Counting Induced Subgraphs: An Algebraic Approach to #W[1]-Hardness

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MFCS 2019

Problem Π

Given . . . ,

Exists . . .

How fast can we solve problem Π?

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poly-time

(nice)

P

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P

not faster than

SAT, CLIQUE, . . .

NP-hard

Problem Π

Given . . . ,

Exists . . .

What can we do about an **NP-hard** problem Π?

Problem Π
Parameter k

Given ...
Exists ...

How fast can we solve **NP-hard** problem Π parametrized in k?

$f(k) \cdot n^c$

(nice)

FPT

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not faster than
k-Clique

W[1]-hard

Problem $\#\Pi$
Parameter k

Given ...
Count ...

How fast can we solve problem $\#\Pi$ parametrized in k?

$f(k) \cdot n^c$
(nice)

FPT

not faster than
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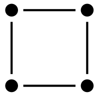
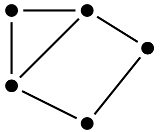
#W[1]-hard

Counting Induced Subgraphs

#IndSub Parameter: $|V(H)|$

Given graphs H and G, how often does H appear as an induced subgraph in G?





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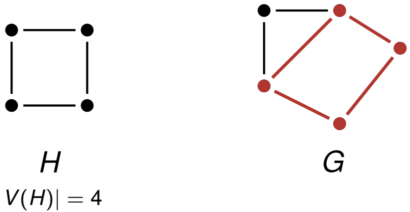
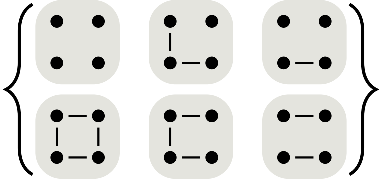


$$\#\mathrm{IndSub}(H \to G) = 1$$

Counting Induced Subgraphs with Property Φ

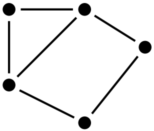
#IndSub(Φ) Parameter: k

Given graph G, how many induced subgraphs of G with k vertices satisfy Φ?



Φ = bipartite

$k = 4$



G

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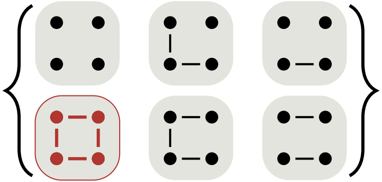

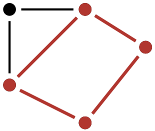

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$$\#\mathrm{IndSub}(\Phi, k \to G) = \underbrace{\#\mathrm{IndSub}(\square \to G)}_{1} + \cdots$$

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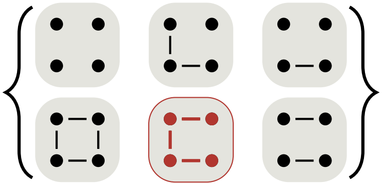
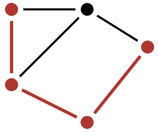


$\Phi = \text{bipartite}$



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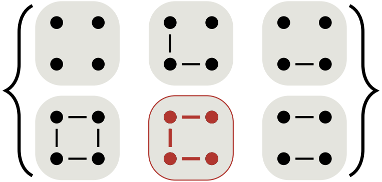

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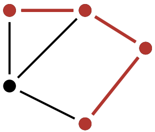

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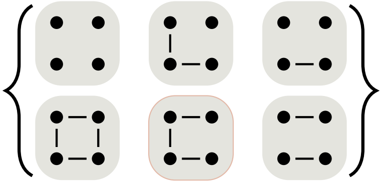

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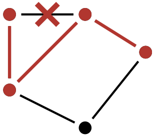

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$$\#\mathrm{IndSub}(\Phi, k \to G) = \underbrace{\#\mathrm{IndSub}(\square \to G)}_{1} + \underbrace{\#\mathrm{IndSub}(\sqsubset \to G)}_{2} = 3$$

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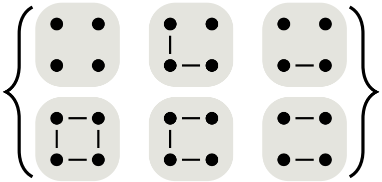
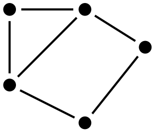
$\Phi_k :=$ 



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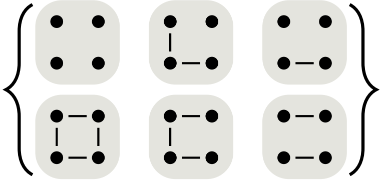

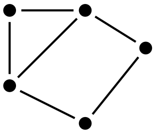

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Known Results

#IndSub(Φ) Parameter: k

Given graph G, compute $\sum_{H \in \Phi_k}$ #IndSub($H \to G$).

- #W[1]-hard if
  - $\Phi =$ connectivity [Jerrum, Meeks '15]
  - Φ includes "low edge-densities" [Jerrum, Meeks '15]
  - Φ is "co-monotone" and (edge-)minimal elements have high treewidth [Meeks '16]
  - $\Phi =$ # of edges is odd/even [Jerrum, Meeks '17]
- Either FPT or #W[1]-hard for every Φ [Curticapean, Dell, Marx '17]

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Theorem [CDM’17] (simplified)

Computing $\sum_F a_{\Phi,k}(F) \cdot \#\mathrm{Hom}(F \to G)$ is FPT if every graph F with $a_{\Phi,k} \neq 0$ has bounded treewidth; and #W[1]-hard otherwise.

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$\rightsquigarrow$ Topological interpretation of $a_{\Phi,k}(C_k)$ possible [Roth, Schmitt '18]

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Main Result

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Theorem (special case)

For Φ that is **non-trivial on bipartite graphs** and **closed under taking subgraphs**, #IndSub(Φ) is #W[1]-hard and cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ (under ETH).

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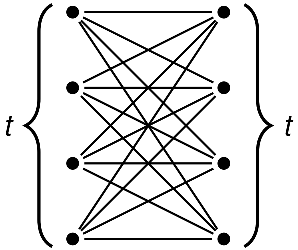

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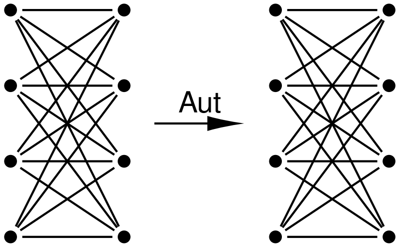

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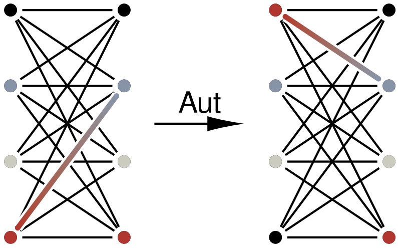

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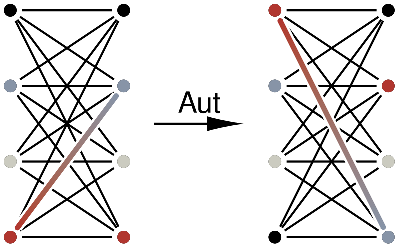

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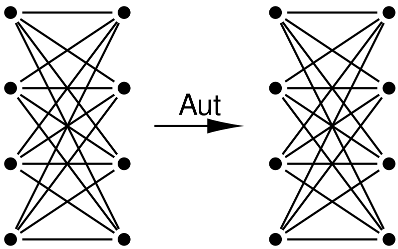

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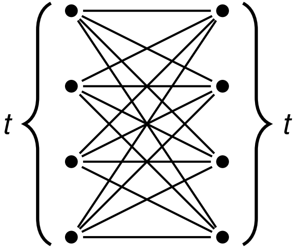

$C_{t,t}$

$2t$ vertices; t^2 edges

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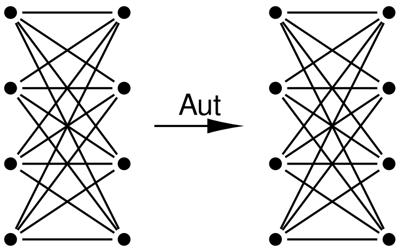
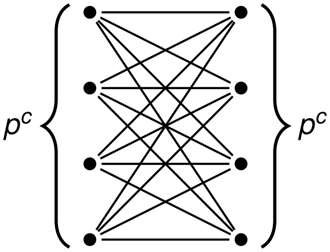


C_{p^c,p^c} is edge-transitive

Can map any edge to any other edge via automorphism



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Given graph G, compute $\sum_{H\in\Phi_k}$ #IndSub($H \to G$) $= \sum_F a_{\Phi,k}(F) \cdot$ #Hom($F \to G$).

Using colored variants of the problems, we obtain:

Lemma

For the biclique $C_{t,t}$, we have

$$a_{\Phi,k}(C_{t,t}) = \sum_{S\subseteq E(C_{t,t})} \Phi(C_{t,t}[S]) \cdot (-1)^{\#E(C_{t,t})-\#S}$$

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Lemma [Rivest, Vuillemin '76/'77]

For the biclique $C_{t,t}$, where $t = p^c$ is a prime power,
the alternating enumerator $\tilde{\chi}(\Phi, C_{t,t})$ of Φ and $C_{t,t}$ satisfies

$$|a_{\Phi,k}(C_{t,t})| = |\tilde{\chi}(\Phi, C_{t,t})| \equiv |\Phi(C_{t,t}[\emptyset]) - \Phi(C_{t,t})| \pmod{p}$$

ALGEBRA

Main Results

#IndSub(Φ) — Parameter: k

Given graph G, compute $\sum_{H\in\Phi_k}$ #IndSub($H \to G$) $= \sum_F a_{\Phi,k}(F) \cdot$ #Hom($F \to G$).

Main Theorem

For all properties Φ where $\mathcal{K} := \{\Phi(C_{t,t}[\emptyset]) \neq \Phi(C_{t,t}) \mid t = p^c\}$ is infinite,
the problem #IndSub(Φ) is #W[1]-hard.
If $\mathcal{K}$ is dense, #IndSub(Φ) cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ (under ETH).

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Corollary

For Φ that is **non-trivial on bipartite graphs** and **closed under taking subgraphs**,
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Examples for Applications

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For the property Φ = Eulerian, the problem #IndSub(Φ) is #W[1]-hard and cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ (under ETH).

Examples for Applications

#IndSub(Φ) — Parameter: k

Given graph G, compute $\sum_{H \in \Phi_k} \#\mathrm{IndSub}(H \to G) = \sum_F a_{\Phi,k}(F) \cdot \#\mathrm{Hom}(F \to G)$.

Main Theorem

For all properties Φ where $\mathcal{K} := \{\Phi(C_{t,t}[\emptyset]) \neq \Phi(C_{t,t}) \mid t = p^c\}$ is infinite, the problem #IndSub(Φ) is #W[1]-hard.
If $\mathcal{K}$ is dense, #IndSub(Φ) cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ (under ETH).

Corollary

For the property Φ = Eulerian, the problem #IndSub(Φ) is #W[1]-hard and cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ (under ETH).

- Every graph $C_{t,t}$ for $t = 2^c$ has an Euler cycle; disconnected graphs do not.

Examples for Applications

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If $\mathcal{K}$ is dense, #IndSub(Φ) cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ (under ETH).

Corollary

For any of the properties $\Phi \in \{\text{disconnected}, \text{planar}, \text{non-hamiltonian}\}$, the problem #IndSub(Φ) **restricted to bipartite input graphs** is #W[1]-hard and cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ (under ETH).

Open Problems

Corollary

For Φ that is **non-trivial on bipartite graphs** and **closed under taking subgraphs**, #IndSub(Φ) is #W[1]-hard and cannot be solved in time $f(k)\cdot|V(G)|^{o(k)}$ (under ETH).

- What about properties on general graphs?
  $\rightsquigarrow$ Also #W[1]-hard?
- What about properties that are not closed under taking subgraphs?
  $\rightsquigarrow$ Also #W[1]-hard?

Are there other edge-transitive graphs
that have a prime power number of edges?

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that have a prime power number of edges?

There are, but not many.

Theorem

Any connected, edge-transitive graph $G = (V, E)$ with $|E| = p^c$ is

- bipartite, or
- vertex-transitive and a subgraph of W_t, where $t = p^{c'}$.

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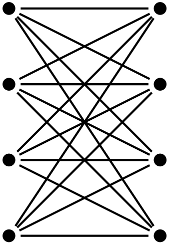

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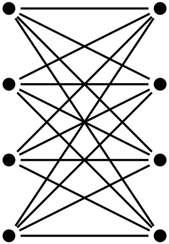

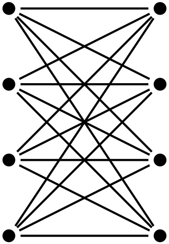

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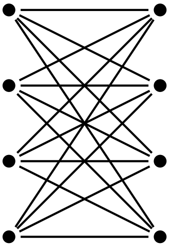
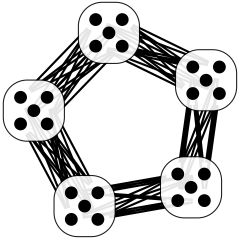
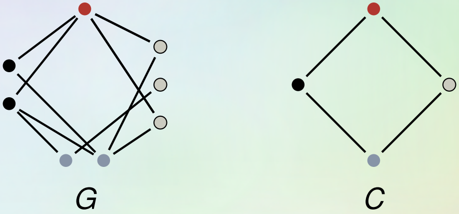




Colorful Graphs

C-Colored Graph

A graph G is C-colored, if there is a homomorphism from G to C.



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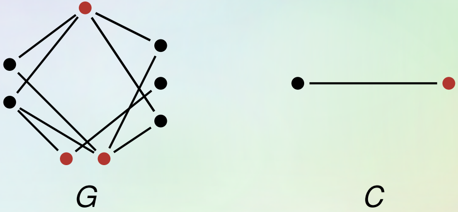

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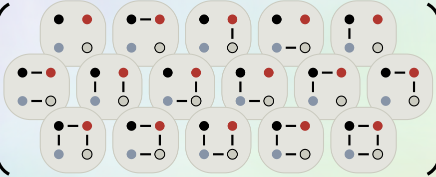
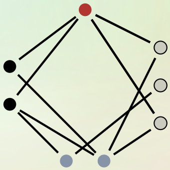
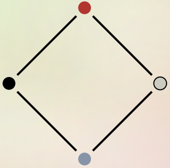
Given C-colored graph G, compute $\sum_{H \in \Phi_k}$ #cp-IndSub$(H \to G) = \sum_F a_{\Phi,k}(F) \cdot$ #cp-Hom$(F \to G)$.

$\Phi_k :=$ {  }

Φ = bipartite
$k = |V(C)| = 4$



G



C

Introducing Colors

Theorem

$\#\textsc{cp-IndSub}(\Phi) \preceq^{T}_{\mathrm{fpt}} \#\textsc{IndSub}(\Phi)$

- Given C-colored graph G and oracle for $\#\textsc{IndSub}(\Phi, |V(C)| \to \star)$
- May assume: Every color of C appears in G
- Split G according to color:

$$\text{cp-IndSub}(\Phi \to G) = \left\{ S \subseteq \binom{V(G)}{|V(C)|} \;\middle|\; c(S) = [|V(C)|] \wedge \Phi(G[S]) = 1 \right\}$$

- Inclusion-Exclusion finishes the proof (G_J: delete all colors J from G):

$$\#\text{cp-IndSub}(\Phi \to G) = \sum_{J \subseteq [|V(C)|]} (-1)^{\#J} \cdot \#\text{IndSub}(\Phi, k \to G_J)$$

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Using Colors

#cp-IndSub(Φ) Parameter: $k = |V(C)|$

Given C-colored graph G, compute $\sum_{H \in \Phi_k}$ #cp-IndSub($H \to G$) $= \sum_F a_{\Phi,k}(F) \cdot$ #cp-Hom($F \to G$).

$$\sum_{H \in \Phi_k} \#\text{cp-IndSub}(H \to G) = \sum_{\substack{H \\ |V(H)|=k}} \Phi(H) \cdot \#\text{cp-IndSub}(H \to G)$$

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$$= \sum_{S\subseteq E(C)} \Phi(C[S]) \cdot \sum_{J\subseteq E(C)\setminus S} (-1)^{\#J} \#\text{cp-Sub}(C[S \cup J] \to G)$$

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\sum_{H \in \Phi_k} \#\text{cp-IndSub}(H \to G) &= \sum_{S \subseteq E(C)} \Phi(C[S]) \cdot \#\text{cp-IndSub}(C[S] \to G) \\
&= \sum_{S \subseteq E(C)} \Phi(C[S]) \cdot \sum_{J \subseteq E(C) \setminus S} (-1)^{\#J} \#\text{cp-Sub}(C[S \cup J] \to G) \\
&= \sum_{S \subseteq E(C)} \Phi(C[S]) \cdot \sum_{J \subseteq E(C) \setminus S} (-1)^{\#J} \#\text{cp-Hom}(C[S \cup J] \to G)
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&= \left(\sum_{S \subseteq E(C)} \Phi(C[S]) \cdot (-1)^{\#E(C) - \#S} \right) \#\text{cp-Hom}(C \to G) + \cdots
\end{aligned}$$

Using Colors

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&= \left(\sum_{S\subseteq E(C)} \Phi(C[S]) \cdot (-1)^{\#E(C)-\#S} \right) \#\text{cp-Hom}(C \to G) + \cdots \\
&= \left(\tilde{\chi}(\Phi, C) \cdot (-1)^{\#E(C)} \right) \#\text{cp-Hom}(C \to G) + \cdots
\end{aligned}$$

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Observation

For C-colored graph G, we have

$$|a_{\Phi,k}(C)| = |\tilde{\chi}(\Phi, C)|.$$

Using Colors

#cp-IndSub(Φ) Parameter: $k = 2t$

Given $C_{t,t}$-colored graph G, compute $\sum_{H\in\Phi_k}$#cp-IndSub($H\to G$)$=\sum_F a_{\Phi,k}(F)\cdot$#cp-Hom($F\to G$)

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$$= \left(\tilde{\chi}(\Phi, C) \cdot (-1)^{\#E(C)}\right) \#\text{cp-Hom}(C \to G) + \cdots$$

Lemma [Rivest, Vuillemin '76/'77]

For the biclique $C_{t,t}$, where $t = p^c$ is a prime power,
the alternating enumerator $\tilde{\chi}(\Phi, C_{t,t})$ of Φ and $C_{t,t}$ satisfies

$$|a_{\Phi,k}(C_{t,t})| = |\tilde{\chi}(\Phi, C_{t,t})| \equiv |\Phi(C[\emptyset]) - \Phi(C)| \pmod p.$$

ALGEBRA

Removing Colors

Theorem

$\#\textsc{Hom}(\mathcal{H}) \preceq^{T}_{\mathrm{fpt}} \#\textsc{cp-Hom}(\mathcal{H})$

- Fix a graph H. Given graph G and oracle for $\#\textsc{cp-Hom}(H \to \star)$
- Compute: $\#\textsc{Hom}(H \to G)$
- Create $k = |V(H)|$ copies of $G \rightsquigarrow V_1, \dots, V_k$
- Connect $u \in V_i$ to $v \in V_j$ if $\{u, v\} \in E$ and $\{i, j\} \in H$

Removing Colors

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Main Result, again

Main Theorem

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$$\#\textsc{Hom}(\mathcal{K}) \preccurlyeq^{T}_{\text{fpt}} \#\textsc{cp-Hom}(\mathcal{K}) \preccurlyeq^{T}_{\text{fpt}} \#\textsc{cp-IndSub}(\Phi) \preccurlyeq^{T}_{\text{fpt}} \#\textsc{IndSub}(\Phi).$$

Main Result, again

Main Theorem

For property Φ, define $\mathcal{K} := \{\Phi(C_{t,t}[\emptyset]) \neq \Phi(C_{t,t}) \mid t = p^c\}$. We have

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Corollary

$\#\textsc{IndSub}(\Phi)$ is #W[1]-hard whenever $\mathcal{K}$ has unbounded treewidth.

Navigation