Analog and Digital Communications - Formulae and Identities

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Contents

1 Signals and Systems

1.1 Basics

1.1.1 Normalized Energy

The normalized energy content E of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1}$$

1.1.2 Average Power

The normalized average power P of a signal $x(t)$ is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2}$$

1.1.3 Dirac Delta Function

The unit impulse function (or Dirac delta function) $\delta(t)$ is defined as

$$\delta(t) = \int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \tag{3}$$

where $\phi(t)$ is any test function continuous at $t = 0$. The unit impulse function is a *generalized function*.

1.1.4 Derivatives of Generalized Functions

The derivative $g'(t)$ of a generalized function $g(t)$ is defined by

$$\int_{-\infty}^{\infty} g'(t)\phi(t)dt = -\int_{-\infty}^{\infty} g(t)\phi'(t)dt \tag{4}$$

1.1.5 Complex Fourier Series

The Fourier series for a signal $x(t)$ is defined as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{5}$$

where ω_0 is the fundamental angular frequency. The Fourier coefficients c_n are given by

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt \tag{6}$$

A plot of $|c_n|$ vs ω is called the amplitude spectrum. A plot of θ_n (the phase constants of c_n) vs ω is called the phase spectrum. Together these are referred to as the frequency spectra.

1.1.6 Parseval's Theorem

Parseval's theorem states that for a periodic signal $x(t)$

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}|x(t)|^2dt = \sum_{n=-\infty}^{\infty}|c_n|^2 \tag{7}$$

1.1.7 Fourier Transform

The Fourier transform, $\mathscr{F}$, of a signal $x(t)$ is given by

$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{8}$$

1.1.8 Inverse Fourier Transform

The inverse Fourier transform of $X(\omega)$, $\mathscr{F}^{-1}$, is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{9}$$

1.2 Properties of the Fourier Transform

$x(t) \longleftrightarrow X(\omega)$ denotes a Fourier transform pair.

1.2.1 Linearity

$$a_1x_1(t) + a_2x_2(t) \longleftrightarrow a_1X_1(\omega) + a_2X_2(\omega) \tag{10}$$

1.2.2 Time Shifting

$$x(t-t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0} \tag{11}$$

1.2.3 Frequency Shifting

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0) \tag{12}$$

1.2.4 Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a}) \tag{13}$$

1.2.5 Time Reversal

$$x(-t) \longleftrightarrow X(-\omega) \tag{14}$$

1.2.6 Duality

$$X(t) \longleftrightarrow 2\pi x(-\omega) \tag{15}$$

1.2.7 Differentiation

Time differentiation

$$x'(t) = \frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega) \tag{16}$$

Frequency differentiation

$$(-jt)x(t) \longleftrightarrow X'(\omega) = \frac{d}{d\omega}X(\omega) \tag{17}$$

1.2.8 Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega) \tag{18}$$

1.2.9 Modulation Theorem

$$x(t)cos(\omega_0 t) \longleftrightarrow \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0) \tag{19}$$

1.3 Convolutions and Correlation

The convolution of two signals $x_1(t)$ and $x_2(t)$ is

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \tag{20}$$

1.3.1 Time Convolution Theorem

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)X_2(\omega) \tag{21}$$

1.3.2 Frequency Convolution Theorem

$$x_1(t)x_2(t) \longleftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega) \tag{22}$$

1.3.3 Cross-Correlation

The cross correlation $R_{12}(\tau)$ of signals $x_1(t)$ and $x_2(t)$ is defined by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt \tag{23}$$

1.3.4 Autocorrelation

The autocorrelation is defined as the cross-correlation of a signal $x_1(t)$ with itself, $R_{11}(\tau)$.

1.3.5 Energy Spectral Density

The energy spectral density S_{11} of a signal $x_1(t)$ is given by

$$S_{11}(\omega) = \mathscr{F}[R_{11}(\tau)] = \int_{-\infty}^{\infty} R_{11}(\tau) e^{-j\omega\tau} d\omega \tag{24}$$

1.4 Linear Time-Invariant Systems

Linear time-invariant (linear time-invariant) systems have several properties, as follows. Suppose $\mathcal{F}$ is an operator representing the action of a system with output $y(t)$.

1.4.1 Additivity

$$\mathcal{F}[x_1(t) + x_2(t)] = \mathcal{F}[x_1(t)] + \mathcal{F}[x_2(t)] \tag{25}$$

1.4.2 Homogeneity

$$\mathcal{F}[ax(t)] = a\mathcal{F}[x(t)] \tag{26}$$

1.4.3 Time-Invariance

$$\mathcal{F}[x(t - t_0)] = y(t - t_0) \tag{27}$$

1.4.4 Impulse Response

The impulse response $h(t)$ of an LTI system is the response of the system with a delta function input

$$h(t) = \mathcal{F}[\delta(t)] \tag{28}$$

1.4.5 Response to Arbitrary Inputs

The response of an LTI system to an arbitrary input can be expressed in terms of a convolution with the impulse response of the system

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \tag{29}$$

1.4.6 Causality

A signal $x(t)$ is causal if, for $t < 0$, $x(t) = 0$.

1.4.7 Frequency Response

Using the time convolution theorem (21) on the response of an LTI system (29), we find that

$$Y(\omega) = X(\omega) H(\omega) \tag{30}$$

where $Y(\omega) = \mathscr{F}[y(t)]$ and $H(\omega) = \mathscr{F}[h(t)]$. We refer to $H(\omega)$ as the *frequency response* or *transfer function*.

1.4.8 Input and Output Spectral Densities

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \tag{31}$$

$$\bar{S}_{yy}(\omega) = |H(\omega)|^2 \bar{S}_{xx}(\omega) \tag{32}$$

2 Amplitude Modulation

Modulation is defined as the process by which some characteristic of a carrier signal is varied in accordance with a modulating signal (message signal). There are two basic types of analog modulation; continuous wave (CW) modulation and pulse modulation.

2.1 Continuous-Wave Modulation

In continuous-wave modulation, a sinusoidal signal is used as a carrier signal. The modulated carrier signal $x_c(t)$ is given by

$$x_c(t) = A(t)cos[\omega_c t + \phi(t)] \tag{33}$$

Let $m(t)$ be the message signal.

2.1.1 Double-Sideband Modulation

Double-sideband (DSB) modulation occurs when $m(t) = A(t)$. The DSB modulated signal $x_{DSB}(t)$ is then given by

$$x_{DSB}(t) = m(t)cos(\omega_c t) \tag{34}$$

By applying the modulation theorem (19), we obtain the following

$$X_{DSB}(\omega) = \frac{1}{2}M(\omega - \omega_c) + \frac{1}{2}M(\omega + \omega_c) \tag{35}$$

See figure 1 for an example.

2.1.2 Ordinary Amplitude Modulation

Ordinary amplitude modulation (AM) varies the carrier signal amplitude linearly with respect to the message signal as follows

$$x_c(t) = [1 + k_a m(t)]A(t)cos(\omega_c t) \tag{36}$$

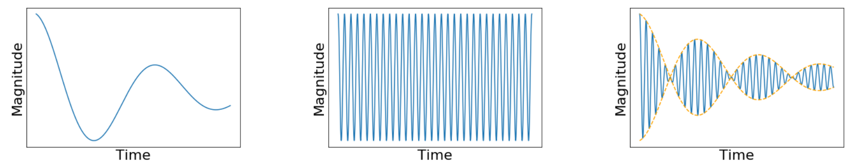
where $|k_a m(t)| < 1$.

2.1.3 Single-Sideband Modulation

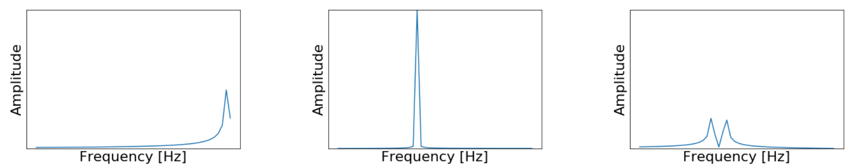
There are two types of single single sideband (SSB) modulation; upper sideband and lower sideband. They can both be generated by phase-shifting the message signal by $\frac{\pi}{2}$, resulting in a modulated signal given as below

$$x_{SSB}(t) = m(t)cos(\omega_c t) \mp \hat{m}(t)sin(\omega_c t) \tag{37}$$

where the difference results in an upper sideband and the sum results in the lower sideband.

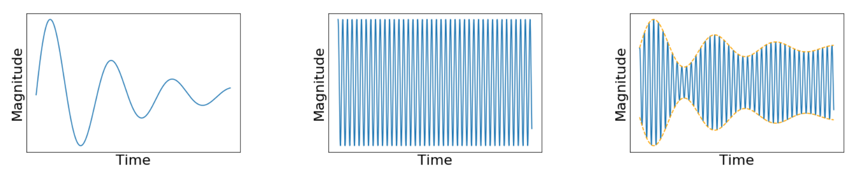


(a) Message signal **(b)** Carrier signal **(c)** Modulated signal

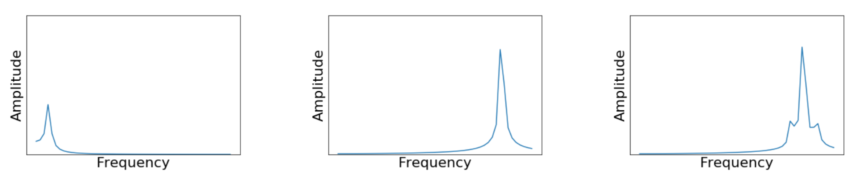


(d) Message signal **(e)** Carrier signal **(f)** Modulated signal

Figure 1: Modulating a message signal onto a carrier signal with DSB modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain. Note how the carrier signal frequency gets split into two smaller peaks. These are referred to as the upper and lower sidebands.



(a) Message signal **(b)** Carrier signal **(c)** Modulated signal

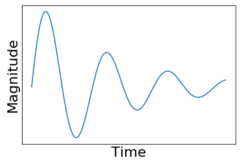


(d) Message signal **(e)** Carrier signal **(f)** Modulated signal

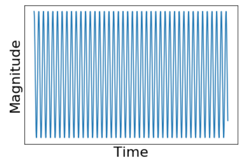
Figure 2: Modulating a message signal onto a carrier signal with AM modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain.

2.1.4 Vestigial-Sideband Modulation

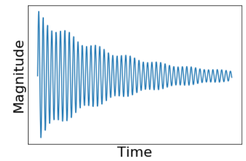
Vestigial-sideband modulation is similar to DSB, except one sideband is attenuated. This is a compromise between DSB and SSB. Vestigial-sideband modulation is performed on top of DSB modulation, with a vestigial-sideband filter



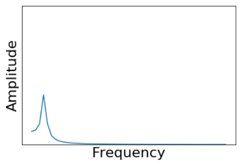
(a) Message signal



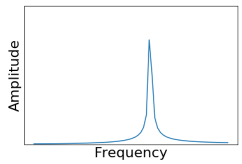
(b) Carrier signal



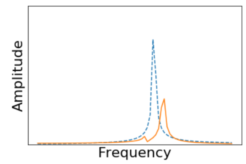
(c) Modulated signal



(d) Message signal



(e) Carrier signal



(f) Modulated signal

Figure 3: Modulating a message signal onto a carrier signal with upper SSB modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain.

applied as the final step.

2.2 Frequency Translation and Mixing

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3 Angle Modulation

Angle modulation includes both phase modulation (PM) and frequency modulation (FM). We again use a sinusoidal carrier signal

$$x_c(t) = Acos[\omega_c t + \phi(t)] \tag{38}$$

However, instead of modifying the amplitude A, we modify the phase angle $\phi(t)$ by making it a function of the message signal $m(t)$.

Let $\theta(t) = \omega_c t + \phi(t)$. Then the instantaneous radian frequency of $x_c(t)$ is given by

$$\omega_{inst} = \frac{d\theta(t)}{dt} = \omega_c + \frac{d\phi(t)}{dt} \tag{39}$$

3.1 Phase Modulation

In phase modulation, we let the phase angle $\phi(t) = k_p m(t)$, where k_p is the phase deviation constant.

3.2 Frequency Modulation

In frequency modulation, we set $\frac{d\phi(t)}{dt} = k_f m(t)$, where k_f is the frequency deviation constant. Integrating over $m(t)$, we find the following expression for $\phi(t)$

$$\phi(t) = k_f \int_{t_0}^{t} m(\lambda) + \phi(t_0) \tag{40}$$

We usually let $t_0 = -\infty$ and $\phi(-\infty) = 0$.

3.3 Sinusoidal Angle Modulation

Sinusoidal message signals result in a phase angle of

$$\phi(t) = \beta sin(\omega_m t) \tag{41}$$

β is referred to as the modulation index and gives the maximum value of phase deviation. β can be expressed as

$$\beta = \frac{\Delta\omega}{\omega_m} \tag{42}$$

where $\Delta\omega$ is the maximum frequency deviation. Substituting (41) into (38) and expressing it as a Fourier series, we obtain

$$x_c(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) cos(\omega_c + n\omega_m)t \tag{43}$$

where $J_n(\beta)$ are Bessel functions of order n. Three things should be noted:

1. The signal is composed of a carrier frequency with infinite harmonic sidebands.
2. The aplitude is dependant on $J_n(\beta)$, which gets smaller with larger values of $|n|$.
3. β also determines the number of significant spectral lines, with smaller β resulting in fewer significant sidebands and greater β resulting in more numerous significant sidebands.

3.4 Bandwidth of Angle Modulated Signals

3.4.1 Sinusoidal Modulation

The bandwidth W_B of a signal with sinusoidal modulation is approximately given as follows

$$W_B \approx 2(\beta + 1)\omega_m \tag{44}$$

3.4.2 Arbitrary Modulation

The Deviation radio D is defined as

$$D = \frac{\Delta\omega}{\omega_M} \tag{45}$$

for a message signal bandwidth ω_M. The bandwidth of the modulated signal is then given as

$$W_B \approx 2(D + 1)\omega_M \tag{46}$$

W_B contains 98% of signal power.

4 Digital Transmission of Analog Signals

4.1 Pulse Code Modulation

Pulse code modulation (PCM) consists of three main steps: sampling, quantisation and encoding. The sampled signal must be *bandwidth limited* - ie it must be composed of a finite number of frequencies.

4.1.1 Pulse Amplitude Modulation

Pulse amplitude modulation (PAM) utilises a periodic train of rectangular pulses as the carrier signal $x_c(t)$. The modulated signal $x_{PAM}(t)$ is then given by the discrete convolution of the message signal $m(t)$ and the carrier signal

$$x_{PAM}(t) = m(t) * x_c(t) = \sum_{n=-\infty}^{\infty} m(nT_s)x_c(t - nT_s) \tag{47}$$

where T_s is the sampling frequency.

4.1.2 Quantising Error

Once the signal has been modulated, thereby quantising in time, its amplitude must also be quantised. There is an error associated with uniform quantisation that is referred to as quantising error or quantising noise. Given an amplitude step size of Δ_A, the quantising error q_e varies uniformly over $[-\frac{\Delta_A}{2}, \frac{\Delta_A}{2}]$. The expected value of q_e^2 is thus

$$\langle q_e^2 \rangle = \frac{1}{\Delta_A} \int_{-\Delta_A/2}^{\Delta_A/2} q_e^2 dq_e = \frac{\Delta_A^2}{12} \tag{48}$$

4.1.3 Non-uniform Quantisation

Oftentimes signals are not evenly distributed over their range of values. In these situations, it is better to use non-uniform quantisation that allows for finer discretisation in ranges of amplitudes where signal values are more likely to be. A typical method of non-uniform quantisation is to first use a non-linear *compression* of the original signal followed by a uniform quantisation of the compressed signal. A common compression used is the μ *law*

$$y = \frac{ln(1 + \mu|m/m_p|)}{ln(1 + \mu)} sgn(m), \qquad \left| \frac{m}{m_p} \right| \leq 1 \tag{49}$$

where μ is a positive constant, m_p is the maximum signal amplitude and $sgn(m) = 1$ if $m > 0$ and -1 otherwise.

Another common compression is the A *law*, given by

$$y = \begin{cases} \frac{A}{1+ln(A)}(\frac{m}{m_p}), & |\frac{m}{m_p}| \leq \frac{1}{A} \\ \frac{(1+ln(A)|m/m_p|)}{1+ln(A)} sgn(m), & \frac{1}{A} \leq |\frac{m}{m_p}| \leq 1. \end{cases} \tag{50}$$

4.1.4 Bandwidth Requirements

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References