# Number Systems

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# Introduction

- A number system defines how a number can be represented using distinct symbols
  - A number can be represented differently in different systems. For example, the two numbers $(2A)_{16}$ and $(52)_8$ both refer to the same quantity, $(42)_{10}$, but their representations are different

- Several number systems have been used in the past and can be categorized into two groups: *positional* and *non-positional* systems. Our main goal is to discuss the positional number systems, but we also give examples of non-positional systems

# Positional Number Systems

- In a positional number system, the position a symbol occupies in the number determines the value it represents
  - In this system, a number represented as:

$$\pm(S_{k-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L})_b$$

has the value of

$$n = \pm S_{k-1} \times b^{k-1} + \cdots + S_1 \times b^1 + S_0 \times b^0 + S_{-1} \times b^{-1} + S_{-L} \times b^{-L}$$

in which $S$ is the set of symbols, $b$ is the *base* (or *radix*) which is equal to the total number of the symbols in the set $S$

  - Notice the radix point (decimal point)

# The decimal system (base 10)

- In this system, the base $b = 10$ and we use ten symbols

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

The symbols in this system are often referred to as *decimal digits* or just *digits*

- A number is written as

$$\pm(S_{k-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L})_{10}$$

- For simplicity, we often drop the parentheses, the base, and the plus sign

$$+(552.23)_{10} \rightarrow 552.23$$

# Integers

- We represent an integer as

$$\pm S_{k-1} \dots S_2 S_1 S_0$$

in which $S_i$ is a digit, $b = 10$ is the base, and $K$ is the number of digits

- The *place values* is the power of the base $(10^0, 10^1, \dots, 10^{K-1})$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $10^{k-1}$ | $10^{k-2}$ | $\cdots$ | $10^2$ | $10^1$ | $10^0$ | Place values |
| $\pm$ | $S_{k-1}$ | $S_{k-2}$ | $\cdots$ | $S_2$ | $S_1$ | $S_0$ | Number |
| | $\downarrow$ | $\downarrow$ | | $\downarrow$ | $\downarrow$ | $\downarrow$ | |

$$N = \pm S_{k-1} \times 10^{k-1} + S_{k-2} \times 10^{k-2} + \cdots + S_2 \times 10^2 + S_1 \times 10^1 + S_0 \times 10^0 \quad \text{Values}$$

# Maximum value and reals

- Sometimes we need to know the maximum value of a decimal integer that can be represented by $K$ digit

$$N_{max} = 10^K - 1$$

- A real (a number with a fractional part) in the decimal system is also familiar. We can represent a real as $\pm S_{k-1} \ldots S_2 S_1 S_0 . S_{-1} S_{-2} \ldots S_{-L}$ and the value is

$$R = \pm \underbrace{S_{K-1} \times 10^{K-1} + \ldots + S_1 \times 10^1 + S_0 \times 10^0}_{\text{Integral part}} + \underbrace{S_{-1} \times 10^{-1} + \ldots + S_{-L} \times 10^{-L}}_{\text{Fractional part}}$$

# The binary system (base 2)

- In this system, the base $b = 2$ and we use only two symbols, $S = \{0, 1\}$. The symbols in this system are often referred to as *binary digits* or *bits*
- We can represent an integer as

$$\pm(S_{k-1} \dots S_2 S_1 S_0)_2$$

in which $S_i$ is a binary digit, $b = 2$ is the base, and $K$ is the number of bits

- What is the corresponding decimal of $(11001)_2$?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $2^{k-1}$ | $2^{k-2}$ | $\bullet\bullet\bullet$ | $2^2$ | $2^1$ | $2^0$ | Place values |
| $\pm$ | $S_{k-1}$ | $S_{k-2}$ | $\bullet\bullet\bullet$ | $S_2$ | $S_1$ | $S_0$ | Number |
| | $\downarrow$ | $\downarrow$ | | $\downarrow$ | $\downarrow$ | $\downarrow$ | |

$$N = \pm\ S_{k-1} \times 2^{k-1} + S_{k-2} \times 2^{k-2} + \bullet\bullet\bullet + S_2 \times 2^2 + S_1 \times 2^1 + S_0 \times 2^0$$ Values

# Maximum value and reals

- The maximum value of a binary integer with $K$ digits is

$$N_{max} = 2^K - 1$$

- A real (a number with a fractional part) in the binary system is represented as $\pm\,(S_{k-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L})_2$ and the value is

$$\begin{array}{cccc} & \text{Integral part} & \bullet & \text{Fractional part} \\ R = \pm & S_{K-1} \times 2^{K-1} \times \dots \times S_1 \times 2^1 \times S_0 \times 2^0 & + & S_{-1} \times 2^{-1} + \dots + S_{-L} \times 2^{-L} \end{array}$$

- What is the corresponding decimal of $(101.11)_2$?

# The hexadecimal system (base 16)

- Base $b = 16$ and we use sixteen symbols to represent a number

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$$

  - The symbols in this system are often referred to as *hexadecimal digits*

- We can represent an integer as

$$\pm(S_{k-1} \ldots S_2 S_1 S_0)_{16}$$

$S_i$ is a hexadecimal digit, $b = 16$ is the base, and $K$ is the number of hexadecimal digits

- What is the corresponding decimal of $(2AE)_{16}$?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $16^{k-1}$ | $16^{k-2}$ | $\cdots$ | $16^2$ | $16^1$ | $16^0$ | Place values |
| $\pm$ | $S_{k-1}$ | $S_{k-2}$ | $\cdots$ | $S_2$ | $S_1$ | $S_0$ | Number |
| | $\downarrow$ | $\downarrow$ | | $\downarrow$ | $\downarrow$ | $\downarrow$ | |

$$N = \pm S_{k-1} \times 16^{k-1} + S_{k-2} \times 16^{k-2} + \cdots + S_2 \times 16^2 + S_1 \times 16^1 + S_0 \times 16^0 \quad \text{Values}$$

# Maximum value and reals

- The maximum value of a hexadecimal integer with $K$ digits is
$$N_{max} = 16^K - 1$$
- Although a real number can be also represented in the hexadecimal system, it is not very common

# The octal system (base 8)

- In this system, the base $b = 8$ and we use eight symbols to represent a number. The set of symbols is

$$S = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

  - The symbols in this system are often referred to as *octal digits*

- We can represent an integer as

$$\pm(S_{k-1} \ldots S_2 S_1 S_0)_8$$

$S_i$ is a octal digit, $b = 8$ is the base, and $K$ is the number of octal digits

- What is the corresponding decimal of $(1256)_8$?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $8^{k-1}$ | $8^{k-2}$ | $\cdots$ | $8^2$ | $8^1$ | $8^0$ | Place values |
| $\pm$ | $S_{k-1}$ | $S_{k-2}$ | $\cdots$ | $S_2$ | $S_1$ | $S_0$ | Number |

$$N = \pm S_{k-1} \times 8^{k-1} + S_{k-2} \times 8^{k-2} + \cdots + S_2 \times 8^2 + S_1 \times 8^1 + S_0 \times 8^0 \quad \text{Values}$$

# Maximum value and reals

- The maximum value of an octal integer with $K$ digits is
$$N_{max} = 2^8 - 1$$
- Although a real number can be also represented in the octal system, it is not very common

# Summary of the four positional systems

**Table 2.1** Summary of the four positional number systems

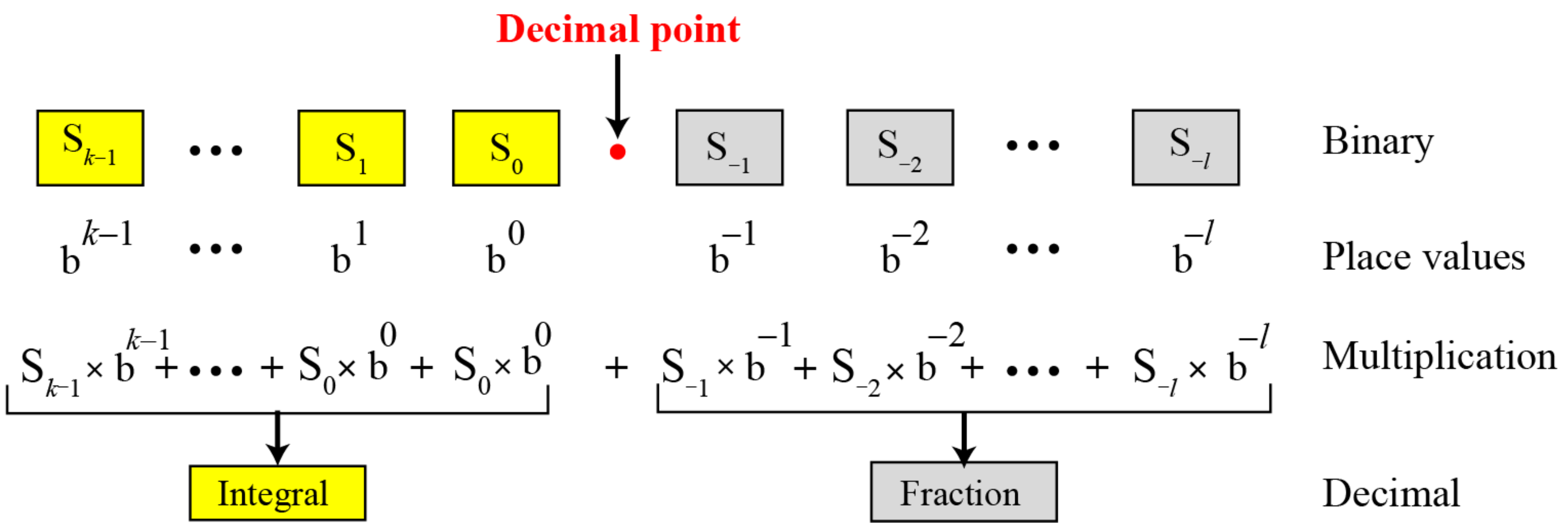
| *System* | *Base* | *Symbols* | *Examples* |
| --- | --- | --- | --- |
| Decimal | 10 | 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 | 2345.56 |
| Binary | 2 | 0, 1 | $(1001.11)_2$ |
| Octal | 8 | 0, 1, 2, 3, 4, 5, 6, 7 | $(156.23)_8$ |
| Hexadecimal | 16 | 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F | $(A2C.A1)_{16}$ |

**Table 2.2** Comparison of numbers in the four systems

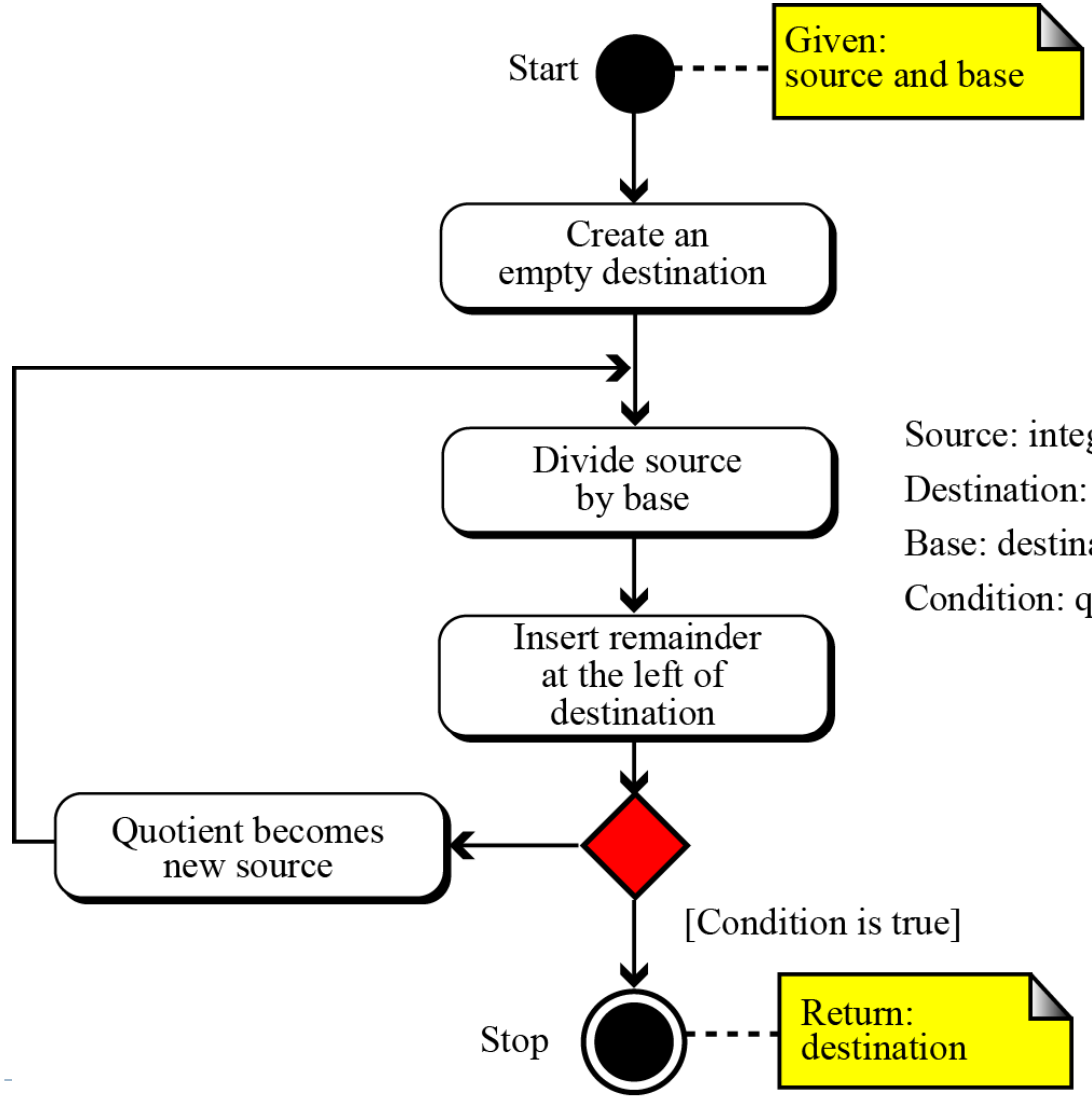
| *Decimal* | *Binary* | *Octal* | *Hexadecimal* |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| 2 | 10 | 2 | 2 |
| 3 | 11 | 3 | 3 |
| 4 | 100 | 4 | 4 |
| 5 | 101 | 5 | 5 |
| 6 | 110 | 6 | 6 |
| 7 | 111 | 7 | 7 |
| 8 | 1000 | 10 | 8 |
| 9 | 1001 | 11 | 9 |
| 10 | 1010 | 12 | A |
| 11 | 1011 | 13 | B |
| 12 | 1100 | 14 | C |
| 13 | 1101 | 15 | D |
| 14 | 1110 | 16 | E |
| 15 | 1111 | 17 | F |

# Conversion - Any base to decimal conversion

- We need to know how to convert a number in one system to the equivalent number in another system
  - What is the corresponding decimal of $(110.11)_2$, $(1A.23)_{16}$, $(23.17)_8$?



# Decimal to any base - integral part




Source: integral part of decimal number

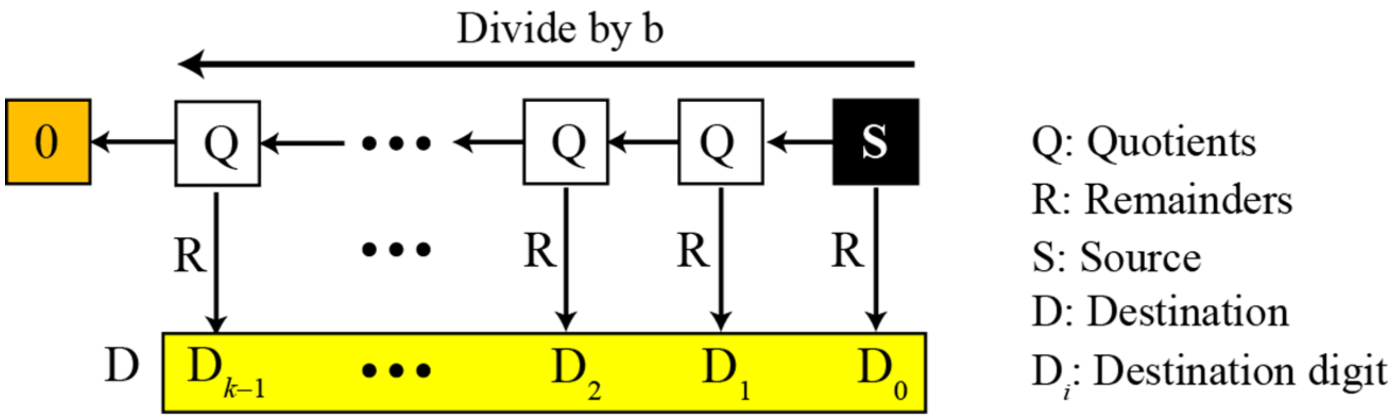
Destination: integral part of converted number

Base: destination base

Condition: quotient is zero

# Decimal to any base - integral part

- Try to convert 35 in decimal to binary

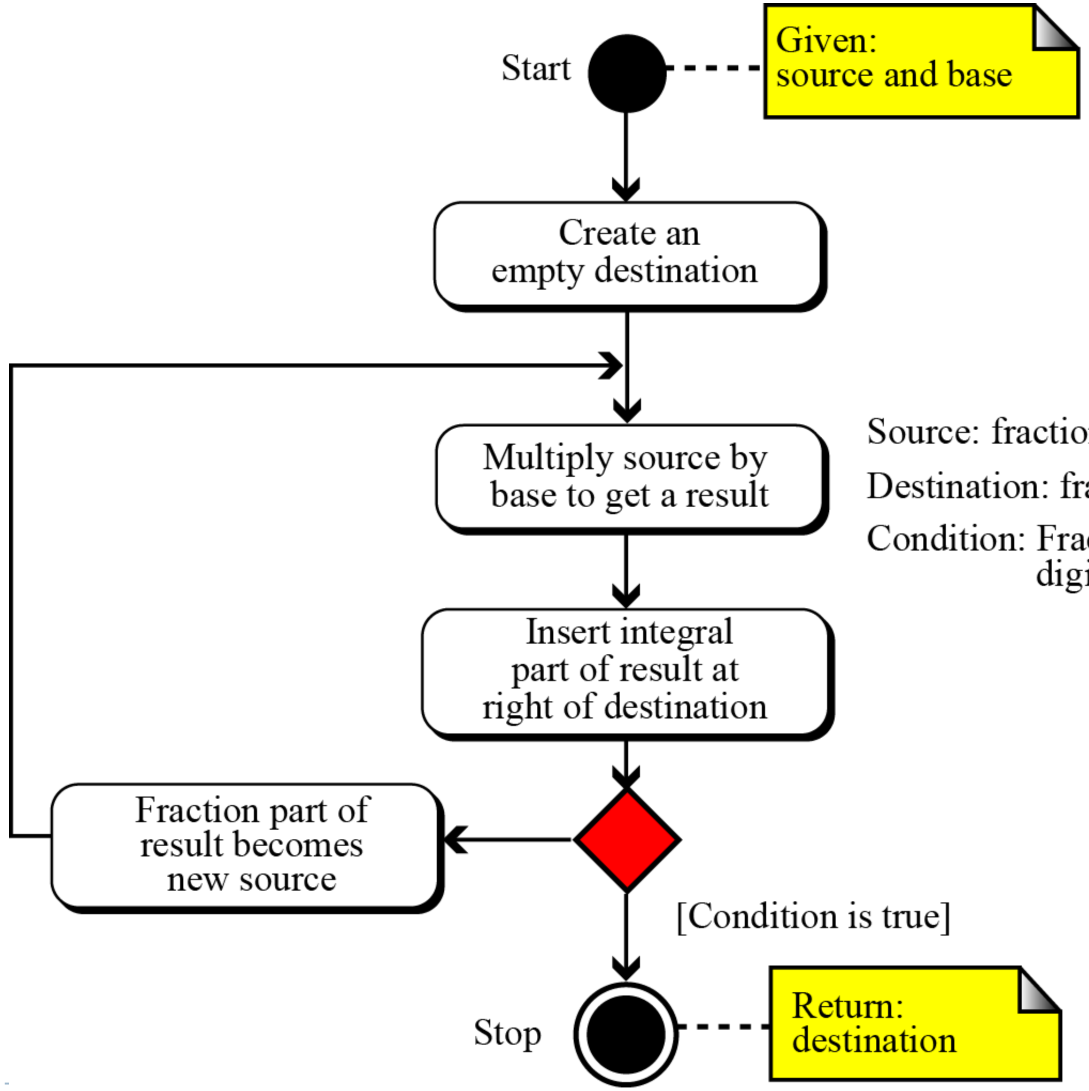


# Decimal to any base - integral part

- Try to convert 126 in decimal to octal system

- Try to convert 126 in decimal to hexadecimal system

# Decimal to any base - fractional part



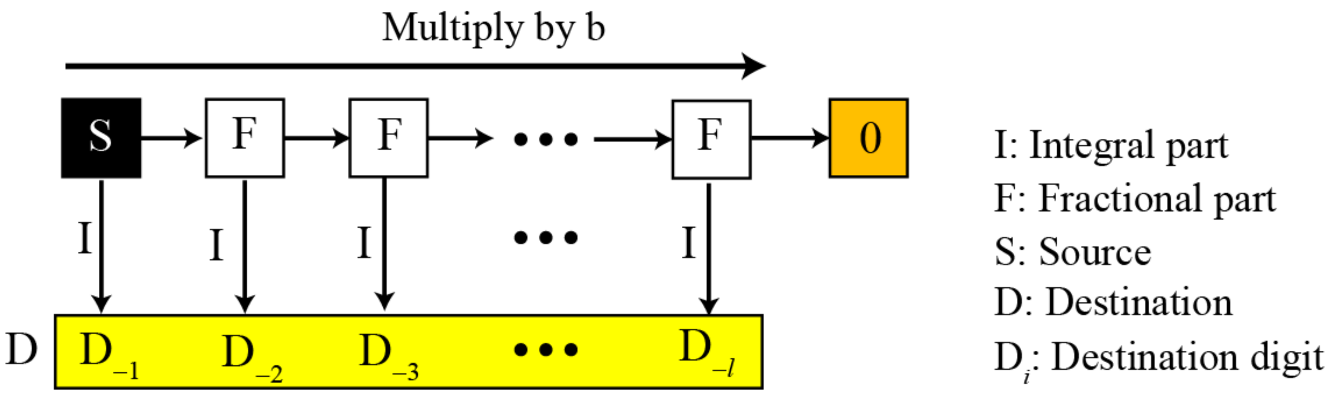

Source: fraction part of decimal number

Destination: fraction part of converted number

Condition: Fraction part is zero or destination digits are enough

# Decimal to any base - fractional part

- Try to convert 0.625 in decimal to binary




Note:
The fraction may never become zero.
Stop when enough digits have been created.

# Decimal to any base - fractional part

- Try to convert 0.634 in decimal to octal using a maximum of four digits

- Try to convert 178.6 in decimal to hexadecimal using only one digit to the right of the decimal point

# Decimal to any base

- An alternative method for converting a small decimal integer (usually less than 256) to binary is to break the number as the sum of numbers that are equivalent to the binary place values shows:

| Place values | $2^7$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|---|---|---|---|---|---|---|---|---|
| Decimal equivalent | 128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 |

- Using the table, we can convert 165 to $(10100101)_2$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Decimal 165 = | 128 | + | 0 | + | 32 | + | 0 | + | 0 | + | 4 | + | 0 | + | 1 |
| Binary | 1 | | 0 | | 1 | | 0 | | 0 | | 1 | | 0 | | 1 |

# Decimal to any base

- A similar method can be used to convert a decimal fraction to binary when the denominator is a power of two:

| Place values | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ |
|---|---|---|---|---|---|---|---|
| Decimal equivalent | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{128}$ |

- Using this table, we convert $\frac{27}{64}$ to $(0.011011)_2$

| Decimal $\frac{27}{64}$ = | 0 | + | $\frac{1}{4}$ | + | $\frac{1}{8}$ | + | 0 | + | $\frac{1}{32}$ | + | $\frac{1}{64}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Binary | 0 | | 1 | | 1 | | 0 | | 1 | | 1 |

# Number of digits

- In a positional number system with base $b$, we can always find the number of digits of an integer using the relation

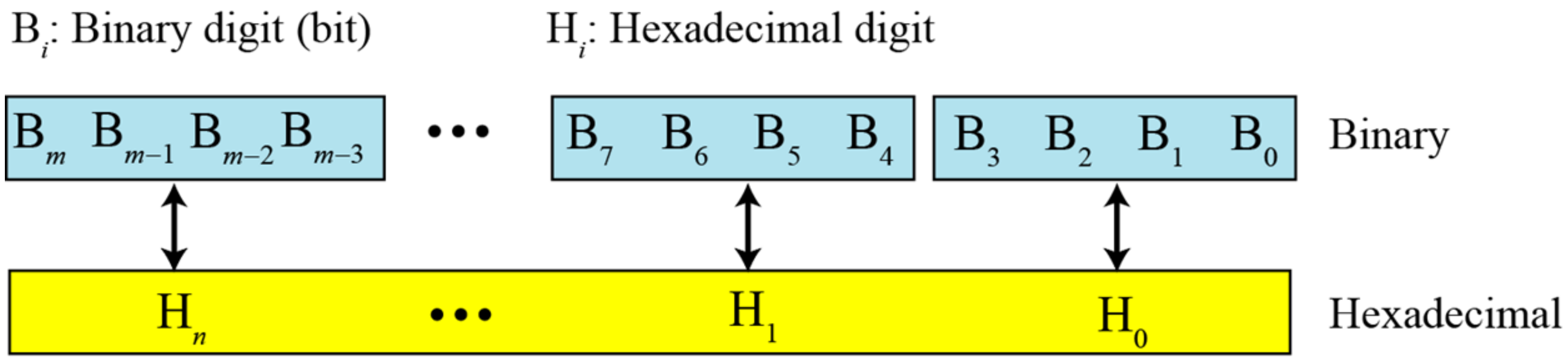
$$K = \lceil \log_b N \rceil$$

Where N is the value of the integer

- For example, try to find the required number of digits in the decimal number 234 in all four systems

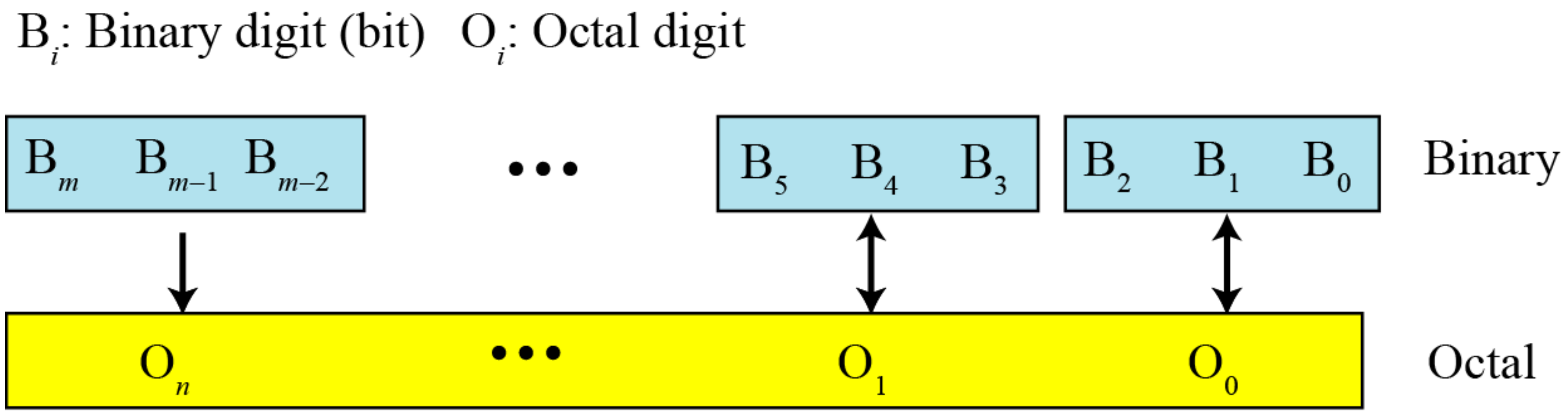
# Binary–hexadecimal conversion

- Try to show the hexadecimal equivalent of the binary number $(10011100010)_2$ and the binary equivalent of $(24C)_{16}$



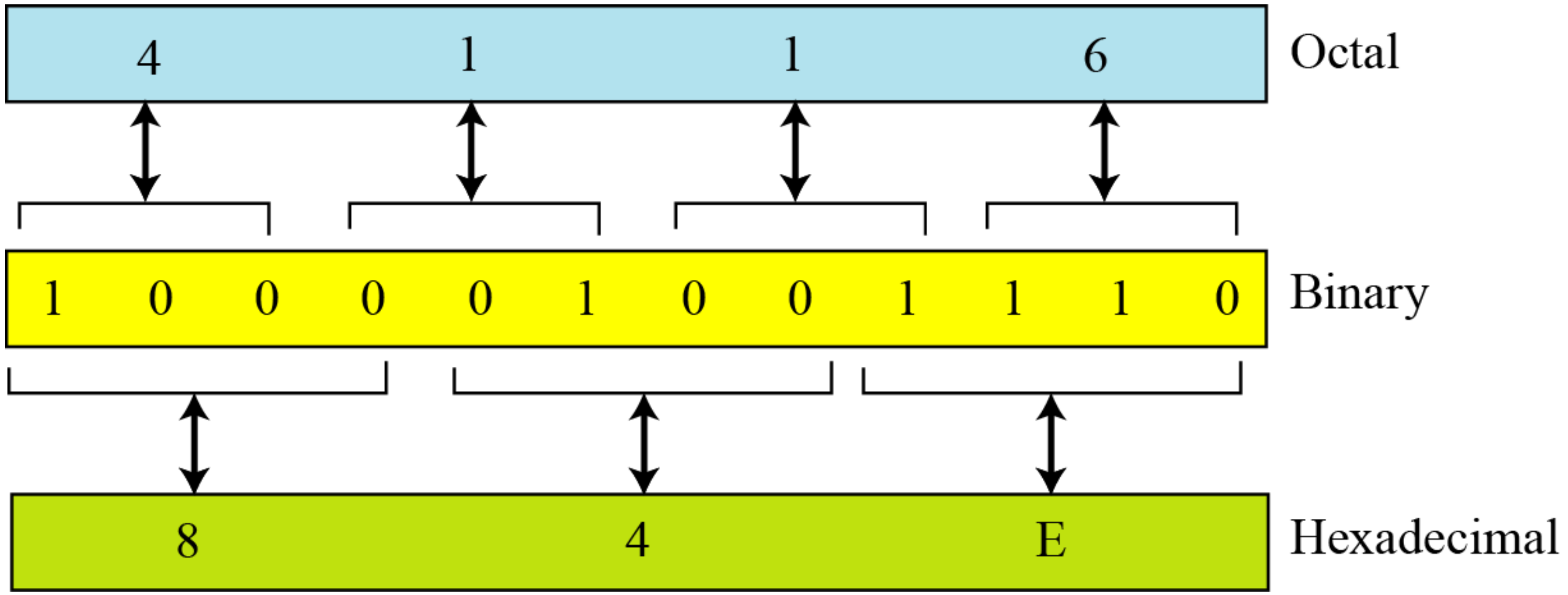
# Binary–octal conversion

- Try to show the octal equivalent of the binary number $(101110010)_2$ and the binary equivalent of $(24)_8$



# Octal–hexadecimal conversion

- We can use the binary system as the intermediate system



# Number of digits from $b_1$ to $b_2$ system

- In general, assume that we are using $K$ digits in base $b_1$ system
  - The maximum number we can represent in this source system is $b_1^K - 1$
  - The maximum number we can represent in  the destination system is $b_2^x - 1$
  - Therefore, $b_2^x - 1 \geq b_1^K - 1 \rightarrow x \geq K \times (\frac{\log b_1}{\log b_2})$ or $x = \left\lceil K \times (\frac{\log b_1}{\log b_2}) \right\rceil$
- Try to find the minimum number of binary digits required to store decimal integers with a maximum of six digits

# Non-positional Number Systems

- A nonpositional number system still uses a limited number of symbols in which each symbol has a value
  - However, the position a symbol occupies in the number normally bears no relation to its value—*the value of each symbol is fixed*
  - In this system, a number is represented as

$$S_{k-1} \ldots S_2 S_1 S_0 . S_{-1} S_{-2} \ldots S_{-L}$$

and it usually has the value

$$n = \pm \underbrace{S_{K-1} + \ldots + S_1 + S_0}_{\text{Integral part}} + \underbrace{S_{-1} + S_{-2} + \ldots + S_{-L}}_{\text{Fractional part}}$$

# Roman number system

- This number system has a set of symbols $S = \{I, V, X, L, C, D, M\}$. The corresponding values are

**Table 2.3** Values of symbols in the Roman number system

| *Symbol* | *I* | *V* | *X* | *L* | *C* | *D* | *M* |
|---|---|---|---|---|---|---|---|
| Value | 1 | 5 | 10 | 50 | 100 | 500 | 1000 |

| | | | | |
|---|---|---|---|---|
| III | → | 1 + 1 + 1 | = | 3 |
| IV | → | 5 – 1 | = | 4 |
| VIII | → | 5 + 1 + 1 + 1 | = | 8 |
| XVIII | → | 10 + 5 + 1 + 1 + 1 | = | 18 |
| XIX | → | 10 + (10 – 1) | = | 19 |
| LXXII | → | 50 + 10 + 10 + 1 + 1 | = | 72 |
| CI | → | 100 + 1 | = | 101 |
| MMVII | → | 1000 + 1000 + 5 + 1 + 1 | = | 2007 |
| MDC | → | 1000 + 500 + 100 | = | 1600 |