Analog and Digital Communications - Formulae and Identities

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Contents

1 Signals and Systems

1.1 Basics

1.1.1 Normalized Energy

The normalized energy content E of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1}$$

1.1.2 Average Power

The normalized average power P of a signal $x(t)$ is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2}$$

1.1.3 Dirac Delta Function

The unit impulse function (or Dirac delta function) $\delta(t)$ is defined as

$$\delta(t) = \int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \tag{3}$$

where $\phi(t)$ is any test function continuous at $t = 0$. The unit impulse function is a *generalized function.*

1.1.4 Derivatives of Generalized Functions

The derivative $g'(t)$ of a generalized function $g(t)$ is defined by

$$\int_{-\infty}^{\infty} g'(t)\phi(t)dt = -\int_{-\infty}^{\infty} g(t)\phi'(t)dt \tag{4}$$

1.1.5 Complex Fourier Series

The Fourier series for a signal $x(t)$ is defined as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{5}$$

where ω_0 is the fundamental angular frequency. The Fourier coefficients c_n are given by

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t} dt \tag{6}$$

A plot of $|c_n|$ vs ω is called the amplitude spectrum. A plot of θ_n (the phase constants of c_n) vs ω is called the phase spectrum. Together these are referred to as the frequency spectra.

1.1.6 Parseval's Theorem

Parseval's theorem states that for a periodic signal $x(t)$

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{7}$$

1.1.7 Fourier Transform

The Fourier transform, $\mathscr{F}$, of a signal $x(t)$ is given by

$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \tag{8}$$

1.1.8 Inverse Fourier Transform

The inverse Fourier transform of $X(\omega)$, $\mathscr{F}^{-1}$, is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega \tag{9}$$

1.2 Properties of the Fourier Transform

$x(t) \longleftrightarrow X(\omega)$ denotes a Fourier transform pair.

1.2.1 Linearity

$$a_1x_1(t)+a_2x_2(t) \longleftrightarrow a_1X_1(\omega)+a_2X_2(\omega) \quad (10)$$

1.2.2 Time Shifting

$$x(t-t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0} \quad (11)$$

1.2.3 Frequency Shifting

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0) \quad (12)$$

1.2.4 Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a}) \quad (13)$$

1.2.5 Time Reversal

$$x(-t) \longleftrightarrow X(-\omega) \quad (14)$$

1.2.6 Duality

$$X(t) \longleftrightarrow 2\pi x(-\omega) \quad (15)$$

1.2.7 Differentiation

Time differentiation

$$x'(t) = \frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega) \quad (16)$$

Frequency differentiation

$$(-jt)x(t) \longleftrightarrow X'(\omega) = \frac{d}{d\omega}X(\omega) \quad (17)$$

1.2.8 Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega) \quad (18)$$

1.3 Convolutions and Correlation

The convolution of two signals $x_1(t)$ and $x_2(t)$ is

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \quad (19)$$

1.3.1 Time Convolution Theorem

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)X_2(\omega) \quad (20)$$

1.3.2 Frequency Convolution Theorem

$$x_1(t)x_2(t) \longleftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega) \quad (21)$$

1.3.3 Cross-Correlation

The cross correlation $R_{12}(\tau)$ of signals $x_1(t)$ and $x_2(t)$ is defined by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt \quad (22)$$

1.3.4 Autocorrelation

The autocorrelation is defined as the cross-correlation of a signal $x_1(t)$ with itself, $R_{11}(\tau)$.

1.3.5 Energy Spectral Density

The energy spectral density S_{11} of a signal $x_1(t)$ is given by

$$S_{11}(\omega) = \mathscr{F}[R_{11}(\tau)] = \int_{-\infty}^{\infty} R_{11}(\tau)e^{-j\omega\tau}d\omega \quad (23)$$

1.4 Linear Time-Invariant Systems

Linear time-invariant (linear time-invariant) systems have several properties, as follows. Suppose $\mathcal{F}$ is an operator representing the action of a system with output $y(t)$.

1.4.1 Additivity

$$\mathcal{F}[x_1(t) + x_2(t)] = \mathcal{F}[x_1(t)] + \mathcal{F}[x_2(t)] \tag{24}$$

1.4.2 Homogeneity

$$\mathcal{F}[ax(t)] = a\mathcal{F}[x(t)] \tag{25}$$

1.4.3 Time-Invariance

$$\mathcal{F}[x(t - t_0)] = y(t - t_0) \tag{26}$$

1.4.4 Impulse Response

The impulse response $h(t)$ of an LTI system is the response of the system with a delta function input

$$h(t) = \mathcal{F}[\delta(t)] \tag{27}$$

1.4.5 Response to Arbitrary Inputs

The response of an LTI system to an arbitrary input can be expressed in terms of a convolution with the impulse response of the system

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{28}$$

1.4.6 Causality

A signal $x(t)$ is causal if, for $t < 0$, $x(t) = 0$.

1.4.7 Frequency Response

Using the time convolution theorem (20) on the response of an LTI system (28), we find that

$$Y(\omega) = X(\omega)H(\omega) \tag{29}$$

where $Y(\omega) = \mathscr{F}[y(t)]$ and $H(\omega) = \mathscr{F}[h(t)]$. We refer to $H(\omega)$ as the *frequency response* or *transfer function.*

1.4.8 Input and Output Spectral Densities

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \tag{30}$$

$$\bar{S}_{yy}(\omega) = |H(\omega)|^2 \bar{S}_{xx}(\omega) \tag{31}$$

2 Amplitude Modulation