Statistics for MFEs – Problem Set 3

Professor Martin Lettau

Due January 3, 2024, 2:00pm PST, to be submitted via bCourses

1. DGS ch. 6.2, problem 14

2. DGS ch. 6.3, problem 6

3. Illustrate the Law of Large Numbers and the Central Limit Theorem for the t_ν-distribution for different values of ν. Write a Python notebook for the following simulation for a given value of ν. Start with $\nu = 100$.

   (a) Draw M i.i.d. random samples of length N from the t_ν distribution. Start with $M = 500$ and $N = 10,000$ but you should experiment with different values.

   (b) For each sample $m = 1, \ldots, M$, compute the sample means for the first $n = 5, 10, 100, 500, 1000, 10000$ draws of the sample.

   (c) For each n, compute the mean, standard deviation and variance of the sample means across the M samples. This yields means, standard deviations, and variances for each of the six different values of n. Report your results in a table.

   (d) For each n, plot the distributions of the M sample means.

   (e) Describe how these simulations relate to the LLN and the CLT. Be careful to distinguish the implications of the LLN and the CLT.

   (f) Repeat steps (a) to (e) for $\nu = 10, 5, 2, 1, 0.5$.

   (g) How does the value of ν affect the limiting behavior of the sample mean?

4. DGS ch. 7.5 Problem 12

5. DGS ch. 7.6 Problem 16

6. Consider the uniform distribution $X \sim U(0, \theta)$. We showed in class that the MLE of θ is $\widehat{\theta}_{ML} = \max(X_1, ..., X_n)$.

   (a) Show that the MM estimator of θ is $\widehat{\theta}_{MM} = 2\bar{X}$, where $\bar{X}$ is the sample mean.

   (b) It can be shown that the theoretical means and variances for the MM and MLE estimators are

$$\begin{aligned}
\mathrm{E}[\hat{\theta}_{MM}] &= \theta_0 \\
\mathrm{Var}[\hat{\theta}_{MM}] &= \frac{\theta_0^2}{3n} \\
\mathrm{E}[\hat{\theta}_{MLE}] &= \frac{n}{n+1}\theta_0 \\
\mathrm{Var}[\hat{\theta}_{MLE}] &= \frac{\theta_0^2}{n(n+2)}.
\end{aligned}$$

Are the MM and ML estimators unbiased and/or consistent? What are their asymptotic distributions?

(c) Set the true θ to 1. Repeat the following steps for $n = 20, 100, 1000$:

   i. Use a random number generator to draw 10,000 samples of length n from $U(0, \theta)$ and compute the MLE and MM estimators of θ for each of the 10,000 samples.

   ii. Plot the histograms of the 10,000 MLE and MM estimators, respectively.

   iii. Compute the mean, bias and standard error of the MLE and MM estimators in the simulated data.

(d) For each n, compare the properties of the MLE and MM estimators in these simulations to their theoretical distributions.

(e) Which properties of the estimators in the simulated data are expected, i.e. close to their theoretical properties, and which ones are not?

(f) STRICTLY OPTIONAL: If the properties of the simulated estimators contradict the theoretical properties, the assumptions underlying the theoretical results must be violated. Which assumption(s) are violated in this case? (Note: We did not cover the precise assumptions required for the theorems about the properties of the MM and MLE estimators in class. To answer this question, you have to study the "regularity" conditions in the theorems.)

Recommended Exercise - NOT graded

Reproduce the plots and tables for the Procter & Gamble and Unilever example in the lecture notes. Update the plots and tables using the most current date for which the data is available.

Data set: link.