

Viewing Graph Solvability in Structure from Motion

Federica Arrigoni

Politecnico di Milano (Italy) – federica.arrigoni@polimi.it



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Outline

- Introduction
- Calibrated Case
- Uncalibrated Case
- Calibrated vs Uncalibrated
- Conclusion

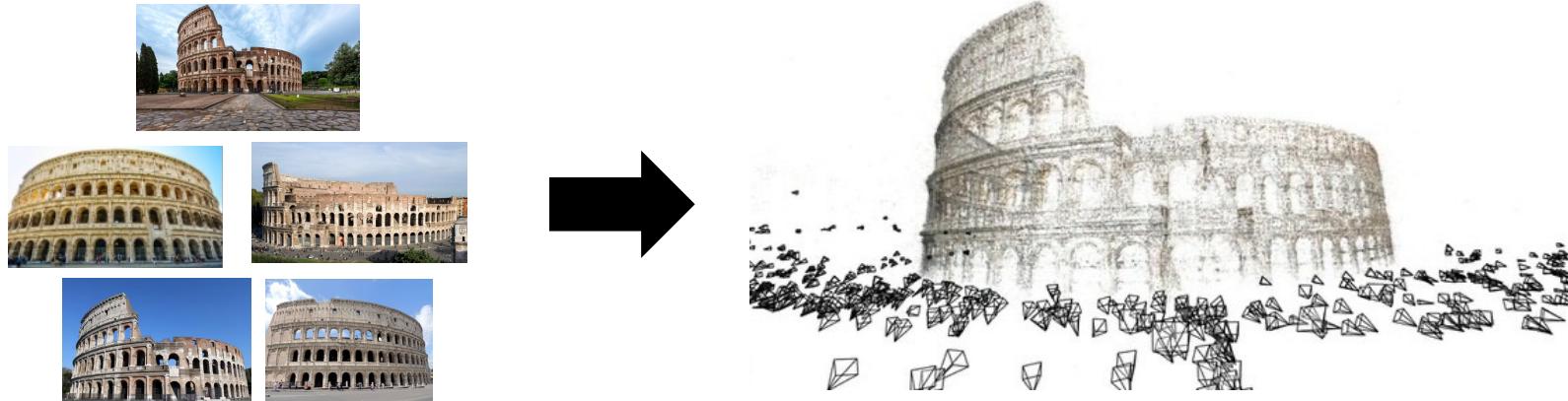
Outline

- **Introduction**
- Calibrated Case
- Uncalibrated Case
- Calibrated vs Uncalibrated
- Conclusion

Introduction

The goal of **structure from motion** (SfM) is to recover both camera motion and scene structure, starting from point correspondences in multiple images:

- camera motion = camera matrices/poses;
- scene structure = 3D coordinates of points.



■ O. Ozyesil, V. Voroninski, R. Basri, A. Singer. *A survey of structure from motion*. Acta Numerica (2017).

Introduction

Formally, the task is to compute **camera matrices** P_i and **coordinates of 3D points** M_j starting from image points m_{ij} such that the following equation is best satisfied:

$$m_{ij} \simeq P_i M_j$$

Projection of point j in image i
KNOWN

Projection matrix of camera i
UNKNOWN

3D coordinates of point j
UNKNOWN

In the calibrated case, calibration matrices are known and projection matrices consist of **rotations** and **translations**: $P_i = K_i[R_i \ t_i]$

$$= K_i [R_i \ t_i]$$

Known Unknown

Introduction

*Is 3D reconstruction **unique**?*



The solution is defined (at least) up to a global **projective transformation**:

$$m_{ij} \simeq P_i M_j = P_i \underbrace{Q Q^{-1}}_{\text{identity}} M_j = \underbrace{P_i Q}_{\text{new cameras}} \underbrace{Q^{-1} M_j}_{\text{new points}}$$

If cameras are calibrated, then the reconstruction ambiguity is represented (at least) by a global **rotation, translation and scale**.

Introduction

The task of solvability is to analyse the **ambiguities** inherent to the SfM problem:

- single transformation → well-posed problem ✓
- multiple transformations → ill-posed problem ✗

There are many ways to approach SfM!

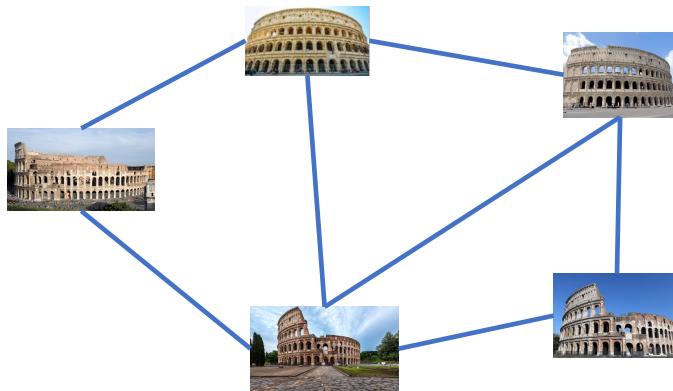


Here we focus on a framework that recovers **camera motion** from two-view relationships only (no points):

- Essential matrix (calibrated)
- Fundamental matrix (uncalibrated)

Introduction

The problem can be represented as a **viewing graph**:

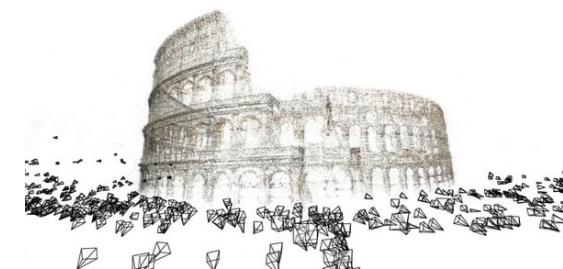
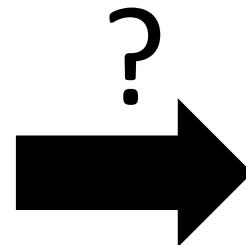
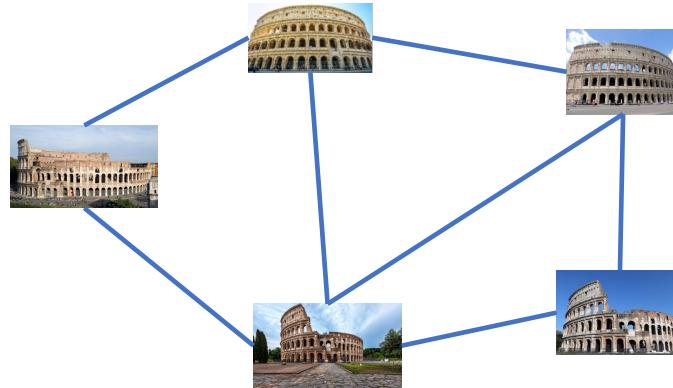


- Nodes = cameras/images
- Edges = two-view relations

Levi & Werman. *The viewing graph*. CVPR 2003.

Introduction

For which graphs do we have a **well-posed** problem?



- ✓ A graph is called **solvable** if and only if the available two-view relationships **uniquely** (up to a single transformation) determine the cameras → *unique solution*
- ✗ Otherwise it is called **non solvable** → *multiple (infinitely many) solutions*

Introduction

Here we focus on **solvability** only (*we do not address reconstruction*).

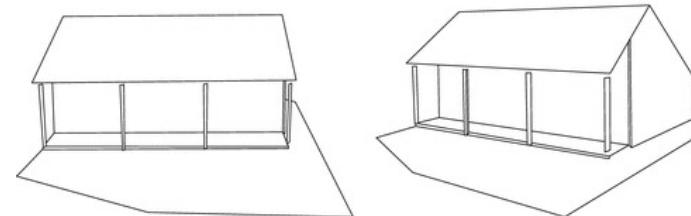
	Calibrated	Uncalibrated
Solvability	■ Arrigoni & Fusiello. <i>Bearing-based network localizability: a unifying view</i> . IEEE TPAMI (2019).	■ Levi & Werman. <i>The viewing graph</i> . CVPR 2003. ■ Rudi, Pizzoli & Pirri. <i>Linear solvability in the viewing graph</i> . ACCV 2011. ■ Trager, Osserman, & Ponce. <i>On the solvability of viewing graphs</i> . ECCV 2018. ■ Arrigoni, Fusiello, Ricci & Pajdla. <i>Viewing graph solvability via cycle consistency</i> . ICCV (2021).
Reconstruction	■ Ozyesil, Voroninski, Basri & Singer. <i>A survey of structure from motion</i> . Acta Numerica (2017).	■ Kasten, Geifman, Galun & Basri. <i>GPSfM: global projective SfM using algebraic constraints on multi-view fundamental matrices</i> . CVPR (2019)

It is important to check solvability **before running SfM**:

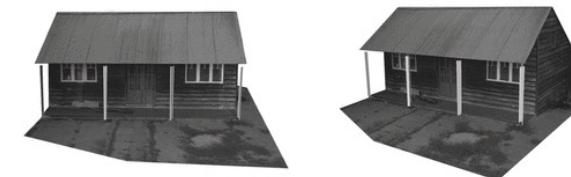
- ✓ If the graph is solvable, the SfM problem is well-posed.
- ✗ If the graph is not solvable, the problem is ill-posed: no method will return a useful solution.

Outline

- Introduction
 - **Calibrated Case** ----->
 - Uncalibrated Case
 - Calibrated vs Uncalibrated
 - Conclusion
- Calibration matrix is required in advance
 - Reconstruction is **metric** (up to scale)



Reconstruction

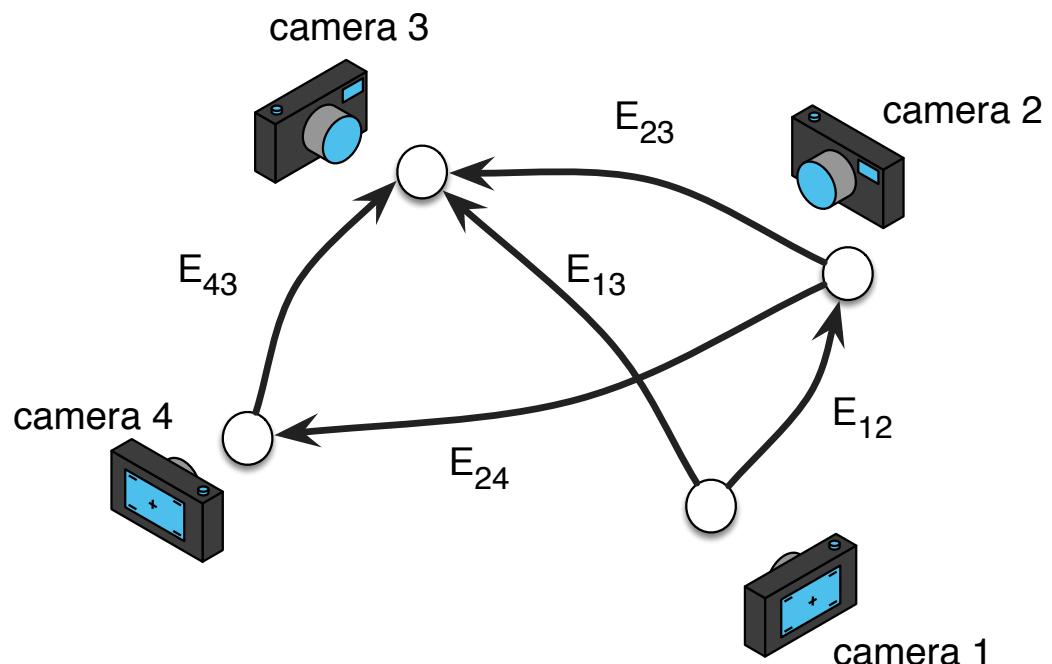


True scene

The Calibrated Case

Problem Formulation

The **viewing graph** is a graph where vertices correspond to cameras and edges represent essential matrices.



Each essential matrix can be decomposed into:

- *Relative rotation R_{ij}*
- *Relative translation t_{ij} (known up to scale)*

The Calibrated Case

Problem Formulation

Solvable graph \Leftrightarrow two-view transformations uniquely (up to a *single* rotation, translation & scale) determine the camera poses.

- We consider a **noiseless-case**
- We split the problem into **rotation and translation**:

$$R_{ij} = R_i R_j^T \quad \Longleftrightarrow \quad -\underbrace{R_i^T \mathbf{t}_{ij}}_{\mathbf{z}_{ij}} = \underbrace{-R_i^T \mathbf{t}_i}_{\mathbf{x}_i} + \underbrace{R_j^T \mathbf{t}_j}_{-\mathbf{x}_j}$$

----->

Consistency constraint between relative and absolute poses

Relative displacement Centre of camera i Centre of camera j

The Calibrated Case

Problem Formulation

Solvable graph \Leftrightarrow two-view transformations uniquely (up to a *single* rotation, translation & scale) determine the camera poses.

- We consider a **noiseless**-case
- We split the problem into **rotation and translation**:

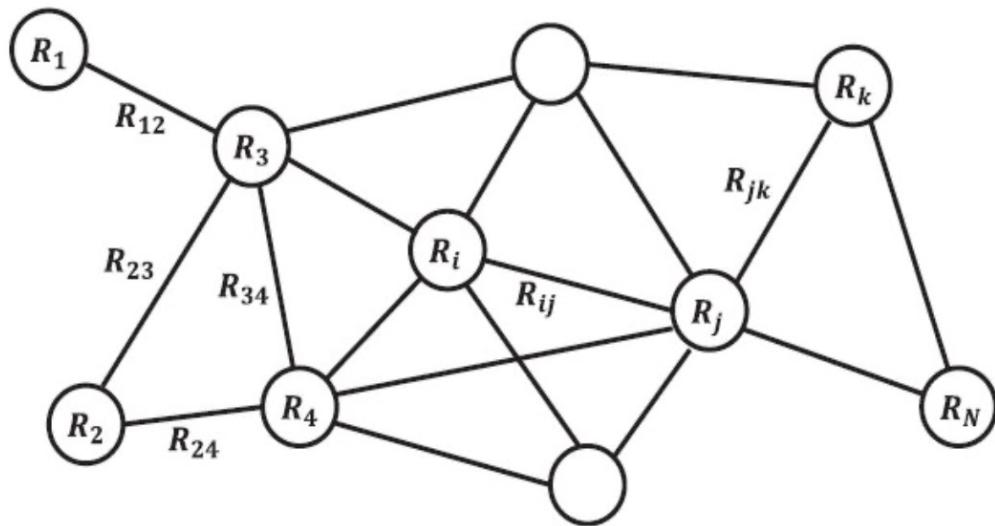
$$\begin{aligned} R_{ij} &= R_i R_j^\top \\ t_{ij} &= -R_i R_j^\top t_j + t_i \end{aligned} \iff \underbrace{-R_i^\top t_{ij}}_{\mathbf{z}_{ij}} = \underbrace{-R_i^\top t_i}_{\mathbf{x}_i} + \underbrace{R_j^\top t_j}_{-\mathbf{x}_j} \quad \xrightarrow{\text{-----}} \quad \begin{array}{l} \text{Consistency constraint} \\ \text{between relative and} \\ \text{absolute poses} \end{array}$$

👉 The magnitude of relative translations are **unknown**: $\|\mathbf{t}_{ij}\| = \|\mathbf{z}_{ij}\| = ?$

The Calibrated Case

Rotations

In which cases can we uniquely (up to a global rotation) recover camera rotations starting from relative rotations? 🤔



Given a **spanning tree**, a solution can be found by setting the root to the identity and propagating the consistency constraint:

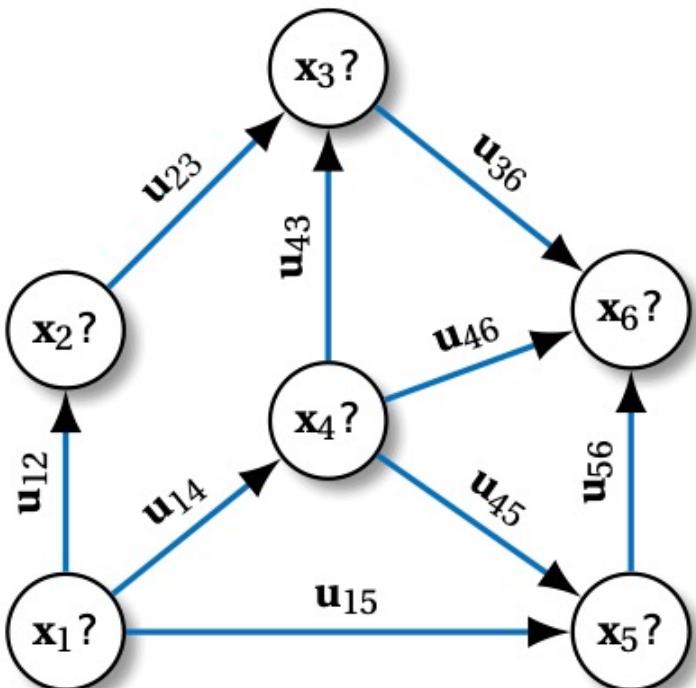
$$R_i = R_{ij}R_j \Leftrightarrow R_{ij} = R_iR_j^T$$

Solvability for rotations \Leftrightarrow **connected** viewing graph

The Calibrated Case

Translations

In which cases can we uniquely (up to translation & scale) recover camera positions from pairwise directions?



- **Nodes** = unknown locations
- **Edges** = known directions

$$\mathbf{u}_{ij} = \frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|} \iff \mathbf{u}_{ij} \times (\mathbf{x}_i - \mathbf{x}_j) = 0$$

A solution can be found from the direction constraint, which is a linear equation!

The Calibrated Case

Translations

Theorem. A graph is solvable if and only if $\text{rank}(S)=3n-4$

Localization
Equation: $Sx=0$

Translation &
scale ambiguity

- ✓ If the viewing graph is **solvable**, then the problem is well-posed.
- ✗ Otherwise, the problem is ill-posed: the **largest solvable component** has to be extracted \Leftrightarrow clustering rows in the null-space of S

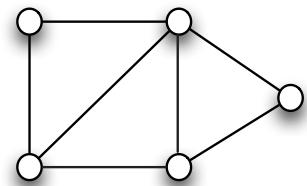
- F. Arrigoni, A. Fusiello. *Bearing-based network localizability: a unifying view*. IEEE TPAMI (2019).
- W. Whiteley. *Matroids from Discrete Geometry*. American Mathematical Society (1997)
- R. Kennedy, K. Daniilidis, O. Naroditsky, C. J. Taylor. *Identifying maximal rigid components in bearing-based localization*. IROS (2012)

The Calibrated Case

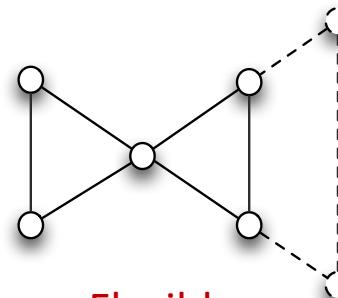
Translations

Solvability for translations \Leftrightarrow **parallel rigid** viewing graph

Definition. A graph is **parallel rigid** when all the configurations with parallel edges differ by translation and scale. Otherwise it is called **flexible**.



Parallel rigid



Flexible



This is a well studied task!

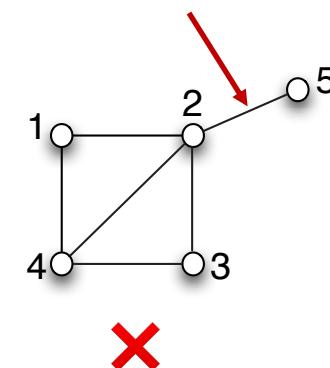
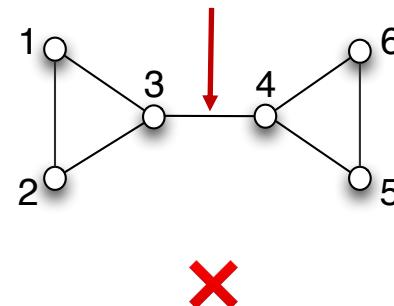
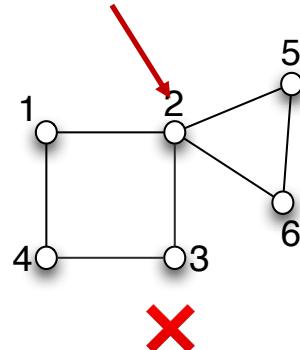
■ O. Ozyesil, A. Singer. Robust camera location estimation by convex programming. CVPR (2015).

The Calibrated Case

Translations

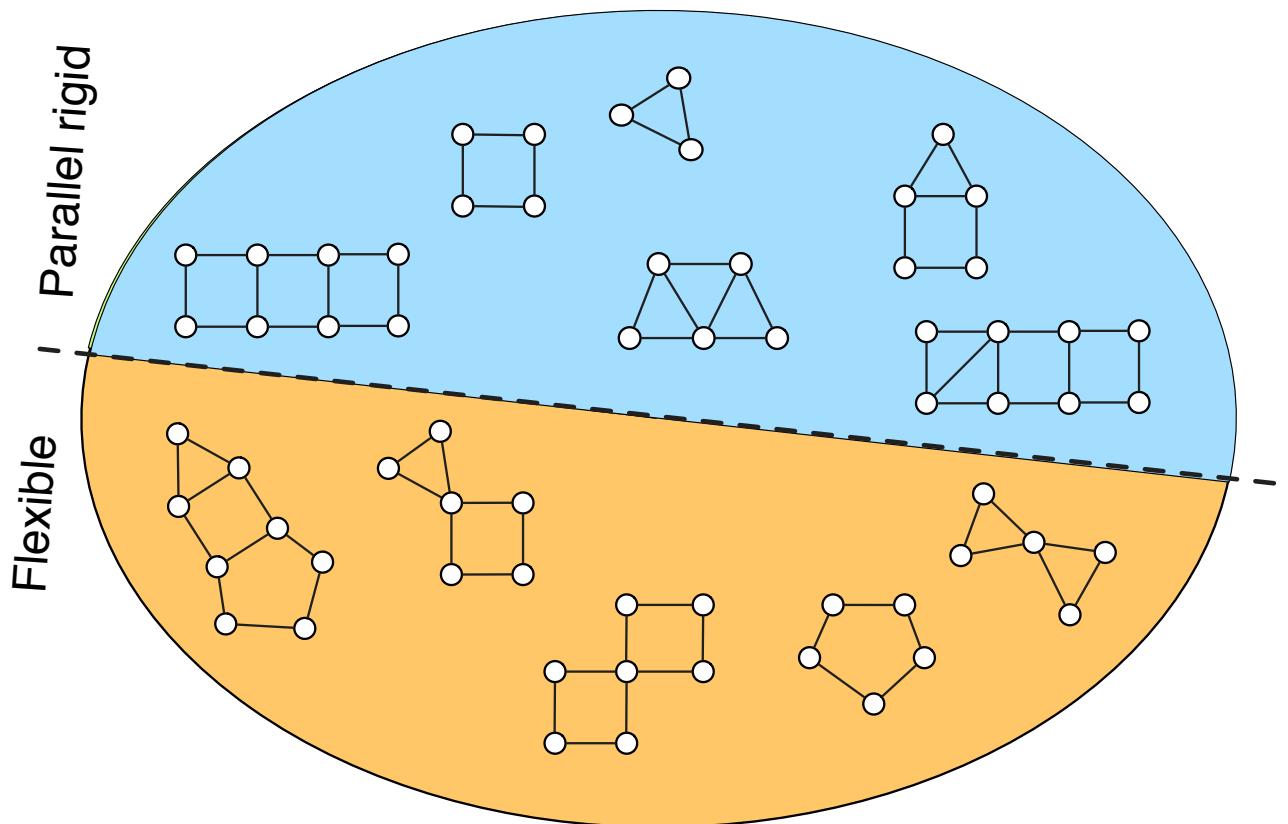
A parallel-rigid graph must satisfy the following **necessary conditions**:

- it has at least **(3n-4)/2 edges**
- It is **bridgeless** (i.e., it remains connected after removing any edge).
- It is **biconnected** (i.e. it does not have **articulation points** meaning that it remains connected after removing any node).



The Calibrated Case

Examples



- A single cycle of **length 3 or 4** is parallel rigid, whereas longer cycles are flexible
- Union of rigid graphs with a common edge is also rigid \Rightarrow **sufficient conditions**

The Calibrated Case Examples

Dataset	nodes	% edges	rigid	articulation	bridges
Arts Quad	5530	2	✗	30	10
Piccadilly	2508	10	✗	59	62
Roman Forum	1134	11	✗	28	28
Union Square	930	6	✗	60	68
Vienna Cathedral	918	25	✗	19	20
Alamo	627	50	✗	17	19
Notre Dame	553	68	✓	–	–
Tower of London	508	19	✗	19	19
Montreal N. Dame	474	47	✗	7	7
Yorkminster	458	26	✗	9	10
Madrid Metropolis	394	31	✗	17	15
NYC Library	376	29	✗	17	18
Piazza del Popolo	354	40	✗	8	9
Ellis Island	247	67	✗	6	7

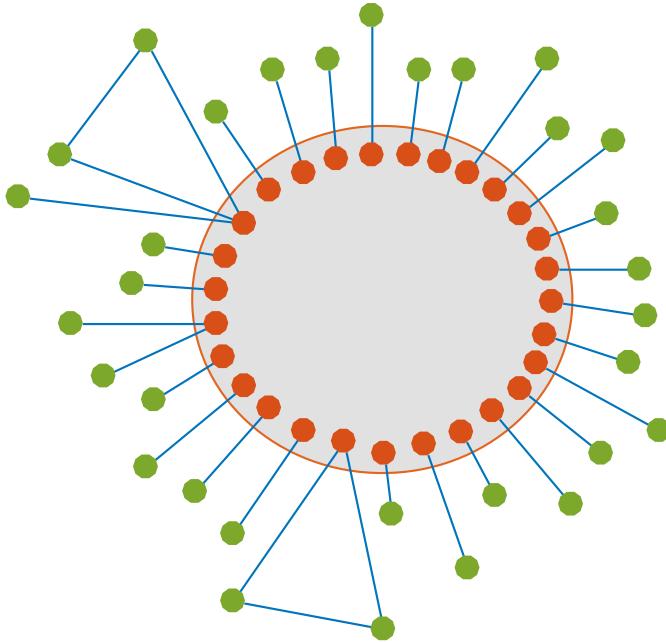
Cornell ArtsQuad <http://vision.soic.indiana.edu/projects/disco/>

1DSfM datasets <http://www.cs.cornell.edu/projects/1dsfm/>

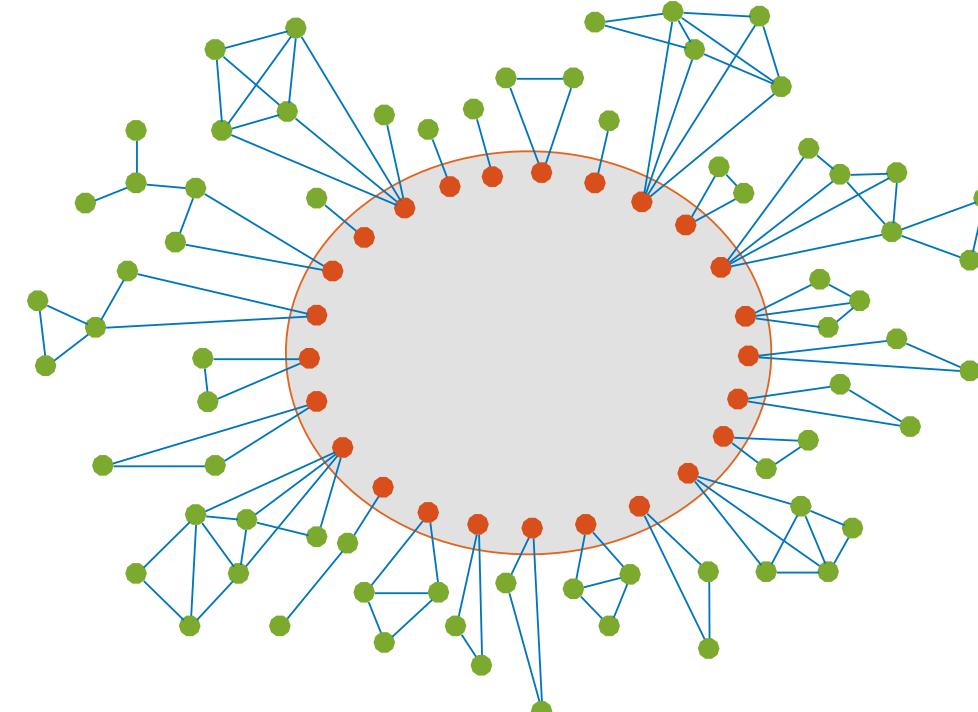
The Calibrated Case

Examples

Simplified representation: edges outside the largest rigid component are drawn.



Roman Forum



Arts Quad

The Calibrated Case

Summary

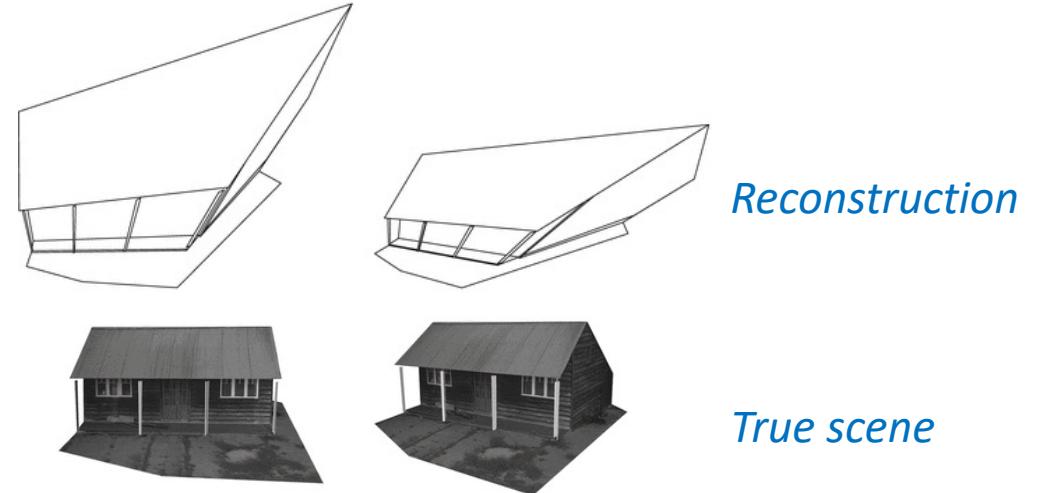
Solvability for rotations \Leftrightarrow **connected** viewing graph

Solvability for translations \Leftrightarrow **parallel rigid** viewing graph

- Parallel rigidity can be tested from the rank of a **linear system**.
- Maximal components can be extracted from the **null-space** of such a system.
- **Large-scale** datasets can be processed. 😊

Outline

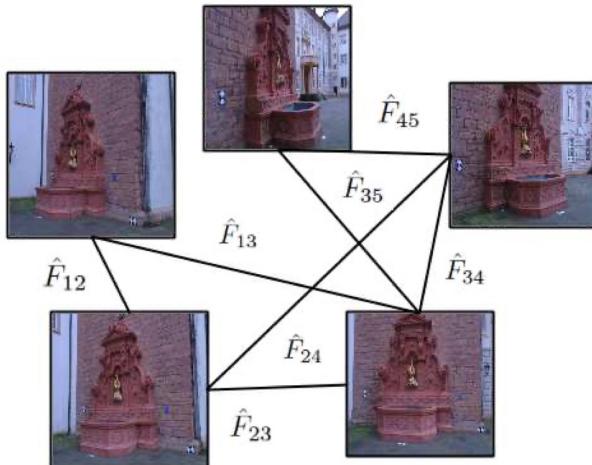
- Introduction
 - Calibrated Case
 - **Uncalibrated Case** ----->
 - Calibrated vs Uncalibrated
 - Conclusion
- No assumptions
 - Reconstruction is **projective**



The Uncalibrated Case

Problem Formulation

The **viewing graph** is a graph where vertices correspond to cameras and edges represent fundamental matrices.



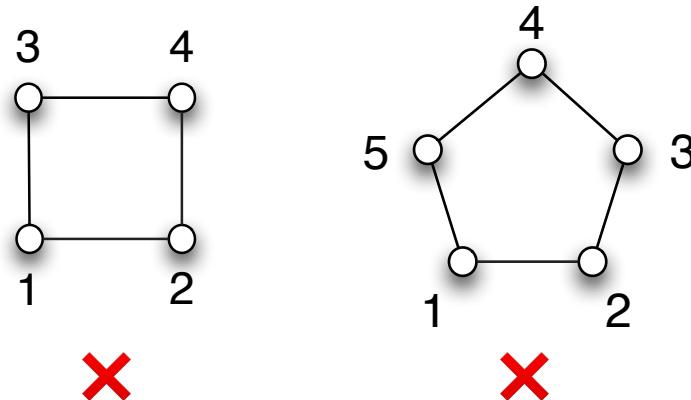
- Solvability depends on the **graph** and **camera centres** only.
- It can be reduced to a property of the graph only if we assume **generic** centres.

Solvable graph \Leftrightarrow it uniquely (up to a *single* projective transformation) determines a projective configuration of cameras.

The Uncalibrated Case

Necessary Conditions

- A solvable graph has **at least $(11n-15)/7$ edges**.
- In a solvable graph, **all the vertices have degree at least two and no two adjacent vertices have degree two** (if $n > 3$).

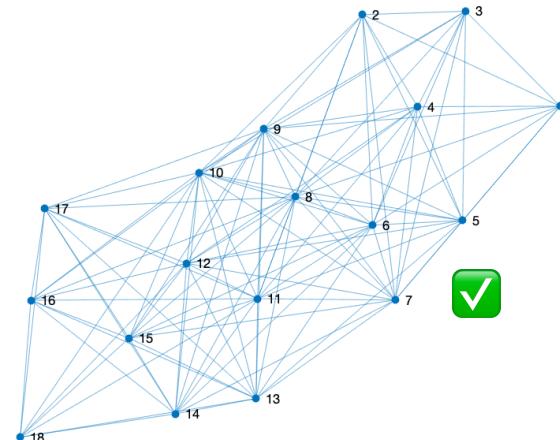
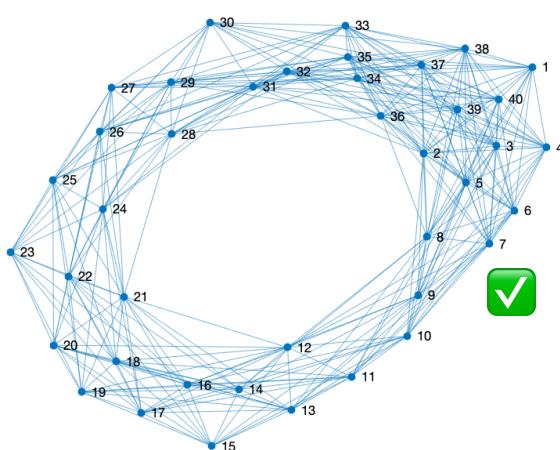


- M. Trager, B. Osserman, and J. Ponce. *On the solvability of viewing graphs*. ECCV 2018.
- N. Levi and M. Werman. *The viewing graph*. CVPR 2003

The Uncalibrated Case

Sufficient Conditions

- **Triangulated** graphs are solvable
- **Constructive** approaches are also available



■ M. Trager, M. Hebert, and J. Ponce. *The joint image hand-book*. ICCV 2015.

■ A. Rudi, M. Pizzoli, and F. Pirri. *Linear solvability in the viewing graph*. ACCV 2011.

The Uncalibrated Case

Algebraic Characterization

Idea: characterize the set of projective transformations that represent all possible ambiguities of the problem.

First, let us identify the family of transformations that leave a **single camera** fixed.

Proposition. Let P be a camera with centre c . All the solutions to $PG = aP$

for $G \in GL(4, \mathbb{R})$ and $a \in \mathbb{R}_{\neq 0}$ are given by

$$G = aI_4 + \mathbf{cv}^T$$

$$\forall a \in \mathbb{R}_{\neq 0}, \mathbf{v} \in \mathbb{R}^4$$



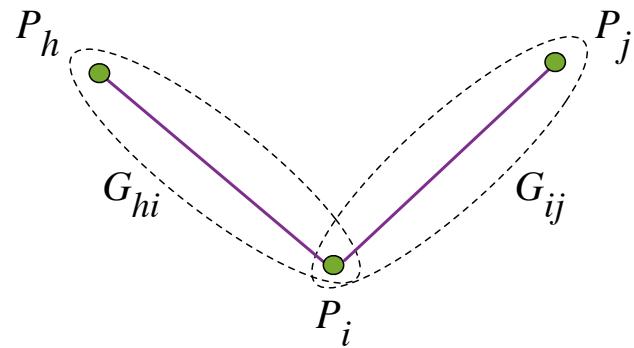
M. Trager, B. Osserman, and J. Ponce. *On the solvability of viewing graphs*. ECCV 2018.

The Uncalibrated Case

Algebraic Characterization

What happens when we have **multiple cameras**, represented as a viewing graph? 🤔

Let us assign an unknown projective transformation G_{ij} to every edge, and let us consider two edges (h, i) and (i, j) with a common vertex i .



Compatibility Condition

$$G_{hi}G_{ij}^{-1} = a_{hij}I_4 + \mathbf{c}_i\mathbf{v}_{hij}^\top$$

$G_{hij} \in GL(4)$ is unknown

$a_{hij} \in \mathbb{R}_{\neq 0}$ and $\mathbf{v}_{hij} \in \mathbb{R}^4$ are unknown

$\mathbf{c}_i \in \mathbb{R}^4$ is known (camera center)

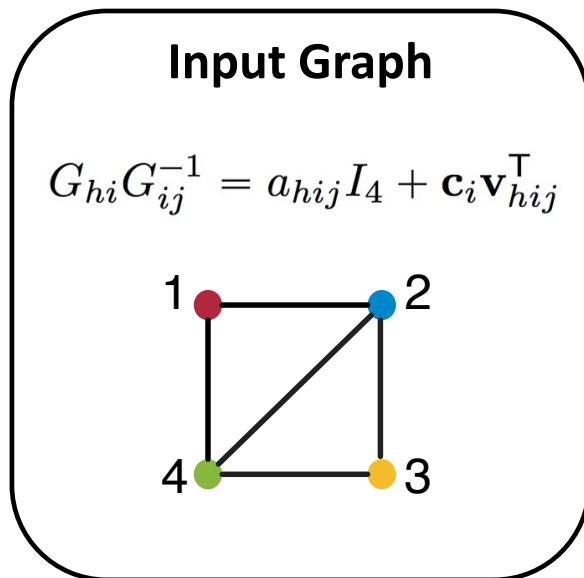
Solvable graph $\Leftrightarrow G_{ij} = s_{ij} H$

Single projective transformation

■ M. Trager, B. Osserman, and J. Ponce. *On the solvability of viewing graphs*. ECCV 2018.

The Uncalibrated Case

Algebraic Characterization



- **Polynomial system of equations with many unknowns**

$G_{hi} \in GL(4)$ is unknown

$a_{hij} \in \mathbb{R}_{\neq 0}$ and $\mathbf{v}_{hij} \in \mathbb{R}^4$ are unknown

$\mathbf{c}_i \in \mathbb{R}^4$ is known (camera center)

(h, i) and (i, j) are adjacent edges



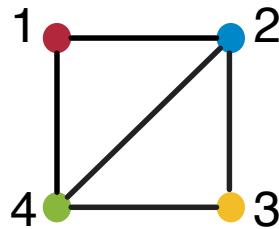
M. Trager, B. Osserman, and J. Ponce. *On the solvability of viewing graphs*. ECCV 2018.

The Uncalibrated Case

Reduced Formulation

Input Graph

$$G_{hi}G_{ij}^{-1} = a_{hij}I_4 + \mathbf{c}_i \mathbf{v}_{hij}^T$$



- **Polynomial system of equations with many unknowns**

$G_{hi} \in GL(4)$ is unknown

$a_{hij} \in \mathbb{R}_{\neq 0}$ and $\mathbf{v}_{hij} \in \mathbb{R}^4$ are unknown

$\mathbf{c}_i \in \mathbb{R}^4$ is known (camera center)

(h, i) and (i, j) are adjacent edges

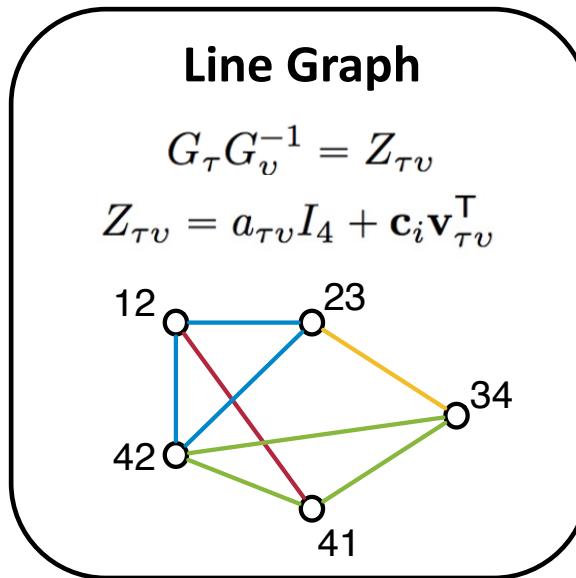
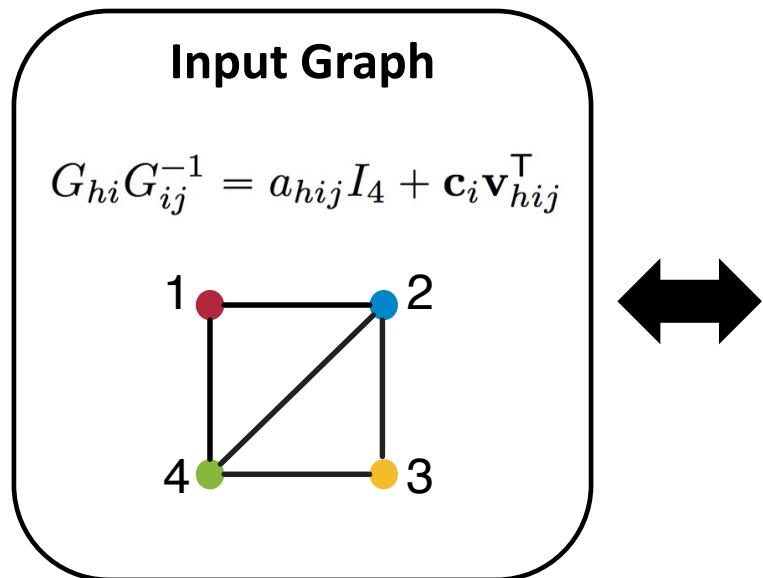
■ M. Trager, B. Osserman, and J. Ponce. *On the solvability of viewing graphs*. ECCV 2018.

- It is possible **eliminate variables** 😊

■ Arrigoni, Fusello, Ricci & Pajdla. *Viewing graph solvability via cycle consistency*. ICCV (2021).

The Uncalibrated Case

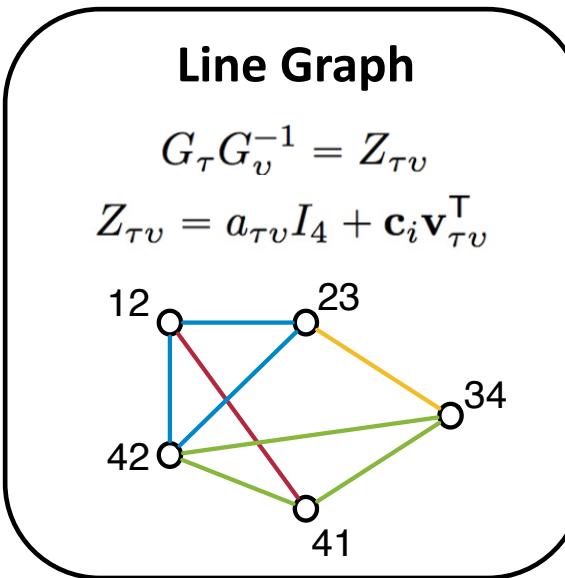
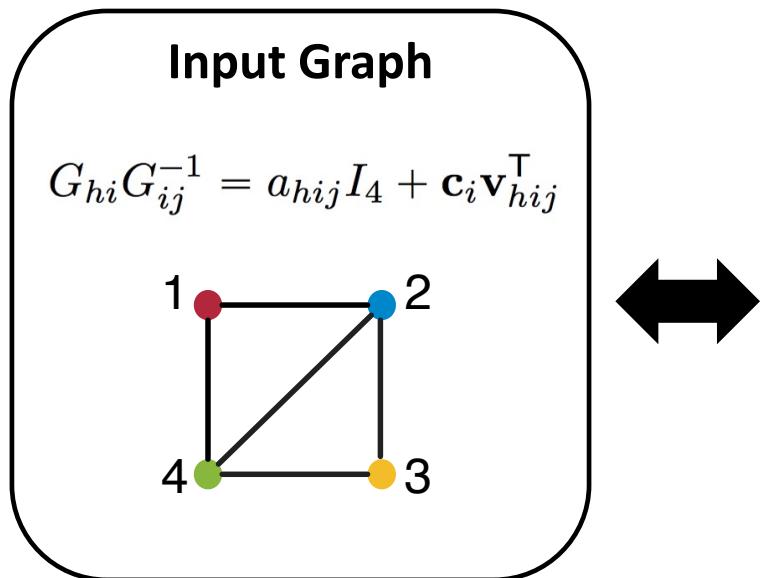
Reduced Formulation



- Each node is an edge in the input graph;
- Two nodes are linked if the corresponding edges are adjacent in the input graph.
- There is one equation for each edge in the line graph.

The Uncalibrated Case

Reduced Formulation



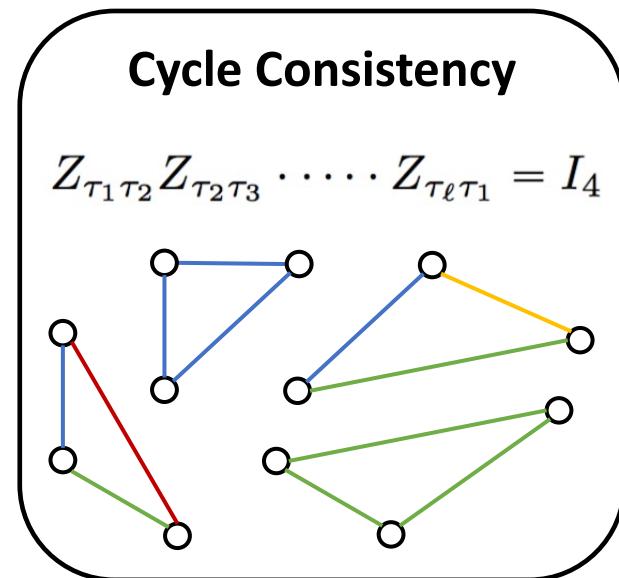
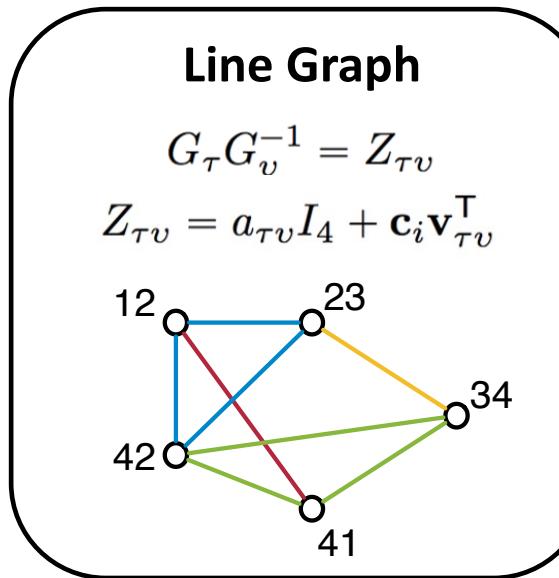
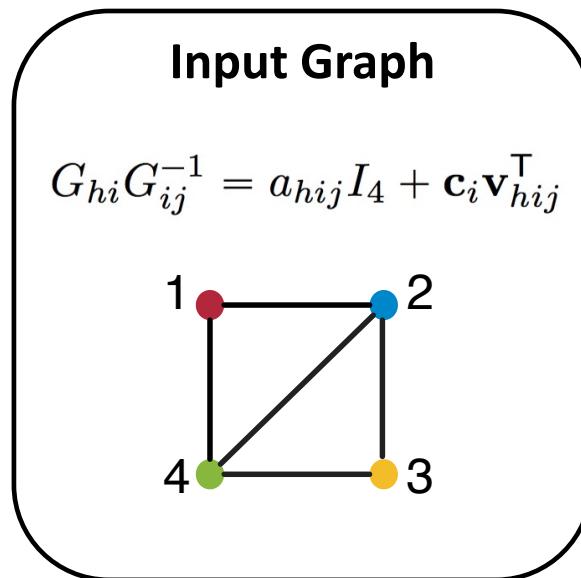
- Each node is an edge in the input graph;
- Two nodes are linked if the corresponding edges are adjacent in the input graph.
- There is one equation for each edge in the line graph.

How can we **eliminate the G variables?** 🤔

Idea: $Z_{12,23} \cdot Z_{23,42} \cdot Z_{42,12} = G_{12} \underbrace{G_{23}^{-1} G_{23}}_I \underbrace{G_{42}^{-1} G_{42}}_I G_{12}^{-1} = I$

The Uncalibrated Case

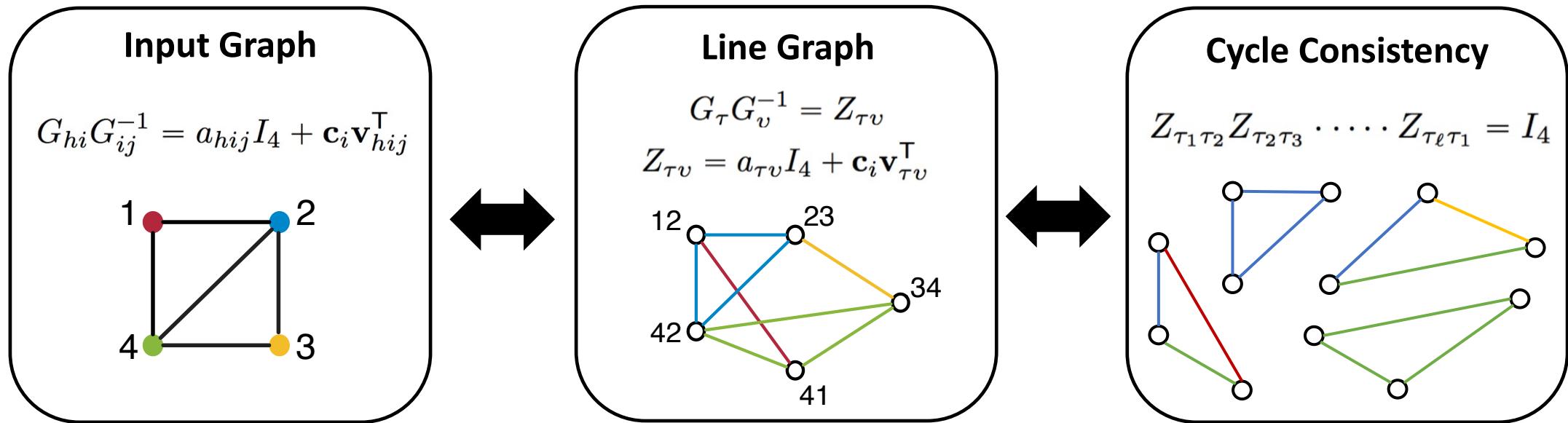
Reduced Formulation



cycle consistency (on all cycles) \Leftrightarrow cycle consistency (on a basis)

The Uncalibrated Case

Reduced Formulation



The formulation can be simplified via a **change of variables**: $\mathbf{u}_{\tau\nu} = \mathbf{v}_{\tau\nu}/\alpha_{\tau\nu}$
⇒ For a solvable graph, we have exactly 1 solution (no ambiguities)

The Uncalibrated Case Algorithm

Algorithm 1 Viewing Graph Solvability

Input: undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Output: solvable or not solvable

1. randomly sample the camera centres
2. compute the line graph $\mathcal{L}(\mathcal{G})$
3. compute a cycle consistency basis for $\mathcal{L}(\mathcal{G})$
4. set up equations
5. compute the number s of real solutions

if $s = 1$ **then**

solvable 

else

not solvable 

end if

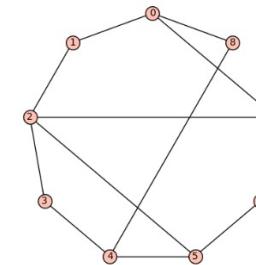
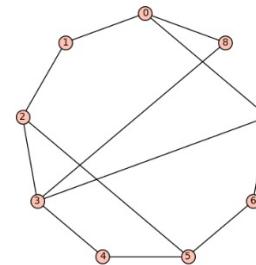
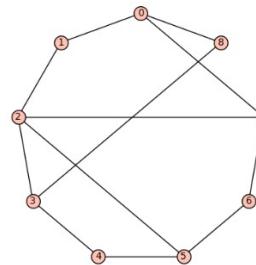
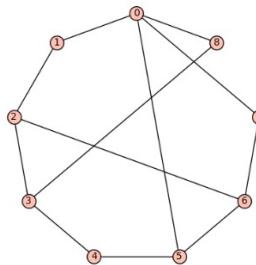
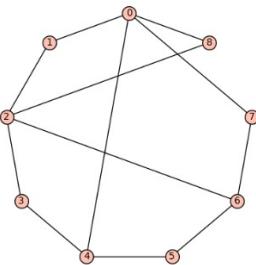
Gröbner basis
(symbolic computation)

<https://github.com/federica-arrigoni/solvability>

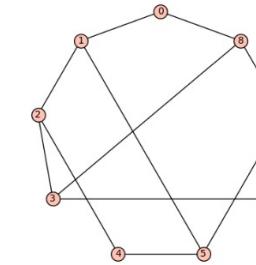
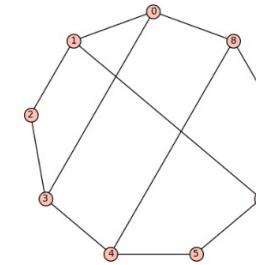
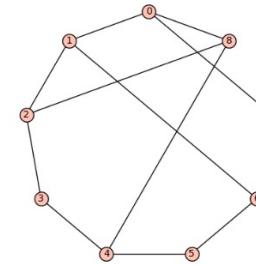
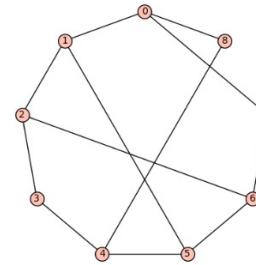
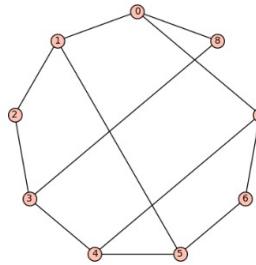
The Uncalibrated Case

Examples

Minimal viewing graphs with 9 vertices



Solvable ✓



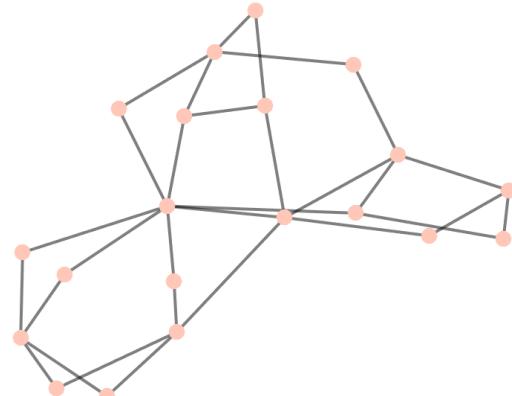
Not solvable

The Uncalibrated Case

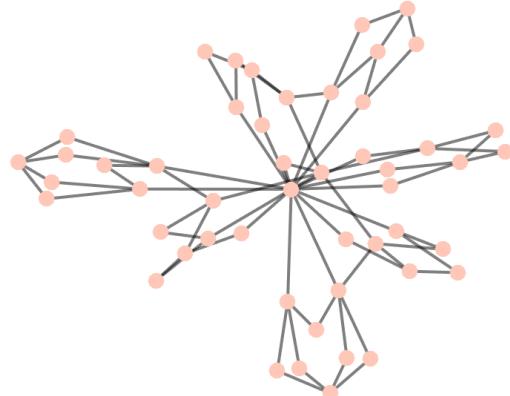
Examples

Execution times on minimal graphs

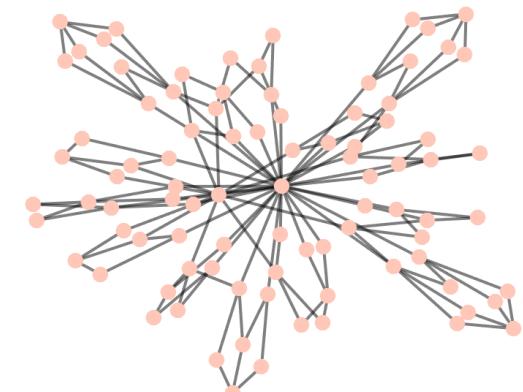
Nodes	10	20	30	40	50	60	70	80	90
Time	1.6 s	9 s	93 s	3 min	15 min	35 min	1 h	≈ 2 h	> 4 h



Solvable graph with 20 nodes



Solvable graph with 50 nodes

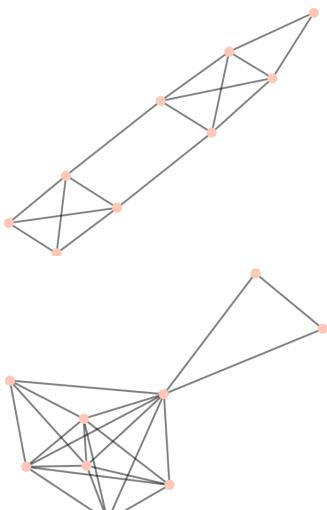


Solvable graph with 90 nodes

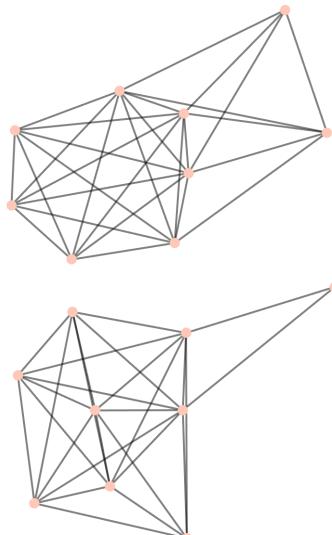
The Uncalibrated Case

Examples

Subgraphs with 9 nodes sampled from
real structure-from-motion viewgraphs



Unsolvable



Solvable

Data set	Solvable			Unsolvable		
	by suff.	by Alg. 1	Tot.	by nec.	by Alg. 1	Tot.
Alcatraz Courtyard	200	0	200	0	0	0
Buddah Tooth	178	20	198	2	0	2
Pumpkin	169	22	191	8	1	9
Skansen Kronan	179	8	187	13	0	13
Tsar Nikolai I	196	0	196	4	0	4
Alamo	136	16	152	48	0	48
Ellis Island	136	30	166	34	0	34
Gendarmenmarkt	128	11	139	61	0	61
Madrid Metropolis	88	28	116	84	0	84
Montreal Notre Dame	140	12	152	48	0	48
Notre Dame	165	18	183	17	0	17
NYC Library	110	19	129	71	0	71
Piazza del Popolo	105	22	127	73	0	73
Piccadilly	109	23	132	68	0	68
Roman Forum	114	28	142	58	0	58
Tower of London	123	18	141	59	0	59
Trafalgar	86	16	102	98	0	98
Union Square	74	19	93	107	0	107
Vienna Cathedral	122	8	130	70	0	70
Yorkminster	116	14	130	70	0	70
Cornell Arts Quad	76	23	99	101	0	101

The Uncalibrated Case

Summary

- Thanks to cycle consistency, **less unknowns** are involved than previous work:

	#Eq.	#Var.										
Our formulation	64	36	64	40	112	63	112	67	192	100	208	109
Trager et al.	128	120	144	141	224	198	240	219	352	286	384	312

- It is possible to classify **previously undecided** viewing graphs and extend solvability testing up to minimal graphs **with 90 nodes**.
- *Larger/denser graphs can not be processed* 😞

Outline

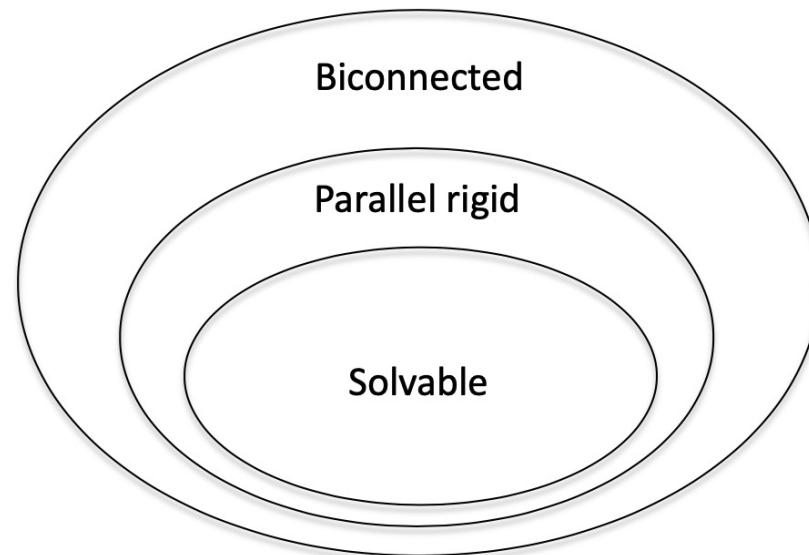
- Introduction
- Calibrated Case
- Uncalibrated Case
- **Calibrated vs Uncalibrated** ----->
- Conclusion



Uncalibrated solvability \Rightarrow calibrated solvability

Calibrated vs Uncalibrated

Proposition. *A solvable (uncalibrated) graph is parallel rigid.*



Expected result!



Well-posed with uncalibrated cameras
⇒ well-posed with calibrated cameras

■ Arrigoni, Fusiello, Rizzi, Ricci & Pajdla. *Revisiting viewing graph solvability: an effective approach based on cycle consistency*. TPAMI (2022).

Calibrated vs Uncalibrated

Proposition. *A solvable (uncalibrated) graph is parallel rigid.*

Proof [sketch].

Parallel rigid graph \Leftrightarrow for any partition of the edges: $\sum_{i=1}^k (3|\mathcal{V}_i| - 4) \geq 3n - 4$

Solvable graph \Rightarrow for any partition of the edges: $\sum_{i=1}^k (11|\mathcal{V}_i| - 15) \geq 11n - 15$



Only necessary condition!

Unknown if the opposite holds

■ Arrigoni, Fusiello, Rizzi, Ricci & Pajdla. *Revisiting viewing graph solvability: an effective approach based on cycle consistency*. TPAMI (2022).

Outline

- Introduction
- Calibrated Case
- Uncalibrated Case
- Calibrated vs Uncalibrated
- Conclusion

Conclusion

	Calibrated	Uncalibrated
Formulation	Linear system	Polynomial system
Datasets	Large-scale	Small-scale
Interpretation	Connected + Parallel rigid	?
Components	Null-space computation	?



“Solved”



Open issues

References

- F. Arrigoni, A. Fusiello, R. Rizzi, E. Ricci & T. Pajdla. *Revisiting viewing graph solvability: an effective approach based on cycle consistency*. IEEE TPAMI (2022).
- F. Arrigoni, A. Fusiello, E. Ricci & T. Pajdla. *Viewing graph solvability via cycle consistency*. ICCV (2021). **Best paper honourable mention** 
- F. Arrigoni & A. Fusiello. *Bearing-based network localizability: a unifying view*. IEEE TPAMI (2019).

Thank you for your attention!

Viewing Graph Solvability in Structure from Motion

Federica Arrigoni

Politecnico di Milano (Italy) – federica.arrigoni@polimi.it



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