Notes for Mathematics for Physics: A Guided Tour for Graduate Students

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Contents

1 Calculus of variations

A functional J is a map $J : C^{\infty}(\mathcal{R}^n) \to \mathcal{R}$. We restrict ourselves to functionals of the form

$$J[y] = \int_{x_1}^{x_2} F(x, y, y^{(1)}, y^{(2)}, ...)dx \tag{1}$$

where $y^{(n)}$ denotes the n^{th} derivative of y with respect to x.

1.1 Functionals of y and $y^{(1)}$

For $J = \int F dx$, where f depends only on x, y and y', we show how to derive the Euler-Lagrange equation and find the y that optimises J.

Let ϵ be a small, arbitrary real number, η an arbitrary function of x and y an arbitrary function of x in C^{∞}. A small perturbation in y is given by $y = y^* + \epsilon\eta$, where y^* is the unperturbed y. Also let $\delta J = J[y^* + \epsilon\eta] - J[y^*]$ and $\delta F = \{F(x, y, y^{(1)}) - F(x, y^*, y^{*(1)})\}$, where δ is the variation operator. The change in J associated with going from y to $y^* + \epsilon\eta$ is given by

$$\delta J = \int_{x_1}^{x_2} \delta F dx \tag{2}$$

A necessary condition for the minimisation of J is $\delta J = 0$. We then use a Taylor expansion of δF around $\epsilon = 0$, discarding all terms second order and above.

$$\delta F \approx \epsilon\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y^{(1)}}\eta^{(1)}\right) \tag{3}$$

Using integration by parts, we can find an alternate expression for $\frac{\partial F}{\partial y^{(1)}}\eta^{(1)}$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(1)}}\eta^{(1)}dx = \frac{\partial F}{\partial y^{(1)}}\eta\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta dx$$

As we assume $\epsilon = 0$ at x_1 and x_2, $\frac{\partial F}{\partial y^{(1)}}\eta\bigg|_{x_1}^{x_2} = 0$. So we have

$$\frac{\partial F}{\partial y^{(1)}}\eta^{(1)} = -\frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta$$

We substitute this into (3) to obtain

$$\delta F \approx \epsilon\eta\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\right)$$

Substituting this expression into (2) results in

$$\int_{x_1}^{x_2} \delta F dx = \int_{x_1}^{x_2} \epsilon\eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y^{(1)}} \right) dx = 0$$

Supposing that η and ϵ are not identically zero, we conclude that

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y^{(1)}} = 0 \tag{4}$$

(4) is known as an *Euler-Lagrange equation.* The expression on the left hand side is referred to as the functional derivative of δJ with respect to y.

1.2 Functionals of higher order derivatives

If F is a function of higher order derivatives of y, e.g. $y^{(5)}$ or $y^{(26)}$, then the Euler-Lagrange equation is extended as follows. Suppose F is a function of the N^{th} order derivative of y. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} + \sum_{n=1}^{N} (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}} \right) = 0 \tag{5}$$

1.3 Functionals of multiple functions

When F is a function of multiple functions y_i and their derivatives y_{ix}, where each y_i is a function of x, then we get a separate Euler-Lagrange equation for each y_i

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y_i^{(1)}} = 0 \tag{6}$$

1.4 Functionals of multiple functions and higher order derivatives

Combining the previous two sections, the Euler-Lagrange equations for functionals of multiple functions y_i and higher order derivatives $y_i^{(n)}$ are given by

$$\frac{\partial F}{\partial y_i} + \sum_{n=1}^{N} (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y_i^{(n)}} \right) = 0 \tag{7}$$

for each i corresponding to a y_i.

1.5 Functionals of multiple functions, multiple independent variables and higher order derivatives

The next logical step is to introduce multiple independent variables. Let μ_m be indices spanning the independent variables x_{μ_m}. Let $y_i^{(n)_{\mu_m}}$ denote the n^{th} partial derivative of y_i with respect to x_{μ_m}. Given a total of M independant variables, functionals of multiple functions y_i and higher order derivatives $y_i^{(n)_{\mu_m}}$, the Euler-Lagrange equation for each y_i is as follows

$$\frac{\partial F}{\partial y_i} + \sum_{n=1}^{N} \sum_{\mu_1 \leq \ldots \leq \mu_n} (-1)^n \frac{d^n}{dx_{\mu_1} \ldots dx_{\mu_n}} \left(\frac{\partial F}{\partial y_i^{(1)_{\mu_1}} \ldots \partial y_i^{(1)_{\mu_n}}} \right) = 0 \tag{8}$$

Note that μ_m is an arbitrary reference to any x_m. Hence x_{μ_1} may be equal to $x_{\mu 2}$ even though $x_1 \neq x_2$. For 3 independent variables x_1, x_2 and x_3, and $n = 2$, we have that

$$\sum_{\mu_1 \leq \ldots \leq \mu_2} dx_{\mu_1} dx_{\mu_2} = dx_1^2 + dx_1 dx_2 + dx_1 dx_3 + dx_2^2 + dx_2 dx_3 + dx_3^2$$

For the same example, but with $n = 1$, we have

$$\sum_{\mu_1 \leq \ldots \leq \mu_1} dx_{\mu_1} dx_{\mu_1} = dx_1^2 + dx_2^2 + dx_3^2$$

1.6 Constraints

We will frequently want to find the stationary point of δJ subject to certain constraints.

1.6.1 Integral constraints

The simplest constraints are integral constraints

$$K = \int_{x_1}^{x_2} G(x, y, y^{(1)}, y^{(2)}, \ldots) dx \tag{9}$$

where K is a constant and G is a functional.

To incorporate this into our Euler-Lagrange equations, we define F_c as follows

$$F_c = F + \lambda G$$

and work with F_c in the same way we would with F, subject to the constraints being satisfied. e.g. (8) becomes

$$\frac{\partial F_c}{\partial y_i} + \sum_{n=1}^{N} \sum_{\mu_1 \leq ... \leq \mu_n} (-1)^n \frac{d^n}{dx_{\mu_1} ... dx_{\mu_n}} \left(\frac{\partial F_c}{\partial y_i^{(1)\mu_1} ... \partial y_i^{(1)\mu_n}} \right) = 0$$

and our solutions must also satisfy (9).

Given multiple functionals G_m in integral constraints, F_c is given by

$$F_c = F + \sum_m \lambda_m G_m$$

and solutions to the Euler-Lagrange equations must satisfy the constraints.

1.6.2 Holonomic constraints

Equivalence constraints not involving derivatives of functions that can be expressed without an integral are referred to as holonomic constraints. They take the form

$$K = G(x, y) \tag{10}$$

where G and K are defined as in the previous section. We now define F_c as

$$F_c = F + \int_{x_1}^{x_2} \lambda(x) G(x, y) dx \tag{11}$$

where $\lambda(x)$ is a Lagrange multiplier. Take note that our Lagrange multiplier is now dependant on our independent variables. Given F_c, we proceed in the same manner as for integral constraints.

1.6.3 Non-holonomic constraints

Non-holonomic constraints are equivalence constraints that involve derivatives of functions. They are not covered.

1.7 The first integral

The quantity $I = F - y^{(1)} \frac{\partial f}{\partial y^{(1)}}$ is called the *first integral.* Suppose F is directly dependent only on y and $y^{(1)}$. We then have the following

$$\begin{aligned} \frac{d}{dx}\left(F - y^{(1)} \frac{\partial F}{\partial y^{(1)}}\right) &= y^{(1)} \frac{\partial F}{\partial y} + y^{(2)} \frac{\partial F}{\partial y^{(1)}} - y^{(2)} \frac{\partial F}{\partial y^{(1)}} - y^{(1)} \frac{d}{dx}\left(\frac{\partial F}{\partial y^{(1)}}\right) \\ &= y^{(1)} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y^{(1)}}\right)\right) \end{aligned} \tag{12}$$

The bracketed expression on the second line is the Euler-Lagrange equation and hence is set to 0. We therefore have the result that the first integral I is equal 0 when F has no direct dependence on x.

For multiple functions y_i the first integral becomes

$$I = F - \sum_i y_i^{(1)} \frac{\partial F}{\partial y_i^{(1)}} \tag{13}$$

1.8 Boundary conditions

1.9 Examples

1.9.1 Lagrangian mechanics

Lagrangian mechanics is a formulation of classical mechanics that uses functionals composed of the difference between kinetic and potential energy, as opposed to the traditional approach based on Newton's laws. The Lagrangian $\mathcal{L}$ is given as follows

$$\mathcal{L} = T - V \tag{14}$$

where T is the kinetic energy and V is the potential energy.

We are free to write T and V in terms of whatever variables we would like. Typically the independent variable is time and functions dependent on time are denoted by q_i.

1.10 Variable end points

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References

[1] R.E. Hunt, *Chapter 1 variational methods*, 2002.

[2] R.S. Schechter, *The variational method in engineering*, McGraw-Hill chemical engineering series, McGraw-Hill, 1967.

[3] M. Stone and P. Goldbart, *Mathematics for physics: A guided tour for graduate students*, Cambridge University Press, United States, 2009.

2 Function spaces

3 Linear ordinary differential equations

4 Linear differential operators

5 Green functions