JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 94, No. 1, pp. 29-54, JULY 1997

Rates of Convergence of Ordinal Comparison for Dependent Discrete Event Dynamic Systems[1]

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Communicated by Y. C. Ho

Abstract. Recent research has demonstrated that ordinal comparison, i.e., comparing relative orders of performance measures, converges much faster than the performance measures themselves do. Sometimes, the rate of convergence can be exponential. However, the actual rate is affected by the dependence among systems under consideration. In this paper, we investigate convergence rates of ordinal comparison for dependent discrete event dynamic systems. Although counterexamples show that positive dependence is not necessarily helpful for ordinal comparison, there does exist some dependence that increases the convergence rate of ordinal comparison. It is shown that positive quadrant dependence increases the convergence rate of ordinal comparison, while negative quadrant dependence decreases the rate. The results of this paper also show that the rate is maximized by using the scheme of common random numbers, a widely-used technique for variance reduction.

Key Words. Discrete event dynamic systems, ordinal comparison, convergence rate, correlation.

1. Introduction

Let us consider the following design problem. We are given a set of N possible designs, indexed by θ_n, $n=1, 2, \ldots, N$, of a discrete event dynamic system. Our task is to choose the best design based on a performance measure $J(\theta)$. We say that design θ_i is better than design θ_j if $J(\theta_i) > J(\theta_j)$.

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Let $\Theta = \{\theta_n, n=1, 2, \ldots, N\}$. Then, the problem is to find the $\theta^* \in \theta$ such that

$$\theta^* = \arg\max_{\theta \in \Theta} J(\theta).$$

The difficulty is that the best designs for the problems dealt in this paper have to be chosen without exactly knowing the values of performance measures.

Discrete event dynamic systems (DEDS) usually have complex structures and are analytically intractable. Closed-form formulas of performance measures are unavailable for most DEDS except for some special cases. A common approach is to use simulation or trajectory observation to obtain an estimate of $J(\theta)$. Let $h(\theta, t)$ denote an estimate of $J(\theta)$ based on simulation or observation for a time period of t. Then, $h(\theta, t)$ is a random variable. We need to choose the true best design θ^* based on the noisy information $h(\theta, t)$.

It is known that an estimate $h(\theta, t)$ converges slowly, if it does, as time t goes to infinity. In general, the rate of convergence of such a value estimate is at best $O(1/\sqrt{t})$; see Refs. 1–2. Simulations or observations aiming at accurately estimating performance measures are costly and time consuming. A natural question is how to avoid such a value-estimation procedure and yet still to find the true best design.

Recent research has shown that the relative order of $\{h(\theta, t)\}$ converges fast. For some DEDS, the rate of convergence of ordinal comparison can be exponential (Ref. 3). Besides ordinal comparison, ordinal optimization in a much broader sense has been demonstrated to be a promising approach for selecting a good design from a finite number of designs (Refs. 4–9) and even for finding the best design over a continuous design space (Rep. 10–13). If our goal is to find the best or good designs, rather than to find an accurate estimate of the best performance value, which is true in many practical situations, it is advantageous to use ordinal comparison for selecting the best design.

This paper is concerned with the effect of correlation dependence on ordinal comparison, especially correlation among different designs of a DEDS. Such dependence occurs in simulation or sample path reconstruction. We are particularly interested in the effect of correlation dependence on the rate of convergence, to be defined later, for ordinal comparison.

Although counterexamples (see Refs. 3, 14) show that positive correlation is not necessarily helpful for ordinal comparison, experimental as well as theoretical results for some special cases have shown that there does exist some dependence that increases the chance of selecting the true best design. However, the class of problems for which an explicit dependence is known to be advantageous is very limited. For a performance measure of averaging i.i.d. random variables with normal distributions, Ho et al. in

Ref. 8 show that the probability of correct selection is an increasing function of the correlation coefficients, implying that positive dependence helps ordinal comparison and negative dependence does the opposite. Glasserman and Vakili in Ref. 14 prove that, for the case of additive performance measures of finite-state Markov chains, association (a type of positive dependence) indeed increases the rate of convergence. Yang and Nelson in Ref. 15 show that the schemes of common random numbers and control variates are helpful in obtaining better confidence intervals for various selection procedures when the performance measure is averaging i.i.d. random variables with normal distributions. Reference 16 provides a tutorial on some important issues of simulation design for order selection.

In this paper, we investigate the effect of various correlations on ordinal comparison. Our goal is to identify the type of correlation that increases or maximizes the convergence of ordinal comparison. The starting point is a notion of indicator process which characterizes the dynamical behavior of ordinal comparison. It is shown that positive quadrant dependence, stronger than positive correlation, increases the convergence rate of the indicator process while negative quadrant dependence decreases the rate. It turns out that the scheme of common random numbers maximizes this rate. Therefore, we find again the power of common random numbers in ordinal comparison.

2. Common Random Numbers

When comparing alternative designs, it has been suggested that simulation experiments under different designs be performed with the same stream of random numbers to improve the accuracy of comparison, widely known as the scheme of common random numbers (CRN). Indeed, it has been observed experimentally, and was proved theoretically in some cases, that the CRN scheme is often effective for variance reduction (Ref. 17). Since the CRN scheme is intuitively appealing and easily implementable, it is perhaps the most popular method for variance reduction.

A function $g(X_1, X_2, \ldots, X_n): R^n \to R$ is said to be superadditive if, for any $X_i \le X'_i$, $X_j \le X'_j$,

$$\begin{aligned} &g(\ldots, X'_i, \ldots, X_j, \ldots) + g(\ldots, X_i, \ldots, X'_j, \ldots) \\ &\le g(\ldots, X_i, \ldots, X_j, \ldots) + g(\ldots, X'_i, \ldots, X'_j, \ldots). \end{aligned}$$

If $g(X_1, X_2, \ldots, X_n)$ is absolutely continuous, then its second-order derivatives exist a.e.; g is superadditive if and only if

$$\partial^2 g(X_1, X_2, \ldots, X_n)/\partial X_i\, \partial X_j \ge 0, \qquad \text{for any pair } i \ne j.$$

The effectiveness of the CRN scheme for variance reduction lies in the following result due to Refs. 18–19.

Theorem 2.1. Let X_i, $i=1, 2, \ldots, n$, be random variables with marginal distributions $F_i(x_i)$, $i=1, 2, \ldots, n$, and joint distribution $H(x_1, x_2, \ldots, x_n)$. Assume that $g(X_1, X_2, \ldots, X_n)$ is right-continuous and superadditive and that $E[g(X_1, X_2, \ldots, X_n)]$ is finite for all $H(x_1, x_2, \ldots, x_n)$. Then, $E[g(X_1, X_2, \ldots, X_n)]$ is maximized by choosing the joint distribution

$$H(x_1, x_2, \ldots, x_n) = \min_i F_i(x_i). \tag{1}$$

Let u be a random variable uniform on [0, 1]. Then, the joint distribution (1) can be realized by choosing $\{X_i\}$ according to (inversion method)

$$X_i = F_i^{-1}(u), \qquad i=1, 2, \ldots, n,$$

where the inverse $F^{-1}(u)$ of a function $F(x)$ is defined as

$$F^{-1}(u) = \inf\{x \in R \mid F(x) \geq u\}.$$

Therefore, Theorem 2.1 says that $E[g(X_1, X_2, \ldots, X_n)]$ is maximized by sampling all X_i using the scheme of CRN. If $g = X_1 X_2$, then the CRN scheme maximizes the covariance of X_1 and X_2, thus minimizing the variance of $X_1 - X_2$.

3. Indicator Processes for Ordinal Comparison

Without loss of generality, we assume that the N designs are indexed in such a way that $\infty > J(\theta_1) > J(\theta_2) > J(\theta_3) > \cdots > J(\theta_N) > -\infty$. For convenience, we denote by $\Theta_b = \Theta - \{\theta_1\}$ the set of bad designs. To find the best design based on simulation or trajectory observation over [0, t], we use the following empirical best as an estimate of the true best design:

$$\tilde{\theta}_t = \arg\max_{\theta \in \Theta} h(\theta, t). \tag{2}$$

Experimental results have shown that (2) can quickly allocate the true best design; see Refs. 4, 9. Intuitively, this implies that the relative order of performance measures converges fast.

In order to characterize the convergence of ordinal comparison, or the convergence of (2), we consider the following indicator process:

$$I(t) = \begin{cases} 1, & \text{if } h(\theta_1, t) \geq \max_{\theta \in \Theta_b} h(\theta, t), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then, $I(t)$ equals 1 if the observed best design is the true best design and equals 0 otherwise.

If the estimator $h(\theta, t)$ is consistent for every $\theta \in \Theta$ in the sense that

$$\lim_{t\to\infty} h(\theta, t) = J(\theta), \qquad \text{a.s.},$$

then

$$\lim_{t\to\infty} \text{Prob}[I(t)=1]=1;$$

that is, the true best design can be found with probability one as time goes to infinity. In this paper, we only consider consistent estimators of the type (31) in the Appendix. We are mainly concerned with the convergence rate of $I(t)$. Since $I(t)$ is a stochastic process taking only two values (1 and 0), the standard measure for its rate of convergence is the rate at which the square root of the mean squared error $(E[(I(t)-1)^2])^{1/2}$ goes to 0. Since

$$(E[(I(t)-1)^2])^{1/2} = (\text{Prob}[I(t)=0])^{1/2},$$

it is sufficient for us to examine the convergence rate of $\text{Prob}[I(t)=0]$. For convenience, we also define, for any $\theta_i \in \theta_b$,

$$I_i(t) = \begin{cases} 1, & \text{if } h(\theta_1, t) \geq h(\theta_i, t), \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Then, $I_i(t)$ is the indicator process for the comparison between the best design and the design θ_i.

In this paper, we are interested in the effect of correlation on the rate of convergence of the indicator process or of the probability $\text{Prob}[I(t)=0]$. Particularly, we would like to identify the type of correlation that increases or maximizes the rate. To this end, we need to have an expression of the rate. We find that the concept of large deviation is useful. The following definition is adopted from Ref. 20.

Definition 3.1. Let $\{X_\epsilon\}$ be a family of random variables defined on a state space $\mathscr{X}$. We say that $\{X_\epsilon\}$ satisfies the large deviation principle with a rate function ϕ if, for every closed set $A \subset \mathscr{X}$,

$$\limsup_{\epsilon\to 0} \epsilon \log \text{Prob}[X_\epsilon \subset A] \leq -\inf_{\lambda\in A} \phi(\lambda),$$

and for every open set $B \subset \mathscr{X}$,

$$\liminf_{\epsilon\to 0} \epsilon \log \text{Prob}[X_\epsilon \subset B] \geq -\inf_{\lambda\in B} \phi(\lambda),$$

where the mapping $\phi: \mathscr{X} \to [0, \infty]$ is lower semicontinuous; i.e.,

$$\liminf_{\lambda_n \to \lambda} \phi(\lambda_n) \geq \phi(\lambda), \qquad \text{for all } \lambda \in \mathscr{X}.$$

Now, define the logarithmic moment generating function

$$\Lambda_\epsilon(s) = \log E[e^{sX_\epsilon}],$$

and suppose that the limit

$$\Lambda(s) = \lim_{\epsilon \to 0} \epsilon \Lambda_\epsilon(s/\epsilon)$$

exists pointwise. The following theorem from Ref. 20 gives an expression for the rate function ϕ.

Theorem 3.1. Assume that $\{X_\epsilon\}$ satisfies the large deviation principle with a rate function ϕ. If ϕ is convex and, for all $c \in R^+$, the set $\{\lambda: \phi(\lambda) \leq c\}$ is a compact subset of $\mathscr{X}$, then ϕ is the Fenchel–Legendre transform of $\Lambda(s)$, namely,

$$\phi(\lambda) = \sup_s \{\lambda s - \Lambda(s)\}.$$

If in addition,

$$\inf_{\lambda \in A^o} \phi(\lambda) = \inf_{\lambda \in \bar{A}} \phi(\lambda), \tag{5}$$

for $A \subset \mathscr{X}$, where A^o and $\bar{A}$ are the interior and closure of A respectively, then we know from Definition 3.1 and Theorem 3.1 that

$$\lim_{\epsilon \to 0} \epsilon \log \text{Prob}[X_\epsilon \subset A] = -\inf_{\lambda \in \bar{A}} \phi(\lambda). \tag{6}$$

The previous equation gives an expression for the convergence rate of $\text{Prob}[X_\epsilon \subset A]$ as ϵ goes to zero.

Assumption 3.1. For any $\theta \in \Theta_b$, the following conditions are satisfied for all possible correlations between $h(\theta, t)$ and $h(\theta_1, t)$:

(i) For all $t \geq 0$, $\log E[e^{st(h(\theta,t) - h(\theta_1,t))}]$ exists (might be $+\infty$) for all $s \geq 0$ and is finite in $[0, s_0)$ for some $s_0 > 0$.

(ii) The limit

$$\Lambda(s, \theta_1, \theta) = \lim_{t \to \infty} (1/t) \log E[e^{st(h(\theta,t) - h(\theta_1,t))}] \tag{7}$$

exists, is continuous, and is finite in $[0, s_0)$. We denote

$$C(\theta_1, \theta) = -\inf_{s \in [0,\infty)} \Lambda(s, \theta_1, \theta).$$

(iii) The process $\{h(\theta, t) - h(\theta_1, t)\}$ satisfies the large deviation principle with a convex rate function ϕ for which the set $\{\lambda: \phi(\lambda) \leq c\}$ is compact for all $c \in R^+$.

Remark 3.1. Assumption 3.1 is a matter of technicality. It is known to be satisfied for the following additive performance measure:

$$h(\theta, t) = (1/t) \sum_{i=1}^{t} L(X_i(\theta)),$$

when either (a) or (b) holds:

(a) $\{(X_i(\theta_1), X_i(\theta))\}$ is a sequence of i.i.d. random variables; in this case,

$$\Lambda(s, \theta_1, \theta) = \log E[e^{s(L(X_1(\theta)) - L(X_1(\theta_1)))}]$$

under the assumption that it is finite;

(b) $\{(X_i(\theta_1), X_i(\theta))\}$ is a finite-state irreducible Markov chain with transition probability matrix $[p((x, y), (x', y'))]$. In this case,

$$\Lambda(s, \theta_1, \theta) = \log \rho(\theta_1, \theta),$$

where $\rho(\theta_1, \theta)$ is the Perron–Frobenius maximal eigenvalue of the irreducible matrix $[p((x, y), (x', y'))\, e^{s(L(x') - L(y'))}]$.

Other cases where Assumption 3.1 is satisfied can be found in Ref. 20. Assumption 3.1 allows us to have a characterization of the convergence rate of the indicator process.

Theorem 3.2. Under Assumption 3.1, the convergence rate of $I_i(t)$ is given by

$$C(\theta_1, \theta_i) = -\lim_{t \to \infty} (1/t) \log \mathrm{Prob}[I_i(t) = 0]. \tag{8}$$

Proof. According to (4),

$$\begin{aligned}\mathrm{Prob}[I_i(t) = 0] &= \mathrm{Prob}[h(\theta_1, t)] \\ &= \mathrm{Prob}[h(\theta_i, t) - h(\theta_1, t) > 0].\end{aligned}$$

Choose

$$A = (0, \infty) \quad \text{and} \quad X_\epsilon = h(\theta_i, t) - h(\theta_1, t).$$

Assumption 3.1 implies that Theorem 3.1 applies. The continuity assumption of $\Lambda(s, \theta_1, \theta_i)$ in Assumption 3.1(ii) guarantees that (5) holds. Therefore, we know from (6) that

$$-\lim_{t\to\infty}(1/t)\log \mathrm{Prob}[I_i(t)=0]=\inf_{\lambda\in[0,\infty)}\phi(\lambda).$$

Moreover, because of the special form of A,

$$\inf_{\lambda\in[0,\infty)}\phi(\lambda)=\inf_{\lambda\in[0,\infty)}\sup_{s\geq 0}\{\lambda s-\Lambda(s,\theta_1,\theta_i)\}=-\inf_{s\in[0,\infty)}\Lambda(s,\theta_1,\theta_i),$$

which is exactly $C(\theta_1, \theta_i)$. □

If (8) holds, then $C(\theta_1, \theta_i)\geq 0$. If $C(\theta_1, \theta_i)>0$, then

$$\mathrm{Prob}[I_i(t)=0]\approx e^{-C(\theta_1,\theta_i)t}, \qquad \text{for large } t.$$

The following result reduces the multiple comparison to the comparison of only two designs.

Theorem 3.3. The rate at which $\mathrm{Prob}[I(t)=0]$ tends toward zero is the same as that of

$$\max_{i:\,\theta_i\in\Theta_b}\mathrm{Prob}[I_i(t)=0];$$

in other words, it converges at the slowest rate of $I_i(t)$.

Proof. First, we have

$$\mathrm{Prob}[h(\theta_1,t)\geq\max_{\theta\in\Theta_b}h(\theta,t)]\leq\min_{\theta\in\Theta_b}\mathrm{Prob}[h(\theta_1,t)\geq h(\theta,t)].$$

Therefore,

$$\begin{aligned}\mathrm{Prob}[I(t)=0]&=1-\mathrm{Prob}[h(\theta_1,t)\geq\max_{\theta\in\Theta_b}h(\theta,t)]\\&\geq 1-\min_{\theta\in\Theta_b}\mathrm{Prob}[h(\theta_1,t)\geq h(\theta,t)]\\&=\max_{i:\,\theta_i\in\Theta_b}\mathrm{Prob}[I_i(t)=0].\end{aligned}$$

On the other hand, $\mathrm{Prob}[I(t)=0]$ is bounded from above by

$$\mathrm{Prob}[h(\theta_1,t)<\max_{\theta\in\Theta_b}h(\theta,t)]\leq\sum_{i=2}^{N}\mathrm{Prob}[I_i(t)=0],$$

which converges to zero at the rate of $\max_{i:\theta_i\in\Theta_b}\mathrm{Prob}[I_i(t)=0]$. The combination of the upper and lower bounds gives Theorem 3.3. □

Therefore, to maximize the convergence rate of the indicator process $I(t)$, it is sufficient to maximize the rate of convergence of all $I_i(t)$, which according to Theorem 3.2 is equivalent to maximizing $C(\theta_1, \theta)$ simultaneously for all $\theta \in \Theta_b$.

According to our convention,

$$J(\theta_1) > J(\theta_2) > \cdots > J(\theta_N).$$

Since θ_2 is the "closest" to the true best design θ_1, it is natural to conjecture that

$$\max_{\theta \in \Theta_b} \operatorname{Prob}[I_i(t) = 0] = \operatorname{Prob}[I_2(t) = 0], \tag{9}$$

which however, is not necessarily true, as illustrated by the following simple example.

Example 3.1. Define three independent sequences of i.i.d. random variables $\{X_i(\theta_j)\}, j = 1, 2, 3$, by

$$X_i(\theta_1) \sim N(3, 0), \qquad X_i(\theta_2) \sim N(2, 0), \qquad X_i(\theta_3) \sim N(1, \sigma),$$
$$\sigma > 0, \quad i = 1, 2, 3, \ldots,$$

and three corresponding averaging performance measures

$$h(\theta_j, t) = (1/t) \sum_{i=1}^{t} X_i(\theta_j), \qquad j = 1, 2, 3.$$

Then,

$$E[h(\theta_1, t)] > E[h(\theta_2, t)] > E[h(\theta_3, t)].$$

Since

$$\operatorname{Prob}[I_2(t) = 0] = 0, \qquad \operatorname{Prob}[I_3(t) = 0] > 0,$$

it is clear that

$$\max_{i=2,3} \operatorname{Prob}[I_i(t) = 0] \neq \operatorname{Prob}[\mathrm{I}_2(\mathrm{t}) = 0].$$

This shows that (9) is not necessarily true.

Many performance estimates in DEDS have asymptotic normal distribution. Therefore, the normal distribution has been used to approximate the distribution of estimates. In this case, we can determine some further properties of ordinal comparison. Assume that $\{h(\theta_i, t), \theta_i \in \Theta\}$ are jointly normally distributed. Then,

$$\text{Prob}[I_i(t)=0]=1-\Phi,\left((J(\theta_1)-J(\theta_i))/\sqrt{\text{Var}[h(\theta_1,t)-h(\theta_i,t]}\right), \tag{10}$$

where

$$\Phi(x)=(1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-x^2/2}\,dx.$$

Since $J(\theta_1)-J(\theta_i)>0$, it is clear from (10) that:

(i) Equation (9) holds if and only if

$$[J(\theta_1)-J(\theta_2)]/\sqrt{\text{Var}[h(\theta_1,t)-h(\theta_2,t)]}$$
$$\leq[J(\theta_1)-J(\theta_i)]/\sqrt{\text{Var}[h(\theta_1,t)-h(\theta_i,t)]},\qquad \forall i=2,3,\ldots,N,$$

which means that Eq. (9) depends not only on the difference $J(\theta_1)-J(\theta_i)$, but also on the variance.

(ii) $\text{Prob}[I_i(t)=0]$ is monotone decreasing in the covariance $\text{Cov}[h(\theta_1,t),h(\theta_i,t)]$, which has also been observed in Ref. 8.

(iii) $\text{Prob}[I_i(t)=0]$ is minimized when $\text{Cov}[h(\theta_1,t),h(\theta_i,t)]$ reaches its maximal value, which can be achieved by the scheme of common random numbers.

4. Effect of Dependence on Ordinal Comparison

4.1. Association.

Definition 4.1. See Ref. 21. The random variables $X_1, X_2, \ldots, X_n$ are said to be associated if

$$\text{Cov}[f(X_1,X_2,\ldots,X_n),g(X_1,X_2,\ldots,X_n)]\geq 0,$$

for any nondecreasing functions $f(\cdot)$, $g(\cdot)$: $R^n\to R$ (a function is said to be nondecreasing if it is nondecreasing in each of its arguments).

It is clear that association is stronger than positive correlation. For any $\theta\in\Theta_b$ and any t, if $h(\theta_1,t)$ and $h(\theta,t)$ are associated, then

$$E[e^{s(h(\theta,t)-h(\theta_1,t))}]\leq E[e^{sh(\theta,t)}]E[e^{-sh(\theta_1,t)}], \tag{11}$$

for all t. Therefore, $C(\theta_1,\theta)$ is greater when $h(\theta_1,t)$ and $h(\theta,t)$ are associated than when they are independent (Ref. 14). Ho et al. in Ref. 8 show that positive correlation is always helpful for ordinal comparison in the case of an averaging sequence of i.i.d. random variables with normal distributions. In fact, they show that the probability of correct selection is monotone increasing in the correlation coefficients. For more general cases, counterexamples show that positive correlation does not necessarily increase the probability of correct selection; see Refs. 3, 14.

Association has the following property.

Lemma 4.1. See Ref. 22. The random variables $X \in R$ and $Y \in R$ are associated if and only if

$$\text{Prob}[X \in S_1, Y \in S_2] \geq \text{Prob}[X \in S_1]\,\text{Prob}[Y \in S_2],$$

for any closed sets $S_1, S_2 \subset R$.

4.2. Quadrant Dependence.

Definition 4.2. See Ref. 23. Two random variables $X \in R$ and $Y \in R$ are said to be positively quadrant dependent if

$$\text{Prob}[X \geq x, Y \geq y] \geq \text{Prob}[X \geq x]\,\text{Prob}[Y \geq y], \qquad \forall x, y, \tag{12}$$

and negatively quadrant dependent if the inequality sign is reversed.

Positive quadrant dependence has the following properties.

Lemma 4.2. See Ref. 23. If X and Y are positively quadrant dependent, then:

(i) X and $-Y$ are negatively quadrant dependent;
(ii) for any nondecreasing functions $f(X)$ and $g(Y)$,

$$E[f(X)g(Y)] \geq E[f(X)]E[g(Y)]. \tag{13}$$

If X and Y are associated, then (12) holds according to Lemma 4.1. Thus, positive quadrant dependence is weaker than association. It is clear from Lemma 4.2 that X and Y are positively correlated if they are positively quadrant dependent. As for the role of positive quadrant dependence in ordinal comparison, we have the following theorem.

Theorem 4.1. Suppose that, for every pair i and j, $h(\theta_i, t)$ and $h(\theta_j, t)$ are positively quadrant dependent. Then, the rate of convergence of ordinal comparison is faster than when $h(\theta_i, t)$ and $h(\theta_j, t)$ are independent.

Proof. The statement follows from choosing

$$f(X) = e^{sh(\theta,t)}, \quad g(Y) = -e^{-sh(\theta_1,t)}, \qquad \forall s \geq 0,$$

in (13). □

What other types of correlation can increase the rate? In order to see what kind of answer we should expect, consider the problem of comparing two performance measures of averaging i.i.d. random variables. Let

$$h(X, t) = (1/t) \sum_{i=1}^{t} f(X_i), \qquad h(Y, t) = (1/t) \sum_{i=1}^{t} g(Y_i),$$

in which $\{(X_i, Y_i)\}$ is a sequence of i.i.d. random vectors. For each i, X_i and Y_i are marginally distributed according to $X \in R$ and $Y \in R$, respectively. Then, according to Remark 3.1, the convergence rate of comparison between X and Y is

$$-\inf_{s \geq 0} \log E[e^{s(f(X) - g(Y))}].$$

In this case, we can prove the following result.

Theorem 4.2. The inequality

$$\inf_{s \geq 0} \log E[e^{s(f(X) - g(Y))}] \leq \inf_{s \geq 0} \log E[e^{sf(X)}] E[e^{-sg(Y)}] \tag{14}$$

holds for all nondecreasing, right-continuous functions f and g if and only if X and Y are positively quadrant dependent.

Proof. Since positive quadrant dependence guarantees (14), we only need to prove the "only if" part. Let $1 > \epsilon > 0$, and let x_0 and y_0 be constants. Construct

$$f(x) = \log f_1(x), \qquad g(y) = -\log g_1(y),$$

with

$$f_1(x) = \begin{cases} 1, & \text{if } x \geq x_0, \\ [(1-\epsilon)/\epsilon](x - x_0) + 1, & \text{if } x_0 - \epsilon \leq x < x_0, \\ \epsilon, & \text{if } x < x_0 - \epsilon, \end{cases}$$

$$g_1(y) = \begin{cases} 1, & \text{if } y < y_0, \\ -[(1-\epsilon)/\epsilon](y - y_0) + 1, & \text{if } y_0 < y_0 + \epsilon, \\ \epsilon, & \text{if } y \geq y_0 + \epsilon. \end{cases}$$

Then, f and g are continuous, nondecreasing. If (14) holds, we must have

$$\inf_{s \geq 0} \log E[f_1^s(X) g_1^s(Y)] \leq \inf_{s \geq 0} \log E[f_1^s(X)] E[g_1^s(Y)]. \tag{15}$$

Note that, for every $1 > \epsilon > 0$,

$$0 < f_1 \leq 1 \quad \text{and} \quad 0 < g_1 \leq 1.$$

Therefore, the infimum is achieved on both sides of (15) at $s=\infty$, which gives

$$\mathrm{Prob}[X \geq x_0,\ Y < y_0] \leq \mathrm{Prob}[X \geq x_0]\ \mathrm{Prob}[Y < y_0],$$

or equivalently,

$$\mathrm{Prob}[X \geq x_0,\ Y \geq y_0] \geq \mathrm{Prob}[X \geq x_0]\ \mathrm{Prob}[Y \geq y_0].$$

The last inequality implies that X and Y are positively quadrant dependent. □

On the other hand, if X and Y are negatively quadrant dependent, we know from Lemma 4.2 that X and $-Y$ are positively quadrant dependent. This implies that X and Y are negatively correlated in the sense that $\mathrm{Cov}[X, Y] \leq 0$ if they are negatively quadrant dependent. Further, for any pair i and j, if $h(\theta_i, t)$ and $\mathrm{h}(\theta_j, t)$ are negatively quadrant dependent, then

$$E[\mathrm{e}^{s(h(\theta,t)-h(\theta_1,t))}] \geq E[\mathrm{e}^{sh(\theta,t)}]E[\mathrm{e}^{-sh(\theta_1,t)}], \qquad \forall s \geq 0.$$

Hence, we have obtained the following result.

Theorem 4.3. Assume that, for any pair i and j, $h(\theta_i, t)$ and $h(\theta_j, t)$ are negatively quadrant dependent. Then, the convergence rate of ordinal comparison is slower than for independent $h(\theta_i, t)$ and $h(\theta_j, t)$.

Theorems 4.2 and 4.3 should be carefully interpreted. It requires the validity of (14) for all nondecreasing, right-continuous functions f and g. For particular functions f and g, it is possible that (14) holds, but X and Y are not positively quadrant dependent. In fact, the following example illustrates a case where the rate of convergence is increased by negative correlation, not by positive quadrant dependence.

Example 4.1. Consider two random variables X and Y with the following joint distribution:

		Y		
		1	2	3
X	1	0.1	0.05	0.65
	2	0.0	0.0	0.1
	3	0.0	0.05	0.05

Then, X and Y are negatively correlated, since

$$E[XY]-E[X]E[Y]=-0.01<0.$$

However, direct verification shows that

$$\inf_{s\in[0,\infty)} E[e^{s(X-Y)}]=0.4201<\inf_{s\in[0,\infty)} E[e^{sX}]E[e^{-sY}]=0.4431.$$

This shows that a negative correlation does not necessarily decrease the convergence rate.

In Example 4.1,

$$\text{Prob}[X\leq 1,\ Y\leq 1]=0.1>\text{Prob}[X\leq 1]\ \text{Prob}[Y\leq 1]=0.08,$$

$$\text{Prob}[X\leq 1,\ Y\leq 2]=0.15<\text{Prob}[X\leq 1]\ \text{Prob}[Y\leq 2]=0.16.$$

Thus, X and Y are neither positively quadrant dependent nor negatively quadrant dependent.

Let X and Y be two random variables with respective cdf $F(x)$ and $G(y)$. Assume that they are generated by the CRN scheme, i.e.,

$$X=F^{-1}(u), \qquad Y=G^{-1}(u),$$

where u is uniform on [0, 1]. Then, for any $x,\ y\in R$,

$$\begin{aligned}\text{Prob}[X\geq x,\ Y\geq y]&=\min\{1-F(x),\ 1-G(y)\}\\&\geq\text{Prob}[X\geq x]\ \text{Prob}[Y\geq y];\end{aligned}$$

in other words, X and Y are positively quadrant dependent. Thus, the CRN scheme implies positive quadrant dependence.

Generally speaking, the CRN scheme does not imply association. To see this point, note that the inequality

$$\text{Prob}[X\in S_1,\ Y\in S_2]\geq\text{Prob}[X\in S_1]\ \text{Prob}[Y\in S_2]$$

may fail if S_1 and S_2 are chosen in such a manner that

$$\text{Prob}[X\in S_1,\ Y\in S_2]=0,$$

while

$$\text{Prob}[X\in S_1]\ \text{Prob}[Y\in S_2]>0.$$

This is a case where $X=u$ and $Y=u$, u is uniform on [0, 1], and $S_1=[0.1, 0.2]$, $S_2=[0.3, 0.4]$.

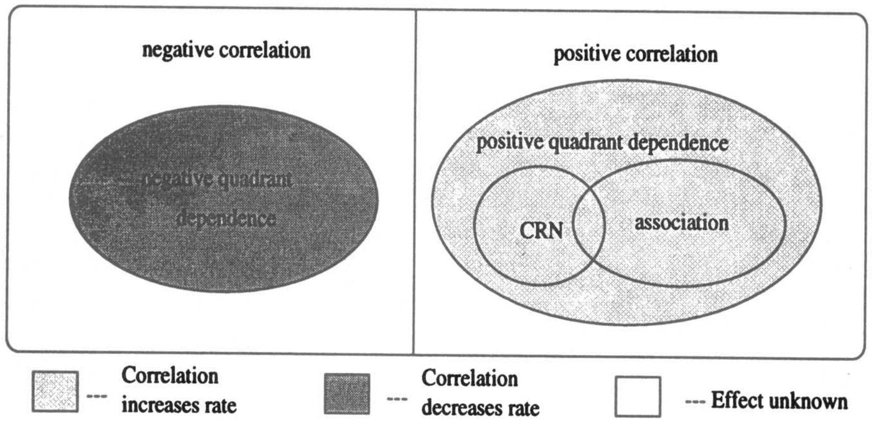

Fig. 1. Effect of correlation on the convergence rate of ordinal comparison.

Figure 1 on the summarizes the effect of correlation on the rate of convergence of ordinal comparison.

5. Maximization of the Convergence Rate of Ordinal Comparison

As stated in the previous section, negative correlation does not necessarily slow down the convergence of comparison, and positive quadrant dependence, stronger than positive correlation, always yields a better rate of comparison than independent simulations. In this section, we consider the simulation design that maximizes the rate of convergence of the indicator process. The results are also helpful for sample path reconstruction in performance analysis.

If Assumption 3.1 is satisfied when $h(\theta, t)$ is replaced by $\max_{\theta\in\Theta_b} h(\theta, t)$, then the convergence rate of $\text{Prob}[I(t)=0]$ is

$$-\inf_{s\in[0,\infty)} \lim_{t\to\infty} (1/t) \log E[e^{st(\max_{\theta\in\Theta_b} h(\theta,t)-h(\theta_1,t))}]. \tag{16}$$

Therefore, maximizing the rate is equivalent to maximizing (16). It is sufficient to minimize

$$E[e^{st(\max_{\theta\in\Theta_b} h(\theta,t)-h(\theta_1,t))}], \qquad \text{for all } s, t\geq 0.$$

However, the function

$$e^{st(\max_{\theta\in\Theta_b} h(\theta,t)-h(\theta_1,t))}$$

does not exhibit the properties that we need such as superadditivity. Fortunately, Theorem 3.3 converts the problem of multiple comparison into comparing only two designs.

We consider DEDS in the framework of generalized semi-Markov processes (GSMP). The detailed formulation is given in the Appendix. We restrict our discussion to a case that is easily realizable in simulation and that has explicit solution. We assume that the parameter θ occurs in the clock sample $Y_\alpha(n, \theta)$ only. We allow only possible correlations among $Y_\alpha(n, \theta_1)$ and $Y_\alpha(n, \theta)$ for any $\alpha \in \mathscr{A}$ and all $n \geq 1$. We do not allow any other form of cross correlation. This implies, according to the sampling scheme (30) in the Appendix, that $\{(u_\alpha(n, \theta_1), u_\alpha(n, \theta_2), \ldots, u_\alpha(n, \theta_N))\}$ is a set of independent random vectors running over $\alpha \in \mathscr{A}$ and $n \geq 1$. We want to find the correlation among $u_\alpha(n, \theta_1), u_\alpha(n, \theta_2), \ldots, u_\alpha(u, \theta_N)$, for every fixed pair α and n, that maximizes the rate of ordinal comparison. Although it seems quite restrictive, this scheme can be easily realized and is most useful.

In DEDS the sample performance measure $h(\theta, t)$ of the form (32) is continuous and monotone in τ_n for every n. Typically in DEDS, τ_n is a monotone function with respect to clock samples $\{Y_\alpha(i, \theta)\}$. As a result, $h(\theta, t)$ is a monotone function with respect to clock samples $Y_\alpha(n, \theta)$. In (30), $Y_\alpha(n, \theta)$ is always a nondecreasing function of u_α (n, θ). Therefore, if $h(\theta, t)$ is nondecreasing in $Y_\alpha(n, \theta)$, then it is nondecreasing in $u_\alpha(n, \theta)$. For some event $\beta \in \mathscr{A}$, if $h(\theta, t)$ is nonincreasing in $Y_\beta(n, \theta)$, then it is not nondecreasing in $u_\beta(n, \theta)$. In the latter case, we replace $u_\beta(n, \theta)$ in (30) by $1 - u_\beta(n\ \theta)$. Then, $h(\theta, t)$ is again a nondecreasing function of $u_\beta(n, \theta)$. Therefore, without loss of generality, we always assume that, for every $\theta \in \Theta$, $h(\theta, t)$ is nondecreasing in $u_\alpha(n, \theta)$ for all $\alpha \in \mathscr{A}$ and all $n \geq 1$.

Theorem 5.1. Assume that $h(\theta, t)$ is right-continuous, nondecreasing in $u_\alpha(n, \theta)$ for every t, every $\alpha \in \mathscr{A}$, and every $n \geq 1$. If Assumption 3.1 is satisfied, then the rate of convergence of the indicator process is maximized by choosing

$$u_\alpha(n, \theta) = v_\alpha\ (n), \tag{17}$$

where $\{v_\alpha(n)\}$ is a set of i.i.d. random numbers uniform on [0, 1]; in other words, the rate is maximized by using the scheme of CRN.

Proof. For any $t, s \geq 0$, and $\theta \in \theta_b$, consider the function

$$-e^{st(h(\theta,t) - h(\theta_1,t))}. \tag{18}$$

It is superadditive in $(h(\theta, t), h(\theta_1, t))$. Since $h(\theta, t)$ is nondecreasing in $u_\alpha(n, \theta)$ for all $\theta \in \Theta$ including θ_1, (18) is superadditive in $(u_\alpha(n, \theta_1),$

$u_{a}, (n, \theta_2), \ldots, u_a(n, \theta_N))$. It is also assumed that $h(\theta, t)$ is a continuous function of $u_a(n, \theta)$. Therefore, Theorem 2.1 applies and the scheme (17) maximizes

$$E[-e^{st(h(\theta,t)-h(\theta_1,t))}],$$

or equivalently minimizes

$$\Lambda(s, \theta_1, \theta) = (1/t) \log E[e^{st(h(\theta,t)-h(\theta_1,t))}]$$

simultaneously for all $s \geq 0$, all $\theta \in \Theta_b$. This means that, since according to (8) the convergence rate of $\mathrm{Prob}[I_i(t) = 0]$ is

$$C(\theta_1, \theta) = -\inf_{s \in [0,\infty)} \lim_{t \to \infty} \Lambda(s, \theta_1, \theta),$$

the rate is maximized by (17). Then, it follows immediately from Theorem 3.3 that (17) maximizes the convergence rate of $I(t)$. □

Case of G/G/1 Queues. To illustrate the applicability of Theorem 5.1, let us consider the G/G/1 FCFS queue in which there are two events: a= arrival and b=departure. Denote by $x(t, \theta)$ the number of customers in the system. The performance measure is the mean queue length in steady state, i.e.,

$$J(\theta) = \lim_{t \to \infty} (1/t) \int_0^t x(s, \theta)\, ds, \qquad \text{a.s.},$$

which exists under the stability condition; i.e., the arrival rate is less than the service rate. Then,

$$h(\theta, t) = (1/t) \int_0^t x(s, \theta)\, ds.$$

Let $Y_a(n, \theta)$ and $Y_b(n, \theta)$ be the sample interarrival time between the nth and the $(n-1)$th customers and the sample service time of the nth customer, respectively. Define $T_n(\theta)$ as the system time of the nth event time, $T_0(\theta) = 0$; and let $D_n(\theta)$ be the departure time of nth arrival. Then, the Lindley equation gives

$$T_n(\theta) = \{T_{n-1}(\theta) - Y_a(n, \theta)\}^+ + Y_b(n, \theta), \tag{19}$$

$$D_n(\theta) = T_n(\theta) + \sum_{i=1}^{n} Y_a(i, \theta). \tag{20}$$

In this case, $h(\theta, t)$ can be expressed as

$$h(\theta, t) = \sum_{n:\, D_n(\theta) \le t} T_n(\theta) + \sum_{n:\, D_n(\theta) > t} \left(t - \sum_{i=1}^{n} Y_a(i, \theta)\right). \tag{21}$$

We see from (19)–(21) that $h(\theta, t)$ is continuous in $Y_a(n, \theta)$, $Y_b(n, \theta)$, nondecreasing in $Y_b(n, \theta)$, and nonincreasing in $Y_a(n, \theta)$. Let $F_a(t, \theta)$ and $F_b(t, \theta)$ be the distribution of interarrival times and the distribution of service times. Assume that they are continuous. Let the interarrival times and the service times be generated as

$$Y_a(n, \theta) = F_a^{-1}(1 - u_a(n, \theta)\, \theta), \qquad Y_b(n, \theta) = F_b^{-1}(u_b(n, \theta), \theta),$$

where $u_a(n, \theta)$ and $u_b(n, \theta)$ are uniform on [0, 1]. Then, $h(x, t)$ is continuous, nondecreasing in both $u_a(n, \theta)$ and $u_b(n, \theta)$. Therefore, the CRN scheme maximizes the rate of ordinal comparison, according to Theorem 5.1, if Assumption 3.1 is satisfied.

The following example, modified from an example of Ref. 24, illustrates the difference in the rates of convergence for independent simulations and simulations using the CRN scheme.

Example 5.1. Consider an M/G/1 FCFS queue with arrival rate λ. The service times have the Weibull distribution

$$G(x) = 1 - e^{-(\gamma x)^\alpha}, \qquad x \ge 0,$$

for constants γ, $\alpha > 0$. Let the design parameter be $\theta = (\lambda, \gamma, \alpha)$. The performance measure is the mean queue length in steady state. We consider five designs with the following parameters:

θ	θ_1	θ_2	θ_3	θ_4	θ_5
λ	1.0	0.5	0.5	1.0	0.7
α	1.3	2.0	1.0	1.0	2.5
γ	2.0	0.75	1.0	1.6	1.0
$J(\theta)$	0.78	1.12	1.0	1.64	1.21

Since usually the smaller the queue length, the better the design, we compare $-J(\theta)$. The true best design is θ_1.

In the simulation of this M/G/1 queue, the interarrival times and the service times are generated as

$$Y_a(n, \theta) = -(1/\lambda) \log(u_a(n, \theta)),$$

$$Y_b(n, \theta) = (1/\gamma)\, e^{(1/\alpha) \log(-\log(1 - u_b(n, \theta)))},$$

where $u_a(n, \theta)$ and $u_b(n, \theta)$ are uniform on [0, 1]. We wrote a program in C and run it on Sun SparcServer 4/470. The simulation performances for independent simulations and for simulation using the CRN scheme are shown respectively in Figs. 2 and 3. Note that the displayed time is the real queue time, not the simulation time. The simulation was completed within seconds.

Figures 2 and 3 show clearly the advantage of the scheme of CRN. It can significantly increase the convergence rate of ordinal comparison.

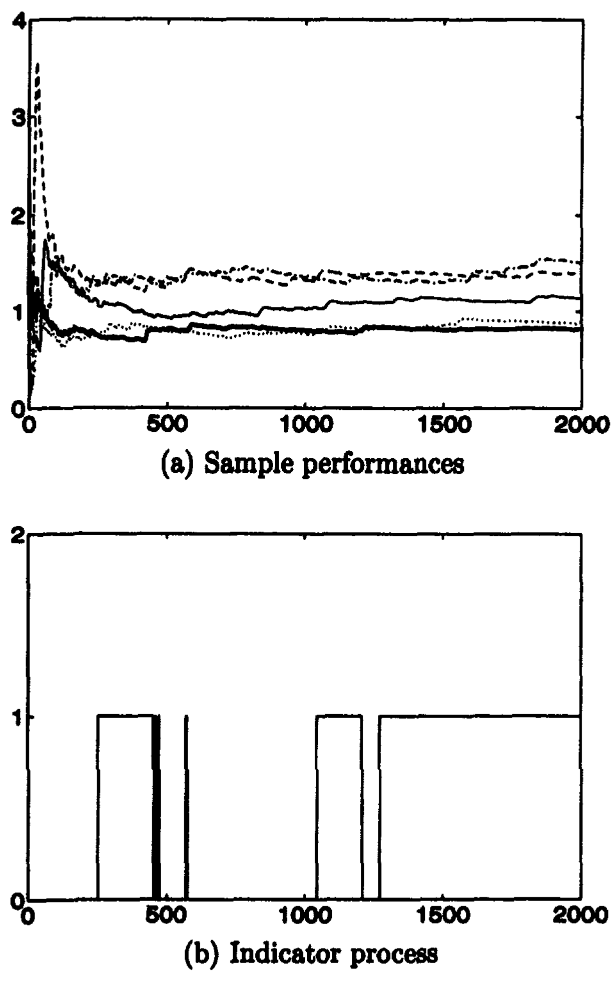

Fig. 2. Independent simulations.

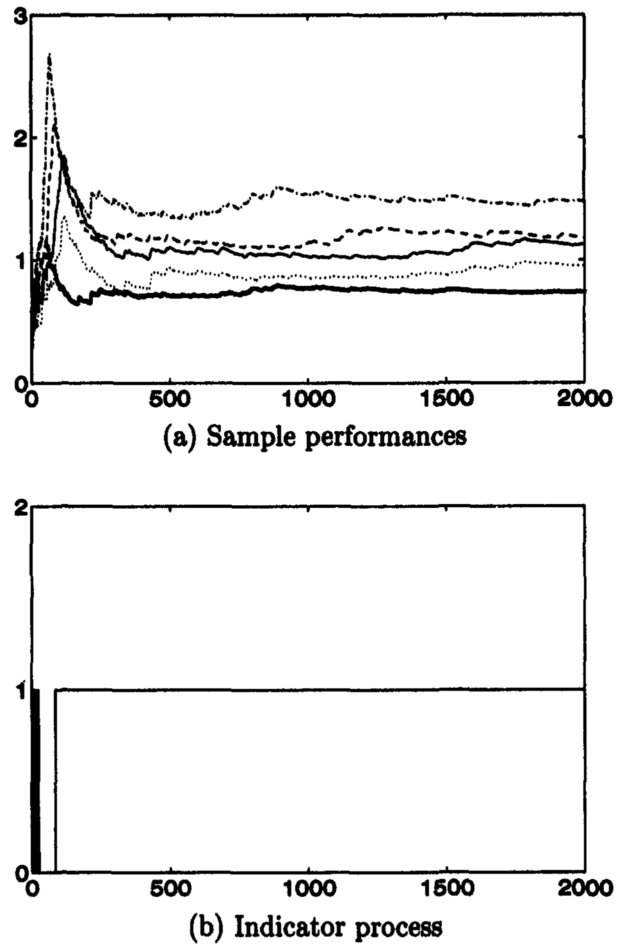

Fig. 3. Simulations using CRN.

6. Comparing Designs of Markov Chains

Consider a Markov chain $\{X(n, \theta) \in R\}$ with initial distribution $\pi_0(x, \theta)$ and transition probability $p(x, y, \theta)$ from x to y. The Markov chain $\{X(n, \theta) \in R\}$ is said to be stochastically monotone if, for every y, every $\theta \in \Theta$,

$$H(x, y, \theta) = P(x, (-\infty, y], \theta) = \sum_{y' \leq y} p(x, y', \theta)$$

is a nonincreasing function of x. Stochastic monotone Markov chains (SMMC) were introduced by Daley in Ref. 25 and have been intensively studied in performance analysis.

Reference 24 proves that the scheme of CRN can provide guaranteed variance reduction for SMMC. In this section, we further show that the CRN scheme actually maximizes the rate of convergence for SMMC. For Markov chains, we consider the following counterpart of the sampling scheme (30) (Ref. 24):

$$X(0, \theta)=\inf\{y|\pi_0(y, \theta)\geq u(0, \theta)\}, \tag{22}$$

$$X(n, \theta)=\inf\{y|H(X(n-1, \theta), y, \theta)\geq u(n, \theta)\}, \tag{23}$$

where $\{u(n, \theta)\}$ is a sequence of i.i.d. random numbers uniform on [0, 1]. We consider additive performance measures of the form

$$h(\theta, t)=(1/t)\sum_{n=1}^{t} L(X(n, \theta), \theta).$$

Theorem 6.1. Assume that $\{X(n, \theta)\in R\}$ is a finite-state, irreducible SMMC and that $L(x, \theta)$ is nondecreasing, right-continuous in x. Then, the rate of convergence of the indicator process is maximized by

$$u(n, \theta)=v(n), \qquad n=0, 1, \ldots, \tag{24}$$

where $\{v(n)\}$ is a sequence of i.i.d. random numbers uniform on [0, 1]; that is, the rate is maximized by the use of the CRN scheme.

Proof. According to Remark 3.1, Assumption 3.1 is satisfied for finite-state, irreducible Markov chains (Ref. 20). Furthermore, for SMMC under the sampling scheme (22)–(23) with nondecreasing, right-continuous $L(x, \theta)$, $h(\theta, t)$ is nondecreasing and right-continuous in $u(n, \theta)$ for all $n\geq 0$ and every $\theta\in\Theta$. This implies that

$$-e^{st(h(\theta,t)-h(\theta_1,t))}$$

is right-continuous and superadditive in $u(n, \theta)$, $u(n, \theta_1)$. Therefore, (24) minimizes

$$E[e^{st(h(\theta,t)-h(\theta_1,t))}],$$

for all $n\geq 0$ and every $\theta\in\Theta$, thus maximizing the rate $C(\theta_1, \theta)$. □

The statement of Theorem 6.1 is valid for countable-state Markov chains provided Assumption 3.1 is satisfied.

7. Appendix: Generalized Semi-Markov Process Framework

This Appendix is intended to introduce the DEDS and their performance measures under consideration. Although heavy notations are involved, most of it can be skipped by accepting the performance measure form (32).

We first introduce some necessary notations. In this paper, θ always represents the design parameter.

$\mathscr{S}$ = state space (finite or countable);

$\mathscr{A}$ = finite set of all possible events;

$\Gamma(x)$ = set of active events when the system is in state $x \in \mathscr{S}$; we assume that $\Gamma(x) \subset \mathscr{A}$ is not empty for any $x \in \mathscr{S}$;

τ_n = epoch of the nth state transition, $\tau_n \in R^+$;

α_n = nth event, $\alpha_n \in \mathscr{A}$;

x_n = nth state visited by the process, $x_n \in \mathscr{S}$; $c_n(\alpha)$ = at τ_n^+, the time remaining until event α occurs, provided $\alpha \in \Gamma(x_n)$; $c_n(\alpha) \in R^+$ is called the remaining clock reading for event α;

$N_\alpha(n)$ = number of instances of α among $\alpha_1, \alpha_2, \ldots, \alpha_n$;

$\mathscr{O}(x'; x, \alpha) = \Gamma(x') \cap \{\Gamma(x) - \{\alpha\}\} \subset \mathscr{A}$, the set of old events following a transition from x to x' triggered by event α;

$\mathscr{N}(x'; x, \alpha) = \Gamma(x') \backslash \{\Gamma(x) - \{\alpha\}\} \subset \mathscr{A}$, the set of new events at such a transition;

$\{Y_\alpha(n, \theta)\}$ = a set of random variables that defines new clock samples; for each $\alpha \in \mathscr{A}$, $Y_\alpha(n, \theta) \in R^+$ denotes the nth new clock sample for event α according to the distribution $F_\alpha(t, \theta)$.

Let $\{x_n\}$ be recursively defined by the following equations:

$$\tau_n = \tau_{n-1} + \min\{c_{n-1}(\alpha) | \alpha \in \Gamma(x_{n-1})\}, \tag{25}$$

$$\alpha_n = \{\alpha' \subset \Gamma(x_{n-1}) | c_{n-1}(\alpha') = \min\{c_{n-1}(\alpha) | \alpha \in \Gamma(x_{n-1})\}\}, \tag{26}$$

$$x_n = \Psi(x_{n-1}, \alpha_n), \tag{27}$$

$$c_n(\alpha) = \begin{cases} c_{n-1}(\alpha) - (\tau_n - \tau_{n-1}), & \text{if } \alpha \in \mathscr{O}(x_n; x_{n-1}, \alpha_n), \\ Y_\alpha(N_\alpha(n-1) + 1, \theta), & \text{if } \alpha \in \mathscr{N}(x_n; x_{n-1}, \alpha_n), \\ +\infty, & \text{if } \alpha \notin \Gamma(x_n), \end{cases} \tag{28}$$

$$N_\alpha(n) = \begin{cases} N_\alpha(n-1) + 1, & \text{if } \alpha = \alpha_n, \\ N_\alpha(n-1), & \text{otherwise.} \end{cases} \tag{29}$$

Define $x(t, \theta)=x_n$ on $t \in [\tau_n, \tau_{n+1})$. Then, the stochastic process $\{x(t, \theta), t \geq 0\}$ is called a generalized semi-Markov process (GSMP). It is right-continuous in t.

A GSMP can be simulated in the following way. For an initial state $x_0 \in \mathscr{S}$, there are an event set $\Gamma(x_0)$ and a set of new clock samples $\{c_0(\alpha)= Y_\alpha(0, \theta) | \alpha \in \Gamma(x_0)\}$. Generally, at τ_{n-1}^+, the state is x_{n-1}, the set of possible events is $\Gamma(x_{n-1})$, and the corresponding set of remaining clock readings is $\{c_{n-1}(\alpha) | \alpha \in \Gamma(x_{n-1})\}$. The event or event set with the shortest remaining clock reading happens first and thus is the next triggering event. Equations (25) and (26) determine the next event epoch and the triggering event. The event triggers a state transition from x_{n-1} to x_n according to (27). In the case where several events happen simultaneously, we assume that ties are broken up arbitrarily, e.g., by a fixed order of events. The state transition is determined by $\Psi(x, \alpha)$ upon the occurrence of event α, which can be deterministic or probabilistic. Equation (28) updates clock readings. Clock readings for old events in $\mathscr{O}(x_n; x_{n-1}, \alpha_n)$ continue to run. Clock readings for new events in $\mathscr{N}(x_n; x_{n-1}, \alpha_n)$ are set to new clock samples. The last equation [Eq. (29)] simply determines which clock sample is to be picked up. It is not necessary for the simulation itself. Any clock samples are usable as long as they are independent and have the desired distribution. However, since we are interested in simulation design, we do need to specify each sample. After (29), the simulation finishes one cycle and goes to (25) for the next one. The simulation process continues until a stop criterion is satisfied; e.g., a simulation budget is reached or a special event has occurred.

We also need to specify the set $\{Y_\alpha(n, \theta)\}$. For each $\alpha \in \mathscr{A}$ and every $n \geq 0$, we assume that the new clock sample $Y_\alpha(n, \theta)$ is generated according to the following inversion method:

$$Y_\alpha(n, \theta) = F_\alpha^{-1} u_\alpha(n, \theta), \theta), \tag{30}$$

where $u_\alpha(n, \theta)$ for different α and n defines a two-dimensional array of independent random numbers uniform on [0, 1]. The conclusions of the paper remain valid if (30) is replaced by any $Y_\alpha(n, \theta) = \psi(u_\alpha(n, \theta), \theta)$, provided that $\psi(x, \theta)$ is nondecreasing in x.

Example 7.1. To illustrate the concept of GSMP, consider a GI/G/1 queue with first-come-first-serve (FCFS) discipline. In this system, there are two events: a = customer arrival, b = customer departure. Let the interarrival or service times between successive customers be i.i.d. with distribution $F_a(t, \theta)$ [or $F_b(t, \theta)$, correspondingly]. The interarrival times and the service times are independent. Let $x(t, \theta)$ be the number of customers at time t.

Then, for this system,

$$\mathscr{S}=\{0, 1, 2, \ldots\}, \qquad \mathscr{A}=\{a, b\},$$

$$\Gamma(x)=\{a, b\}, \qquad \text{if } x>0,$$

$$\Gamma(x)=\{a\}, \qquad \text{if } x=0,$$

$$\Psi(x, \alpha)=\begin{cases} x+1, & \text{if } \alpha=a, \\ x-1, & \text{if } \alpha=b. \end{cases}$$

The new clock samples are

$$Y_a(n, \theta)=F_a^{-1}(u_a(n, \theta), \theta), \qquad Y_b(n, \theta)=F_b^{-1}(u_b(n, \theta), \theta).$$

The significance of GSMP lies in its generality. It is convenient for simulation and performance analysis. We use GSMP as the model of DEDS. The trajectory of a DEDS is meant as a sample path of the corresponding GSMP.

Next, we specify the form of the performance measure. In this paper, we are concerned with infinite-horizon performance. Particularly, we consider a performance measure $J(\theta)$ of the form

$$J(\theta)=\lim_{t\to\infty}(1/t)\int_0^t L(x(s, \theta), \theta)\, ds, \qquad \text{a.s.,} \tag{31}$$

where $L(x(t, \theta), \theta)\in R$ is a real-valued, continuous function of the state $x(t, \theta)$. For the problems considered in this paper, we assume that the limit in (31) exists for any sample trajectory of the DEDS. A natural estimate of $J(\theta)$, based on simulation or observation of length t, is the empirical average

$$h(\theta, t)=(1/t)\int_0^t L(x(s, \theta), \theta)\, ds.$$

An estimator which satisfies (31) is said to be consistent. The consistency of $h(\theta, t)$ implies that we have a better and better estimate as $t\to\infty$.

Let $e(t)$ be the total number of events that occur on $[0, t]$, and let $\tau_0=0$. Then, another form of $h(\theta, t)$ is

$$\begin{aligned} h(\theta, t)&=(1/t)\left[\sum_{n=0}^{e(t)-1} L(x_n, \theta)\times(\tau_{n+1}-\tau_n)+L(x_{e(t)}, \theta)\times(t-\tau_{e(t)})\right] \\ &=(1/t)\left[\sum_{n=1}^{e(t)} (L(x_{n-1}, \theta)-L(x_n, \theta))\tau_n+tL(x_{e(t)}, \theta)\right]. \end{aligned} \tag{32}$$

The form (32) implies that $h(\theta, t)$ is continuous in event epochs for a fixed sequence of states, but it usually jumps when the state changes from one state to another.

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