Notes for Mathematics for Physics: A Guided Tour for Graduate Students

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1 Calculus of variations

A functional J is a map $J : C^\infty(\mathcal{R}) \to \mathcal{R}$. We restrict ourselves to functionals of the form

$$J[y] = \int_{x_1}^{x_2} F(x, y, y^{(1)}, y^{(2)}, ...)dx \tag{1}$$

where $y^{(n)}$ denotes the n^{th} derivative of y with respect to x.

1.1 Minimising functionals of y and y_x

For $J = \int F dx$, where f depends only on x, y and y', we show how to derive the Euler-Lagrange equation and find the y that optimises J.

Let ϵ be a small, arbitrary real number, η an arbitrary function of x and y an arbitrary function of x in C^∞. A small perturbation in y is given by $y = y^* + \epsilon\eta$, where y^* is the unperturbed y. Also let $\delta J = J[y^* + \epsilon\eta] - J[y^*]$ and $\delta F = \{F(x, y, y^1) - F(x, y^*, y^{*(1)})\}$, where δ is the variation operator. The change in J associated with going from y to $y^* + \epsilon\eta$ is given by

$$\delta J = \int_{x_1}^{x_2} \delta F dx \tag{2}$$

A necessary condition for the minimisation of J is $\delta J = 0$. We then use a Taylor expansion of δF around $\epsilon = 0$, discarding all terms second order and above.

$$\delta F \approx \epsilon\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y^{(1)}}\eta^{(1)}\right) \tag{3}$$

Using integration by parts, we can find an alternate expression for $\frac{\partial F}{\partial y^{(1)}}\eta^{(1)}$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(1)}}\eta^{(1)} dx = \frac{\partial F}{\partial y^{(1)}}\eta\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta dx$$

As we assume $\epsilon = 0$ at x_1 and x_2, $\frac{\partial F}{\partial y^{(1)}}\eta\bigg|_{x_1}^{x_2} = 0$. So we have

$$\frac{\partial F}{\partial y^{(1)}}\eta^{(1)} = -\frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta$$

We substitute this into (3) to obtain

$$\delta F \approx \epsilon\eta\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\right)$$

Substituting this expression into (2) results in

$$\int_{x_1}^{x_2} \delta F dx = \int_{x_1}^{x_2} \epsilon\eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y^{(1)}} \right) dx = 0$$

Supposing that η and ϵ are not identically zero, we conclude that

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y^{(1)}} = 0 \tag{4}$$

(4) is known as the *Euler-Lagrange equation.* The expression on the left hand side is referred to as the functional derivative of δJ with respect to y.

1.2 Functionals of higher order derivatives

If F is a function of higher order derivatives of y, e.g. $y^{(5)}$ or $y^{(26)}$, then the Euler-Lagrange equation is extended as follows. Suppose F is a function of the n^{th} order derivative of y. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} + \sum_{i=1}^{n} (-1)^i \frac{d^i}{dx^i} \left(\frac{\partial F}{\partial y^{(i)}} \right) = 0 \tag{5}$$

1.3 Minimising functionals of multiple functions

When F is a function of multiple functions y_i and their derivatives y_{ix}, where each y_i is a function of x, then we get a separate Euler-Lagrange equation for each y_i

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y_i^{(1)}} = 0 \tag{6}$$

1.4 Functionals of multiple functions and higher order derivatives

Combining the previous two sections, the Euler-Lagrange equations for functionals of multiple functions y_i and higher order derivatives $y_i^{(n)}$ are given by

$$\frac{\partial F}{\partial y_i} + \sum_{k=1}^{n} (-1)^k \frac{d^k}{dx^k} \left(\frac{\partial F}{\partial y_i^{(k)}} \right) = 0 \tag{7}$$

for each i corresponding to a y_i.

1.5 Examples

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1.6 Variable end points

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References

[1] R.S. Schechter, *The variational method in engineering*, McGraw-Hill chemical engineering series, McGraw-Hill, 1967.

2 Function spaces

3 Linear ordinary differential equations

4 Linear differential operators

5 Green functions