Notes for Mathematics for Physics: A Guided Tour for Graduate Students

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Contents

1 Calculus of variations

A functional J is a map $J : C^\infty(\mathcal{R}^n) \to \mathcal{R}$. We restrict ourselves to functionals of the form

$$J[y] = \int_{x_1}^{x_2} F(x, y, y^{(1)}, y^{(2)}, ...)dx \qquad (1)$$

where $y^{(n)}$ denotes the nth derivative of y with respect to x.

1.1 Functionals of y and $y^{(1)}$

For $J = \int F dx$, where f depends only on x, y and y', we show how to derive the Euler-Lagrange equation and find the y that optimises J.

Let ϵ be a small, arbitrary real number, η an arbitrary function of x and y an arbitrary function of x in C^∞. A small perturbation in y is given by $y = y^* + \epsilon\eta$, where y^* is the unperturbed y. We require that $y^*(x_1) = y(x_1)$ and $y^*(x_2) = y(x_2)$, ie $\eta(x_1) = \eta(x_2) = 0$. Let $\delta J = J[y^* + \epsilon\eta] - J[y^*]$ and $\delta F = \{F(x, y, y^{(1)}) - F(x, y^*, y^{*(1)})\}$, where δ is the variation operator. The change in J associated with going from y to $y^* + \epsilon\eta$ is given by

$$\delta J = \int_{x_1}^{x_2} \delta F dx \qquad (2)$$

A necessary condition for the minimisation of J is $\delta J = 0$. We then use a Taylor expansion of δF around $\epsilon = 0$, discarding all terms second order and above.

$$\delta F \approx \epsilon\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y^{(1)}}\eta^{(1)}\right) \qquad (3)$$

Using integration by parts, we can find an alternate expression for $\frac{\partial F}{\partial y^{(1)}}\eta^{(1)}$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(1)}}\eta^{(1)}dx = \frac{\partial F}{\partial y^{(1)}}\eta\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta dx$$

As we assume $\epsilon = 0$ at x_1 and x_2, $\frac{\partial F}{\partial y^{(1)}}\eta\Big|_{x_1}^{x_2} = 0$. So we have

$$\frac{\partial F}{\partial y^{(1)}}\eta^{(1)} = -\frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta$$

We substitute this into (3) to obtain

$$\delta F \approx \epsilon\eta\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\right)$$

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Substituting this expression into (2) results in

$$\int_{x_1}^{x_2} \delta F dx = \int_{x_1}^{x_2} \epsilon\eta\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\right)dx = 0$$

Supposing that η and ϵ are not identically zero, we conclude that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}} = 0 \tag{4}$$

(4) is known as an *Euler-Lagrange equation*. The expression on the left hand side is the *functional derivative* of J with respect to y, $\frac{\delta J}{\delta y}$.

1.2   Functionals of higher order derivatives

If F is a function of higher order derivatives of y, e.g. $y^{(5)}$ or $y^{(26)}$, then the Euler-Lagrange equation is extended as follows. Suppose F is a function of the Nth order derivative of y. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} + \sum_{n=1}^{N}(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0 \tag{5}$$

1.3   Functionals of multiple functions

When F is a function of multiple functions y_i and their derivatives y_{ix}, where each y_i is a function of x, then we get a separate Euler-Lagrange equation for each y_i

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i^{(1)}} = 0 \tag{6}$$

1.4   Functionals of multiple functions and higher order derivatives

Combining the previous two sections, the Euler-Lagrange equations for functionals of multiple functions y_i and higher order derivatives $y_i^{(n)}$ are given by

$$\frac{\partial F}{\partial y_i} + \sum_{n=1}^{N}(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y_i^{(n)}}\right) = 0 \tag{7}$$

for each i corresponding to a y_i.

1.5 Functionals of multiple functions, multiple independent variables and higher order derivatives

The next logical step is to introduce multiple independent variables. Let μ_m be indices spanning the independent variables x_{μ_m}. Let $y_i^{(n)\mu_m}$ denote the nth partial derivative of y_i with respect to x_{μ_m}. Given a total of M independant variables, functionals of multiple functions y_i and higher order derivatives $y_i^{(n)\mu_m}$, the Euler-Lagrange equation for each y_i is as follows

$$\frac{\partial F}{\partial y_i} + \sum_{n=1}^{N} \sum_{\mu_1 \leq ... \leq \mu_n} (-1)^n \frac{d^n}{dx_{\mu_1}...dx_{\mu_n}} \left(\frac{\partial F}{\partial y_i^{(1)\mu_1}...\partial y_i^{(1)\mu_n}} \right) = 0 \qquad (8)$$

Note that μ_m is an arbitrary reference to any x_m. Hence x_{μ_1} may be equal to $x_{\mu 2}$ even though $x_1 \neq x_2$. For 3 independent variables x_1, x_2 and x_3, and $n = 2$, we have that

$$\sum_{\mu_1 \leq ... \leq \mu_2} dx_{\mu_1} dx_{\mu_2} = dx_1^2 + dx_1 dx_2 + dx_1 dx_3 + dx_2^2 + dx_2 dx_3 + dx_3^2$$

For the same example, but with $n = 1$, we have

$$\sum_{\mu_1 \leq ... \leq \mu_1} dx_{\mu_1} dx_{\mu_1} = dx_1^2 + dx_2^2 + dx_3^2$$

1.6 Constraints

We will frequently want to find the stationary point of δJ subject to certain constraints.

1.6.1 Integral constraints

The simplest constraints are integral constraints

$$K = \int_{x_1}^{x_2} G(x, y, y^{(1)}, y^{(2)}, ...)dx \qquad (9)$$

where K is a constant and G is a functional.

To incorporate this into our Euler-Lagrange equations, we define F_c as follows

$$F_c = F + \lambda G$$

and work with F_c in the same way we would with F, subject to the constraints being satisfied. e.g. (8) becomes

$$\frac{\partial F_c}{\partial y_i} + \sum_{n=1}^{N} \sum_{\mu_1 \le \dots \le \mu_n} (-1)^n \frac{d^n}{dx_{\mu_1}\dots dx_{\mu_n}} \left(\frac{\partial F_c}{\partial y_i^{(1)\mu_1}\dots \partial y_i^{(1)\mu_n}} \right) = 0$$

and our solutions must also satisfy (9).

Given multiple functionals G_m in integral constraints, F_c is given by

$$F_c = F + \sum_m \lambda_m G_m$$

and solutions to the Euler-Lagrange equations must satisfy the constraints.

1.6.2   Holonomic constraints

Equivalence constraints not involving derivatives of functions that can be expressed without an integral are referred to as holonomic constraints. They take the form

$$K = G(x,y) \tag{10}$$

where G and K are defined as in the previous section. We now define F_c as

$$F_c = F + \int_{x_1}^{x_2} \lambda(x) G(x,y) dx \tag{11}$$

where $\lambda(x)$ is a Lagrange multiplier. Take note that our Lagrange multiplier is now dependant on our independent variables. Given F_c, we proceed in the same manner as for integral constraints.

1.6.3   Non-holonomic constraints

Non-holonomic constraints are equivalence constraints that involve derivatives of functions. They are not covered.

1.7   The first integral

The quantity $I = F - y^{(1)}\frac{\partial f}{\partial y^{(1)}}$ is called the *first integral*. Suppose F is directly dependent only on y and $y^{(1)}$. We then have the following

$$\frac{d}{dx}\left(F - y^{(1)}\frac{\partial F}{\partial y^{(1)}} \right) = y^{(1)}\frac{\partial F}{\partial y} + y^{(2)}\frac{\partial F}{\partial y^{(1)}} - y^{(2)}\frac{\partial F}{\partial y^{(1)}} - y^{(1)}\frac{d}{dx}\left(\frac{\partial F}{\partial y^{(1)}} \right)$$

$$= y^{(1)}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y^{(1)}} \right) \right)$$

$$\tag{12}$$

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The bracketed expression on the second line is the Euler-Lagrange equation and hence is set to 0. We therefore have the result that the first integral I is equal 0 when F has no direct dependence on x.

For multiple functions y_i the first integral becomes

$$I = F - \sum_i y_i^{(1)} \frac{\partial F}{\partial y_i^{(1)}} \tag{13}$$

1.8   Free boundary value problems

When an end-point is free to vary, eg x_2 is not specified, then we must also determine x_2 as part of the optimisation solution.

The general procedure is to proceed as if x_2 is fixed and find the extremal y. x_2 can then be determined by solving

$$\mathcal{L}(x_2, y(x_2), y'(x_2), ...) = 0 \tag{14}$$

for x_2. This is referred to as the *natural boundary condition* for x_2. A corresponding condition applies to determine x_1, should it also not be predetermined.

1.9   Examples

1.9.1   Example 1: Brachistochrone

Suppose a bead is sliding down a wire from a height of y_0 to y_1 along a length of wire with start and end x coordinates being x_0 and x_1. We want to know the shape of the wire that results in the particle moving from x_0 to x_1 as quickly as possible under the force of gravity alone.

The time taken by the bead to slide down the wire is given by the distance travelled divided by the velocity of the bead. Using the fact that $T + V = constant$ and choosing a reference point of V such that the *constant* $= 0$ allows us to derive an expression for the velocity, $v = \sqrt{2gy}$. The distance travelled is simply the integral of ds, hence our functional is

$$\begin{aligned} t_{total} &= \int_{y_0}^{y_1} \frac{ds}{\sqrt{2gy}} \\ &= \int_{y_0}^{y_1} \sqrt{\frac{1 + y'^2}{2gy}} dt \end{aligned} \tag{15}$$

We thus have

$$F(y, y') = \sqrt{\frac{1 + y'^2}{2gy}} \tag{16}$$

Using the Euler-Lagrange equation and solving the resulting differential equation yields a solution of

$$x = at - asin(t)$$
$$y = a - acos(t) \tag{17}$$

where a is some constant chosen to satisfy the boundary conditions below

$$(x_0, y_0) = (at - asin(0), a - acos(0))$$
$$(x_1, y_1) = (at - asin(t_{total}), a - acos(t_{total})) \tag{18}$$

1.9.2 Interlude: Lagrangian mechanics

Lagrangian mechanics is a formulation of classical mechanics that uses functionals composed of the difference between kinetic and potential energy, as opposed to the traditional approach based on Newton's laws. The Lagrangian $\mathcal{L}$ is given as follows

$$\mathcal{L} = T - V \tag{19}$$

where T is the kinetic energy and V is the potential energy.

We are free to write T and V in terms of whatever variables we would like. Typically the independent variable is time and functions dependent on time are denoted by q_i.

The Principle of Least Action states that the time evolution of a physical system between time t_0 and t_1 minimises the *action* of the system. The action $\mathcal{S}$ is defined as

$$\mathcal{S} = \int_{t_0}^{t_1} \mathcal{L} dt \tag{20}$$

By minimising the action using the calculus of variations we can therefore find the equations describing the time evolution of a system.

1.9.3 Example 2: Constrained optimisation

Suppose a chain is fixed to two walls and we would like to determine the shape in which it will hang. We can do this by minimising the potential energy using a functional. The potential energy at each point in the chain is given by $PE =$

$\mu g y(x)ds$, where μ is the density of the chain per unit length, g is the force of gravity and $y(x)$ is the height. Integrating over the length of the chain gives us the total potential energy V, and is a functional that we can minimise with respect to y.

$$V = g\mu \int_a^b y\sqrt{1 - y'^2}dx \tag{21}$$

We then have

$$F(y, y') = y\sqrt{1 - y'^2} \tag{22}$$

The associated Euler-Lagrange equation is as follows

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = \sqrt{1 - y'^2} - \tag{23}$$

THE ABOVE IS UNFINISHED

References

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2 Function spaces

2.1 Vector spaces

A vector space consists of a set V (whose elements are referred to as vectors), a field $\mathbb{F}$ (whose elements are called scalars) and two operators:

1. A vector addition operation $+$ such that for $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$.

2. A scalar multiplication operation such that for $\lambda \in \mathbb{F}$ and $\mathbf{u} \in V$, $\lambda \mathbf{u} \in V$.

which satisfy the following conditions

1. For $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \tag{24}$$

2. There exists some $\mathbf{0} \in V$ such that for $\forall \mathbf{v} \in V$,
$$\mathbf{v} + \mathbf{0} = \mathbf{v} \tag{25}$$

3. For $\forall \mathbf{v} \in V$, there exists some $-\mathbf{v} \in V$ such that
$$\mathbf{v} + -\mathbf{v} = \mathbf{v} - \mathbf{v} = \mathbf{0} \tag{26}$$

4. For $a, b \in \mathbb{F}$ and $\mathbf{v} \in V$, $(ab)\mathbf{v} = a(b\mathbf{v})$.

5. For $\forall \mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$.

6. For $a, b \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$,
$$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u} \tag{27}$$
   and
$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \tag{28}$$

2.2 Functions as vectors

Taking functions to be infinite-dimensional vectors, we can take the vectors in a vector space to be functions. We refer to vector spaces of functions as function spaces. We use this to build up Hilbert spaces. An example of a vector space of functions is $C^n[a, b]$ - the set of all functions in $[a, b]$ with continuous n^{th} order derivatives.

2.3 Inner products

An inner product associates two functions $f, g \in V$ with a real number and is denoted by $\langle f, g \rangle$. It obeys three axioms:

1. Bilinearity
$$\langle cf + dg, h \rangle = c\langle f, h \rangle + d\langle g, h \rangle \tag{29}$$
$$\langle f, cg + dh \rangle = c\langle f, g \rangle + d\langle f, h \rangle \tag{30}$$

2. Symmetry

$$\langle f, g \rangle = \langle g, f \rangle \tag{31}$$

3. Non-negative

$$\langle f, g \rangle \geq 0 \tag{32}$$

with $\langle f, f \rangle = 0$ iff $f = 0$.

2.4   Norms

The norm of a function space provides a measure of the 'size' of its elements. The norm of some function space V is a function $||.|| : V \longrightarrow \mathbb{F}$ satisfying the following conditions:

1. $||f|| \geq 0||$, with $||f|| = 0$ iff $f = 0$,
2. $||f + g|| \leq ||f|| + ||g||$,
3. $||\lambda f|| = |\lambda|||f||$ for constant λ.

We can obtain a norm from an inner product by taking the pth root of the inner product of some function f with itself: $\langle f, f \rangle^{1/p}$. Note that *a norm is not necessarily linear*, so the set of all norms built from inner products is only a subset of all possible norms.

The space $L^p[a, b]$ is defined to be a function space over the interval $[a, b]$, function set $F[a, b]$, for $1 \leq p < \infty$ and a norm of

$$||f||_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p} \tag{33}$$

$||f||_p$ must be finite for $\forall f \in F$.

The elements of $L^p[a, b]$ are equivalence classes of functions, with functions in the same equivalence class differing by a function of norm 0. Note that a function can have a norm of zero despite not being identically 0 if it is only non-zero at a finite number of points. This violates the first condition for being a norm, but forget that for now.

2.5   Metric spaces

Let V be a set. A metric on V is a function $d : V \times V \longrightarrow \mathbb{R}$ such that for all $f, g, h \in V$ we have

1. Non-negativity: $0 \leq d(f, g) < \infty$

2. Uniqueness: $d(f, g) = 0$ if and only if $f = g$.

3. Symmetry: $d(f, g) = d(g, f)$

4. The triangle inequality: $d(f, g) \leq d(f, h) + d(h, g)$

2.6   From inner products to norms, and norms to metrics

An inner product $\langle , \rangle$ can be converted into a norm $||.||$ by setting $||f|| = \langle f, f \rangle$. $||.||$ is referred to as the *induced norm*.

A norm $||.||$ can be converted to a metric d by setting $d(f, g) = ||f - g||$. d is referred to as the *induced metric*.

2.7   Convergence of functions

Sequence of functions f_n may converge to f *pointwise*, *uniformly* or *in the mean*.

2.7.1   Pointwise convergence

For all x and all $\epsilon > 0$, there exists some N such that, if $n \geq N$, then $|f_n - f| < \epsilon$.

2.7.2   Uniform convergence

For all $\epsilon > 0$ there exists some N such that, for all x and $n > N$, $sup|f_n(x) - f(x)| < \epsilon$.

2.7.3   Convergence in the mean

For all $\epsilon > 0$ there exists some N such that $\int |f_n(x) - f(x)| dx < \epsilon$ for $n > N$.

2.8   Banach spaces

A normed function space is *complete* if every Cauchy sequence in it converges to an element contained in the normed function space. A complete normed function space is called a *Banach space*. $L^p[a, b]$ spaces are complete if the integrals in the L^p norm are taken to be Lebesgue integrals.

2.9   Hilbert spaces

A Hilbert space is a *complete* inner product space. A Banach space can be turned into a Hilbert space by inducing an inner product from the norm. e.g. if we define the inner product as

$$\langle f, g \rangle = \int_a^b f^* g \, dx \tag{34}$$

then the $L^2[a, b]$ norm can be written as

$$||f||_2 = \sqrt{\langle f, f \rangle} \tag{35}$$

A Banach space differs from a Hilbert space in that a Banach space does not necessarily have an inner product, whereas a Hilbert space has an inner product which induces a norm. $\langle f, g \rangle$ is a continuous functional of f and g. A set of functions $\{u_n\}$ is orthonormal if $\langle u_n, u_m \rangle = \delta_{nm}$. An orthonormal set of functions $\{u_n\}$ within some Hilbert space is *complete* if any element f inside the Hilbert space can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x) \tag{36}$$

2.9.1 Defining functions in a Hilbert space

Functions within a Hilbert space H are defined in terms of the convergence of sequences within H.

$$f \equiv \lim_{n \to \infty} f_n \tag{37}$$

Note that the limit, and hence f, may not exist.

2.9.2 Parseval's theorem

Given a representation of f with a complete orthonormal representation, the norm of f is given by

$$||f|| = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \tag{38}$$

2.10 Orthogonal polynomials

Let $L_w^2[a, b]$ be a real Hilbert space with a real inner product

$$\langle u, v \rangle_w = \int_a^b w(x) u(x) v(x) dx \tag{39}$$

where $u(x), v(x)$ are polynomials and $w(x)$ is a weighting function.

Using the *Gram-Schmidt procedure*, we recursively define sets of orthonormal polynomials from the set of polynomials $\{1, x, x^2, ...x^n, ...\}$. We begin with $P_0 = 1/||\mathbf{1}||_w$ where $||\mathbf{1}||_w = \sqrt{\int_a^b 1 * w(x) dx}$. Higher orders of $P_n(x)$ are defined recursively as follows

$$P_{n+1}(x) = \frac{x P_n(x) - \sum_{i=0}^{n} P_i(x) \langle P_i, x P_n \rangle_w}{||x P_n(x) - \sum_{i=0}^{n} P_i(x) \langle P_i, x P_n \rangle_w||_w} \tag{40}$$

Where the numerator represents a construction of the next orthogonal polynomial by multiplying the previous one by x to increase its order and then subtracting all non-orthogonal components of it, with respect to previous orthonormal polynomials defined. The denominator is simply a normalisation.

For $n \neq k$, P_n is orthogonal to all other P_k, leading to the following

$$\langle P_n, P_m \rangle_w = \delta_{nm} \tag{41}$$

These polynomials obey a three-term recurrence relation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x) \tag{42}$$

2.10.1 Legendre polynomials

On the interval $[a, b]$, with $a = -1, b = 1$ and $w(x) = 1$, the *Legendre polynomials* are given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \tag{43}$$

Note that we do not normalise them by the scalar product, but instead set $P_n(1) = 1$.

Their inner product is

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{nm} \tag{44}$$

and their three-term recurrence relation is

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{45}$$

The Legendre polynomials form a complete set of functions for expanding functions in $[-1, 1]$.

2.10.2 Hermite polynomials

Hermite polynomials are defined on $(-\infty, \infty)$ with $w(x) = e^{-x^2}$. They are given as follows

$$H_n(x) = \frac{d^n}{dt^n} e^{-x^2 - (x-t)^2} \Big|_{t=0} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{46}$$

Their associated inner product is

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm} \tag{47}$$

and the associated three-term recurrence relation is

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x) \tag{48}$$

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2.10.3 Tchebychef polynomials

Tchebychef polynomials are defined over $[-1, 1]$ and $w(x) = (1 - x^2)^{\pm 1/2}$. Tchebychef polynomials of the first kind are given by

$$T_n(x) = cos(ncos^{-1}(x)) \tag{49}$$

and Tchebychef polynomials of the second kind are given by

$$U_{n-1}(x) = \frac{sin(ncos^{-1}(x))}{sin(cos^{-1}(x))} = \frac{1}{n}T_n'(x) \tag{50}$$

Both Tchebechef polynomials obey the same recurrence relation

$$2xT_n = T_{n+1} + T_{n-1} \tag{51}$$

The inner products of Tchebychef polynomials are as below

$$\int_0^\pi T_n T_m dx = h_n \delta_{nm} \tag{52}$$

$$\int_0^\pi U_n U_m dx = \frac{\pi}{2}\delta_{nm} \tag{53}$$

Where $h_n = \pi$ for $n = 0$ and $h_n = \pi/2$ for $n > 0$.

2.11 Linear operators

Linear operators in function spaces are analogous to linear operators on matrices.

An operator A operating on a function f can calculated as follows

$$g(x) = \int_a^b A(x, y)f(y)dy \tag{54}$$

In this case, A is an *integral kernel*.

Suppose we have an operator $A(x, y) = \delta(x - y)$. The action of A on f is

$$f(x) = \int_a^b \delta(x - y)f(y)dy \tag{55}$$

$\delta(x - y)$ is then clearly the identity operator.

2.12 Bibliography

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3 Linear ordinary differential equations

4 Linear differential operators

5 Green functions