



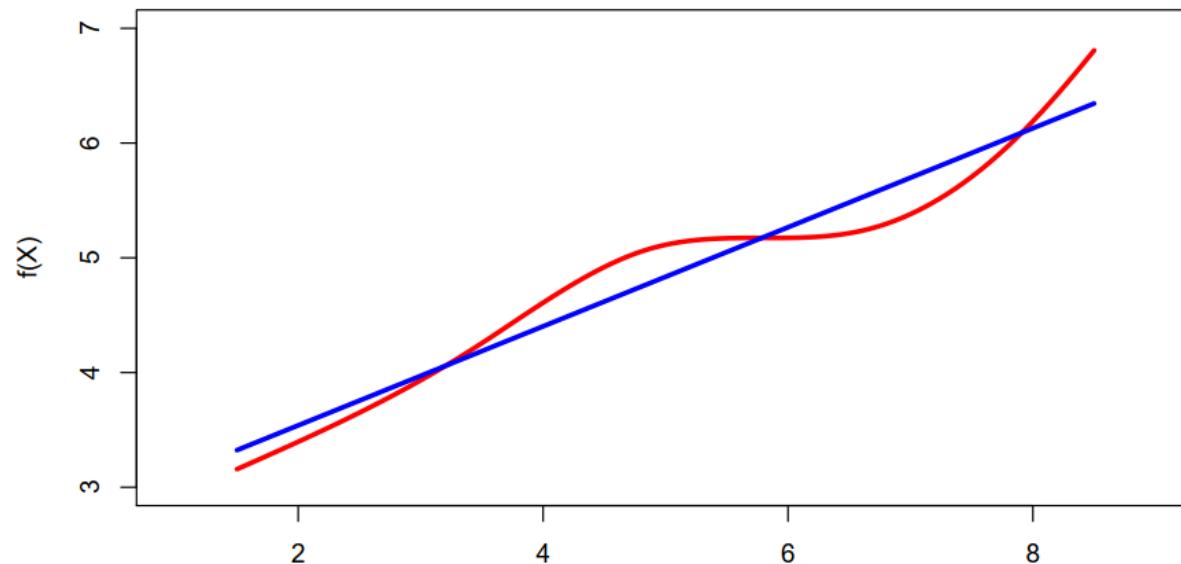
# Regression

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# Linear regression

- ▶ Linear regression is a simple approach to supervised learning. It assumes that the dependence of  $Y$  on  $X_1, X_2, \dots, X_p$  is linear
- ▶ True regression functions are never linear!



- ▶ Although it may seem overly simplistic,<sup>x</sup> linear regression is extremely useful both conceptually and practically

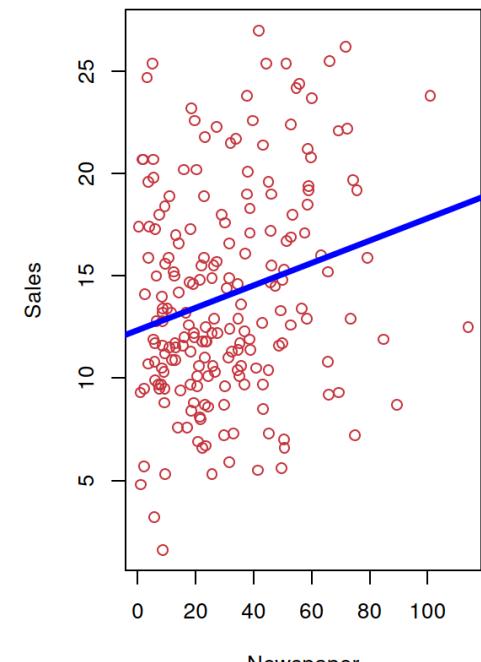
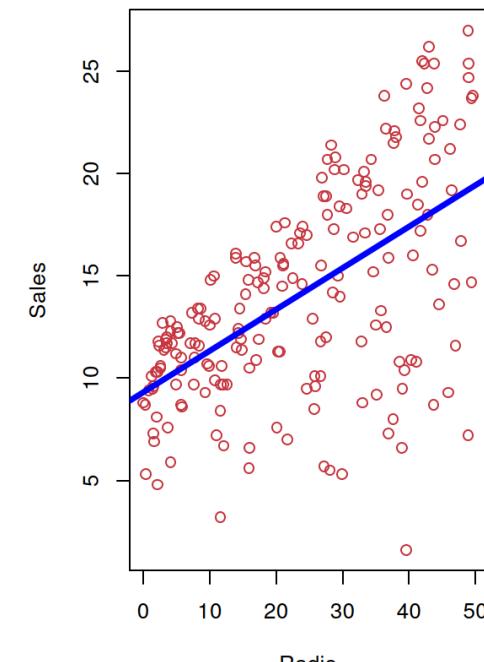
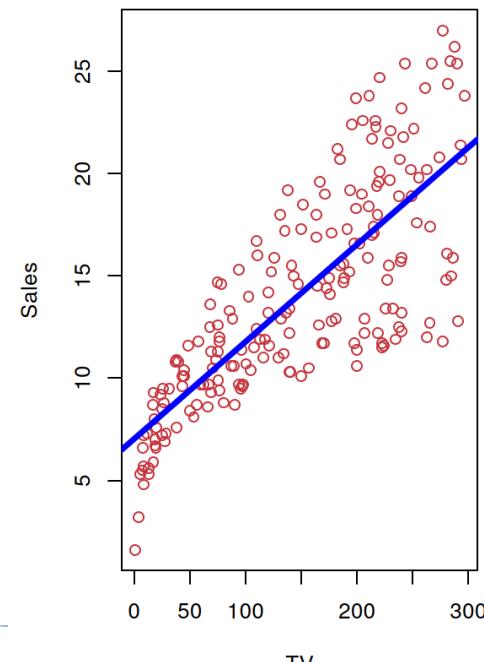
# Linear regression for the advertising data

- ▶ Consider the advertising data, questions we might ask:

1. Is there a *relationship* between advertising budget and sales?
2. How *strong* is the relationship between advertising budget and sales?
3. Which media contribute to sales?
4. How *accurately* can we predict future sales?
5. Is the relationship linear?
6. Is there *synergy* among the advertising media?



<https://www.kaggle.com/datasets/yasserh/advertising-sales-dataset>



# (1) Simple linear regression using a single predictor $X$

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- ▶ We assume a model

$$Y = \beta_0 + \beta_1 X + \epsilon,$$

where  $\beta_0$  and  $\beta_1$  are two unknown constants that represent the intercept and slope, also known as *coefficients* or *parameters*, and  $\epsilon$  is the error term which is assumed to be i.i.d. that follows the normal distribution. (LINE)

$$Y \sim N(\beta_0 + \beta_1 X, \sigma^2), \quad \sigma^2 = \text{Var}(\epsilon)$$
$$Y|X \sim N(\beta_0 + \beta_1 X, \sigma^2), \quad \sigma^2 = \text{Var}(\epsilon)$$

- ▶ Given  $\beta_0$  and  $\beta_1$  for the model coefficients, we predict future sales using

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x,$$

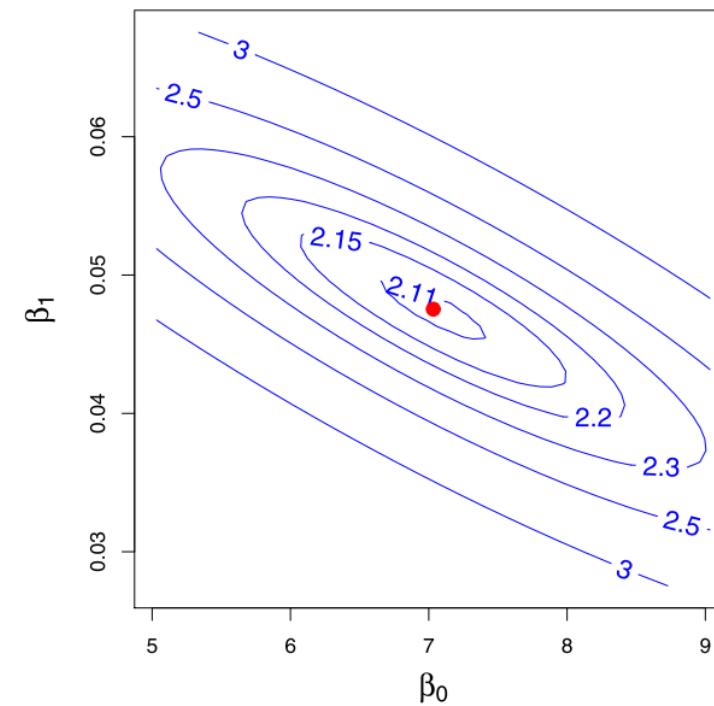
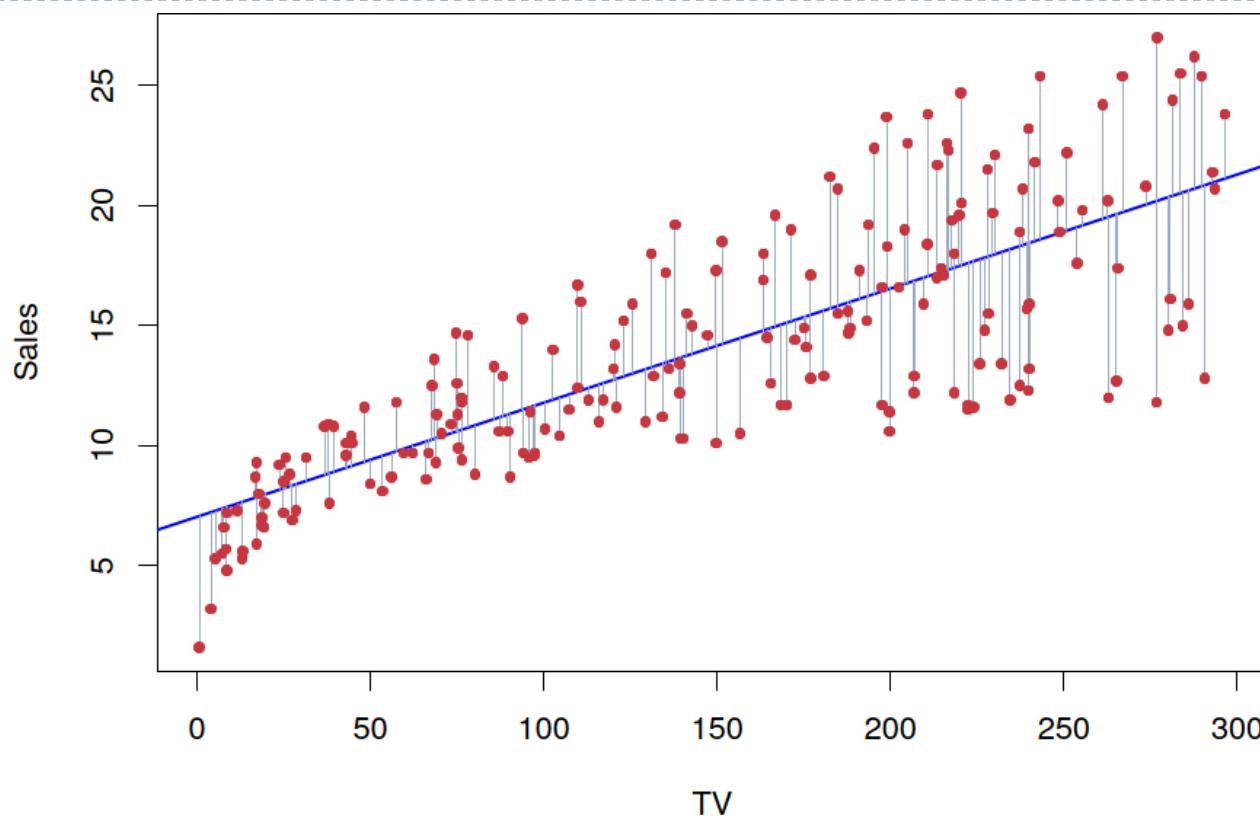
where  $\hat{y}$  indicates a prediction of  $Y$  on the basis of  $X = x$ . The hat symbol denotes an estimated value

# Estimation of the parameters by least squares

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- ▶ There are many ways of measuring closeness. We use *least squares* here
  - ▶ Let  $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$  be the prediction for  $Y$  based on the  $i$ th value of  $X$ . Then  $e_i = y_i - \hat{y}_i$  represents the  $i$ th *residual*
  - ▶ We define *the residual sum of squares (RSS)* as
$$\begin{aligned} RSS &= e_1^2 + e_2^2 + \cdots + e_n^2 \\ &= (y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1)^2 + (y_2 - \hat{\beta}_0 - \hat{\beta}_1 x_2)^2 + \cdots + (y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)^2 \end{aligned}$$
- ▶ The least squares approach (Maximum likelihood) chooses  $\hat{\beta}_0$  and  $\hat{\beta}_1$  to minimize the RSS. The minimizing values can be shown to be
$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
  - ▶ where  $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$  and  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$  are the sample mean
  - ▶ Scale does not affect the estimation of  $\hat{y}$

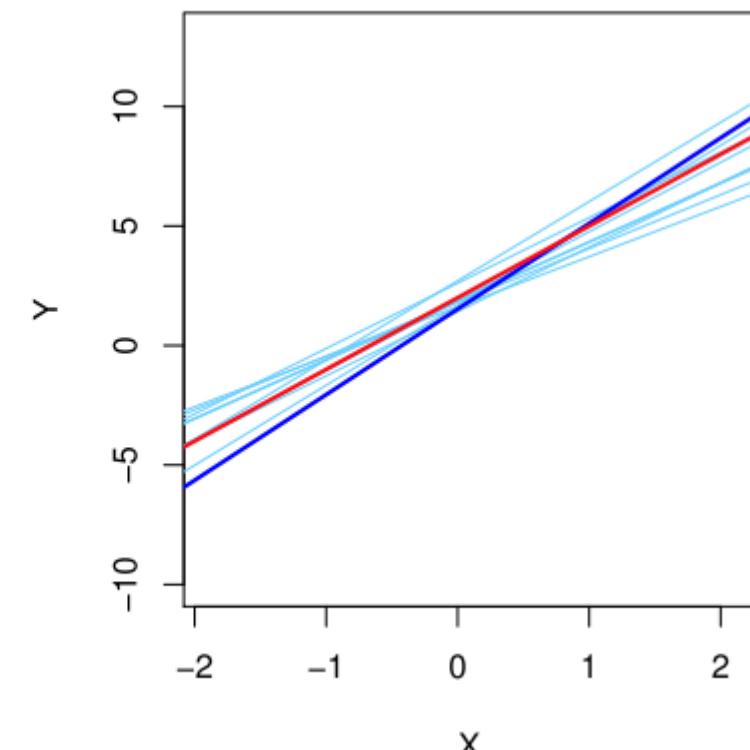
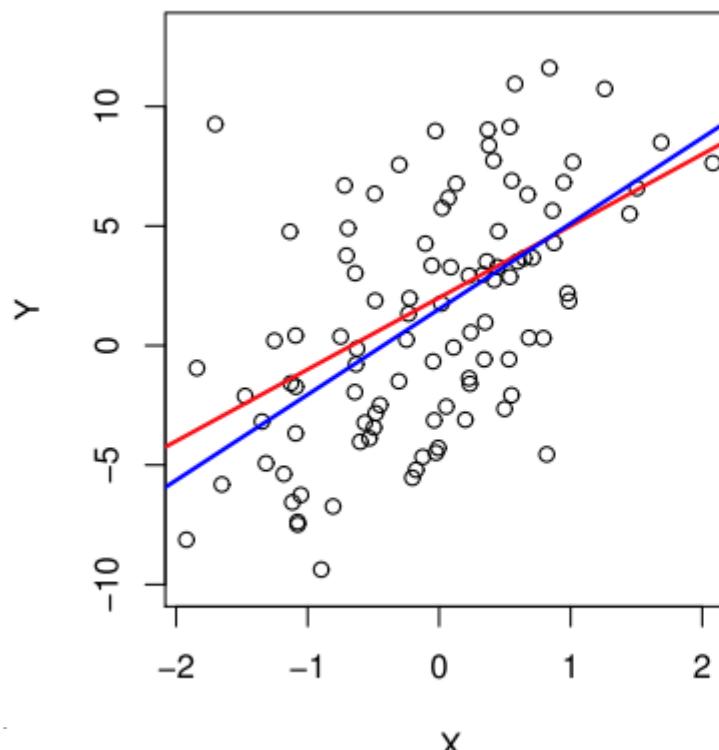
## Example: Advertising data



- ▶ The least squares fit for the regression of sales onto TV. In this case, a linear fit captures the essence of the relationship, although it is deficient in some part of the plot

# Assessing the Accuracy of the Coefficient Estimates

- ▶  $Y = 2 + 3X + \epsilon$ 
  - ▶ Red line indicates the *population regression line*
  - ▶ If we want to estimate the population mean and the standard error
    - ▶  $\hat{\mu} = \frac{1}{n} \sum x_i$
    - ▶  $Var(\hat{\mu}) = SE(\hat{\mu})^2 = \frac{\sigma^2}{n}$
    - ▶  $\sigma$  is standard deviation of  $x_i$ 's



# Assessing the Accuracy of the Coefficient Estimates

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- ▶ The standard error of an estimator reflects how it varies under repeated sampling. We have

$$SE(\hat{\beta}_1)^2 = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}, SE(\hat{\beta}_0)^2 = \sigma^2 \left[ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right]$$

- ▶ where  $\sigma^2 = Var(\epsilon)$ ,  $\hat{\sigma}^2 = \frac{RSS}{n-2} = \frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2 = (RSE)^2$
- ▶ These standard errors can be used to compute confidence intervals
  - ▶ A 95% confidence interval is defined as a range of values such that with 95% probability, the range will contain the true unknown value of the parameter. It has the form

$$\hat{\beta}_1 \pm 2 \times SE(\hat{\beta}_1)$$

## Confidence intervals — continued

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- ▶ That is, there is approximately a 95% chance that the interval

$$[\hat{\beta}_1 - 2 \cdot SE(\hat{\beta}_1), \hat{\beta}_1 + 2 \cdot SE(\hat{\beta}_1)]$$

will contain the true value of  $\beta_1$  (under a scenario where we got repeated samples like the present sample)

- ▶ For the advertising data, the 95% confidence interval for  $\beta_1$  is [0.042, 0.053] and for  $\beta_0$  is [6,130, 7,935]
  - ▶ In the absence of advertising, sales will on average fall somewhere between 6,130 and 7,935 units
  - ▶ For each \$1,000 increase in television advertising, there will be an average increase in sales of between 42 and 53 units

## Hypothesis testing

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- ▶ Standard errors can also be used to perform *hypothesis tests* on the coefficients
- ▶ The most common hypothesis test involves testing the *null hypothesis* of  
 $H_0$  : There is no relationship between  $X$  and  $Y$   
versus the alternative hypothesis  
 $H_a$  : There is some relationship between  $X$  and  $Y$
- ▶ Mathematically, this corresponds to testing  
 $H_0 : \beta_1 = 0$  versus  $H_a : \beta_1 \neq 0$   
since if  $\beta_1 = 0$  then the model reduces to  $Y = \beta_0 + \epsilon$ , and  $X$  is not associated with  $Y$

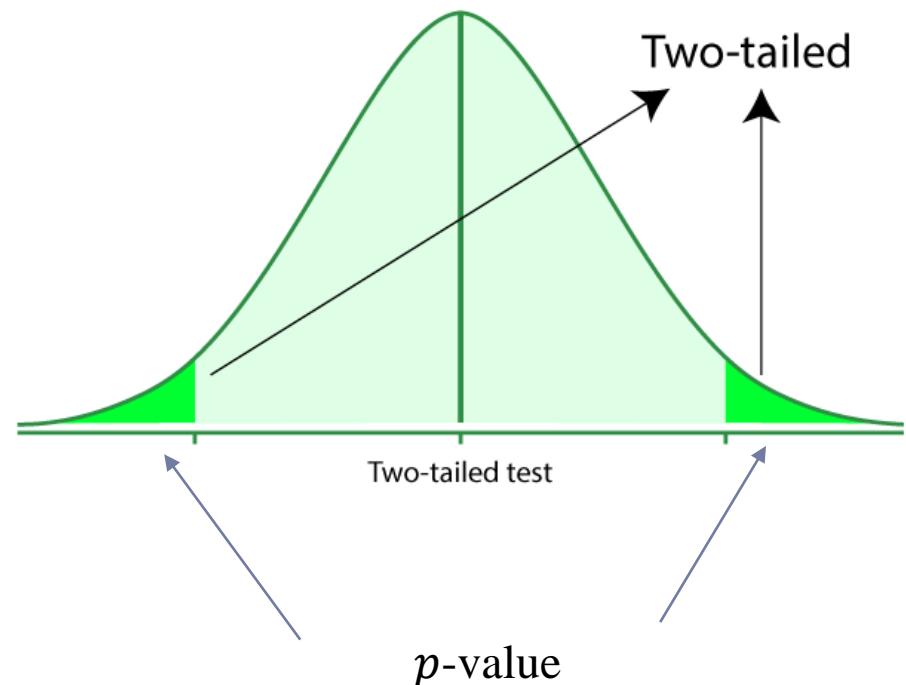
# Hypothesis testing — continued

- To test the null hypothesis, we compute a t-statistic, given by

$$t = \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)}$$

- This will have a  $t$ -distribution with  $n - 2$  degrees of freedom, assuming  $\beta_1 = 0$
- Using statistical software, it is easy to compute the probability of observing any value equal to  $|t|$  or larger. We call this probability the  $p$ -value

<https://www.geeksforgeeks.org/difference-between-one-tailed-and-two-tailed-tests/>



# Assessing the Overall Accuracy of the Model

- We compute the *Residual Standard Error* (RSE) (smaller is better)

$$RSE = \sqrt{\frac{1}{n-2} RSS} = \sqrt{\frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2} = \hat{\sigma}$$

where the residual sum-of-squares is  $RSS = \sum_{i=1}^n (y_i - \hat{y}_i)^2$

- R-squared* or fraction of variance explained is (larger is better)

$$R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}$$

where  $TSS = \sum_{i=1}^n (y_i - \bar{y})^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 + \sum_{i=1}^n (\hat{y}_i - \bar{y})^2$  is the total sum of squares

- It can be shown that in this simple linear regression setting that  $R^2 = r^2$ , where  $r$  is the correlation between  $X$  and  $Y$  (Exercise 7):

$$r = Cor(X, Y) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$$

## Advertising data results

	Coefficient	Std. error	t-statistic	p-value
Intercept	7.0325	0.4578	15.36	< 0.0001
TV	0.0475	0.0027	17.67	< 0.0001

Quantity	Value
Residual standard error	3.26
$R^2$	0.612
F-statistic	312.1

$$F = \frac{(TSS - RSS)/p}{RSS/(n - p - 1)} \sim F_{p, n-p-1}$$

# Extension of simple linear regression

Simple regression of **sales** on **radio**

	Coefficient	Std. error	t-statistic	p-value
Intercept	9.312	0.563	16.54	< 0.0001
radio	0.203	0.020	9.92	< 0.0001

Simple regression of **sales** on **newspaper**

	Coefficient	Std. error	t-statistic	p-value
Intercept	12.351	0.621	19.88	< 0.0001
newspaper	0.055	0.017	3.30	0.00115

## (2) Multiple Linear Regression

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- ▶ Instead of fitting three simple linear regression model for each predictor
- ▶ Here our model is

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \epsilon$$

- ▶ We interpret  $\beta_j$  as the average effect on  $Y$  of a one unit increase in  $X_j$ , *holding all other predictors fixed*. In the advertising example, the model becomes

$$Sales = \beta_0 + \beta_1 \times TV + \beta_2 \times radio + \beta_3 \times newspaper + \epsilon$$

## Estimation and Prediction for Multiple Regression

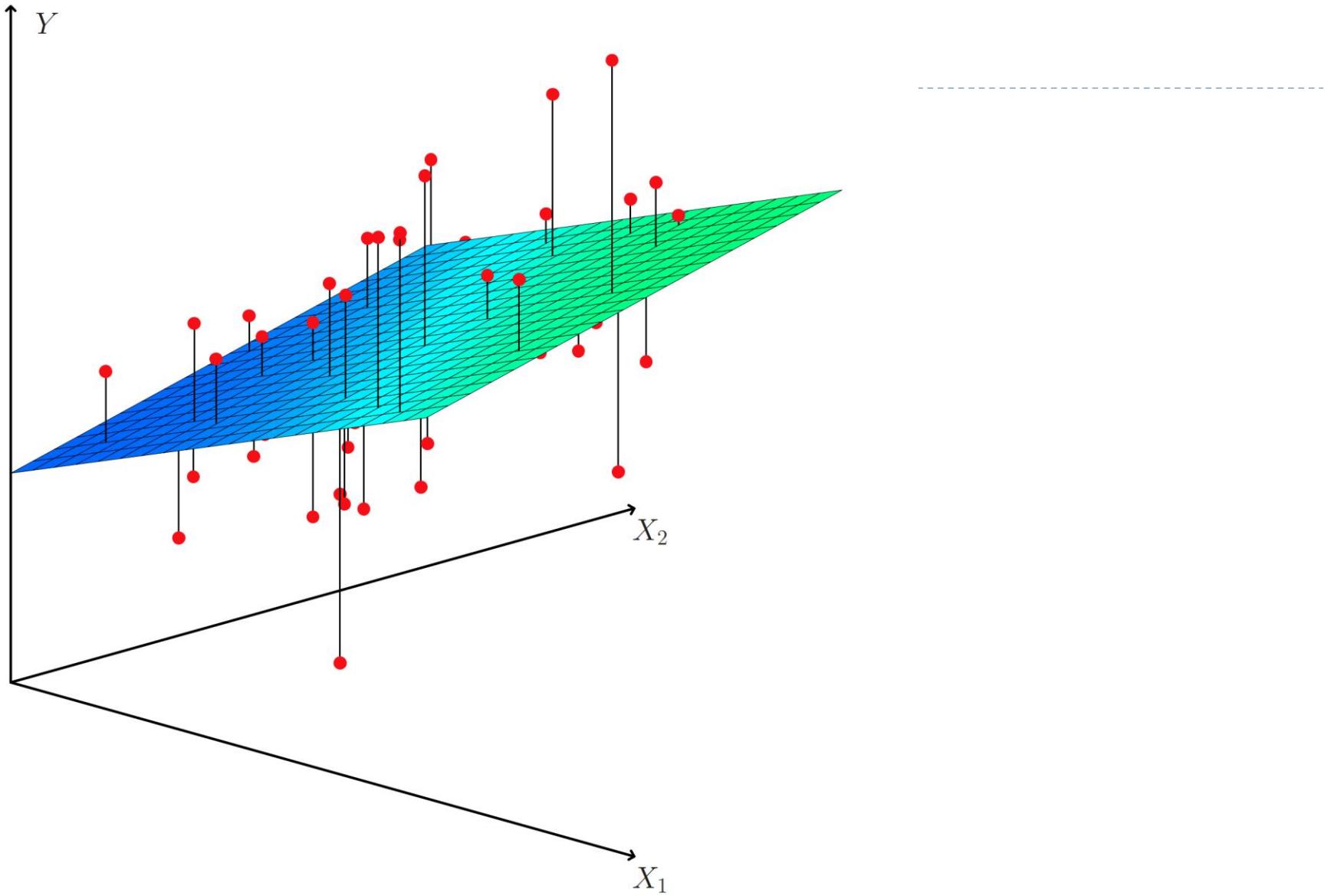
- Given estimates  $\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_p$ , we can make predictions using the formula

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \dots + \hat{\beta}_p x_p$$

- We estimate  $\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_p$  as the value that minimize the sum of squared residuals

$$RSS = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \dots - \hat{\beta}_p x_{ip})^2$$

- This is done using standard statistical software. The values  $\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_p$  that minimize RSS are the multiple least squares regression coefficient estimates.
- The standard error and t-statistic can also be obtained with linear algebra (see ESL)



# Results for advertising data

	Coefficient	Std. error	t-statistic	p-value
Intercept	2.939	0.3119	9.42	< 0.0001
TV	0.046	0.0014	32.81	< 0.0001
radio	0.189	0.0086	21.89	< 0.0001
newspaper	-0.001	0.0059	-0.18	0.8599

Correlation matrix

	TV	radio	newspaper	sales
TV	1.0000	0.0548	0.0567	0.7822
radio		1.0000	0.3541	0.5762
newspaper			1.0000	0.2283
sales				1.0000



<https://www.causeweb.org/cause/resources/fun/cartoons/ice-cream-sales-and-shark-sightings>

## Some important questions

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1. Is at least one of the predictors  $X_1, X_2, \dots, X_p$  useful in predicting the response?
2. Do all the predictors help to explain  $Y$ , or is only a subset of the predictors useful?
3. How well does the model fit the data?
4. Given a set of predictor values, what response value should we predict, and how accurate is our prediction?

## Is at least one predictor useful?

- For the first question, we can use the F-statistic

$H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$  versus  $H_a : \text{at least one } \beta_j \neq 0$

$$F = \frac{(TSS - RSS)/p}{RSS/(n - p - 1)} \sim F_{p, n-p-1}$$

- Note if linear model assumption is hold,  $E\left\{\frac{RSS}{n-p-1}\right\} = \sigma^2$  and if  $H_0$  hold,  $E\{(TSS -$

Quantity	Value
Residual standard error	1.69
$R^2$	0.897
F-statistic	570

## Is at least one predictor useful?

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- ▶ To examine whether a particular set of  $q$  variables are zeros or not

$$H_0 : \beta_{p-q+1} = \beta_{p-q+2} = \cdots = \beta_p = 0$$

$$\text{we use } F = \frac{(RSS_0 - RSS)/q}{RSS/(n-p-1)}$$

We fit a second model that uses all the variables except those last  $q$  to get  $RSS_0$

- ▶ In the previous table of simple linear regression,  $q = 1$
- ▶ It seems likely that if any one of the  $p$ -values for the individual variables is very small, then at least one of the predictors is related to the response. Why should we care about the overall F-statistics?
  - ▶ P-hacking!

## Deciding on the important variables

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- ▶ If  $p > n$ , we cannot fit the multiple linear regression model using least squares and, therefore, can not use F-statistics!
- ▶ On the other hand, if we conclude on the basis of that  $p$ -value that at least one of the predictors is related to the response, then it is natural to wonder which are the guilty ones!
- ▶ The most direct approach is called all subsets or best subsets regression: we compute the least squares fit for all possible subsets and then choose between them based on some criterion that *balances training error with model size*
  - ▶ However, we often can't examine all possible models, since they are  $2^p$  of them; for example, when  $p = 40$ , there are over a billion models

## Deciding on the important variables - Forward selection

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1. Begin with the null model — a model that only contains an intercept
2. Fit  $p$  simple linear regressions and add to the null model the variable that results in the lowest RSS
3. Add to that model the variable that results in the lowest RSS among all two-variable models
4. Continue until some stopping rule is satisfied, for example when all remaining variables have a  $p$ -value above some threshold



## Deciding on the important variables - Backward selection

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1. Start with all variables in the model
2. Remove the variable with the largest  $p$ -value — that is, the variable that is the least statistically significant
3. The new  $(p - 1)$  –variable model is fit, and the variable with the largest  $p$  –value is removed
4. Continue until a stopping rule is reached. For instance, we may stop when all remaining variables have a significant  $p$ -value defined by some significance threshold

## Deciding on the important variables - Mixed selection

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1. Begin with the null model, and as with forward selection, we add the variable that provides the best fit
2. If at any point the  $p$ -value for one of the variables in the model rises above a certain threshold, then we remove that variable from the model
3. Continue to perform these forward and backward steps until all variables in the model have a sufficiently low  $p$ -value, and all variables outside the model would have a large  $p$ -value if added to the model

## Deciding on the important variables - Mixed selection

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- ▶ Mixed selection can remedy the following situations
  - ▶ Backward selection cannot be used if  $p > n$ , while forward selection can always be used
  - ▶ Forward selection is a greedy approach and might include variables early that later become redundant
- ▶ Later, we discuss more systematic criteria for choosing an “optimal number” of members in the path of models produced by forward stepwise selection
- ▶ These include Mallow’s  $C_p$ , Akaike information criterion (AIC), Bayesian information criterion (BIC), adjusted  $R^2$  and Cross-validation (CV)

## Model fit and prediction

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- ▶ It can be shown that  $R^2 = \text{Cor}(Y, \hat{Y})$  in this case
  - ▶  $R^2$  will always increase when more variables are added to the model
- ▶ We compute the Residual Standard Error

$$RSE = \sqrt{\frac{1}{n-p-1} RSS} = \sqrt{\frac{1}{n-p-1} \sum_{i=1}^n (y_i - \hat{y}_i)^2} = \hat{\sigma}$$

- ▶ How close between  $\hat{Y}$  and  $f(X)$  can be quantified by the *confidence interval*
  - ▶ Note that even if we knew  $f(X)$ —that is, even if we knew the true values for  $\beta_0, \beta_1, \dots, \beta_p$ —the response cannot be predicted perfectly because of  $\epsilon$  (irreducible error)
  - ▶ For the new prediction, how much will  $Y$  vary from  $\hat{Y}$ ? We use *prediction intervals* to answer this question

# Confidence interval versus prediction interval

- ▶ Prediction intervals are always wider than confidence intervals, because they incorporate both the error in the estimate for  $f(X)$  (the reducible error) and the uncertainty as to how much an individual point will differ from the population regression plane (the irreducible error)

$$C.I. = \hat{y}_i \pm t_{\alpha/2} \hat{\sigma} \sqrt{\left( \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)}$$

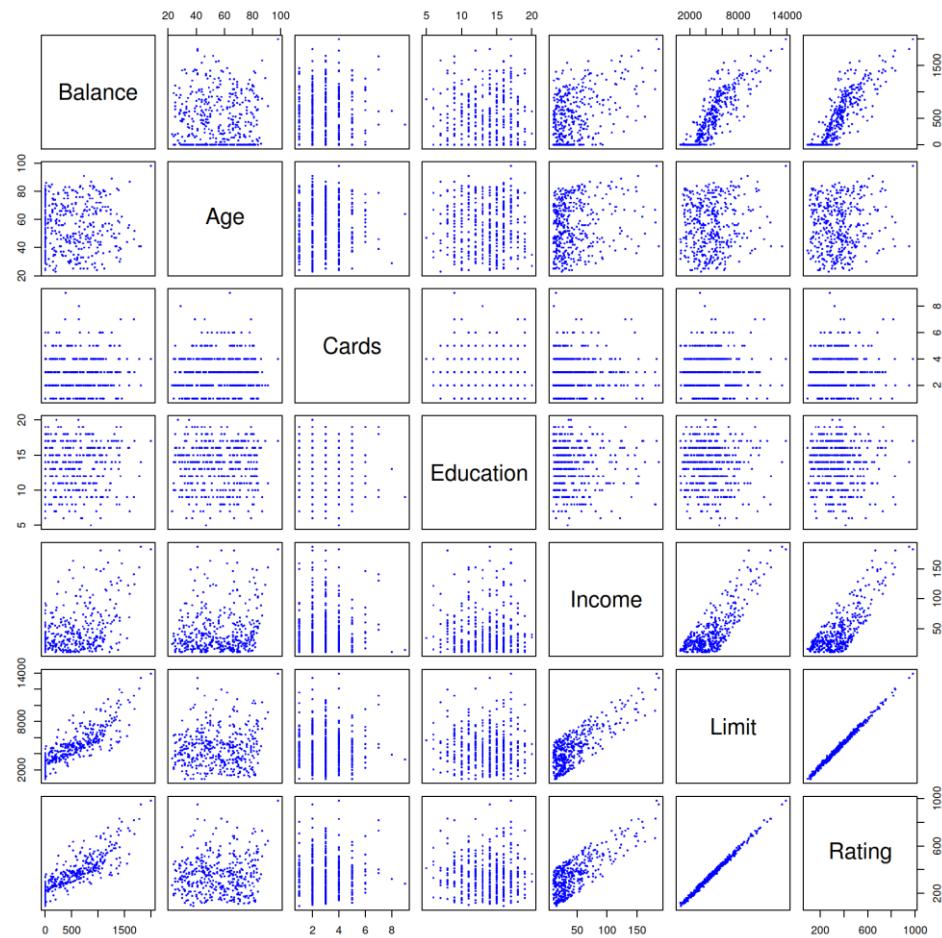
$$P.I. = \hat{y}_i \pm t_{\alpha/2} \hat{\sigma} \sqrt{\left( 1 + \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)}$$

- ▶ For multiple linear regression see [here](#)

## (3) Other Considerations in the Regression Model

### ▶ Qualitative Predictors

- ▶ Some predictors are not quantitative but are qualitative, taking a discrete set of values
- ▶ These are also called *categorical* predictors or *factor* variables
- ▶ In the [credit card dataset](#), in addition to the 7 quantitative variables shown, there are four qualitative variables: **own** (house ownership), **student** (student status), **status** (marital status), and **region** (East, West or South)



## Qualitative Predictors — continued

- ▶ Example: investigate differences in credit card balance between a person who has a house or not, ignoring the other variables. We create a *dummy variable*

$$x_i = \begin{cases} 1 & \text{if } i\text{th person owns a house} \\ 0 & \text{if } i\text{th person does not own a house} \end{cases}$$

- ▶ Resulting model:

$$y_i = \beta_0 + \beta_1 x_1 + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person owns a house} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person does not own a house} \end{cases}$$

	Coefficient	Std. error	t-statistic	p-value
Intercept	509.80	33.13	15.389	< 0.0001
own [Yes]	19.73	46.05	0.429	0.6690

## Qualitative predictors with more than two levels

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- With more than two levels, we create additional dummy variables. For example, for the **region** variable, we create two dummy variables. The first could be

$$x_{i1} = \begin{cases} 1 & \text{if } i\text{th person is from the South} \\ 0 & \text{if } i\text{th person is not from the South} \end{cases}$$

and the second could be

$$x_{i2} = \begin{cases} 1 & \text{if } i\text{th person is from the West} \\ 0 & \text{if } i\text{th person is not from the West} \end{cases}$$

- Then both of these variables can be used in the regression equation, in order to obtain the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person is from the South} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if } i\text{th person is from the West} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person is from the East} \end{cases}$$

## Qualitative predictors with more than two levels

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- There will always be one fewer dummy variable than the number of levels. The level with no dummy variable — East in this example — is known as the *baseline*

	Coefficient	Std. error	t-statistic	p-value
Intercept	531.00	46.32	11.464	< 0.0001
region[South]	-18.69	65.02	-0.287	0.7740
region[West]	-12.50	56.68	-0.221	0.8260

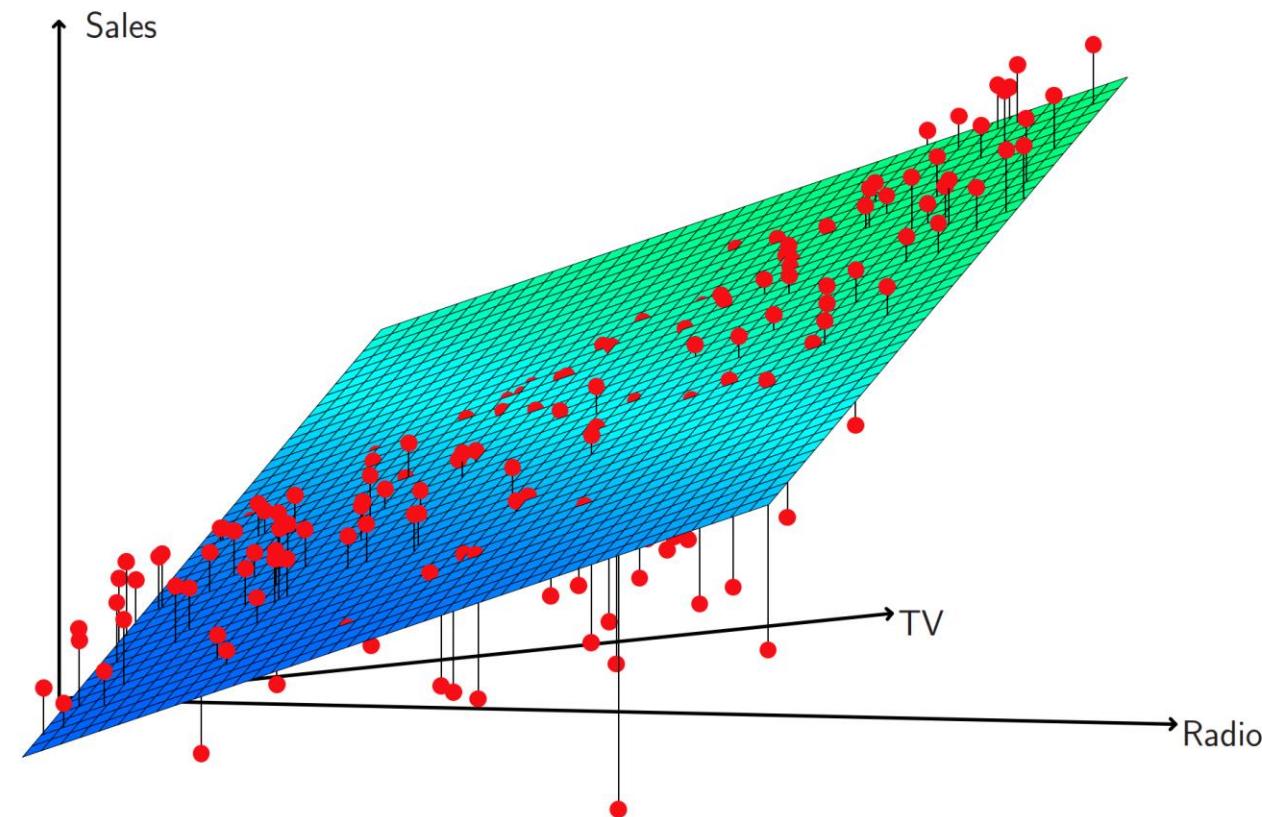


## Extensions of the Linear Model

- ▶ Removing the additive assumption: interactions and nonlinearity
- ▶ Interactions:
  - ▶ In our previous analysis of the Advertising data, we assumed that the effect on sales of increasing one advertising medium is independent of the amount spent on the other media
  - ▶ For example, the linear model
$$sales = \beta_0 + \beta_1 \times TV + \beta_2 \times radio + \beta_3 \times newspaper$$
states that the average effect on sales of a one-unit increase in TV is always  $\beta_1$ , regardless of the amount spent on radio
- ▶ But suppose that spending money on radio advertising actually increases the effectiveness of TV advertising, so that the slope term for TV should increase as radio increases

# Extensions of the Linear Model

- ▶ Given a fixed budget of \$100,000, spending half on radio and half on TV may increase sales!
- ▶ The positive residuals (those visible above the surface) tend to lie along the 45-degree line, where TV and Radio budgets are split evenly. The negative residuals tend to lie away from this line
- ▶ In marketing, this is known as a *synergy effect*, and in statistics, it is referred to as an *interaction effect*



## Modelling interactions — Advertising data

- ▶ The model takes the form

$$\begin{aligned} sales &= \beta_0 + \beta_1 \times TV + \beta_2 \times radio + \beta_3 \times (radio \times TV) + \epsilon \\ &= \beta_0 + (\beta_1 + \beta_3 \times radio) \times TV + \beta_2 \times radio + \epsilon \end{aligned}$$

	Coefficient	Std. error	t-statistic	p-value
Intercept	6.7502	0.248	27.23	< 0.0001
TV	0.0191	0.002	12.70	< 0.0001
radio	0.0289	0.009	3.24	0.0014
TV×radio	0.0011	0.000	20.73	< 0.0001

- ▶ The  $p$ -value for the interaction term  $radio \times TV$  is extremely low, indicating that there is strong evidence for  $H_a: \beta_3 \neq 0$
- ▶ The  $R^2$  for the interaction model is 96.8%, compared to only 89.7% for the model that predicts sales using TV and radio without an interaction term

## Modelling interactions — Advertising data

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- ▶  $(96.8 - 89.7)/(100 - 89.7) = 69\%$  of the variability in sales that remains after fitting the additive model has been explained by the interaction term
  - ▶ The coefficient estimates in the table suggest that an increase in TV advertising of \$1,000 is associated with increased sales of  $(\beta_1 + \beta_3 \times \text{radio}) \times 1000 = 19 + 1.1 \times \text{radio units}$
  - ▶ An increase in radio advertising of \$1,000 will be associated with an increase in sales of  $(\beta_2 + \beta_3 \times \text{TV}) \times 1000 = 29 + 1.1 \times \text{TV units}$
- ▶ Sometimes it is the case that an interaction term has a very small  $p$ -value, but the associated main effects (in this case, TV and radio) do not
  - ▶ The *hierarchy principle*: If we include an interaction in a model, we should also include the main effects, even if the  $p$ -values associated with their coefficients are not significant. Otherwise the interpretation may change



## Interactions between qualitative and quantitative variables

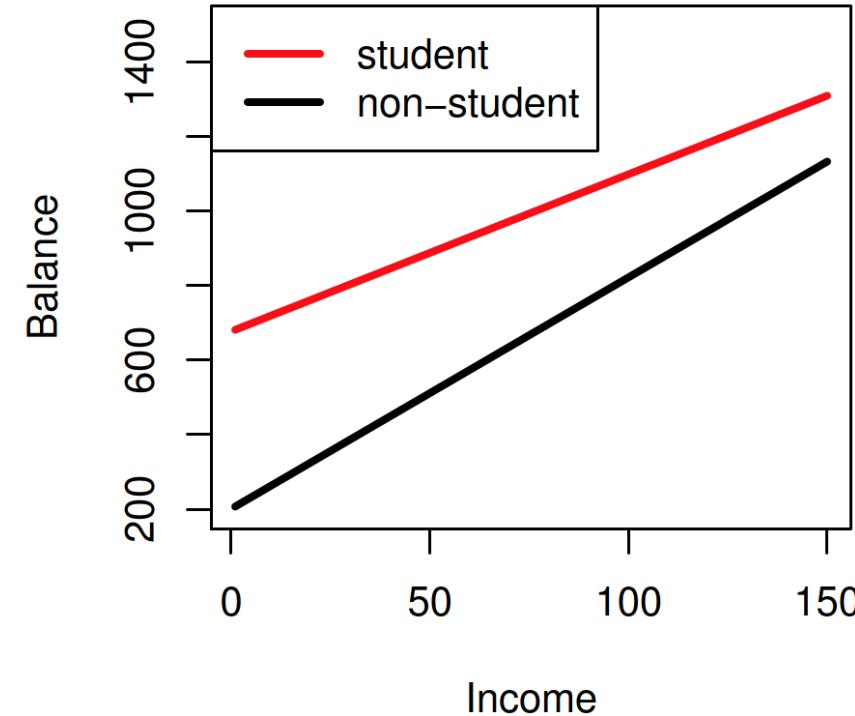
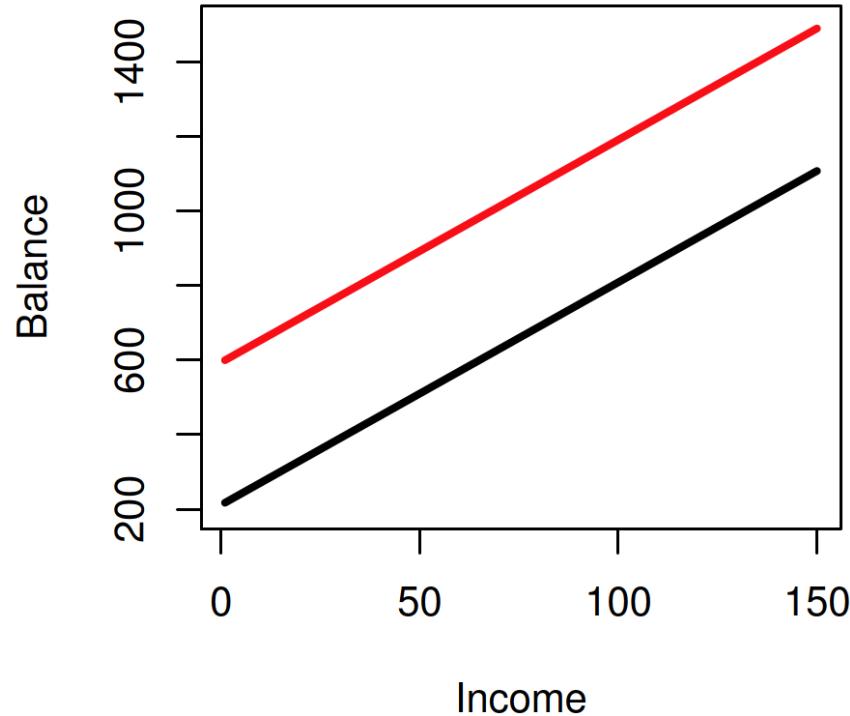
- ▶ Consider the Credit data set, and suppose that we wish to predict balance using income (quantitative) and student (qualitative)
- ▶ Without an interaction term, the model takes the form

$$\begin{aligned} \text{balance}_i &\approx \beta_0 + \beta_1 \times \text{income}_i + \begin{cases} \beta_2 & \text{if } i\text{th person is a student} \\ 0 & \text{if } i\text{th person is not a student} \end{cases} \\ &= \beta_1 \times \text{income}_i + \begin{cases} \beta_0 + \beta_2 & \text{if } i\text{th person is a student} \\ \beta_0 & \text{if } i\text{th person is not a student} \end{cases} \end{aligned}$$

- ▶ With interactions, it takes the form

$$\begin{aligned} \text{balance}_i &\approx \beta_0 + \beta_1 \times \text{income}_i + \begin{cases} \beta_2 + \beta_3 \times \text{income}_i & \text{if } i\text{th person is a student} \\ 0 & \text{if } i\text{th person is not a student} \end{cases} \\ &= \begin{cases} (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \times \text{income}_i & \text{if } i\text{th person is a student} \\ \beta_0 + \beta_1 \times \text{income}_i & \text{if } i\text{th person is not a student} \end{cases} \end{aligned}$$

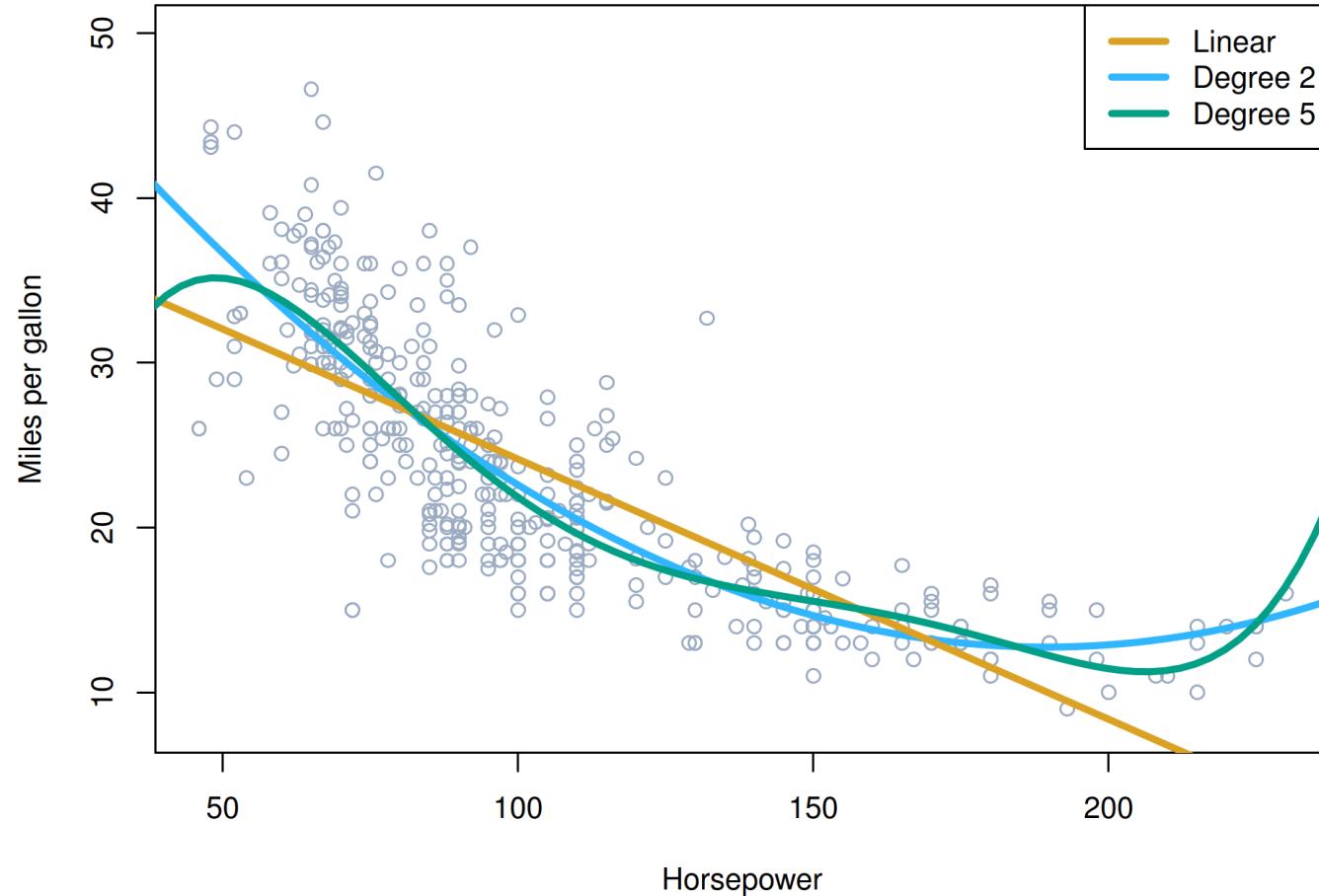
# Interactions between qualitative and quantitative variables



Credit data; Left: no interaction between `income` and `student`  
Right: with an interaction term between `income` and `student`

# Non-linear effects of predictors

- ▶ *Polynomial regression on Auto data*



## Non-linear effects of predictors

- ▶ The figure suggests that

$$mpg = \beta_0 + \beta_1 \times horsepower + \beta_2 \times horsepower^2 + \epsilon$$

may provide a better fit

	Coefficient	Std. error	t-statistic	p-value
Intercept	56.9001	1.8004	31.6	< 0.0001
horsepower	-0.4662	0.0311	-15.0	< 0.0001
horsepower <sup>2</sup>	0.0012	0.0001	10.1	< 0.0001

- ▶ The  $R^2$  of the quadratic fit is 0.688, compared to 0.606 for the linear fit, and the  $p$ -value for the quadratic term is highly significant
- ▶ If including horsepower<sup>2</sup> led to such a big improvement in the model, why not include horsepower<sup>3</sup>, horsepower<sup>4</sup>, or even horsepower<sup>5</sup>?

## (4) Potential Problems

---

- ▶ When we fit a linear regression model to a particular data set, many problems may occur. Most common among these are the following:
  1. Non-linearity of the response-predictor relationships
  2. Correlation of error terms
  3. Non-constant variance of error terms
  4. Outliers
  5. High-leverage points
  6. Collinearity

In practice, identifying and overcoming these problems is as *much an art as a science*

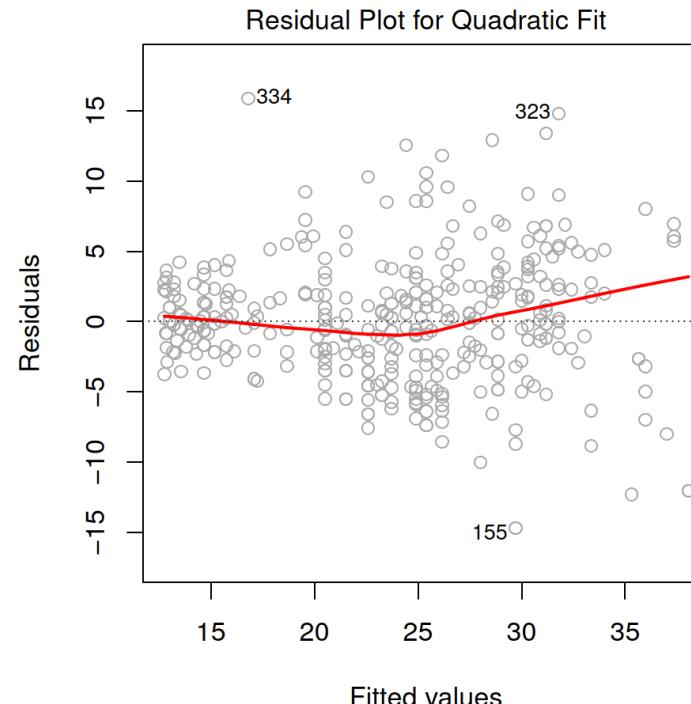
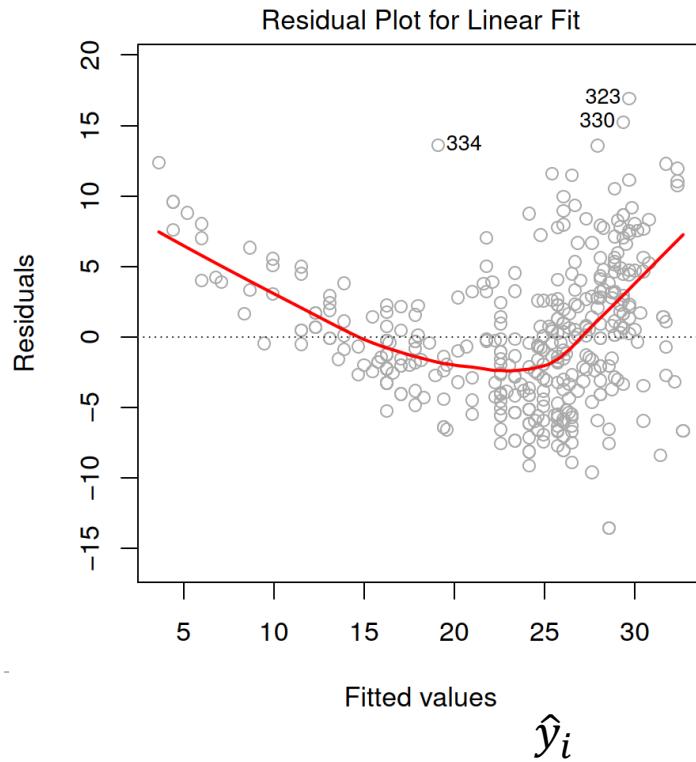
# 1. Non-linearity of the Data

---

- ▶ The linear regression model assumes that there is a straight-line relationship between the predictors and the response
  - ▶ The prediction accuracy of the model can be significantly reduced if it is nonlinear
  - ▶ If the true relationship is far from linear, then the conclusions that we draw are suspect
- ▶ *Residual plots* are a useful graphical tool for identifying non-linearity
  - ▶ Ideally, the residual plot will show no discernible pattern. The presence of a pattern may indicate a problem with some aspects of the linear model

- ▶ The red line is a smooth fit to the residuals, which is displayed in order to make it easier to identify any trends (mpg vs. horsepower)
- ▶ The residuals exhibit a clear U-shape in the left panel, which provides a strong indication of non-linearity in the data
- ▶ In contrast, the right-hand panel displays the residual plot that results from the model, which contains a quadratic term
- ▶ There appears to be little pattern in the residuals, suggesting that the quadratic term improves the fit to the data

$$e_i = y_i - \hat{y}_i$$



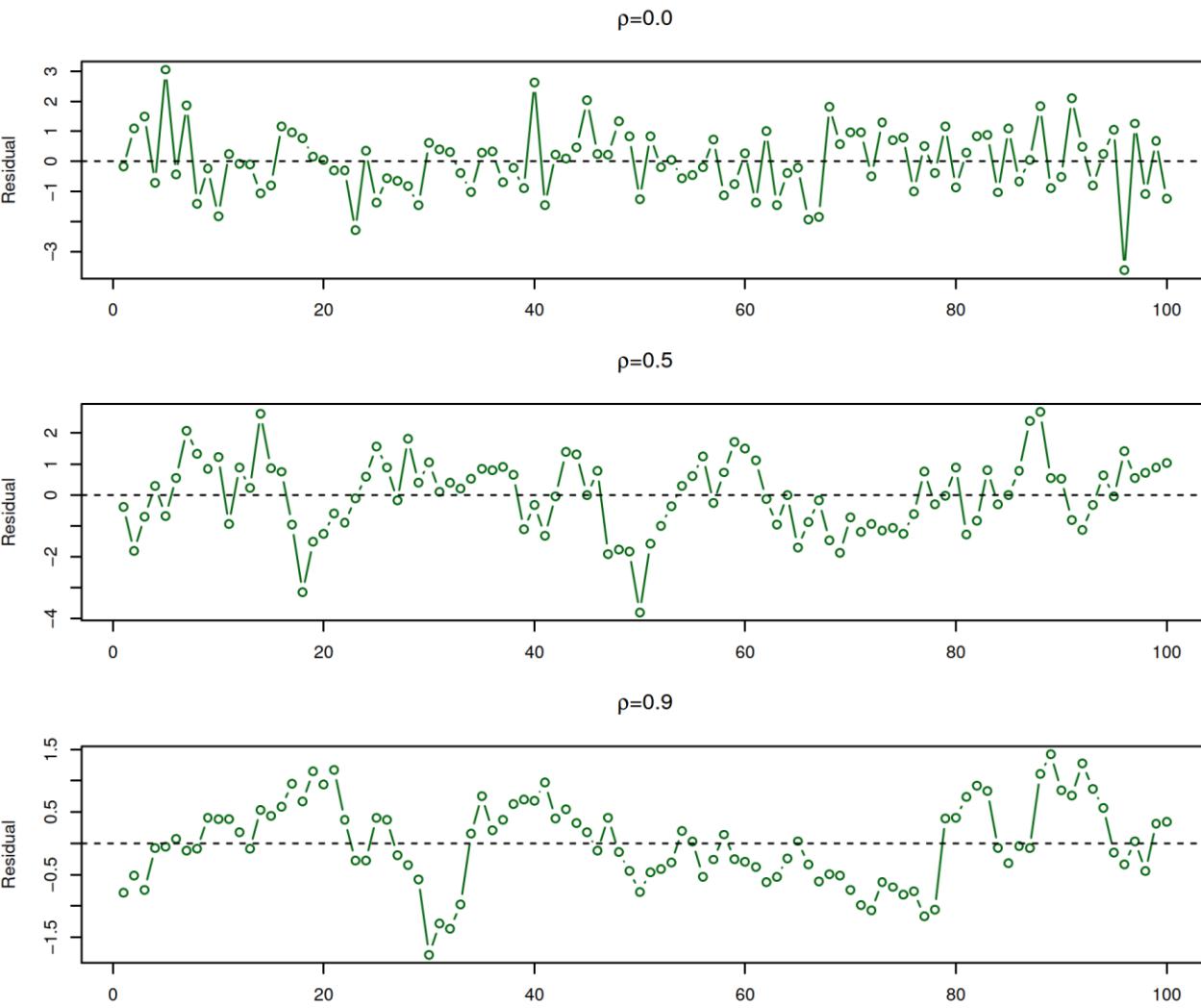
## 2. Correlation of Error Terms

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- ▶ An important assumption of the linear regression model is that the error terms,  $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ , are uncorrelated. What does this mean?
  - ▶ For instance, if the errors are uncorrelated, then the fact that  $\epsilon_i$  is positive provides little or no information about the sign of  $\epsilon_{i+1}$
  - ▶ As an extreme example, suppose we accidentally doubled our data, leading to observations and error terms identical in pairs. If we ignored this, our standard error calculations would be as if we had a sample of size  $2n$ , when in fact we have only  $n$  samples. Our estimated parameters would be the same for the  $2n$  samples as for the  $n$  samples, but the confidence intervals would be narrower by a factor of  $\sqrt{2}$ !
  - ▶ If in fact there is correlation among the error terms, then the estimated standard errors will tend to underestimate the true standard errors. As a result, confidence and prediction intervals will be narrower than they should be

## 2. Correlation of Error Terms

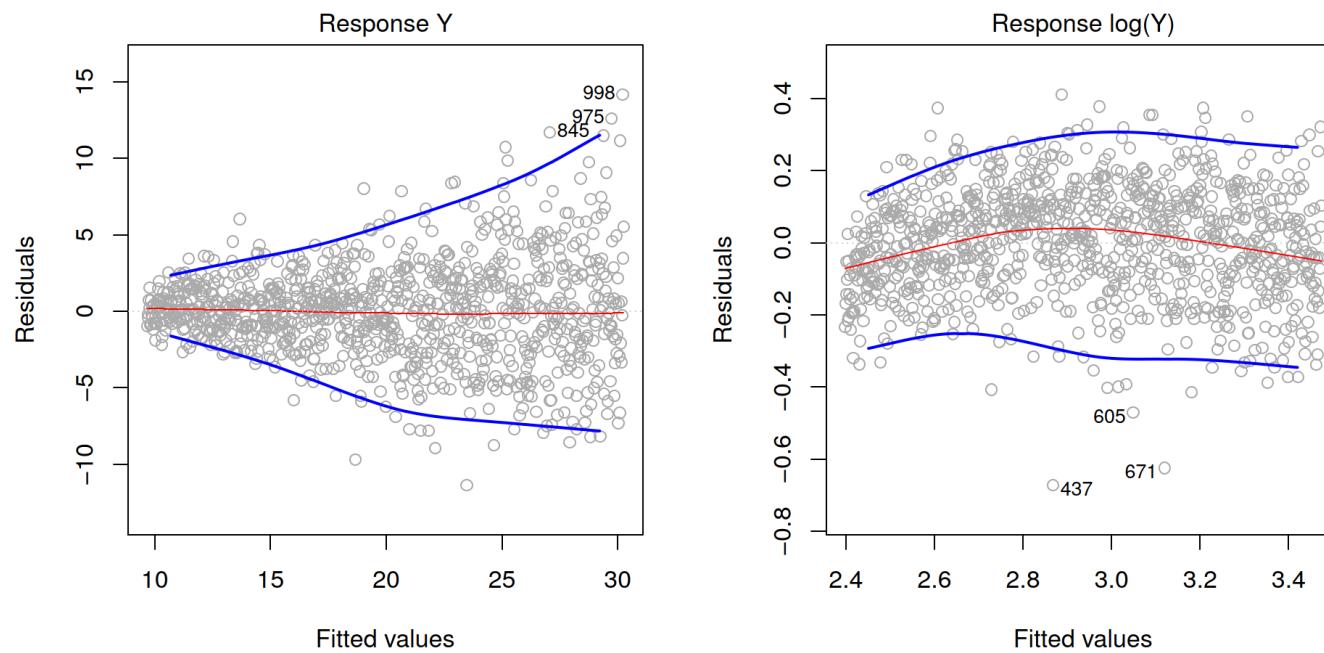
- In the top panel, we see the residuals from a linear regression fit to data generated with *uncorrelated errors*
- The residuals in the bottom panel show a clear pattern in the residuals—adjacent residuals tend to take on similar values
- The center panel illustrates a moderate case in which the residuals had a correlation of 0.5



Observation  
Plots of residuals from simulated time series data sets generated with differing levels of correlation  $\rho$  between error terms for adjacent time point

### 3. Non-constant Variance of Error Terms

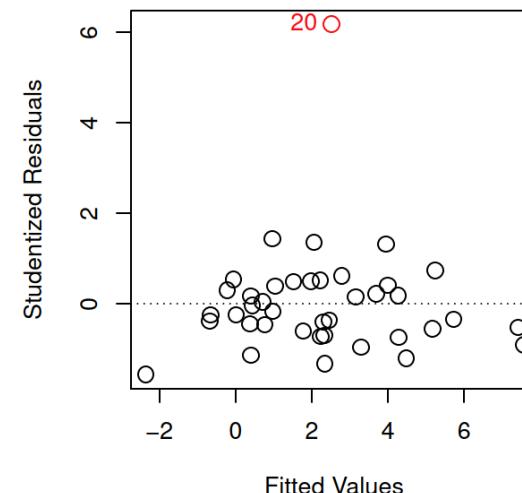
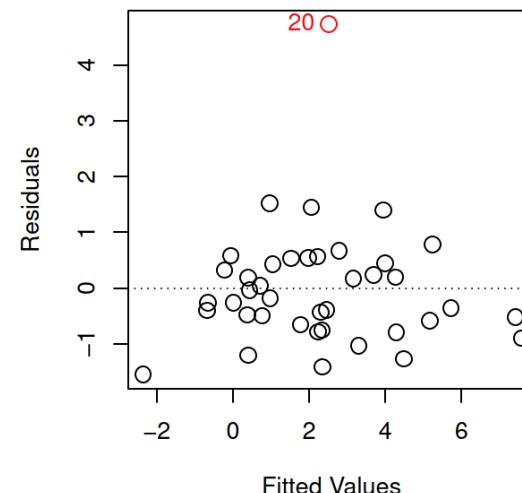
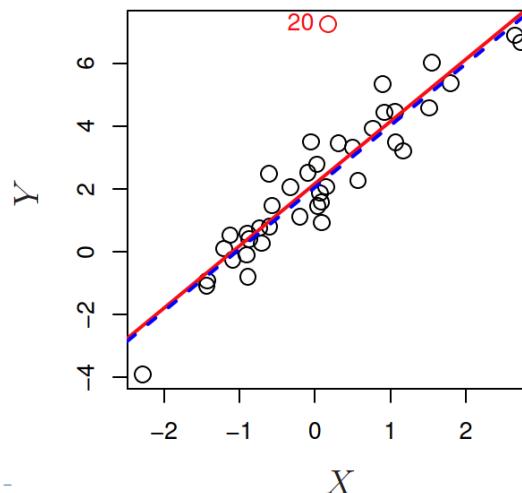
- ▶ It is often the case that the variances of the error terms are non-constant
  - ▶ For instance, the variances of the error terms may increase with the value of the response. One can identify non-constant variances in the errors, or *heteroscedasticity*, from the presence of a funnel shape in the residual plot



The right-hand panel displays the residual plot after transforming the response using  $\log Y$

## 4. Outliers (unusual $y$ )

- ▶ An outlier is a point for which  $y_i$  is far from the value predicted by the model
  - ▶ Outliers can arise for a variety of reasons, such as incorrect recording of an observation during data collection
  - ▶ The red solid line is the least squares regression fit, while the blue dashed line is the least squares fit after removal of the outlier
  - ▶ Observations whose studentized residuals are greater than 3 in absolute value are possible outliers



Studentized  
Residuals

$$t_i = \frac{e_i}{SE(e_i)} = e_i / (\hat{\sigma} \sqrt{1 - h_i})$$

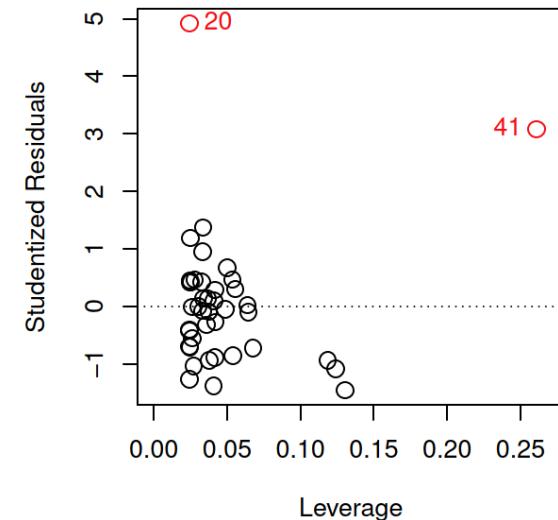
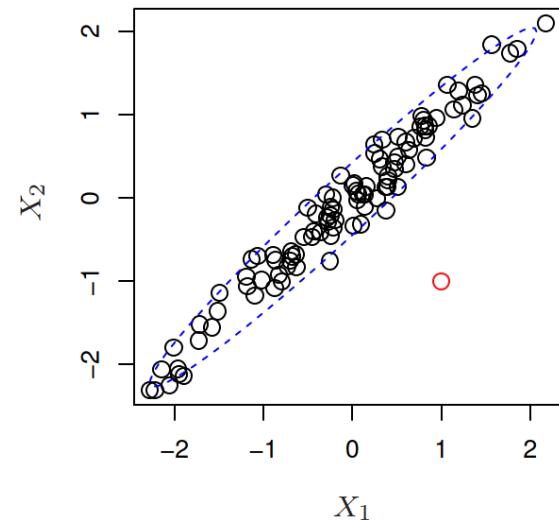
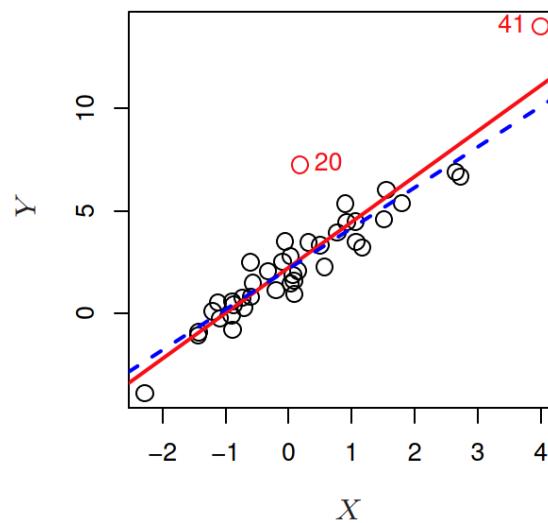
## 4. Outliers

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- ▶ It is typical for an outlier that does not have an unusual predictor value to have little effect on the least squares fit. However, it can cause other problems
  - ▶ For instance, in this example, the RSE is 1.09 when the outlier is included in the regression, but it is only 0.77 when the outlier is removed
  - ▶ Since RSE ( $\sigma$ ) is used to compute all confidence intervals and  $p$ -values, such a dramatic increase caused by a single data point can have implications for the interpretation of the fit
- ▶ If we believe that an outlier has occurred due to an error in data collection or recording, then one solution is to simply remove the observation
  - ▶ However, care should be taken since an outlier may instead indicate a deficiency with the model, such as a missing predictor

## 5. High Leverage Points (unusual $x$ )

- ▶ We just saw that outliers are observations for which the response  $y_i$  is unusual given the predictor  $x_i$ 
  - ▶ In contrast, observations with high leverage have an unusual value for  $x_i$
  - ▶ In order to quantify an observation's leverage, we compute the *leverage statistic*. A large value of this statistic indicates an observation with high leverage



## 5. High Leverage Points

---

- ▶ For a simple linear regression,

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{i'=1}^n (x_{i'} - \bar{x})^2}$$

- ▶ In general, the leverage statistics is always between  $1/n$  and 1, and the average leverage for all the observations is always equal to  $(p + 1)/n$
- ▶ So if a given observation has a leverage statistic that greatly exceeds  $(p + 1)/n$ , then we may suspect that the corresponding point has high leverage
- ▶ A value whose absence would significantly change the regression equation is termed an *influential observation*
  - ▶ Although an influential point will typically have high leverage, a high leverage point is not necessarily an influential point
  - ▶ Comparison between influential and high leverage point

## 5. High Leverage Points

- ▶ Cook's distance is a commonly used estimate of the influence of a data point

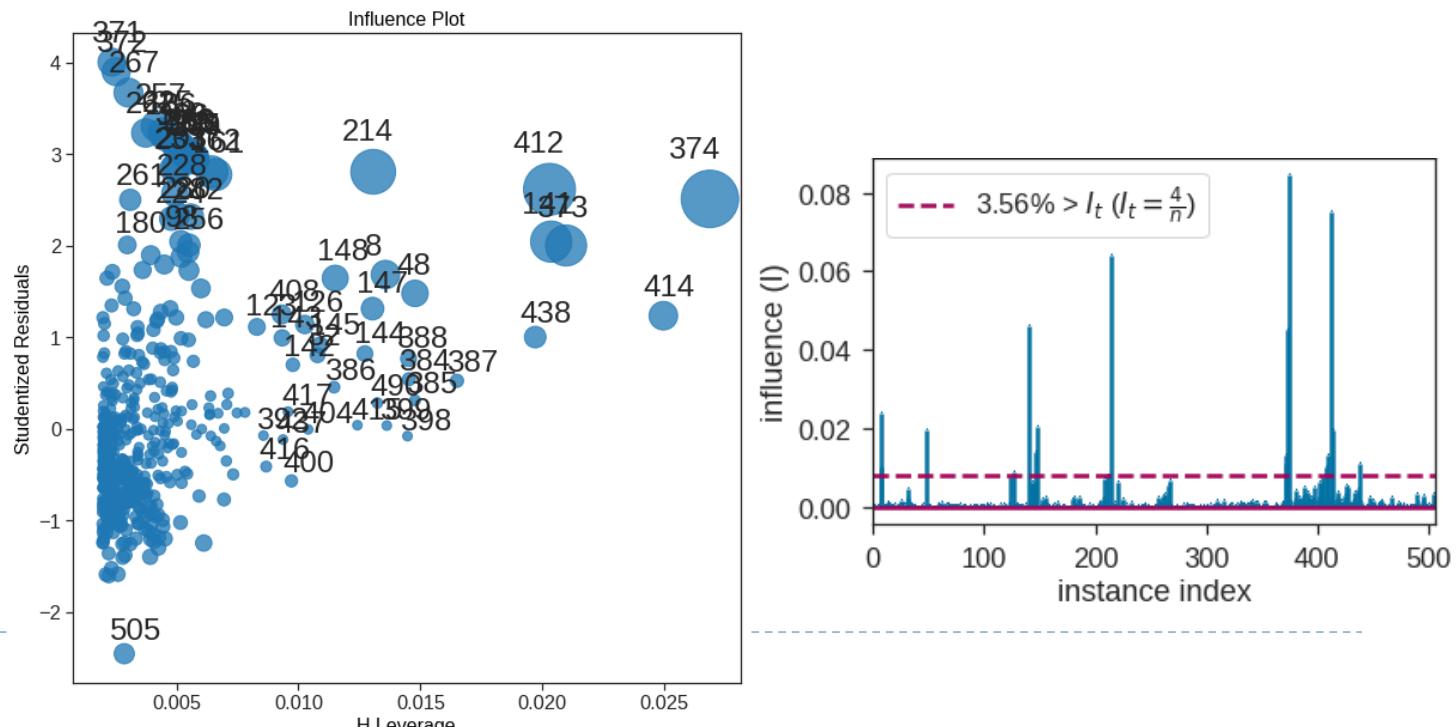
$$\frac{\sum_{j=1}^n (\hat{y}_j - \hat{y}_{j(-i)})}{(p+1) \hat{\sigma}^2} = \frac{1}{p+1} t_i^2 \frac{h_i}{1-h_i}$$

$\hat{y}_{j(-i)}$  is the fitted response value obtained when excluding  $i$

- ▶ An influence plot can be used to analyze data points

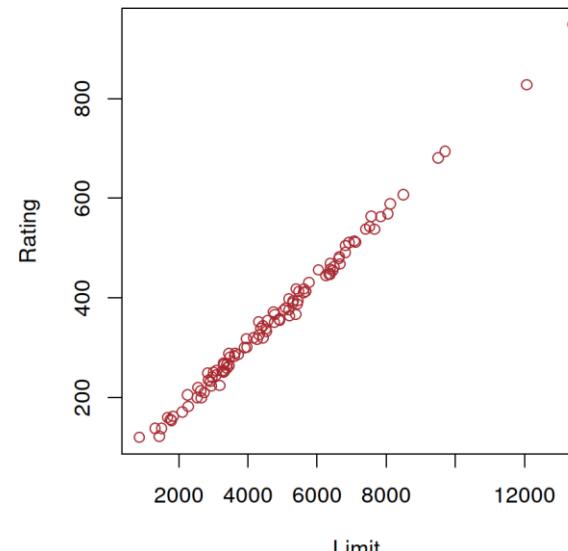
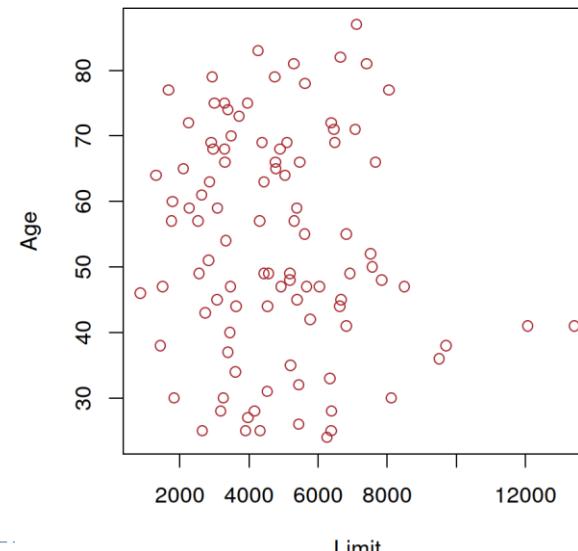
- ▶ How to deal with influential points?

- ▶ It indicates data points that are particularly worth checking for validity
- ▶ It indicates regions of the design space where it would be good to be able to obtain more data points



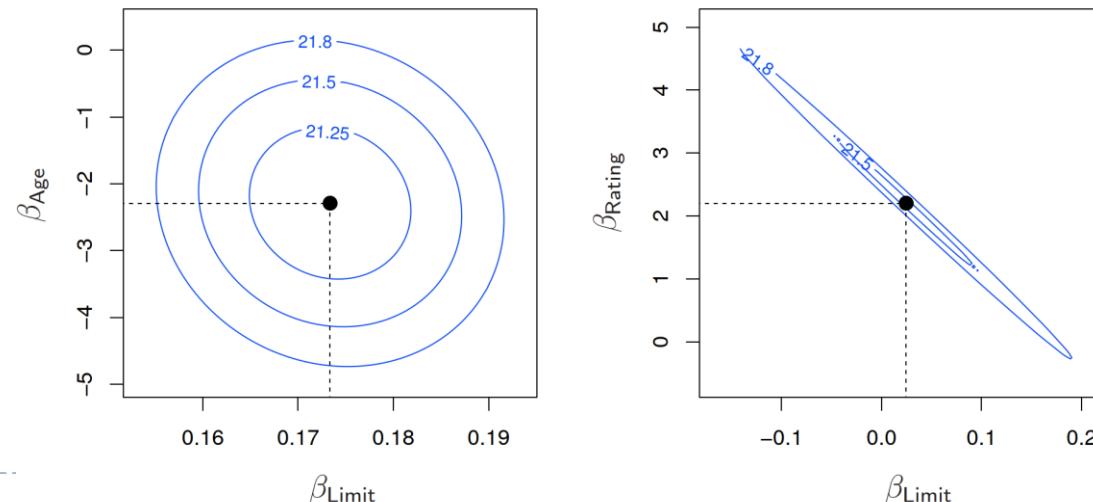
## 6. Collinearity

- ▶ Collinearity refers to the situation in which two or more predictor variables are closely related to one another
  - ▶ The concept of collinearity is illustrated below using the Credit data set
  - ▶ In the left-hand panel, the `limit` and `age` appear to have no obvious relationship
  - ▶ In contrast, in the right-hand panel, the predictor `limit` and `rating` are very highly correlated with each other, and we say that they are collinear



## 6. Collinearity

- ▶ Figure below illustrates some of the difficulties that can result from collinearity
  - ▶ The left-hand panel is a contour plot of the RSS associated with different possible coefficient estimates for the regression of balance on limit and age
  - ▶ Each ellipse represents a set of coefficients that correspond to the same RSS, with ellipses nearest to the center taking on the lowest values of RSS
  - ▶ The black dots and associated dashed lines represent the coefficient estimates that result in the smallest possible RSS—in other words, these are the least squares estimates



## 6. Collinearity

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- ▶ The contours run along a narrow valley; there is a broad range of values for the coefficient estimates that result in equal values for RSS
  - ▶ Hence a small change in the data could cause the pair of coefficient values that yield the smallest RSS—that is, the least squares estimates—to move anywhere along this valley
  - ▶ This results in a great deal of uncertainty in the coefficient estimates and thus reduce the *power* of the hypothesis test
- ▶ The table below compares the coefficient estimates obtained from two separate multiple regression models
  - ▶ In the first regression, both `age` and `limit` are highly significant with very small *p*-values
  - ▶ In the second, the collinearity between `limit` and `rating` has caused the standard error for the `limit` coefficient estimate to increase by a factor of 12 and the *p*-value to increase to 0.701

## 6. Collinearity

- ▶ In other words, the importance of the limit variable has been masked due to the presence of collinearity
- ▶ A simple way is to use a correlation matrix to detect collinearity. Unfortunately, it is possible for collinearity to exist between three or more variables even if no pair of variables has a particularly high correlation

		Coefficient	Std. error	t-statistic	p-value
Model 1	Intercept	−173.411	43.828	−3.957	< 0.0001
	age	−2.292	0.672	−3.407	0.0007
	limit	0.173	0.005	34.496	< 0.0001
Model 2	Intercept	−377.537	45.254	−8.343	< 0.0001
	rating	2.202	0.952	2.312	0.0213
	limit	0.025	0.064	0.384	0.7012

## 6. Collinearity

---

- ▶ A better way to assess *multicollinearity* is to compute the *variance inflation factor (VIF)* using the formula

$$VIF(\hat{\beta}_j) = \frac{1}{1 - R_{X_j|X_{-j}}^2}$$

where  $R_{X_j|X_{-j}}^2$  is the  $R^2$  from a regression of  $X_j$  onto all of the other predictors

- ▶ In the Credit data, a regression of balance on age, rating, and limit indicates that the predictors have VIF values of 1.01, 160.67, and 160.59
  - ▶ As a rule of thumb, a VIF value that exceeds 5 or 10 indicates a problem
- ▶ When faced with the problem of collinearity
  1. Drop one of the problematic variables from the regression since it is redundant
  2. The second solution is to combine the collinear variables together into a single predictor

## (5) Comparison of Linear Regression with K-Nearest Neighbors

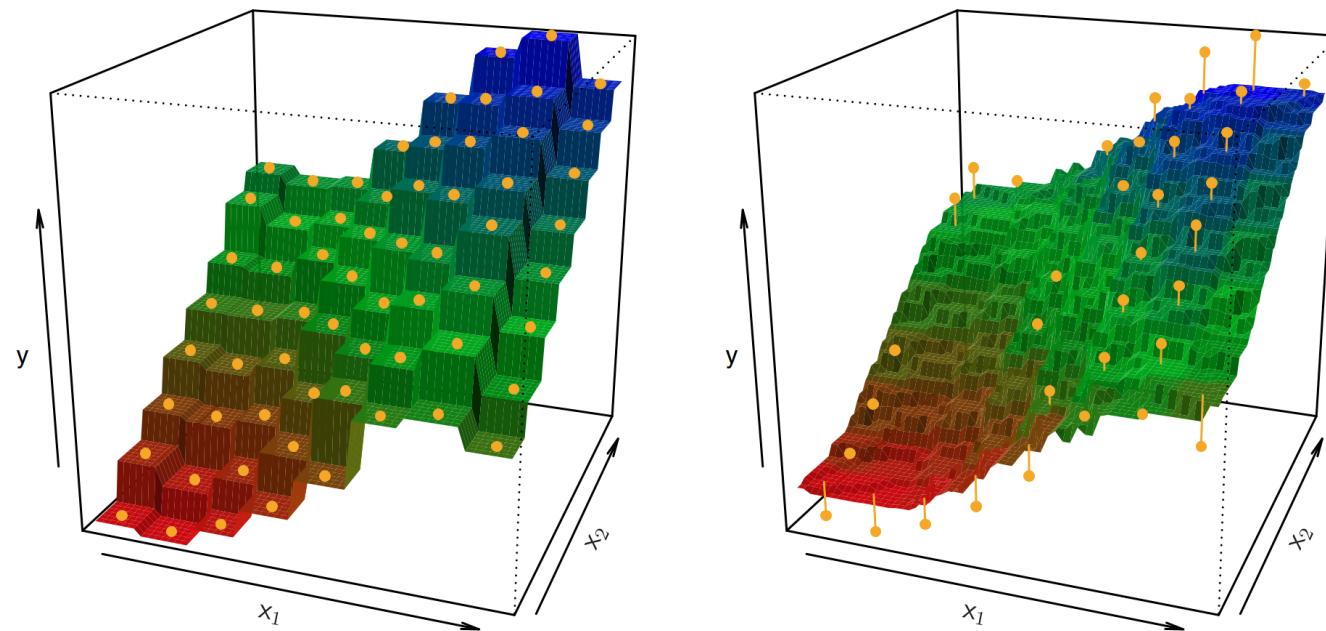
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- ▶ The non-parametric methods do not explicitly assume a parametric form for  $f(X)$  and thereby provide an alternative and more flexible approach
  - ▶ Here, we consider one of the simplest and best-known non-parametric methods, *K-nearest neighbors regression* (KNN regression)
  - ▶ Given a value for  $K$  and a prediction point  $x_0$ , KNN regression first identifies the  $K$  training observations that are closest to  $x_0$ , represented by  $N_0$
  - ▶ It then estimates  $f(x_0)$  using the average of all the training responses in  $N_0$ . In other words,

$$\hat{f}(x_0) = \frac{1}{K} \sum_{x_i \in N_0} y_i$$

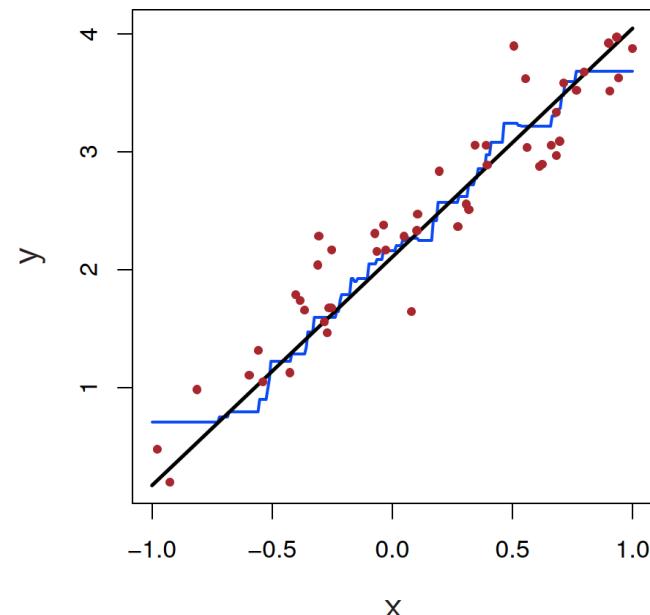
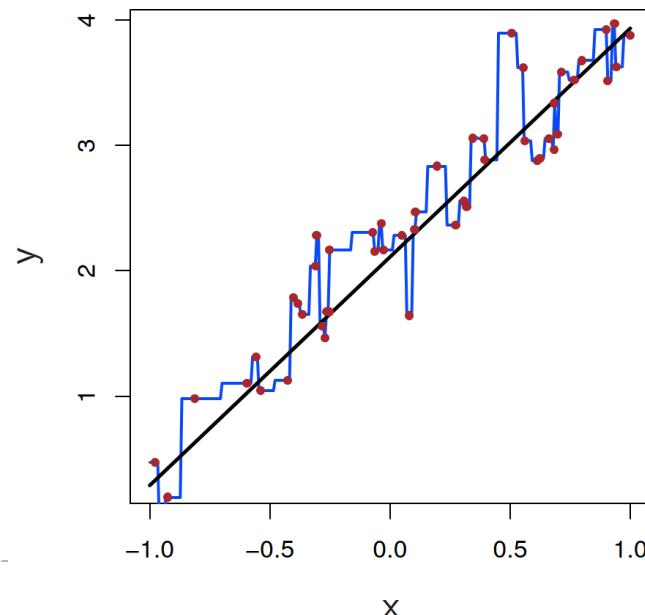
# Comparison of Linear Regression with K-Nearest Neighbors

- ▶ The figure below illustrates two KNN fits on a data set with  $p = 2$  predictors
  - ▶  $K = 1$  is shown in the left-hand panel, while the right-hand panel corresponds to  $K = 9$
- ▶ In general, the optimal value for  $K$  will depend on the bias-variance tradeoff
  - ▶ A small value for  $K$  provides the most flexible fit, which will have low bias but high variance



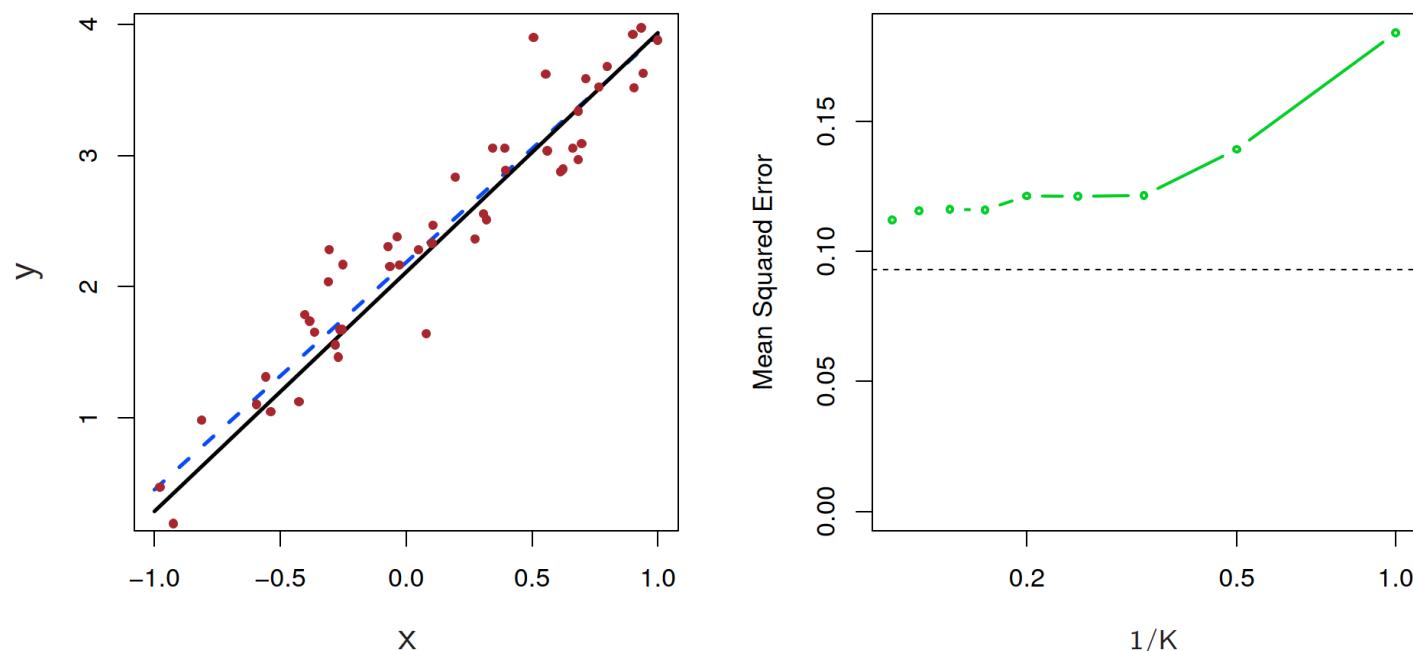
# Comparison of Linear Regression with K-Nearest Neighbors

- ▶ The black solid lines represent  $f(X)$ , while the blue curves correspond to the KNN fits using  $K = 1$  and  $K = 9$  ( $n = 50$ )
- ▶ In this case, the  $K = 1$  predictions are far too variable, while the smoother  $K = 9$  fit is much closer to  $f(X)$
- ▶ However, since the true relationship is linear, it is hard for a non-parametric approach to compete with linear regression: a non-parametric approach incurs a cost in variance that is not offset by a reduction in bias



# Comparison of Linear Regression with K-Nearest Neighbors

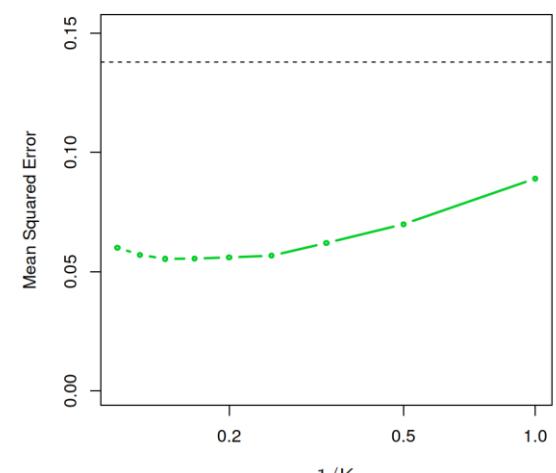
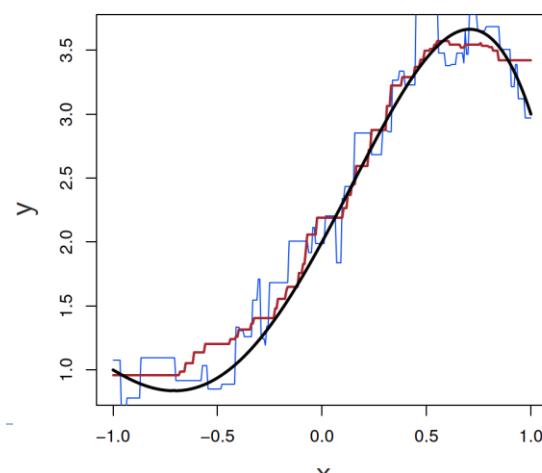
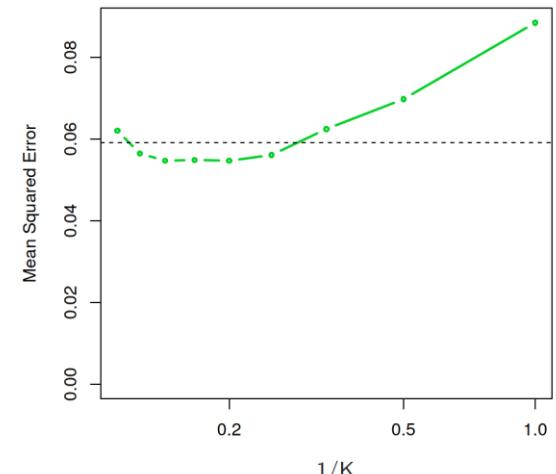
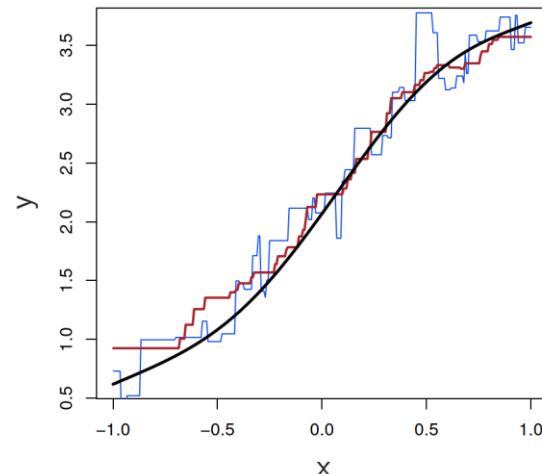
- ▶ The blue dashed line in the left-hand panel of Figure represents the linear regression fit to the same data
  - ▶ In the right-hand panel, the green solid line, plotted as a function of  $1/K$ , represents the test set mean squared error (MSE) for KNN
  - ▶ The KNN errors are well above the black dashed line, which is the test MSE for linear regression. When the value of  $K$  is large, then KNN performs only a little worse than least squares regression in terms of MSE



$$MSE = \frac{RSS}{n}$$

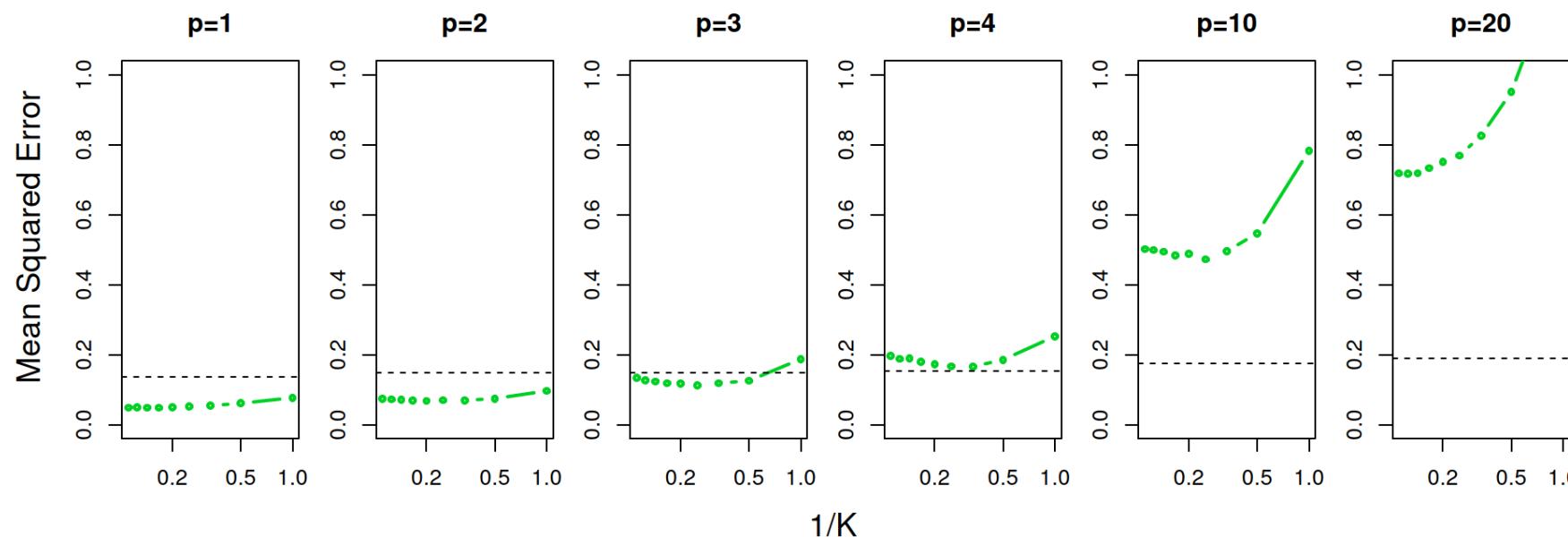
# Comparison of Linear Regression with K-Nearest Neighbors

- In practice, the true relationship between  $X$  and  $Y$  is rarely exactly linear
  - In the top row, the true relationship is close to linear
  - The bottom row illustrates a more substantial deviation from linearity. In this situation, KNN substantially outperforms linear regression for all values of  $K$
  - Note that as the extent of non-linearity increases, there is little change in the test set MSE for the non-parametric KNN method, but there is a large increase in the test set MSE of linear regression



# Comparison of Linear Regression with K-Nearest Neighbors

- ▶ The figure below considers the same strongly non-linear situation, except that we have added additional noise predictors that are not associated with the response
  - ▶ In fact, the increase in dimension has only caused a small deterioration in the linear regression test set MSE, but it has caused more than a ten-fold increase in the MSE for KNN!
  - ▶ This decrease in performance as the dimension increases is a common problem for KNN!



# Generalizations of the Linear Model

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- ▶ In much of the rest of this course, we discuss methods that expand the scope of linear models and how they are fit:
  - ▶ Classification problems: logistic regression, support vector machines
  - ▶ Non-linearity: kernel smoothing, splines and generalized additive models
  - ▶ Interactions: Tree-based methods, bagging, random forests and boosting (these also capture non-linearities)
  - ▶ Regularized fitting: Ridge regression and lasso

# Appendix

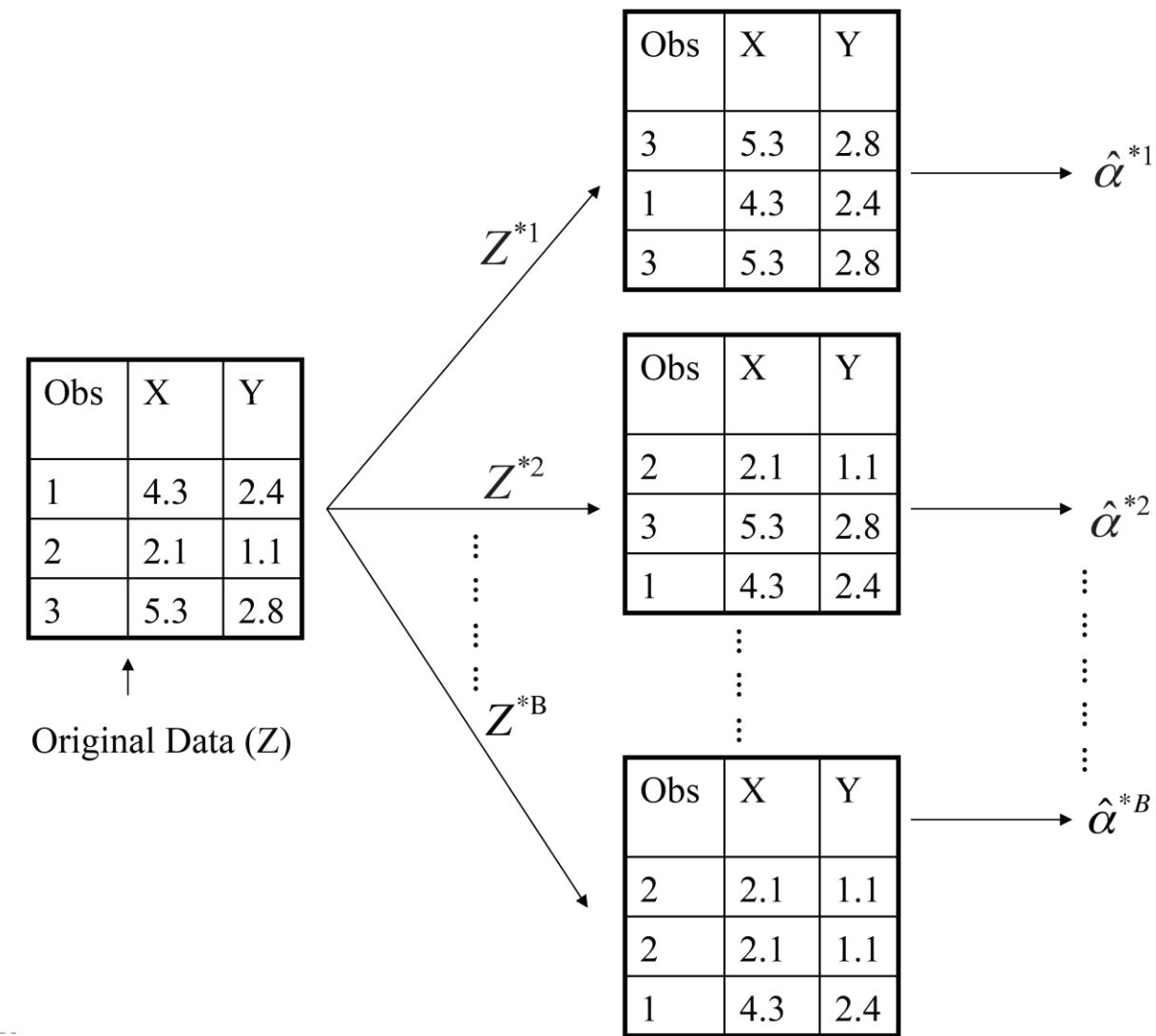
# Reference

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- ▶ About statistics
  - ▶ <http://www.stat.nthu.edu.tw/~swcheng/Teaching/math2820/index.php>
  - ▶ <https://www.youtube.com/playlist?list=PLj6E8qlqmkFvsst4-ww1mrax1D65FQI1m>
  - ▶ <https://www.youtube.com/playlist?list=PLblh5JKOoLUK0FLuzwntyYI10UQFUhsY9>
- ▶ About the proof of linear regression
  - ▶ [https://en.wikipedia.org/wiki/Proofs\\_involving\\_ordinary\\_least\\_squares](https://en.wikipedia.org/wiki/Proofs_involving_ordinary_least_squares)
  - ▶ [https://en.wikipedia.org/wiki/Simple\\_linear\\_regression](https://en.wikipedia.org/wiki/Simple_linear_regression)
- ▶ About the concept of linear regression
  - ▶ <https://online.stat.psu.edu/stat501/lesson/3/3.3>
  - ▶ <https://online.stat.psu.edu/stat415/lesson/8/8.1>
  - ▶ <https://stats.stackexchange.com/questions/85560/shape-of-confidence-interval-for-predicted-values-in-linear-regression>

# Bootstrap for confidence interval

- ▶ A graphical illustration of the bootstrap approach on a small sample containing  $n = 3$  observations
- ▶ Each bootstrap data set contains  $n$  observations, sampled with replacement from the original data set. Each bootstrap data set is used to obtain an estimate of  $\alpha$



# Bootstrap for confidence interval

<https://www.statsmodels.org/stable/examples/notebooks/generated/lowess.html#Confidence-interval>

1. Generate  $n$  “bootstrap sample” data points  $x_i^*, y_i^*$
2. Fit regression using  $x_i^*, y_i^*$
3. Evaluate the regression line on fix  $x$ -grid
4. Repeat step 1-3 for  $B$  times and collect the values in step 3.
5. For each point in the  $x$ -grid, calculate the confidence interval using collected value

```
def lowess_with_confidence_bounds(  
    x, y, eval_x, N=200, conf_interval=0.95, lowess_kw=None  
):  
    """  
    Perform Lowess regression and determine a confidence interval by bootstrap resampling  
    """  
    # Lowess smoothing  
    smoothed = sm.nonparametric.lowess(exog=x, endog=y, xvals=eval_x, **lowess_kw)  
  
    # Perform bootstrap resamplings of the data  
    # and evaluate the smoothing at a fixed set of points  
    smoothed_values = np.empty((N, len(eval_x)))  
    for i in range(N):  
        sample = np.random.choice(len(x), len(x), replace=True)  
        sampled_x = x[sample]  
        sampled_y = y[sample]  
  
        smoothed_values[i] = sm.nonparametric.lowess(  
            exog=sampled_x, endog=sampled_y, xvals=eval_x, **lowess_kw  
        )  
  
    # Get the confidence interval  
    sorted_values = np.sort(smoothed_values, axis=0)  
    bound = int(N * (1 - conf_interval) / 2)  
    bottom = sorted_values[bound - 1]  
    top = sorted_values[-bound]  
  
    return smoothed, bottom, top  
  
# Compute the 95% confidence interval  
eval_x = np.linspace(0, 4 * np.pi, 31)  
smoothed, bottom, top = lowess_with_confidence_bounds(  
    x, y, eval_x, lowess_kw={"frac": 0.1}  
)
```

## CCPR plot or Partial regression plot

---

- ▶ The CCPR plot provides a way to judge the effect of one regressor on the response variable by taking into account the effects of the other independent variables. The partial residuals plot is defined as  $\text{residuals} + \beta_i X_i$  versus  $X_i$ . The component adds  $\beta_i X_i$  versus  $X_i$  to show where the fitted line would lie. Care should be taken if  $X_i$  is highly correlated with any of the other independent variables. If this is the case, the variance evident in the plot will be an underestimate of the true variance.
- ▶ [https://en.wikipedia.org/wiki/Partial\\_residual\\_plot#CCPR\\_plot](https://en.wikipedia.org/wiki/Partial_residual_plot#CCPR_plot)
- ▶ [https://en.wikipedia.org/wiki/Partial\\_regression\\_plot](https://en.wikipedia.org/wiki/Partial_regression_plot) (added variable plot)

## Two quotes by famous Statisticians

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- ▶ “Essentially, all models are wrong, but some are useful” - George Box
- ▶ “The only way to find out what will happen when a complex system is disturbed is to disturb the system, not merely to observe it passively” - Fred Mosteller and John Tukey, paraphrasing George Box

# Interpreting regression coefficients

---

- ▶ The ideal scenario is when the predictors are uncorrelated - a balanced design:
  - ▶ Each coefficient can be estimated and tested separately
  - ▶ Interpretations such as “a unit change in  $X_j$  is associated with a  $\beta_j$  change in  $Y$ , while all the other variables stay fixed”, are possible
- ▶ Correlations amongst predictors cause problems:
  - ▶ The variance of all coefficients tends to increase, sometimes dramatically
  - ▶ Interpretations become hazardous - when  $X_j$  changes, everything else changes
- ▶ *Claims of causality* should be avoided for observational data