revolving C about the x-axis.



Area between the Parametric equation and axis
We know that the area under a curve $y = F(x)$ from a to b is

$$A = \int_a^b F(x)\, dx,$$

where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations

$$x = f(t), \quad y = g(t), \quad t_1 \leq t \leq t_2,$$

then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\, dx = \int_{t_1}^{t_2} g(t) f'(t)\, dt$$

- As t varies from t_1 to t_2, the corresponding values of x vary from $a = f(t_1)$ to $b = f(t_2)$, and we have

$$dx = \frac{dx}{dt} dt = f'(t) dt.$$

- The area under the curve in terms of x and y is

$$A = \int_a^b y \, dx.$$

- By substituting $y = g(t)$ and $dx = f'(t)dt$, the area becomes

$$A = \int_{t_1}^{t_2} g(t) f'(t) \, dt.$$

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Polar coordinates

- You may represent graphs as collections of points (x, y) on the rectangular coordinate system. The corresponding equations for these graphs have been in either rectangular or parametric form.
- In this section you will study a coordinate system called the **polar coordinate system**.
- To form the polar coordinate system in the plane, fix a point O, called the **pole** (or origin), and construct from O an initial ray called the **polar axis**, as shown in Figure 6.

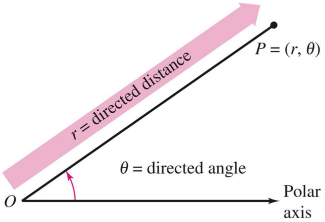


Figure 6: The definition of polar coordinates.

- Then each point P in the plane can be assigned **polar coordinates** (r, θ), as follows

$$r = \text{directed distance from } O \text{ to } P$$
$$\theta = \text{directed angle, counterclockwise from polar axis to segment } \overline{OP}$$

- Figure 7 shows three points on the polar coordinate system.

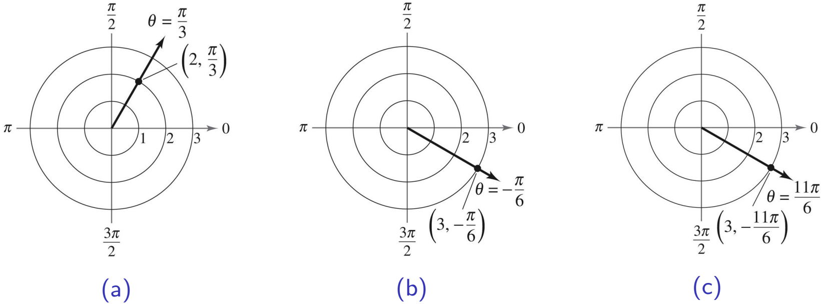


Figure 7: Points $\left(2, \frac{\pi}{3}\right)$, $\left(3, -\frac{\pi}{6}\right)$, $\left(3, \frac{11}{6}\pi\right)$ on the polar coordinates system.

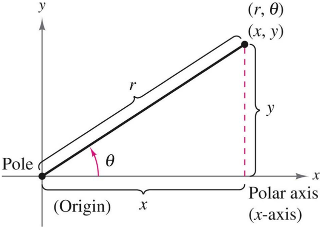
- It is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.
- With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates!
- For instance, coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point [parts (b) and (c) in Figure 7]. Also, because r is a directed distance, coordinates (r, θ) and $(-r, \pi + \theta)$ represent the same point.
- In general, the point (r, θ) can be written as

$$(r, \theta) = (r, \theta + 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta + (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

Coordinate conversion

- To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown below:



- Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$ the definitions of the trigonometric functions imply that $\tan\theta = \frac{y}{x}$, $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

- If $r < 0$, you can show that the same relationships hold.

Theorem 10.10 (Polar-to-rectangular conversion)

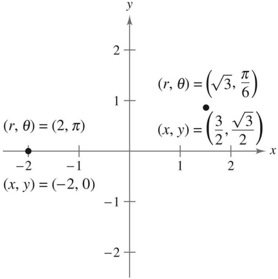
The polar coordinates (r, θ) *of a point are related to the rectangular coordinates* (x, y) *of the point as follows.*

1. $x = r\cos\theta$ *and* $y = r\sin\theta$. **2.** $\tan\theta = \frac{y}{x}$ *and* $r^2 = x^2 + y^2$.

Example 1 (Polar-to-rectangular conversion)

a. Convert the point $(r, \theta) = (2, \pi)$ to rectangular coordinates

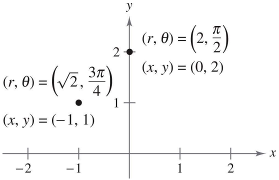
b. Convert the point $(r, \theta) = (\sqrt{3}, \pi/6)$ to rectangular coordinates



Example 2 (Rectangular-to-polar conversion)

a. Convert the point $(x, y) = (-1, 1)$ to polar coordinates

b. Convert the point $(x, y) = (0, 2)$ to polar coordinates



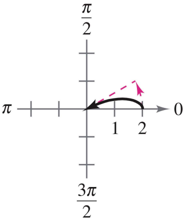
Example 3 (Graphing polar equations)

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

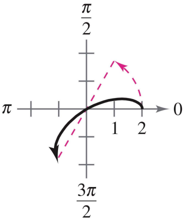
a. $r = 2$ **b.** $\theta = \frac{\pi}{3}$ **c.** $r = \sec\theta$

Example 4 (Sketching a polar graph)

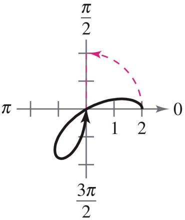
Sketch the graph of $r = 2\cos 3\theta$.



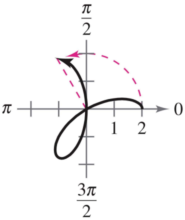
(a) $0 \le \theta \le \pi/6$.



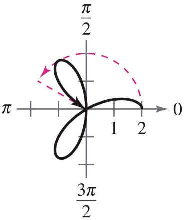
(b) $0 \le \theta \le \pi/3$.



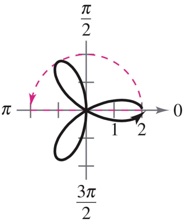
(c) $0 \le \theta \le \pi/2$.



(d) $0 \le \theta \le 2\pi/3$.



(e) $0 \le \theta \le 5\pi/6$.



(f) $0 \le \theta \le \pi$.

Figure 8: Sketching a polar graph.

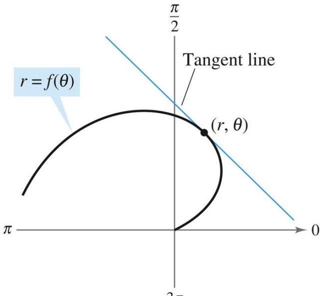
Slope and tangent lines

- To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To find the slope in polar form, use the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

- Using the parametric form of $\mathrm{d}y/\mathrm{d}x$ given in Theorem 10.7, you have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$



Theorem 10.11 (Slope in polar form)

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $\mathrm{d}x/\mathrm{d}\theta \neq 0$ at (r, θ).

- From Theorem 10.11, you can make the following observations.
  1. Solution to $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ yield horizontal tangents, provided that $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$.
  2. Solution to $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$ yield vertical tangents, provided that $\frac{\mathrm{d}y}{\mathrm{d}\theta} \neq 0$.
- If $\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$ are simultaneously 0, no conclusion can be drawn about tangent lines.

Example 5 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \leq \theta \leq \pi$.

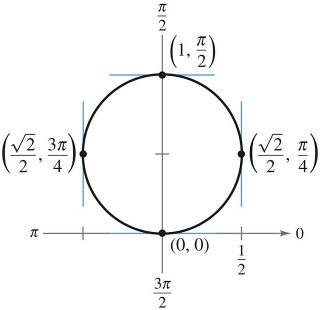


Figure 9: Horizontal and vertical tangent lines of $r = \sin\theta$.

Example 6 (Finding horizontal and vertical tangent lines)

Find the horizontal and vertical tangents to the graph of $r = 2(1 - \cos\theta)$.

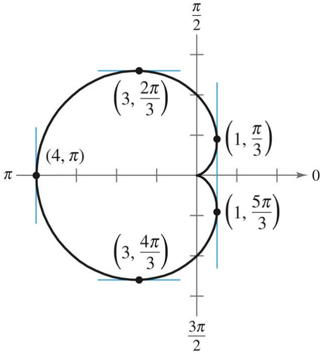


Figure 10: Horizontal and vertical tangent lines of $r = 2(1 - \cos\theta)$.

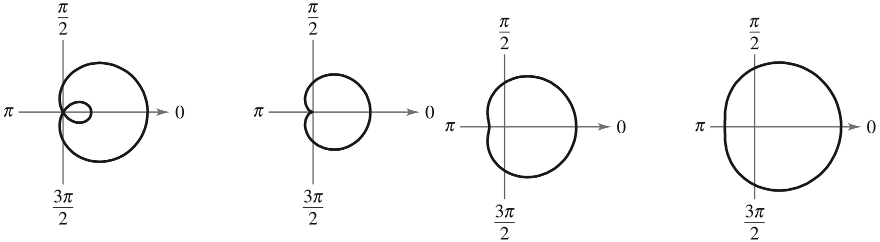
- This graph is called a **cardioid**. Note that both derivatives ($\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$) are 0 when $\theta = 0$.
- Using this information alone, you don't know whether the graph has a horizontal or vertical tangent line at the pole. From Figure 10, however, you can see that the graph has a **cusp** at the pole.

Theorem 10.12 (Tangent lines at the pole)

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Special polar graphs

- Several important types of graphs have equations that are simpler in polar form than in rectangular form.
- For example, the polar equation of a circle having a radius of a and centered at the origin is simply $r = a$. Several other types of graphs that have simpler equations in polar form are shown below.



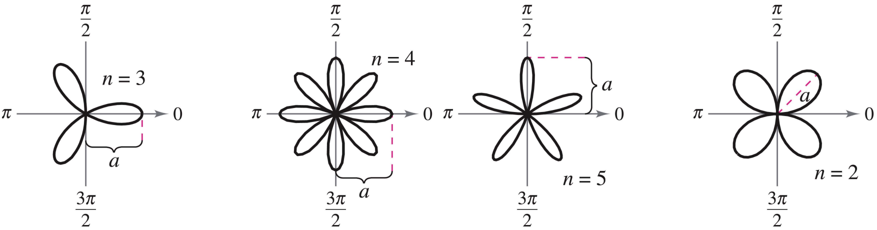
(a) $\frac{a}{b} < 1$.
Limacon with inner loop.

(b) $\frac{a}{b} = 1$.
Cardioid (heart-shaped).

(c) $1 < \frac{a}{b} < 2$.
Dimpled limaçon.

(d) $\frac{a}{b} \geq 2$.
Convex limaçon.

Figure 11: Limacon: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$ $(a > 0, b > 0)$.



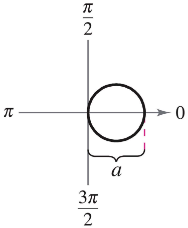
(a) $r = a\cos n\theta$. Rose curve.

(b) $r = a\cos n\theta$. Rose curve.

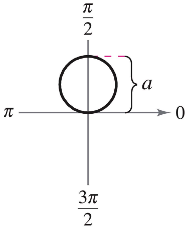
(c) $r = a\sin n\theta$. Rose curve.

(d) $r = a\sin n\theta$. Rose curve.

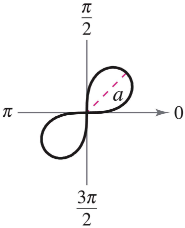
Figure 12: Rose curves: n petals if n is odd, $2n$ petals if n is even ($n \geq 2$).



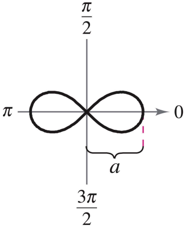
(a) $r = a\cos\theta$.
Circle.



(b) $r = a\sin\theta$.
Circle.



(c) $r^2 = a^2 \sin 2\theta$.
Lemniscate.



(d)
$r^2 = a^2 \cos 2\theta$.
Lemniscate.

Figure 13: Circles and Lemniscate.

Table of Contents

Area of a polar region

- The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area.
- In Figure 14, note that the area of a circular sector of radius r is given by $\frac{1}{2}\theta r^2$ provided θ is measured in radians.

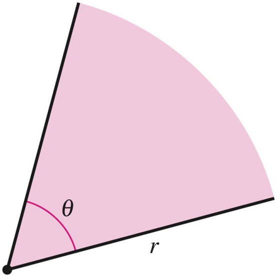


Figure 14: The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.

- Consider the function given by $r = f(\theta)$, where f is continuous and nonnegative on the interval given by $\alpha \le \theta \le \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 15.

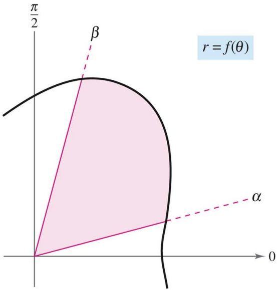


Figure 15: Area in polar coordinates.

- To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = \beta$$

- Then approximate the area of the region by the sum of the areas of the n sectors, as shown in Figure 15.
- Radius of ith sector $= f(\theta_i)$. Central angle of ith sector $= \frac{\beta-\alpha}{n} = \triangle\theta$

$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right) \triangle\theta [f(\theta_i)]^2.$$

- Taking the limit as $n \to \infty$ produces

$$A = \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \triangle\theta = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta.$$

Theorem 10.13 (Area in polar coordinates)

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, \mathrm{d}\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta, \quad 0 < \beta - \alpha \leq 2\pi.$$

Example 1 (Finding the area of a polar region)

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.

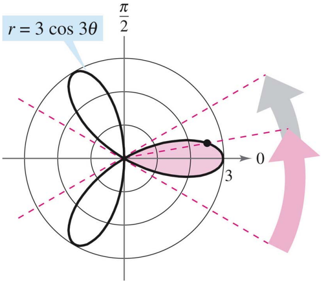


Figure 16: The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$.

Example 2 (Finding the area bounded by a single curve)

Find the area of the region lying between the inner and outer loops of the limacon $r = 1 - 2\sin\theta$.

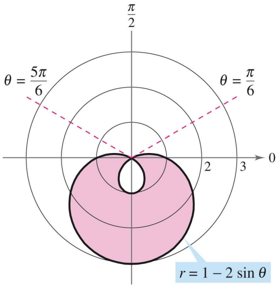


Figure 17: The area between the inner and outer loops is approximately 8.34.

Points of intersection of polar graphs

- Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs.
- For example, consider the points of intersection of the graphs of $r = 1 - 2\cos\theta$ and $r = 1$ as shown in Figure 18.

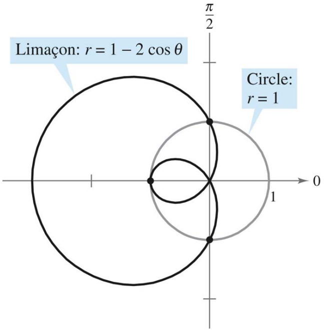


Figure 18: Limacon: $r = 1 - 2\cos\theta$ and three points intersection: $(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$.

- If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

$$r = 1 - 2\cos\theta \qquad 1 = 1 - 2\cos\theta \qquad \cos\theta = 0 \qquad \theta = \frac{\pi}{2},\ \frac{3\pi}{2}$$

- The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$.
- However, from Figure 18 you can see that there is a third point of intersection that did not show up when the two polar equations were solved simultaneously.
- The reason the third point was not found is that it does not occur with the same coordinates in the two graphs!

- On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of $r = 1 - 2\cos\theta$, the point occurs with coordinates $(-1, 0)$.
- You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 19.

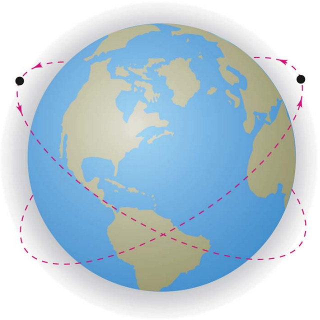


Figure 19: The paths of satellites can cross without causing a collision.

Example 3 (Finding the area of a region between two curves)

Find the area of the region common to the two regions bounded by the following curves.

$$r = -6\cos\theta \qquad \text{Circle} \qquad\qquad r = 2 - 2\cos\theta \qquad \text{Cardioid}$$

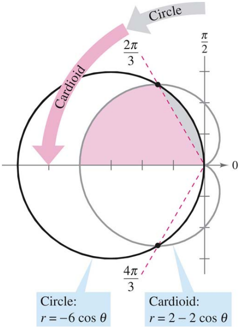


Figure 20: Find the area between circle $r = -6\cos\theta$ and cardioid $r = 2 - 2\cos\theta$.

Arc length in polar form

- The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations.

Theorem 10.14 (Arc length of a polar curve)

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, \mathrm{d}\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta.$$

Example 4 (Finding the length of a polar curve)

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$ as shown in Figure 21.

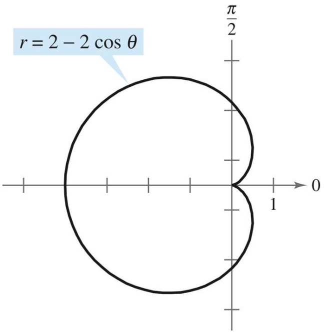


Figure 21: The cardioid $r = 2 - 2\cos\theta$.

Area of a surface of revolution

- The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

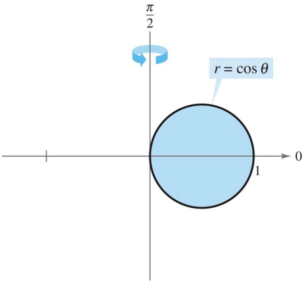
Theorem 10.15 (Area of a surface of revolution)

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line as follows.

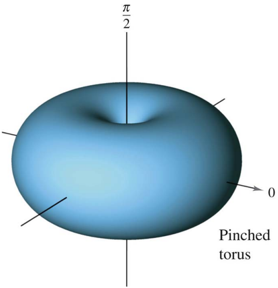
1. $S = 2\pi \int_\alpha^\beta y \,\mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta$
   About the polar axis
2. $S = 2\pi \int_\alpha^\beta x \,\mathrm{d}s = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta$
   About the line $\theta = \frac{\pi}{2}$

Example 5 (Finding the area of a surface of revolution)

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 22.



(a) Circle.



(b) Pinched torus.

Figure 22: Revolving a circle $r = \cos\theta$ around $x = \frac{\pi}{2}$.