The similar diagonalization problem of real symmetric matrices

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from Math and Applied Math (QiangJiJiHua 2001)

June 27, 2022

This essay is aimed to submit the first assignment of the course Mathematics Software by Wang Heyu, Zhejiang University, 2021-2022 short semester. The problem is an essential question Linear Algebra, also have significance in Numerical Linear Algebra. The following proposition provide one of the most important views for studying the decomposition of real symmetric matrix.

1 Propositions

Theorem. A is a real symmetric matrix, then there is an orthogonal matrix Q, such that

$$Q^{-1}AQ = Q^TAQ = D \tag{1}$$

where D is diagonal.

2 Proof

Proof.

At first, we are going to explain that all the eigenvalues of A is real. Assume λ is an eigen value of A and $\exists q s.t. Aq = \lambda q$, then we have:

$$\lambda^2 = q^T\lambda\lambda q = q^TA^TAq = |Aq|^2 > 0 \tag{2}$$

which implies $\lambda \in \mathbb{R}$.

Now we can begin the proof by using induction to the dimension of A, which is assumed to be n:

$n = 1$: Trivial

$n = k \implies n = k+1$: According to the algebra fundamental theorem, there have to be an eigenvalue λ_0 of $A \in \mathbb{R}^{(k+1)\times(k+1)}$, and a vector $q, |q| = 1 s.t. Aq = \lambda_0 q$. By using Schimdt orthogonalization, we can get a standard orthogonal β basis contain q easily, and we can assume

$$[A]_\beta = \begin{pmatrix} \lambda & \\ & A_1 \end{pmatrix} \tag{3}$$

without loss of generality. And there is an orthogonal matrix $Q s.t. [A]_\beta = Q^TAQ$, so

$$A = Q[A]_\beta Q^T \implies A_1 = Q_1A_1Q_1^T \tag{4}$$

then use the induction to A_1, the theorem is proved.