Statistics for MFEs – Problem Set 2

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Due December 20, 2023 2:00pm PST, to be submitted via Canvas

1. Suppose that a person's score X on a mathematics aptitude test is a number between 0 and 1, and that the score Y on a music aptitude test is also a number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores X and Y are distributed according to the following joint PDF:

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & for\ 0 \leq x \leq 1\ and\ 0 \leq y \leq 1, \\ 0 & otherwise. \end{cases}$$

   (a) What proportion of college students obtain a score greater than 0.8 on the mathematics test?

   (b) If a student's score on the music test is 0.3, what is the probability that his score on the mathematics test will be greater than 0.8?

   (c) If a student's score on the mathematics test is 0.3, what is the probability that his score on the music test will be greater than 0.8?

2. Suppose that the PDF of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{1}{n!}x^n e^{-x} & for\ x > 0 \\ 0 & otherwise. \end{cases}$$

   Suppose also that for any given value $X = x (x > 0)$, the n random variables $Y_1, ..., Y_n$ are i.i.d. and the conditional PDF g of each of them is as follows:

$$g(y|x) = \begin{cases} \frac{1}{x} & for\ 0 < y < x, \\ 0 & otherwise. \end{cases}$$

   Determine (a) the marginal joint PDF of $Y_1, ..., Y_n$ and (b) the conditional PDF of X for any given values of $Y_1, ..., Y_n$.

3. Suppose that the PDF of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{1}{2}x & for\ 0 < x < 2, \\ 0 & otherwise. \end{cases}$$

   Determine the PDF of $Y = 4 - X^3$.

4. Construct an example of a distribution for which the mean is finite but the variance is infinite.

5. Let X have the PDF:
$$f(x) = \begin{cases} x^{-2} & if\ x > 1, \\ 0 & otherwise. \end{cases}$$
Prove that the m.g.f. $\psi(t)$ is finite for all $t \leq 0$ but not for $t > 0$.

6. Suppose that X and Y have a continuous joint distribution for which the joint PDF is as follows:
$$f(x, y) = \begin{cases} \frac{1}{3}(x + y) & for\ 0 \leq x \leq 1\ and\ 0 \leq y \leq 2, \\ 0 & otherwise. \end{cases}$$
Determine the value of $\mathrm{Var}(2X - 3Y + 8)$.

7. Let z_1 and z_2 be two independent N(0,1) RVs and
$$\begin{aligned} x &= \mu_x + \sigma_x z_1 \\ y &= \mu_y + \sigma_y \left(\rho z_1 + \sqrt{(1 - \rho^2)} z_2 \right). \end{aligned}$$
   (a) Show that (x, y) are bivariate normal with $x \sim N(\mu_x, \sigma_x^2), y \sim N(\mu_y, \sigma_y^2)$ and covariance $\rho \sigma_x \sigma_y$.
   (b) Show that $\mathrm{E}(y|x) = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x)$.
   (c) Show that $\mathrm{Var}(y|x) = \sigma_y^2 (1 - \rho^2)$.