Analog and Digital Communications - Formulae and Identities

January 12, 2020

Contents

1 Signals and Systems

1.1 Basics

1.1.1 Normalized Energy

The normalized energy content E of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1}$$

1.1.2 Average Power

The normalized average power P of a signal $x(t)$ is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2}$$

1.1.3 Dirac Delta Function

The unit impulse function (or Dirac delta function) $\delta(t)$ is defined as

$$\delta(t) = \int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \tag{3}$$

where $\phi(t)$ is any test function continuous at $t = 0$. The unit impulse function is a *generalized function*.

1.1.4 Derivatives of Generalized Functions

The derivative $g'(t)$ of a generalized function $g(t)$ is defined by

$$\int_{-\infty}^{\infty} g'(t)\phi(t)dt = -\int_{-\infty}^{\infty} g(t)\phi'(t)dt \tag{4}$$

1.1.5 Complex Fourier Series

The Fourier series for a signal $x(t)$ is defined as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{5}$$

where ω_0 is the fundamental angular frequency. The Fourier coefficients c_n are given by

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t} dt \tag{6}$$

A plot of $|c_n|$ vs ω is called the amplitude spectrum. A plot of θ_n (the phase constants of c_n) vs ω is called the phase spectrum. Together these are referred to as the frequency spectra.

1.1.6 Parseval's Theorem

Parseval's theorem states that for a periodic signal $x(t)$

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{7}$$

1.1.7 Fourier Transform

The Fourier transform, $\mathscr{F}$, of a signal $x(t)$ is given by

$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{8}$$

1.1.8 Inverse Fourier Transform

The inverse Fourier transform of $X(\omega)$, $\mathscr{F}^{-1}$, is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{9}$$

1.2 Properties of the Fourier Transform

$x(t) \longleftrightarrow X(\omega)$ denotes a Fourier transform pair.

1.2.1 Linearity

$$a_1x_1(t) + a_2x_2(t) \longleftrightarrow a_1X_1(\omega) + a_2X_2(\omega) \tag{10}$$

1.2.2 Time Shifting

$$x(t - t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0} \tag{11}$$

1.2.3 Frequency Shifting

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0) \tag{12}$$

1.2.4 Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a}) \tag{13}$$

1.2.5 Time Reversal

$$x(-t) \longleftrightarrow X(-\omega) \tag{14}$$

1.2.6 Duality

$$X(t) \longleftrightarrow 2\pi x(-\omega) \tag{15}$$

1.2.7 Differentiation

Time differentiation

$$x'(t) = \frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega) \tag{16}$$

Frequency differentiation

$$(-jt)x(t) \longleftrightarrow X'(\omega) = \frac{d}{d\omega}X(\omega) \tag{17}$$

1.2.8 Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega) \tag{18}$$

1.2.9 Modulation Theorem

$$x(t)cos(\omega_0 t) \longleftrightarrow \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0) \tag{19}$$

1.3 Convolutions and Correlation

The convolution of two signals $x_1(t)$ and $x_2(t)$ is

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \tag{20}$$

1.3.1 Time Convolution Theorem

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)X_2(\omega) \tag{21}$$

1.3.2 Frequency Convolution Theorem

$$x_1(t)x_2(t) \longleftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega) \tag{22}$$

1.3.3 Cross-Correlation

The cross correlation $R_{12}(\tau)$ of signals $x_1(t)$ and $x_2(t)$ is defined by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt \tag{23}$$

1.3.4 Autocorrelation

The autocorrelation is defined as the cross-correlation of a signal $x_1(t)$ with itself, $R_{11}(\tau)$.

1.3.5 Energy Spectral Density

The energy spectral density S_{11} of a signal $x_1(t)$ is given by

$$S_{11}(\omega) = \mathscr{F}[R_{11}(\tau)] = \int_{-\infty}^{\infty} R_{11}(\tau)e^{-j\omega\tau}d\omega \tag{24}$$

1.4 Linear Time-Invariant Systems

Linear time-invariant (linear time-invariant) systems have several properties, as follows. Suppose $\mathcal{F}$ is an operator representing the action of a system with output $y(t)$.

1.4.1 Additivity

$$\mathcal{F}[x_1(t) + x_2(t)] = \mathcal{F}[x_1(t)] + \mathcal{F}[x_2(t)] \tag{25}$$

1.4.2 Homogeneity

$$\mathcal{F}[ax(t)] = a\mathcal{F}[x(t)] \tag{26}$$

1.4.3 Time-Invariance

$$\mathcal{F}[x(t - t_0)] = y(t - t_0) \tag{27}$$

1.4.4 Impulse Response

The impulse response $h(t)$ of an LTI system is the response of the system with a delta function input

$$h(t) = \mathcal{F}[\delta(t)] \tag{28}$$

1.4.5 Response to Arbitrary Inputs

The response of an LTI system to an arbitrary input can be expressed in terms of a convolution with the impulse response of the system

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \tag{29}$$

1.4.6 Causality

A signal $x(t)$ is causal if, for $t < 0$, $x(t) = 0$.

1.4.7 Frequency Response

Using the time convolution theorem (21) on the response of an LTI system (29), we find that

$$Y(\omega) = X(\omega)H(\omega) \tag{30}$$

where $Y(\omega) = \mathscr{F}[y(t)]$ and $H(\omega) = \mathscr{F}[h(t)]$. We refer to $H(\omega)$ as the *frequency response* or *transfer function*.

1.4.8 Input and Output Spectral Densities

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \tag{31}$$

$$\bar{S}_{yy}(\omega) = |H(\omega)|^2 \bar{S}_{xx}(\omega) \tag{32}$$

2 Amplitude Modulation

Modulation is defined as the process by which some characteristic of a carrier signal is varied in accordance with a modulating signal (message signal). There are two basic types of analog modulation; continuous wave (CW) modulation and pulse modulation.

2.1 Continuous-Wave Modulation

In continuous-wave modulation, a sinusoidal signal is used as a carrier signal. The modulated carrier signal $x_c(t)$ is given by

$$x_c(t) = A(t)cos[\omega_c t + \phi(t)] \tag{33}$$

Let $m(t)$ be the message signal.

2.1.1 Double-Sideband Modulation

Double-sideband (DSB) modulation occurs when $m(t) = A(t)$. The DSB modulated signal $x_{DSB}(t)$ is then given by

$$x_{DSB}(t) = m(t)cos(\omega_c t) \tag{34}$$

By applying the modulation theorem (19), we obtain the following

$$X_{DSB}(\omega) = \frac{1}{2}M(\omega - \omega_c) + \frac{1}{2}M(\omega + \omega_c) \tag{35}$$

See figure 1 for an example.

2.1.2 Ordinary Amplitude Modulation

Ordinary amplitude modulation (AM) varies the carrier signal amplitude linearly with respect to the message signal as follows

$$x_c(t) = [1 + k_a m(t)]A(t)cos(\omega_c t) \tag{36}$$

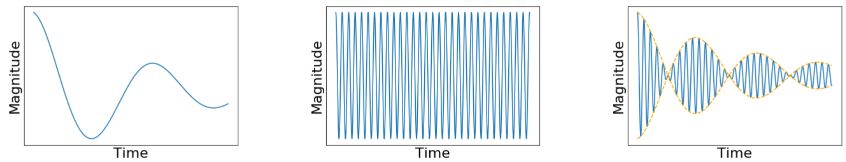
where $|k_a m(t)| < 1$.

2.1.3 Single-Sideband Modulation

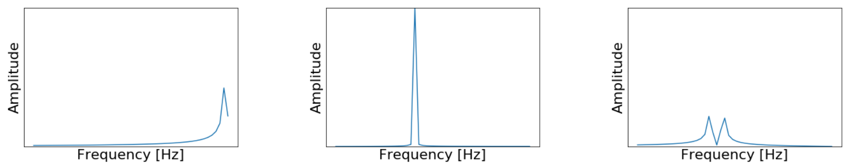
There are two types of single single sideband (SSB) modulation; upper sideband and lower sideband. They can both be generated by phase-shifting the message signal by $\frac{\pi}{2}$, resulting in a modulated signal given as below

$$x_{SSB}(t) = m(t)cos(\omega_c t) \mp \hat{m}(t)sin(\omega_c t) \tag{37}$$

where the difference results in an upper sideband and the sum results in the lower sideband.

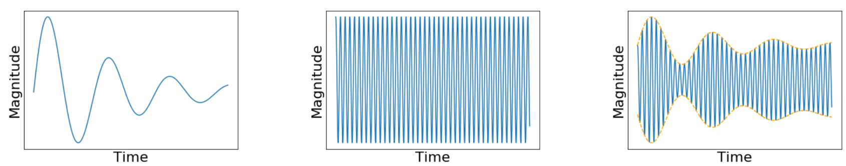


(a) Message signal **(b)** Carrier signal **(c)** Modulated signal

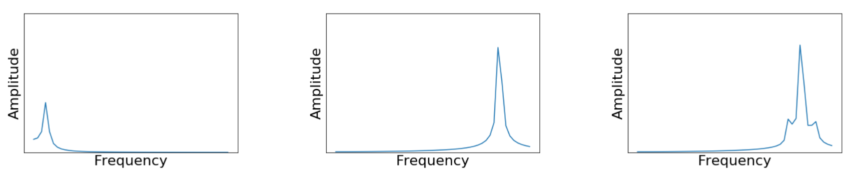


(d) Message signal **(e)** Carrier signal **(f)** Modulated signal

Figure 1: Modulating a message signal onto a carrier signal with DSB modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain. Note how the carrier signal frequency gets split into two smaller peaks. These are referred to as the upper and lower sidebands.



(a) Message signal **(b)** Carrier signal **(c)** Modulated signal

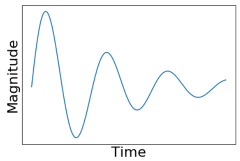


(d) Message signal **(e)** Carrier signal **(f)** Modulated signal

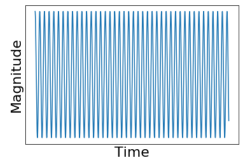
Figure 2: Modulating a message signal onto a carrier signal with AM modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain.

2.1.4 Vestigial-Sideband Modulation

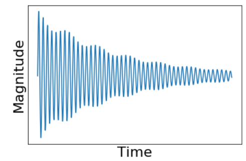
Vestigial-sideband modulation is similar to DSB, except one sideband is attenuated. This is a compromise between DSB and SSB. Vestigial-sideband modulation is performed on top of DSB modulation, with a vestigial-sideband filter



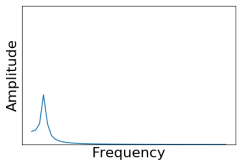
(a) Message signal



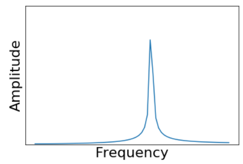
(b) Carrier signal



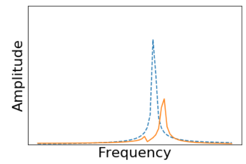
(c) Modulated signal



(d) Message signal



(e) Carrier signal



(f) Modulated signal

Figure 3: Modulating a message signal onto a carrier signal with upper SSB modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain.

applied as the final step.

2.2 Frequency Translation and Mixing

Test

3 Angle Modulation

Angle modulation includes both phase modulation (PM) and frequency modulation (FM). We again use a sinusoidal carrier signal

$$x_c(t) = Acos[\omega_c t + \phi(t)] \tag{38}$$

However, instead of modifying the amplitude A, we modify the phase angle $\phi(t)$ by making it a function of the message signal $m(t)$.

Let $\theta(t) = \omega_c t + \phi(t)$. Then the instantaneous radian frequency of $x_c(t)$ is given by

$$\omega_{inst} = \frac{d\theta(t)}{dt} = \omega_c + \frac{d\phi(t)}{dt} \tag{39}$$

3.1 Phase Modulation

In phase modulation, we let the phase angle $\phi(t) = k_p m(t)$, where k_p is the phase deviation constant.

3.2 Frequency Modulation

In frequency modulation, we set $\frac{d\phi(t)}{dt} = k_f m(t)$, where k_f is the frequency deviation constant. Integrating over $m(t)$, we find the following expression for $\phi(t)$

$$\phi(t) = k_f \int_{t_0}^{t} m(\lambda) + \phi(t_0) \tag{40}$$

We usually let $t_0 = -\infty$ and $\phi(-\infty) = 0$.

3.3 Sinusoidal Angle Modulation

Sinusoidal message signals result in a phase angle of

$$\phi(t) = \beta sin(\omega_m t) \tag{41}$$

β is referred to as the modulation index and gives the maximum value of phase deviation. β can be expressed as

$$\beta = \frac{\Delta\omega}{\omega_m} \tag{42}$$

where $\Delta\omega$ is the maximum frequency deviation. Substituting (41) into (38) and expressing it as a Fourier series, we obtain

$$x_c(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) cos(\omega_c + n\omega_m)t \tag{43}$$

where $J_n(\beta)$ are Bessel functions of order n. Three things should be noted:

1. The signal is composed of a carrier frequency with infinite harmonic sidebands.

2. The aplitude is dependant on $J_n(\beta)$, which gets smaller with larger values of $|n|$.

3. β also determines the number of significant spectral lines, with smaller β resulting in fewer significant sidebands and greater β resulting in more numerous significant sidebands.

3.4 Bandwidth of Angle Modulated Signals

3.4.1 Sinusoidal Modulation

The bandwidth W_B of a signal with sinusoidal modulation is approximately given as follows

$$W_B \approx 2(\beta + 1)\omega_m \tag{44}$$

3.4.2 Arbitrary Modulation

The Deviation radio D is defined as

$$D = \frac{\Delta\omega}{\omega_M} \tag{45}$$

for a message signal bandwidth ω_M. The bandwidth of the modulated signal is then given as

$$W_B \approx 2(D + 1)\omega_M \tag{46}$$

W_B contains 98% of signal power.

4 Digital Transmission of Analog Signals

4.1 Pulse Code Modulation

Pulse code modulation (PCM) consists of three main steps: sampling, quantisation and encoding. The sampled signal must be *bandwidth limited* - ie it must be composed of a finite number of frequencies.

4.1.1 Pulse Amplitude Modulation

Pulse amplitude modulation (PAM) utilises a periodic train of rectangular pulses as the carrier signal $x_c(t)$. The modulated signal $x_{PAM}(t)$ is then given by the discrete convolution of the message signal $m(t)$ and the carrier signal

$$x_{PAM}(t) = m(t) * x_c(t) = \sum_{n=-\infty}^{\infty} m(nT_s)x_c(t - nT_s) \tag{47}$$

where T_s is the sampling period. Once the signal has been modulated, thereby quantising in time, its amplitude must also be quantised. This can be done by slicing the amplitude into discrete, uniformly sized intervals and replacing all signal values within a given interval with the centre value fo the interval.

4.1.2 PAM Uniform Quantising Error

There is an error associated with uniform amplitude quantisation that is referred to as quantising error or quantising noise. Given an amplitude step size of Δ_A, the quantising error q_e varies uniformly over $[-\frac{\Delta_A}{2}, \frac{\Delta_A}{2}]$. The expected value of q_e^2 is thus

$$\langle q_e^2 \rangle = \frac{1}{\Delta_A} \int_{-\Delta_A/2}^{\Delta_A/2} q_e^2 dq_e = \frac{\Delta_A^2}{12} \tag{48}$$

4.1.3 Non-uniform Quantisation

Oftentimes signals are not evenly distributed over their range of values. In these situations, it is better to use non-uniform quantisation that allows for finer discretisation in ranges of amplitudes where signal values are more likely to be. A typical method of non-uniform quantisation is to first use a non-linear *compression* of the original signal followed by a uniform quantisation of the compressed signal. A common compression used is the *μ law*

$$y = \frac{ln(1 + \mu|m/m_p|)}{ln(1+\mu)} sgn(m), \qquad \left|\frac{m}{m_p}\right| \leq 1 \tag{49}$$

where μ is a positive constant, m_p is the maximum signal amplitude and $sgn(m) = 1$ if $m > 0$ and -1 otherwise.

Another common compression is the *A law*, given by

$$y = \begin{cases} \frac{A}{1+ln(A)}(\frac{m}{m_p}), & |\frac{m}{m_p}| \leq \frac{1}{A} \\ \frac{(1+ln(A)|m/m_p|)}{1+ln(A)} sgn(m), & \frac{1}{A} \leq |\frac{m}{m_p}| \leq 1. \end{cases} \tag{50}$$

4.1.4 Bandwidth Requirements

Binary PCM with L levels of quantisation satisfying $L = 2^n$ require $n = log_2(L)$ binary pulses to be transmitted for each sample of the message signal. If the sampling rate is f_s then nf_s binary pulses must be transmitted per second. If the PCM signal is a low-pass signal of bandwidth f_{PCM}, then the required minimum sampling rate is $2f_{PCM}$. In other words

$$f_{PCM} \geq \frac{n}{2} f_s \tag{51}$$

4.1.5 Delta Modulation

Delta modulation (DM) performs analog to digital signal conversion using the *sgn* function. Let $m(t)$ be the analog signal and $\tilde{m}(t)$ be the digital signal. The value of $\tilde{m}(t)$ for the next sampling period of length T_s is given by $\Delta sgn[e(t)]$ for $e(t) = m(t) - \tilde{m}(t)$.

4.1.6 DM Quantising Error

The quantising error in delta modulation is inside the interval $(-\Delta, \Delta)$ and hence the expected value fo the mean squared quantising error is

$$\langle q_e^2 \rangle = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} q_e^2 dq_e = \frac{\Delta^2}{3} \tag{52}$$

4.1.7 Adaptive Delta Modulation

4.2 Signaling Formats

Binary symbols can be transmitted by several different pulse waveforms, given below.

4.2.1 Unipolar Nonreturn-to-Zero

Unipolar nonreturn-to-zero (NRZ) signalling represents 1 using a pulse of constant, non-zero amplitude over an entire bit interval. No signal is transmitted over over a bit interval to communicate 0.

4.2.2 Bipolar NRZ

For a given amplitude A_{NRZ}, 1 is represented by a pulse of amplitude A_{NRZ} and 0 is represented by a pulse of amplitude $-A_{NRZ}$.

4.2.3 Unipolar Return-to-Zero

1 is represented by a positive amplitude pulse that decays to an amplitude of 0 before the end of the bit interval. The absence of a pulse represents 0.

4.2.4 Bipolar RZ

1 is represented by a positive amplitude pulse that decays to an amplitude of 0 before the end of the bit interval. 0 is represented by a negative amplitude pulse that increases to an amplitude of 0 before the end of the bit interval.

4.2.5 Alternate Mark Inversion RZ

1 is represented by interchanging positive and negative pulses of equal amplitude, both returning to 0 before the end fo the bit interval. 0 is represented by no signal.

4.2.6 Split-Phase

Within a bit interval, 1 is represented by a positive pulse followed by a negative pulse, both of the same absolute amplitude. 0 is represented by a negative pulse followed by a positive pulse, again with the same absolute amplitude.

4.3 Time-Division Multiplexing

Time-division multiplexing (TDM) allows several signals to be transmitted over the same frequency by allocating them to particular intervals of time. The required minimum sampling rate is $2f_{TDM}$, for

$$f_{TDM} = \frac{1}{2} n f_s \tag{53}$$

where n is the number of signals and f_s is the sampling frequency of the whole multiplexed signal.

4.4 Pulse Shaping and Intersymbol Interference

Intersymbol interference (ISI) occurs when transmitted pulses spread out and over adjacent time slots. Two ways to address this are pulse shaping and raised-cosine filtering.

4.4.1 Pulse Shaping

Pulse shaping uses pulses with the following shape

$$h(t) = \frac{1}{T_s} \frac{sin(\pi t/T_s)}{\pi t/T_s} \tag{54}$$

4.4.2 Raised-Cosine Filtering

Raised-cosine filtering uses a frequency response as follows

$$H(\omega) = \begin{cases} 1, & 0 \le |\omega| \le (1-\alpha)W \\ \frac{1}{2}\left\{1 - sin\left[\frac{\pi}{2\alpha W}(|\omega| - W)\right]\right\}, & (1-\alpha)W \le |\omega| \le (1+\alpha)W \\ 0, & |\omega| > (1+\alpha)W \end{cases} \tag{55}$$

where $W = \pi T_s$. The corresponding impulse response is

$$h(t) = \frac{1}{T_s}\left(\frac{sin(Wt)}{Wt}\frac{cos(\alpha Wt)}{1-(2\alpha Wt/\pi)^2}\right) \tag{56}$$

The bandwidth required for a pulse transmission rate of $\frac{1}{T}$ is

$$f_B = \frac{1+\alpha}{2T} \tag{57}$$

4.5 Digital Carrier Modulation Systems

When transmitting low frequency binary symbols, we need to transmit them with high frequency waves. There are several ways to do this.

4.5.1 Amplitude-Shift Keying

Amplitude-shift keying (ASK) modulates the signal as follows

$$x_c(t) = \begin{cases} Acos(\omega_c t), & \text{symbol 1} \\ 0, & \text{symbol 0} \end{cases} \tag{58}$$

4.5.2 Frequency-Shift Keying

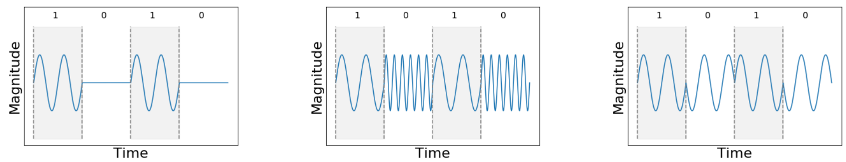
Frequency-shift keying (FSK) modulates the signal as follows

$$x_c(t) = \begin{cases} Acos(\omega_1 t), & \text{symbol 1} \\ Acos(\omega_2 t), & \text{symbol 0} \end{cases} \tag{59}$$

4.5.3 Phase-Shift Keying

Phase-shift keying (PSK) modulates the signal as follows

$$x_c(t) = \begin{cases} Acos(\omega_c t), & \text{symbol 1} \\ Acos(\omega_c t + \pi), & \text{symbol 0} \end{cases} \tag{60}$$



(a) ASK **(b)** FSK **(c)** PSK

Figure 4: Modulating a message signal onto a carrier signal with AM modulation. Plots (a), (b) and (c) display signals in the time domain, whereas (d), (e) and (c) display the signals in the frequency domain.

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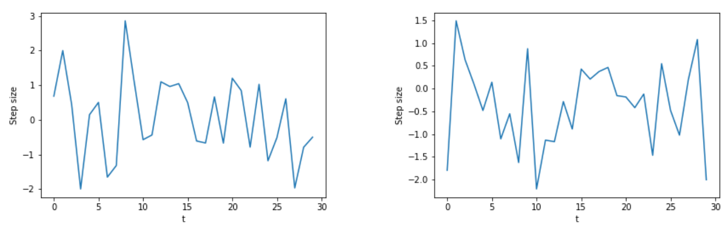
5 Stochastic Processes

5.1 Overview of Stochastic Processes

A stochastic process $\{X_t\}$ is a collection of random variables X_t. A *sample function* represents a single possible realisation of the stochastic process, ie each random variable X_t takes on a single value.

For example, suppose that we have a stochastic process X_t, with $X_t = \mathcal{N}(0,1)$. This is the Wiener process. Two possible sample functions of X_t are given in figure 5.



(a) First sample function.

(b) Second sample function.

Figure 5: Two realisations of sample functions of a Wiener process.

5.2 Autocorrelation, Autocovariance and Joint Density Functions

The first order density $f_X(x,\tau)$ of a stochastic process $\{X_t\}$ is given by

$$f_X(x,\tau) = p(X_t = x | t = \tau) \tag{61}$$

Suppose we have n random variables X_t, then the n^{th} order density functions is defined as follows

$$f_X(x_1, ...x_n, \tau_1, ...\tau_n) = p(X_{t_1} = x_1, ..., X_{t_n} = x_n | t_1 = \tau_1, ..., t_n = \tau_n) \tag{62}$$

The autocorrelation of X_t is given by

$$R_{XX}(t, t-\tau) = E[X_t X_{t-\tau}] \tag{63}$$

for some given number of time lags τ.

The autocovariance is given by

$$C_{XX}(t, t-1) = E[(X_t - \mu_X(t))(X_{t-1} - \mu_X(t-1))] \tag{64}$$

5.3 Stationary Processes

A stochastic process $\{X_t\}$ is *stationary in the strict sense* (SSS) if its statistical properties are invariant with respect to time. In other words

$$f_X(x_1, ...x_n, \tau_1, ...\tau_n) = f_X(x_1, ...x_n, \tau_1 - c, ...\tau_n - c) \tag{65}$$

for all n and where c is some shift in time.

A stochastic process $\{X_t\}$ is *stationary in the wide sense* (WSS - side sense stationary) if the expected value of its density function is invariant with respect to time and if the autocorrelation function of the process depends only on τ. This means that for any given τ, the autocorrelation function is invariant with respect to time.

Two processes $\{X_t\}$ and $\{Y_t\}$ are joint-sense stationary if both $\{X_t\}$ and $\{Y_t\}$ are WSS and their cross-correlation depends only on the time difference τ

$$R_{XY}(t, t+\tau) = E[X_t Y_{t+\tau}] = R_{XY}(\tau) \tag{66}$$

In other words, given τ, the cross correlation between $\{X_t\}$ and $\{Y_t\}$ is time-invariant.

5.4 Ergodicity

The time averaged mean of a sample function $x(t)$ of a random process $\{X_t\}$ is defined as

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt \tag{67}$$

The time averaged autocorrelation of $x(t)$ is defined as

$$\langle x(t)x(t+\tau) \rangle = \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt \tag{68}$$

If $\{X_t\}$ is stationary, then the expected value of the time-averaged mean is equal to the ensemble mean

$$\begin{aligned} E[\langle x(t) \rangle] &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E[x(t)]dt \\ &= \mu_X \\ &= E[X_t] \end{aligned} \tag{69}$$

The expected value of the time-averaged autocorrelation will also be equal to the ensemble autocorrelation

$$
\begin{aligned}
E[\langle x(t)x(t+\tau)\rangle] &= \lim_{T\to\infty}\int_{-T/2}^{T/2} E[x(t)x(t+\tau)]dt \\
&= \lim_{T\to\infty}\int_{-T/2}^{T/2} R_{XX}(\tau)dt \\
&= R_{XX}(\tau)
\end{aligned}
\tag{70}
$$

A stochastic process $\{X_t\}$ is said to be *ergodic in the mean* if

$$\langle x(t)\rangle = E[X(t)] \tag{71}$$

which implies

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \int_{\Omega} xf(x)dx \tag{72}$$

where Ω is the sample space of X_t and $f(x) = \int_{\Theta} f(x,t)dt$ where Θ is the set of all t.

A stochastic process $\{X_t\}$ is *ergodic in the autocorrelation* if

$$\langle x(t)x(t+\tau)\rangle = E[X(t)X(t+\tau)] \tag{73}$$

5.5 Transmission of Stochastic Processes Through Linear Systems

References