# Notes for Mathematics for Physics: A Guided Tour for Graduate Students

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December 30, 2019

## Contents

# 1 Calculus of variations

A functional $J$ is a map $J : C^\infty(\mathcal{R}^n) \to \mathcal{R}$. We restrict ourselves to functionals of the form

$$J[y] = \int_{x_1}^{x_2} F(x, y, y^{(1)}, y^{(2)}, ...)dx \tag{1}$$

where $y^{(n)}$ denotes the $n^{th}$ derivative of $y$ with respect to $x$.

## 1.1 Functionals of $y$ and $y^{(1)}$

For $J = \int F dx$, where $f$ depends only on $x$, $y$ and $y'$, we show how to derive the Euler-Lagrange equation and find the $y$ that optimises $J$.

Let $\epsilon$ be a small, arbitrary real number, $\eta$ an arbitrary function of x and $y$ an arbitrary function of $x$ in $C^\infty$. A small perturbation in $y$ is given by $y = y^* + \epsilon\eta$, where $y^*$ is the unperturbed $y$. Also let $\delta J = J[y^* + \epsilon\eta] - J[y^*]$ and $\delta F = \{F(x, y, y^{(1)}) - F(x, y^*, y^{*(1)})\}$, where $\delta$ is the variation operator. The change in $J$ associated with going from $y$ to $y^* + \epsilon\eta$ is given by

$$\delta J = \int_{x_1}^{x_2} \delta F dx \tag{2}$$

A necessary condition for the minimisation of $J$ is $\delta J = 0$. We then use a Taylor expansion of $\delta F$ around $\epsilon = 0$, discarding all terms second order and above.

$$\delta F \approx \epsilon \left( \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y^{(1)}}\eta^{(1)} \right) \tag{3}$$

Using integration by parts, we can find an alternate expression for $\frac{\partial F}{\partial y^{(1)}}\eta^{(1)}$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y^{(1)}}\eta^{(1)} dx = \frac{\partial F}{\partial y^{(1)}}\eta\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta dx$$

As we assume $\epsilon = 0$ at $x_1$ and $x_2$, $\frac{\partial F}{\partial y^{(1)}}\eta\bigg|_{x_1}^{x_2} = 0$. So we have

$$\frac{\partial F}{\partial y^{(1)}}\eta^{(1)} = -\frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\eta$$

We substitute this into (3) to obtain

$$\delta F \approx \epsilon\eta\bigg(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\bigg)$$

Substituting this expression into (2) results in

$$\int_{x_1}^{x_2} \delta F dx = \int_{x_1}^{x_2} \epsilon\eta\bigg(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}}\bigg)dx = 0$$

Supposing that $\eta$ and $\epsilon$ are not identically zero, we conclude that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y^{(1)}} = 0 \tag{4}$$

(4) is known as an *Euler-Lagrange equation.* The expression on the left hand side is referred to as the functional derivative of $\delta J$ with respect to $y$.

## 1.2 Functionals of higher order derivatives

If $F$ is a function of higher order derivatives of $y$, e.g. $y^{(5)}$ or $y^{(26)}$, then the Euler-Lagrange equation is extended as follows. Suppose $F$ is a function of the $N^{th}$ order derivative of $y$. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} + \sum_{n=1}^{N}(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0 \tag{5}$$

## 1.3 Functionals of multiple functions

When $F$ is a function of multiple functions $y_i$ and their derivatives $y_{ix}$, where each $y_i$ is a function of $x$, then we get a separate Euler-Lagrange equation for each $y_i$

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i^{(1)}} = 0 \tag{6}$$

## 1.4 Functionals of multiple functions and higher order derivatives

Combining the previous two sections, the Euler-Lagrange equations for functionals of multiple functions $y_i$ and higher order derivatives $y_i^{(n)}$ are given by

$$\frac{\partial F}{\partial y_i} + \sum_{n=1}^{N} (-1)^n \frac{d^n}{dx^n} \left( \frac{\partial F}{\partial y_i^{(n)}} \right) = 0 \tag{7}$$

for each $i$ corresponding to a $y_i$.

## 1.5 Functionals of multiple functions, multiple independent variables and higher order derivatives

The next logical step is to introduce multiple independent variables. Let $\mu_m$ be indices spanning the independent variables $x_{\mu_m}$. Let $y_i^{(n)\mu_m}$ denote the $n^{th}$ partial derivative of $y_i$ with respect to $x_{\mu_m}$. Given a total of M independant variables, functionals of multiple functions $y_i$ and higher order derivatives $y_i^{(n)\mu_m}$, the Euler-Lagrange equation for each $y_i$ is as follows

$$\frac{\partial F}{\partial y_i} + \sum_{n=1}^{N} \sum_{\mu_1 \leq \ldots \leq \mu_n} (-1)^n \frac{d^n}{dx_{\mu_1} \ldots dx_{\mu_n}} \left( \frac{\partial F}{\partial y_i^{(1)\mu_1} \ldots \partial y_i^{(1)\mu_n}} \right) = 0 \tag{8}$$

Note that $\mu_m$ is an arbitrary reference to any $x_m$. Hence $x_{\mu_1}$ may be equal to $x_{\mu 2}$ even though $x_1 \neq x_2$. For 3 independent variables $x_1$, $x_2$ and $x_3$, and $n = 2$, we have that

$$\sum_{\mu_1 \leq \ldots \leq \mu_2} dx_{\mu_1} dx_{\mu_2} = dx_{\mu_1}^2 + dx_{\mu_1} dx_{\mu_2} + dx_{\mu_1} dx_{\mu_3} + dx_{\mu_2}^2 + dx_{\mu_2} dx_{\mu_3} + dx_{\mu_3}^2$$

For the sample example, but with $n = 1$, we have

$$\sum_{\mu_1 \leq \ldots \leq \mu_1} dx_{\mu_1} dx_{\mu_1} = dx_{\mu_1}^2 + dx_{\mu_2}^2 + dx_{\mu_3}^2$$

## 1.6 Constraints

We will frequently want to find the stationary point of $\delta J$ subject to certain constraints.

### 1.6.1 Integral constraints

The simplest constraints are integral constraints

$$K = \int_{x_1}^{x_2} G(x, y, y^{(1)}, y^{(2)}, \ldots) dx \tag{9}$$

where $K$ is a constant and $G$ is a functional.

To incorporate this into our Euler-Lagrange equations, we define $F_c$ as follows

$$F_c = F + \lambda G$$

and work with $F_c$ in the same way we would with $F$, subject to the constraints being satisfied. e.g. (8) becomes

$$\frac{\partial F_c}{\partial y_i} + \sum_{n=1}^{N} \sum_{\mu_1 \leq \ldots \leq \mu_n} (-1)^n \frac{d^n}{dx_{\mu_1} \ldots dx_{\mu_n}} \left( \frac{\partial F_c}{\partial y_i^{(1)\mu_1} \ldots \partial y_i^{(1)\mu_n}} \right) = 0$$

and our solutions must also satisfy (9).

Given multiple functionals $G_m$ in integral constraints, $F_c$ is given by

$$F_c = F + \sum_m \lambda_m G_m$$

and solutions to the Euler-Lagrange equations must satisfy the constraints.

### 1.6.2 Holonomic constraints

Equivalence constraints not involving derivatives of functions that can be expressed without an integral are referred to as holonomic constraints. They take the form

$$K = G(x, y) \tag{10}$$

where $G$ and $K$ are defined as in the previous section. We now define $F_c$ as

$$F_c = F + \int_{x_1}^{x_2} \lambda(x) G(x, y) dx \tag{11}$$

where $\lambda(x)$ is a Lagrange multiplier. Take note that our Lagrange multiplier is now dependant on our independent variables. Given $F_c$, we proceed in the same manner as for integral constraints.

### 1.6.3 Non-holonomic constraints

Non-holonomic constraints are equivalence constraints that involve derivatives of functions. They are not covered.

## 1.7 Examples

### 1.7.1 Lagrangian mechanics

## 1.8 Variable end points

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# References


[1] R. E. Hunt, *Chapter 1 variational methods*, 2002.

[2] R.S. Schechter, *The variational method in engineering*, McGraw-Hill chemical engineering series, McGraw-Hill, 1967.

[3] Michael Stone and Paul Goldbart, *Mathematics for physics: A guided tour for graduate students*, Cambridge University Press, United States, 2009.


# 2 Function spaces

# 3 Linear ordinary differential equations

# 4 Linear differential operators

# 5 Green functions