Statistics for MFEs – Problem Set 5

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Due January 17, 2024, 2:00pm PST, to be submitted via Canvas

1. Consider the joint pdf

$$f(x, y) = 3(x^2 + y)/11 \quad \text{for} \quad 0 \le x \le 2, 0 \le y \le 1.$$

   (a) Plot the joint pdf $f(x, y)$.

   (b) Show that the marginal pdf of X is

$$f_x(x) = 3(2x^2 + 1)/22 \quad \text{for} \quad 0 \le x \le 2.$$

   Plot $f_x(x)$.

   (c) Derive the marginal pdf of Y, $f_y(y)$. Plot $f_y(y)$.

   (d) Derive the conditional pdf of Y given X, $f(y|x)$. Plot $f(y|x)$ for $X = 0, 1, 2$.

   (e) Derive the CEF of Y given X.

   (f) Calculate $\mathrm{E}(X), \mathrm{E}(Y), \mathrm{E}(X^2), \mathrm{E}(Y^2), \mathrm{E}(XY), \mathrm{Var}(X), \mathrm{Var}(Y), \mathrm{Cov}(X, Y)$.

   (g) Find the best linear predictor of Y given X.

   (h) Plot the CEF and BLP as a function of X.

2. For the joint pdf in the table below:

	$x = 1$	$x = 2$	$x = 3$
$y = 0$	0.15	0.10	0.15
$y = 1$	0.15	0.30	0.15

   (a) Find the conditional expectation function $\mathrm{E}(Y|X)$.

   (b) Find the best linear predictor $\mathrm{E}^*(Y|X)$.

   (c) Prepare a table that gives $\mathrm{E}(Y|x)$ and $\mathrm{E}^*(Y|x)$ for $x = 1, 2, 3$.

3. In class, we considered the best linear predictor $\mathrm{E}^*(Y|X) = \alpha + \beta X$. More precisely, $\mathrm{E}^*(Y|X)$ should be called the best *affine* predictor since, strictly speaking, a *linear* function has no intercept. To avoid confusion, we call the predictor without a constant a *proportional* predictor $\mathrm{E}(Y|X) = \gamma X$. Denote the γ that minimizes the MSE in the class of proportional predictors as γ^{**} and the optimal prediction $\mathrm{E}^{**}(Y|X) = \gamma^{**} X$.

   (a) Show that $\gamma^{**} = \mathrm{E}(XY)/\,\mathrm{E}(X^2)$.

(b) Is $\mathrm{E}^{**}(Y|X)$ an unbiased predictor? Explain.

(c) Let $e = Y - \gamma^{**}X$. Is $\mathrm{Cov}(X, e) = 0$?

(d) Find the minimized value of $\mathrm{E}(e^2)$.

(e) Compare this $\mathrm{E}(e^2)$ with those that result when the CEF and the BLP are used.

4. For the matrix

$$X' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & -2 & 3 & -5 \end{bmatrix},$$

compute $P = X(X'X)^{-1}X'$ and $M = I - P$.

(a) Verify that $MP = 0$.

(b) Let $Q = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$ and $\tilde{X} = XQ$. Compute the P and M based on $\tilde{X}$ instead of X.