



# Number Systems

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# Introduction

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- ▶ A number system defines how a number can be represented using distinct symbols
  - ▶ A number can be represented differently in different systems. For example, the two numbers  $(2A)_{16}$  and  $(52)_8$  both refer to the same quantity,  $(42)_{10}$ , but their representations are different
- ▶ Several number systems have been used in the past and can be categorized into two groups: *positional* (位置) and *non-positional* (非位置) systems. Our main goal is to discuss the positional number systems, but we also give examples of non-positional systems

# Positional Number Systems

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- ▶ In a positional number system, the position a symbol occupies in the number determines the value it represents

- ▶ In this system, a number is represented as:

$$\pm(S_{k-1} \dots S_2 S_1 S_0 \cdot S_{-1} S_{-2} \dots S_{-L})_b$$

has the value of

$$n = \pm S_{k-1} \times b^{k-1} + \dots + S_1 \times b^1 + S_0 \times b^0 + S_{-1} \times b^{-1} + \dots + S_{-L} \times b^{-L}$$

in which  $S$  is the set of symbols,  $b$  is the *base* (基底) (or *radix*) which is equal to the total number of the symbols in the set  $S$

- ▶ Notice the radix point (decimal point)

## The decimal system (十進位系統) (base 10)

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- ▶ In this system, the base  $b = 10$  and we use ten symbols

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

The symbols in this system are often referred to as *decimal digits* or just *digits*

- ▶ A number is written as

$$\pm(S_{k-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L})_{10}$$

- ▶ For simplicity, we often drop the parentheses, the base, and the plus sign

$$+(552.23)_{10} \rightarrow 552.23$$

# Integers

- ▶ We represent an integer as

$$\pm S_{k-1} \dots S_2 S_1 S_0$$

in which  $S_i$  is a digit,  $b = 10$  is the base, and  $K$  is the number of digits

- ▶ The *place values* (位値) is the power of the base ( $10^0, 10^1, \dots, 10^{K-1}$ )

	$10^{k-1}$	$10^{k-2}$	$\dots$	$10^2$	$10^1$	$10^0$	Place values
$\pm$	$S_{k-1}$	$S_{k-2}$	$\dots$	$S_2$	$S_1$	$S_0$	Number
	↓	↓		↓	↓	↓	
$N =$	$\pm S_{k-1} \times 10^{k-1}$	$+ S_{k-2} \times 10^{k-2}$	$+ \dots$	$+ S_2 \times 10^2$	$+ S_1 \times 10^1$	$+ S_0 \times 10^0$	Values

## Maximum value and reals

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- ▶ Sometimes we need to know the maximum value of a decimal integer that can be represented by  $K$  digit

$$N_{max} = 10^K - 1$$

- ▶ A real (a number with a fractional part) in the decimal system is also familiar. We can represent a real as  $\pm S_{K-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L}$  and the value is

$$R = \pm \underbrace{S_{K-1} \times 10^{K-1} + \dots + S_1 \times 10^1 + S_0 \times 10^0}_{\text{Integral part}} + \underbrace{S_{-1} \times 10^{-1} + \dots + S_{-L} \times 10^{-L}}_{\text{Fractional part}}$$



## The binary system (二進位系統) (base 2)

- ▶ In this system, the base  $b = 2$  and we use only two symbols,  $S = \{0, 1\}$ . The symbols in this system are often referred to as *binary digits* or *bits*
- ▶ We can represent an integer as
$$\pm(S_{k-1} \dots S_2 S_1 S_0)_2$$
in which  $S_i$  is a binary digit,  $b = 2$  is the base, and  $K$  is the number of bits
- ▶ What is the corresponding decimal of  $(11001)_2$ ?

	$2^{k-1}$	$2^{k-2}$	$\dots$	$2^2$	$2^1$	$2^0$	Place values
$\pm$	$S_{k-1}$	$S_{k-2}$	$\dots$	$S_2$	$S_1$	$S_0$	Number
	↓	↓		↓	↓	↓	
$N =$	$\pm S_{k-1} \times 2^{k-1}$	$+ S_{k-2} \times 2^{k-2}$	$+ \dots$	$+ S_2 \times 2^2$	$+ S_1 \times 2^1$	$+ S_0 \times 2^0$	Values

## Maximum value and reals

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- ▶ The maximum value of a binary integer with  $K$  digits is

$$N_{max} = 2^K - 1$$

- ▶ A real (a number with a fractional part) in the binary system is represented as  $\pm (S_{k-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L})_2$  and the value is

$$R = \pm \begin{array}{c} \text{Integral part} \\ S_{K-1} \times 2^{K-1} \times \dots \times S_1 \times 2^1 \times S_0 \times 2^0 \end{array} \bullet \begin{array}{c} \text{Fractional part} \\ S_{-1} \times 2^{-1} + \dots + S_{-L} \times 2^{-L} \end{array} +$$

- ▶ What is the corresponding decimal of  $(101.11)_2$ ?



# The hexadecimal system (十六進位系統) (base 16)

- ▶ Base  $b = 16$  and we use sixteen symbols to represent a number

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$$

- ▶ The symbols in this system are often referred to as *hexadecimal digits*
- ▶ We can represent an integer as
$$\pm(S_{k-1} \dots S_2 S_1 S_0)_{16}$$
 $S_i$  is a hexadecimal digit,  $b = 16$  is the base, and  $K$  is the number of hexadecimal digits
- ▶ What is the corresponding decimal of  $(2AE)_{16}$ ?

$16^{k-1}$	$16^{k-2}$	$\dots$	$16^2$	$16^1$	$16^0$	Place values	
$\pm$	$S_{k-1}$	$S_{k-2}$	$\dots$	$S_2$	$S_1$	$S_0$	Number
	↓	↓		↓	↓	↓	
$N = \pm S_{k-1} \times 16^{k-1} + S_{k-2} \times 16^{k-2} + \dots + S_2 \times 16^2 + S_1 \times 16^1 + S_0 \times 16^0$							Values

## Maximum value and reals

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- ▶ The maximum value of a hexadecimal integer with  $K$  digits is

$$N_{max} = 16^K - 1$$

- ▶ Although a real number can be also represented in the hexadecimal system, it is not very common

## The octal system (八進位系統) (base 8)

- ▶ In this system, the base  $b = 8$  and we use eight symbols to represent a number. The set of symbols is

$$S = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

- ▶ The symbols in this system are often referred to as *octal digits*
- ▶ We can represent an integer as

$$\pm(S_{k-1} \dots S_2 S_1 S_0)_8$$

$S_i$  is a octal digit,  $b = 8$  is the base, and  $K$  is the number of octal digits

- ▶ What is the corresponding decimal of  $(1256)_8$ ?

	$8^{k-1}$	$8^{k-2}$	$\dots$	$8^2$	$8^1$	$8^0$	Place values
$\pm$	$S_{k-1}$	$S_{k-2}$	$\dots$	$S_2$	$S_1$	$S_0$	Number
	↓	↓		↓	↓	↓	
$N =$	$\pm S_{k-1} \times 8^{k-1}$	$+ S_{k-2} \times 8^{k-2}$	$+ \dots +$	$S_2 \times 8^2$	$+ S_1 \times 8^1$	$+ S_0 \times 8^0$	Values

## Maximum value and reals

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- ▶ The maximum value of an octal integer with  $K$  digits is

$$N_{max} = 2^8 - 1$$

- ▶ Although a real number can be also represented in the octal system, it is not very common

# Summary of the four positional systems

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**Table 2.1** Summary of the four positional number systems

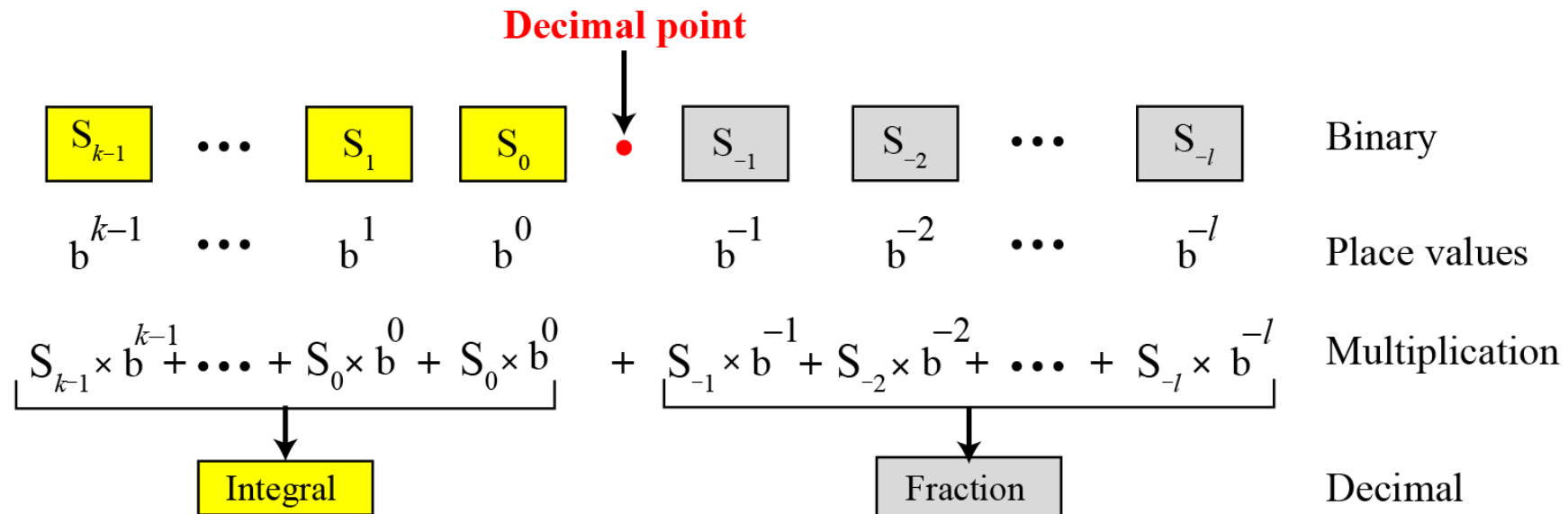
<i>System</i>	<i>Base</i>	<i>Symbols</i>	<i>Examples</i>
Decimal	10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	2345.56
Binary	2	0, 1	$(1001.11)_2$
Octal	8	0, 1, 2, 3, 4, 5, 6, 7	$(156.23)_8$
Hexadecimal	16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F	$(A2C.A1)_{16}$

**Table 2.2** Comparison of numbers in the four systems

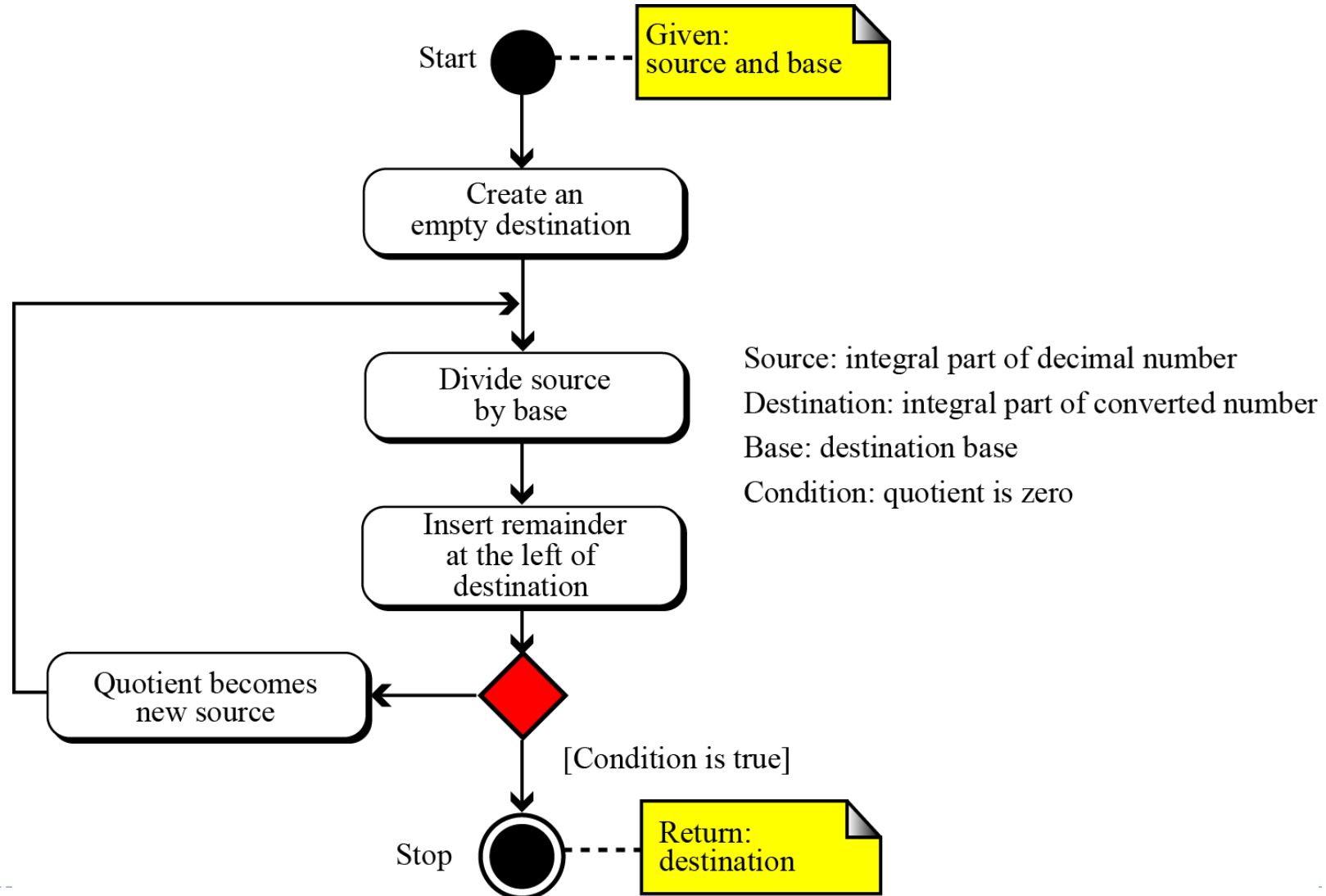
<i>Decimal</i>	<i>Binary</i>	<i>Octal</i>	<i>Hexadecimal</i>
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

# Conversion - Any base to decimal conversion

- ▶ We need to know how to convert a number in one system to the equivalent number in another system
  - ▶ What is the corresponding decimal of  $(110.11)_2$ ,  $(1A.23)_{16}$ ,  $(23.17)_8$ ?



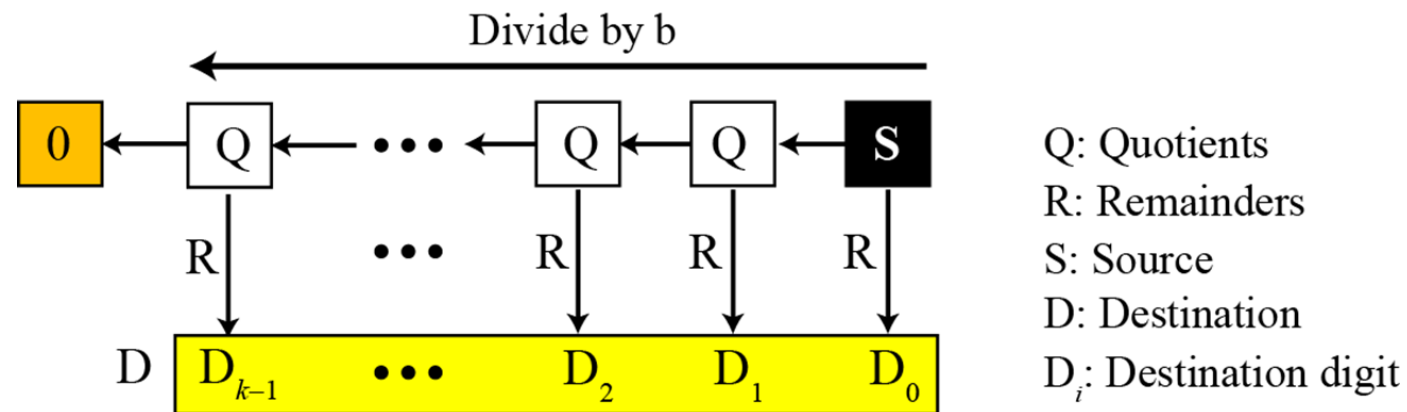
# Decimal to any base - integral part





# Decimal to any base - integral part

- ▶ Try to convert 35 in decimal to binary

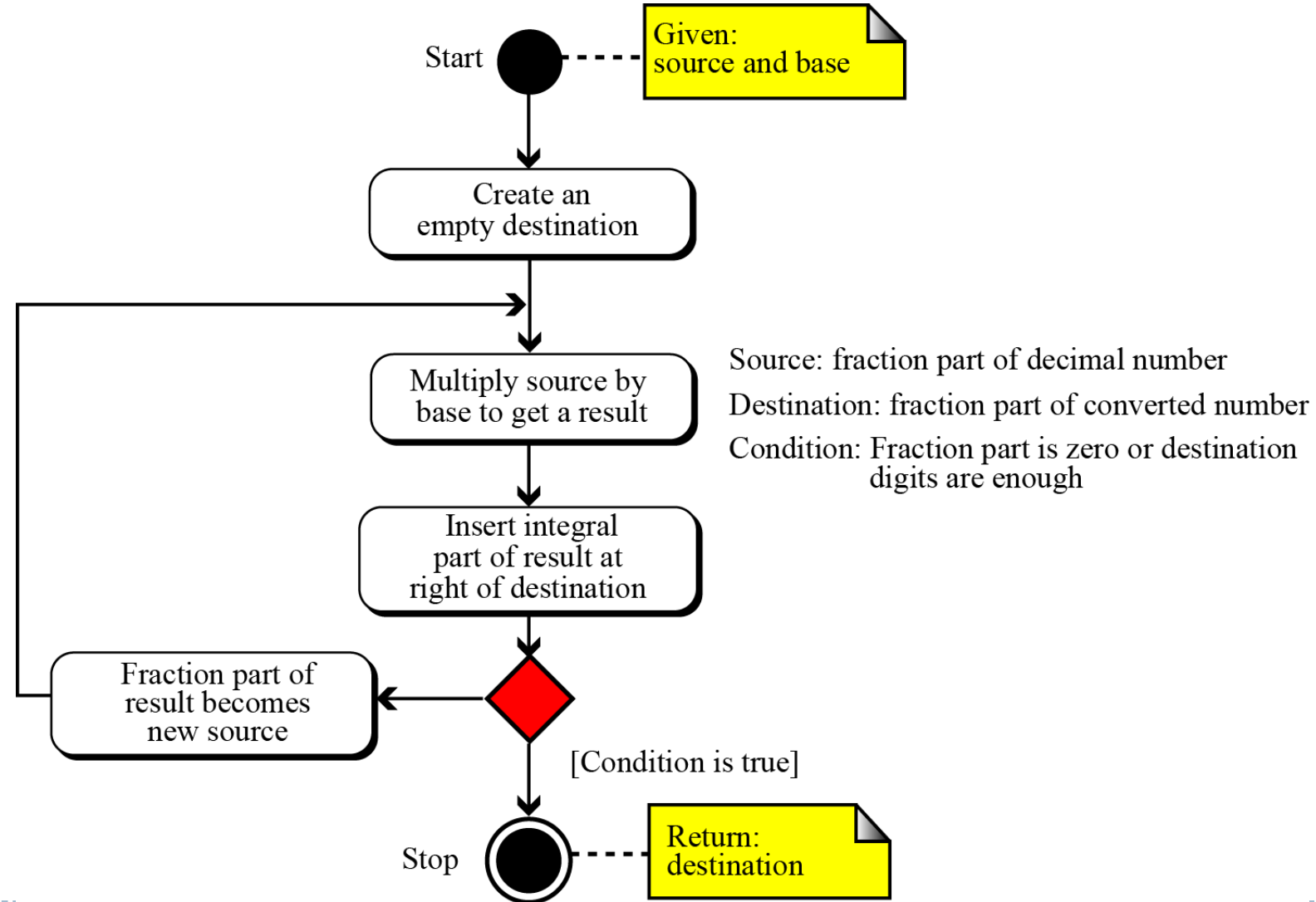


## Decimal to any base - integral part

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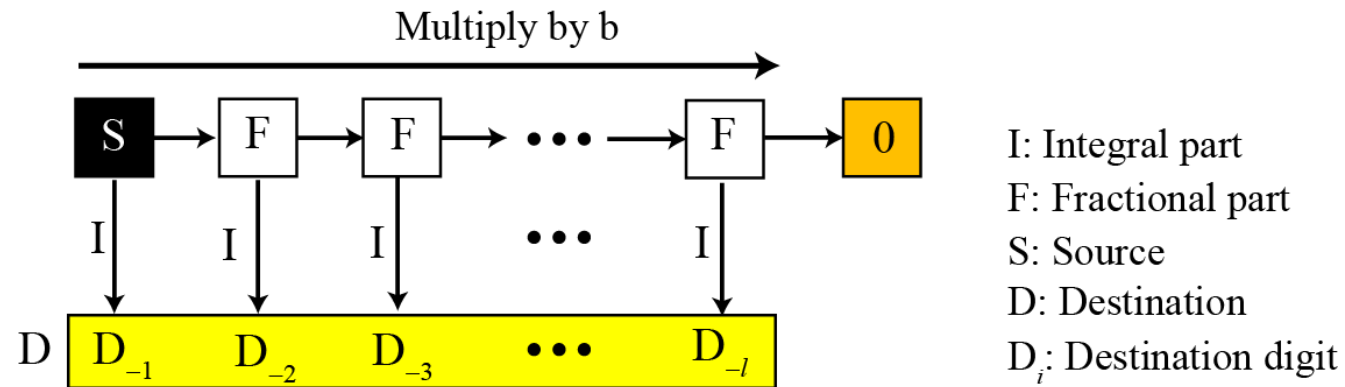
- ▶ Try to convert 126 in decimal to octal system
  
  
  
  
  
  
  
  
  
  
- ▶ Try to convert 126 in decimal to hexadecimal system

# Decimal to any base - fractional part



# Decimal to any base - fractional part

- Try to convert 0.625 in decimal to binary



Note:

The fraction may never become zero.

Stop when enough digits have been created.

## Decimal to any base - fractional part

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- ▶ Try to convert 0.634 in decimal to octal using a maximum of four digits
  
  
  
  
  
  
  
  
  
  
- ▶ Try to convert 178.6 in decimal to hexadecimal using only one digit to the right of the decimal point

## Decimal to any base

- ▶ An alternative method for converting a small decimal integer (usually less than 256) to binary is to break the number as the sum of numbers that are equivalent to the binary place values shows:

Place values	$2^7$	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
Decimal equivalent	128	64	32	16	8	4	2	1

- ▶ Using the table, we can convert 165 to  $(10100101)_2$

Decimal 165 =	128	+	0	+	32	+	0	+	0	+	4	+	0	+	1
Binary	1		0		1		0		0		1		0		1

## Decimal to any base

- ▶ A similar method can be used to convert a decimal fraction to binary when the denominator is a power of two:

Place values	$2^{-1}$	$2^{-2}$	$2^{-3}$	$2^{-4}$	$2^{-5}$	$2^{-6}$	$2^{-7}$
Decimal equivalent	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$

- ▶ Using this table, we convert  $\frac{27}{64}$  to  $(0.011011)_2$

Decimal $\frac{27}{64} =$	0	+	$\frac{1}{4}$	+	$\frac{1}{8}$	+	0	+	$\frac{1}{32}$	+	$\frac{1}{64}$
Binary	0		1		1		0		1		1

## Number of digits

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- ▶ In a positional number system with base  $b$ , we can always find the number of digits of an integer using the relation

$$K = \lceil \log_b N \rceil$$

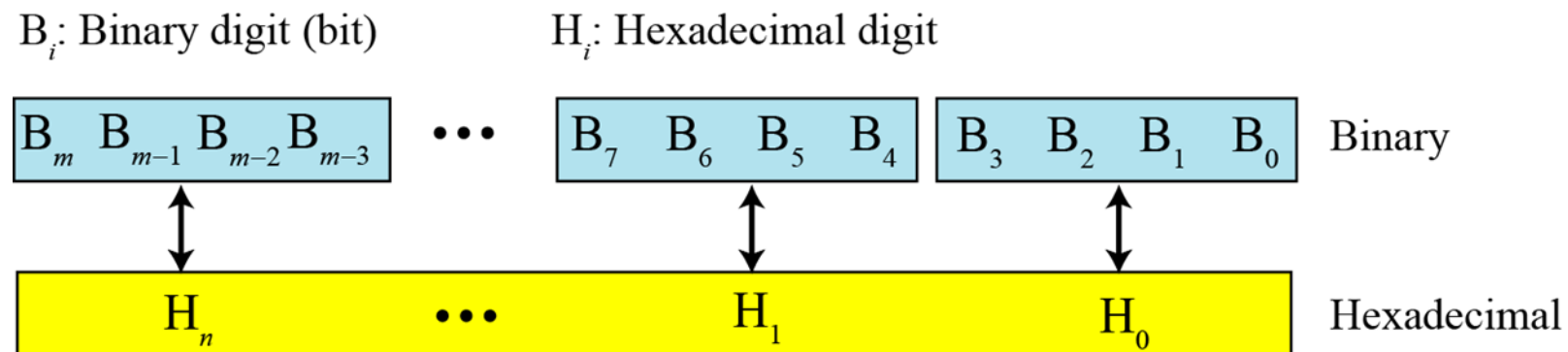
Where  $N$  is the value of the integer

- ▶ For example, try to find the required number of digits in the decimal number 234 in all four systems



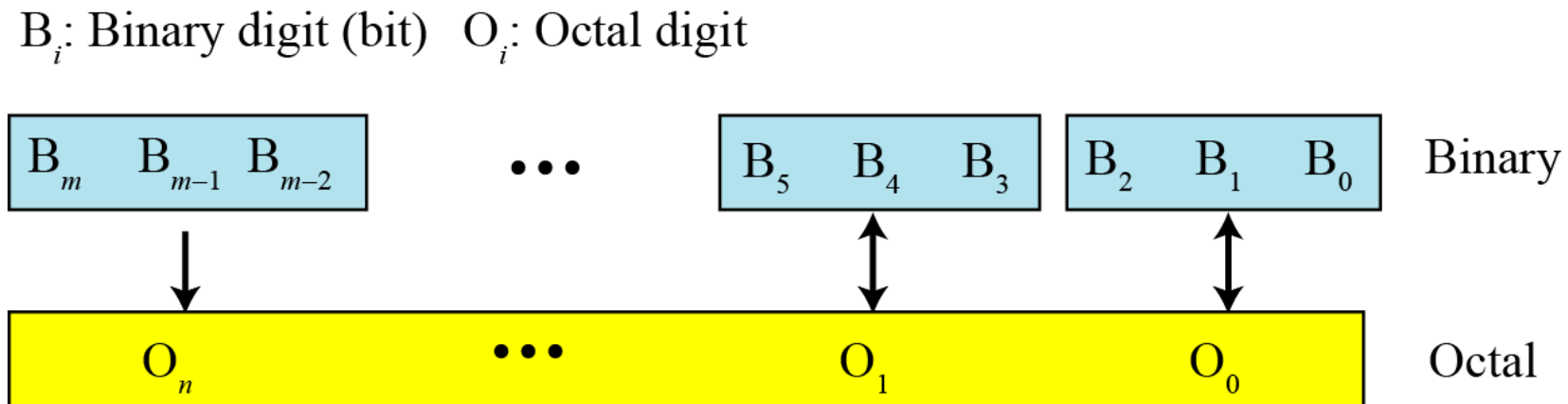
# Binary–hexadecimal conversion

- ▶ Try to show the hexadecimal equivalent of the binary number  $(10011100010)_2$  and the binary equivalent of  $(24C)_{16}$



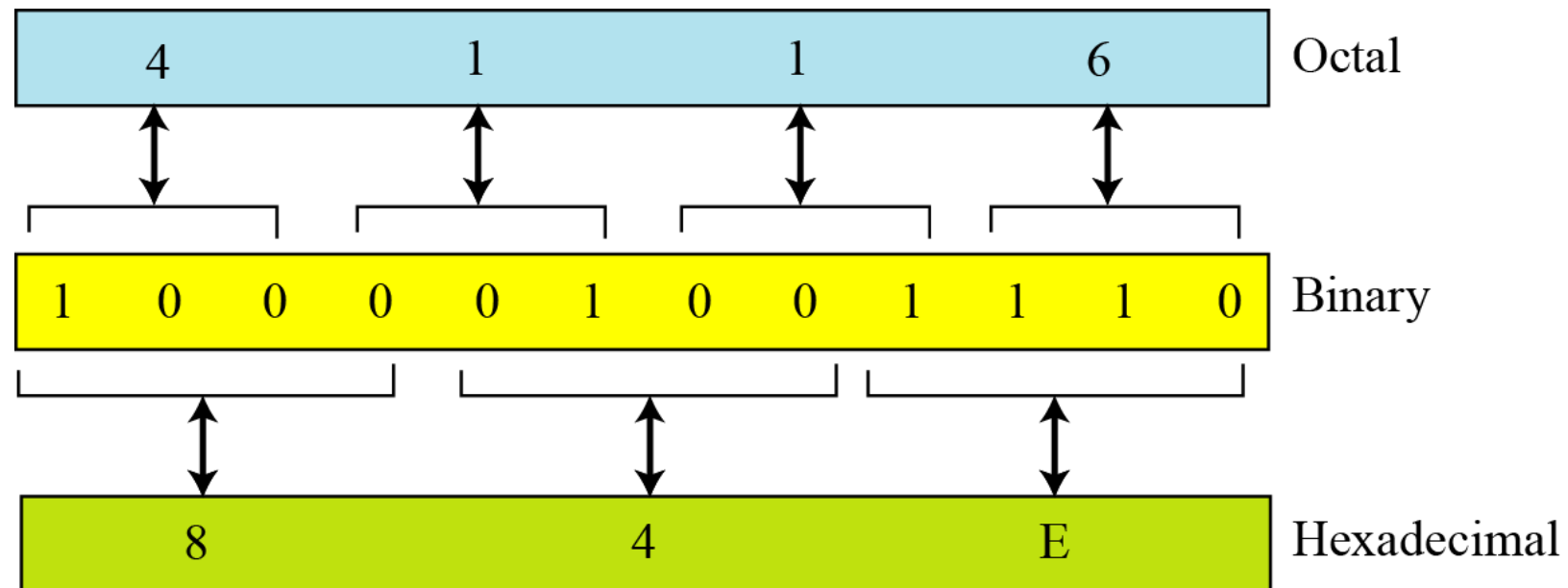
## Binary–octal conversion

- Try to show the octal equivalent of the binary number  $(101110010)_2$  and the binary equivalent of  $(24)_8$



# Octal–hexadecimal conversion

- ▶ We can use the binary system as the intermediate system



## Number of digits from $b_1$ to $b_2$ system

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- ▶ In general, assume that we are using  $K$  digits in base  $b_1$  system
  - ▶ The maximum number we can represent in this source system is  $b_1^K - 1$
  - ▶ The maximum number we can represent in the destination system is  $b_2^x - 1$
  - ▶ Therefore,  $b_2^x - 1 \geq b_1^K - 1 \rightarrow x \geq K \times (\frac{\log b_1}{\log b_2})$  or  $x = \left\lceil K \times (\frac{\log b_1}{\log b_2}) \right\rceil$
- ▶ Try to find the minimum number of binary digits required to store decimal integers with a maximum of six digits

# Non-positional Number Systems

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- ▶ A nonpositional number system still uses a limited number of symbols in which each symbol has a value
  - ▶ However, the position a symbol occupies in the number normally bears no relation to its value—*the value of each symbol is fixed*
  - ▶ In this system, a number is represented as

$$S_{k-1} \dots S_2 S_1 S_0 \cdot S_{-1} S_{-2} \dots S_{-L}$$

and it usually has the value

$$n = \pm \begin{array}{c} \text{Integral part} \\ S_{K-1} + \dots + S_1 + S_0 \end{array} + \begin{array}{c} \text{Fractional part} \\ S_{-1} + S_{-2} + \dots + S_{-L} \end{array}$$

# Roman number system

- ▶ This number system has a set of symbols  $S = \{I, V, X, L, C, D, M\}$ . The corresponding values are

**Table 2.3** Values of symbols in the Roman number system

<i>Symbol</i>	<i>I</i>	<i>V</i>	<i>X</i>	<i>L</i>	<i>C</i>	<i>D</i>	<i>M</i>
Value	1	5	10	50	100	500	1000

III	→	$1 + 1 + 1$	=	3
IV	→	$5 - 1$	=	4
VIII	→	$5 + 1 + 1 + 1$	=	8
XVIII	→	$10 + 5 + 1 + 1 + 1$	=	18
XIX	→	$10 + (10 - 1)$	=	19
LXXII	→	$50 + 10 + 10 + 1 + 1$	=	72
CI	→	$100 + 1$	=	101
MMVII	→	$1000 + 1000 + 5 + 1 + 1$	=	2007
MDC	→	$1000 + 500 + 100$	=	1600