Analog and Digital Communications - Formulae and Identities

December 26, 2019

Contents

1 Signals and Systems

1.1 Basics

1.1.1 Normalized Energy

The normalized energy content E of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1}$$

1.1.2 Average Power

The normalized average power P of a signal $x(t)$ is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2}$$

1.1.3 Dirac Delta Function

The unit impulse function (or Dirac delta function) $\delta(t)$ is defined as

$$\delta(t) = \int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \tag{3}$$

where $\phi(t)$ is any test function continuous at $t = 0$. The unit impulse function is a *generalized function*.

1.1.4 Derivatives of Generalized Functions

The derivative $g'(t)$ of a generalized function $g(t)$ is defined by

$$\int_{-\infty}^{\infty} g'(t)\phi(t)dt = -\int_{-\infty}^{\infty} g(t)\phi'(t)dt \tag{4}$$

1.1.5 Complex Fourier Series

The Fourier series for a signal $x(t)$ is defined as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{5}$$

where ω_0 is the fundamental angular frequency. The Fourier coefficients c_n are given by

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt \tag{6}$$

A plot of $|c_n|$ vs ω is called the amplitude spectrum. A plot of θ_n (the phase constants of c_n) vs ω is called the phase spectrum. Together these are referred to as the frequency spectra.

1.1.6 Parseval's Theorem

Parseval's theorem states that for a periodic signal $x(t)$

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}|x(t)|^2dt = \sum_{n=-\infty}^{\infty}|c_n|^2 \tag{7}$$

1.1.7 Fourier Transform

The Fourier transform, $\mathscr{F}$, of a signal $x(t)$ is given by

$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{8}$$

1.1.8 Inverse Fourier Transform

The inverse Fourier transform of $X(\omega)$, $\mathscr{F}^{-1}$, is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{9}$$

1.2 Properties of the Fourier Transform

$x(t) \longleftrightarrow X(\omega)$ denotes a Fourier transform pair.

1.2.1 Linearity

$$a_1x_1(t) + a_2x_2(t) \longleftrightarrow a_1X_1(\omega) + a_2X_2(\omega) \tag{10}$$

1.2.2 Time Shifting

$$x(t-t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0} \tag{11}$$

1.2.3 Frequency Shifting

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0) \tag{12}$$

1.2.4 Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a}) \tag{13}$$

1.2.5 Time Reversal

$$x(-t) \longleftrightarrow X(-\omega) \tag{14}$$

1.2.6 Duality

$$X(t) \longleftrightarrow 2\pi x(-\omega) \tag{15}$$

1.2.7 Differentiation

Time differentiation

$$x'(t) = \frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega) \tag{16}$$

Frequency differentiation

$$(-jt)x(t) \longleftrightarrow X'(\omega) = \frac{d}{d\omega}X(\omega) \tag{17}$$

1.2.8 Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega) \tag{18}$$

1.2.9 Modulation Theorem

$$x(t)cos(\omega_0 t) \longleftrightarrow \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0) \tag{19}$$

1.3 Convolutions and Correlation

The convolution of two signals $x_1(t)$ and $x_2(t)$ is

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \tag{20}$$

1.3.1 Time Convolution Theorem

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)X_2(\omega) \tag{21}$$

1.3.2 Frequency Convolution Theorem

$$x_1(t)x_2(t) \longleftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega) \tag{22}$$

1.3.3 Cross-Correlation

The cross correlation $R_{12}(\tau)$ of signals $x_1(t)$ and $x_2(t)$ is defined by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt \tag{23}$$

1.3.4 Autocorrelation

The autocorrelation is defined as the cross-correlation of a signal $x_1(t)$ with itself, $R_{11}(\tau)$.

1.3.5 Energy Spectral Density

The energy spectral density S_{11} of a signal $x_1(t)$ is given by

$$S_{11}(\omega) = \mathscr{F}[R_{11}(\tau)] = \int_{-\infty}^{\infty} R_{11}(\tau) e^{-j\omega\tau} d\omega \tag{24}$$

1.4 Linear Time-Invariant Systems

Linear time-invariant (linear time-invariant) systems have several properties, as follows. Suppose $\mathcal{F}$ is an operator representing the action of a system with output $y(t)$.

1.4.1 Additivity

$$\mathcal{F}[x_1(t) + x_2(t)] = \mathcal{F}[x_1(t)] + \mathcal{F}[x_2(t)] \tag{25}$$

1.4.2 Homogeneity

$$\mathcal{F}[ax(t)] = a\mathcal{F}[x(t)] \tag{26}$$

1.4.3 Time-Invariance

$$\mathcal{F}[x(t - t_0)] = y(t - t_0) \tag{27}$$

1.4.4 Impulse Response

The impulse response $h(t)$ of an LTI system is the response of the system with a delta function input

$$h(t) = \mathcal{F}[\delta(t)] \tag{28}$$

1.4.5 Response to Arbitrary Inputs

The response of an LTI system to an arbitrary input can be expressed in terms of a convolution with the impulse response of the system

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \tag{29}$$

1.4.6 Causality

A signal $x(t)$ is causal if, for $t < 0$, $x(t) = 0$.

1.4.7 Frequency Response

Using the time convolution theorem (21) on the response of an LTI system (29), we find that

$$Y(\omega) = X(\omega) H(\omega) \tag{30}$$

where $Y(\omega) = \mathscr{F}[y(t)]$ and $H(\omega) = \mathscr{F}[h(t)]$. We refer to $H(\omega)$ as the *frequency response* or *transfer function*.

1.4.8 Input and Output Spectral Densities

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \tag{31}$$

$$\bar{S}_{yy}(\omega) = |H(\omega)|^2 \bar{S}_{xx}(\omega) \tag{32}$$

2 Amplitude Modulation

Modulation is defined as the process by which some characteristic of a carrier signal is varied in accordance with a modulating signal (message signal). There are two basic types of analog modulation; continuous wave (CW) modulation and pulse modulation.

2.1 Continuous-Wave Modulation

In continuous-wave modulation, a sinusoidal signal is used as a carrier signal. The modulated carrier signal $x_c(t)$ is given by

$$x_c(t) = A(t)cos[\omega_c t + \phi(t)] \tag{33}$$

Let $m(t)$ be the message signal.

2.1.1 Double-Sideband Modulation

Double-sideband (DSB) modulation occurs when $m(t) = A(t)$. The DSB modulated signal $x_{DSB}(t)$ is then given by

$$x_{DSB}(t) = m(t)cos(\omega_c t) \tag{34}$$

By applying the modulation theorem (19), we obtain the following

$$X_{DSB}(\omega) = \frac{1}{2}M(\omega - \omega_c) + \frac{1}{2}M(\omega + \omega_c) \tag{35}$$

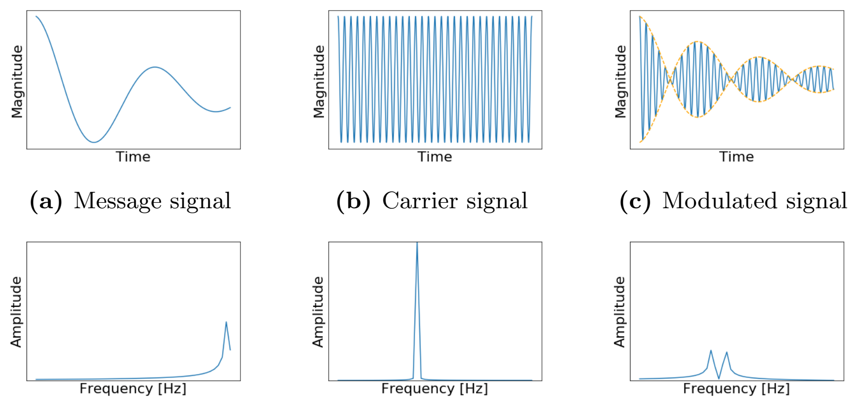
See figure 1 for an example.

2.1.2 Ordinary Amplitude Modulation

2.1.3 Single-Sideband Modulation

2.1.4 Vestigial-Sideband Modulation

References



(a) Message signal **(b)** Carrier signal **(c)** Modulated signal

(d) Message signal **(e)** Carrier signal **(f)** Modulated signal

Figure 1: Modulating a message signal onto a carrier signal with DSB modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain. Note how the carrier signal frequency gets split into two smaller peaks. These are referred to as the upper and lower sidebands.