Signals and Systems - Formulae and Identities

December 23, 2019

1 Basics

1.1

The normalized energy content E of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

1.2

The normalized average power P of a signal $x(t)$ is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

1.3

The unit impulse function (or Dirac delta function) $\delta(t)$ is defined as

$$\delta(t) = \int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)$$

where $\phi(t)$ is any test function continuous at $t = 0$. The unit impulse function is a *generalized function*.

1.4

The derivative $g'(t)$ of a generalized function $g(t)$ is defined by

$$\int_{-\infty}^{\infty} g'(t)\phi(t)dt = -\int_{-\infty}^{\infty} g(t)\phi'(t)dt$$

1.5

The Fourier series for a signal $x(t)$ is defined as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where ω_0 is the fundamental angular frequency. The Fourier coefficients c_n are given by

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt$$

A plot of $|c_n|$ vs ω is called the amplitude spectrum. A plot of θ_n (the phase constants of c_n) vs ω is called the phase spectrum. Together these are referred to as the frequency spectra.

1.6

Parseval's theorem states that for a periodic signal $x(t)$

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

1.7

The Fourier transform, $\mathcal{F}$, of a signal $x(t)$ is given by

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

1.8

The inverse Fourier transform of $X(\omega)$, $\mathcal{F}^{-1}$, is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

2 Properties of the Fourier Transform

$x(t) \longleftrightarrow X(\omega)$ denotes a Fourier transform pair.

2.1 Linearity

$$a_1x_1(t)+a_2x_2(t) \longleftrightarrow a_1X_1(\omega)+a_2X_2(\omega)$$

2.2 Time Shifting

$$x(t-t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0}$$

2.3 Frequency Shifting

$$x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0)$$

2.4 Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$$

2.5 Time Reversal

$$x(-t) \longleftrightarrow X(-\omega)$$

2.6 Duality

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

2.7 Differentiation

Time differentiation

$$x'(t) = \frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega)$$

Frequency differentiation

$$(-jt)x(t) \longleftrightarrow X'(\omega) = \frac{d}{d\omega}X(\omega)$$

2.8 Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$$

3 Convolutions