Gaussian Processes

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1 Gaussian Distributions

If X is a random variable with a univariate Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$, its probability density function $f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right) \tag{1}$$

The univarate Gaussian distribution can be extended to multiple dimensions, where the probability density for some vector $\boldsymbol{x}$ is given by

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{D/2}\sqrt{|\boldsymbol{\Sigma}|}} exp\left(- \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \right) \tag{2}$$

where $\boldsymbol{\mu}$ is a vector of expected values and $\boldsymbol{\Sigma}$ is the covariance matrix. We write $p(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ or $X \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Suppose $\boldsymbol{x}^* = [\boldsymbol{x}, \boldsymbol{x'}]$. We can express $p(\boldsymbol{x}^*)$ as $p(\boldsymbol{x}, \boldsymbol{x'})$, which is given as follows

$$p(\boldsymbol{x}, \boldsymbol{x'}) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_x \\ \boldsymbol{\mu}_{x'} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{xx} & \boldsymbol{\Sigma}_{xx'} \\ \boldsymbol{\Sigma}_{x'x} & \boldsymbol{\Sigma}_{x'x'} \end{bmatrix} \right) \tag{3}$$

where $\boldsymbol{\Sigma}_{xx'}$ is the covariance matrix of $\boldsymbol{x}$ and $\boldsymbol{x'}$.

1.0.1 Conditional Distribution

The conditional distribution $p(\boldsymbol{x}|\boldsymbol{x'})$ is given by

$$p(\boldsymbol{x}|\boldsymbol{x'}) = \mathcal{N}(\boldsymbol{\mu}_{x|x'}, \boldsymbol{\Sigma}_{x|x'}) \tag{4}$$

$$\boldsymbol{\mu}_{x|x'} = \boldsymbol{\mu}_x + \boldsymbol{\Sigma}_{xx'}\boldsymbol{\Sigma}_{x'x'}^{-1}(\boldsymbol{x'} - \boldsymbol{\mu}_{x'}) \tag{5}$$

$$\boldsymbol{\Sigma}_{x|x'} = \boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xx'}\boldsymbol{\Sigma}_{x'x'}^{-1}\boldsymbol{\Sigma}_{x'x} \tag{6}$$

$$\tag{7}$$