Replicate : A Hybrid System Model of Seasonal Snowpack Water Balance

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1 Motivation for Work

Systems with long time steps may be modeled with hybrid automata [2]. This project replicates part of the work involved with modeling the seasonal snowpack water balance and the snow-water-equivalence (SWE). The SWE is important in estimations of the water available for consumption in the winter snowpack. This is of particular importance in the desert west of the USA. As policies are developed concerning water rights it will be useful to have accurate models and simulations.

2 Physical Model

The problem space primarily concerns the Spring thaw. We can assume that there is no additional snowfall, no appreciable sublimation and that there is no heat transfer by convection to the atmosphere. Further we treat this as a bulk with instantaneous heat conduction (assume no thermal resistance). The conversion of the snowpack to run-off is due solely to radiative effects, insolation, heating from the sun, and night time cooling.

2.1 Diurnal Insolation

In the paper the insolation was obtained emperically. For my purpose the solar intensity is modeled as a sine function during the day and clipped for night. Typical daily insolation values are about 3kW-hr/day.

$$u(t)_{solar} = a \sin 2\pi t \tag{1}$$

$$\int_0^{1day} u(t)_{solar} dt = \frac{a}{2\pi} \tag{2}$$

$$\approx 3\text{kW-hr/day} \tag{3}$$

$$a \approx 30,000 kJ/day \tag{4}$$

This is coupled with a constant cooling to produce heating function.

$$u(t)_{space} \approx 0.1\text{kW}/m^2 \tag{5}$$
$$\approx 8600\text{kJ}/m^2\text{-day} \tag{6}$$
$$u(t) = 30,000 \sin 2\pi t - 8600\text{kJ}/m^2\text{-day} \tag{7}$$

2.2 Capacitive Heating

The heat capacity of snow is the same as that of ice.

$$\frac{dT}{dt} = \frac{u(t)}{M_{snow}C_{snow}} \tag{8}$$

This is used only in the Frozen mode as that is the only mode where the temperature changes.

2.3 Melting

The latent heat of fusion for snow is the same as for ice.

$$\frac{dM_{water}}{dt} = -\frac{dM_{ice}}{dt} \tag{9}$$
$$= \frac{u(t)}{L_f} \tag{10}$$

This is not used in the Frozen mode as there is no water present.

2.4 Compaction

Snow is a mixture of ice and air. The specific composition is varied and several models exist. The compaction behavior over time is complex but can be well described by formula. Starting with the half-saturation formula where A (kg/m^2) is the maximum saturation level and B ($days$) is the half-saturation time. Taking the derivative gives the compaction rate.

$$\rho_{snow}(t) = \frac{A}{1 + B/t} \tag{11}$$
$$\frac{d\rho_{snow}(t)}{dt} = \frac{AB}{(B+t)^2} \tag{12}$$
$$\tag{13}$$

Computing the time when a particular density is reached and substituting gives a function for the compaction rate as a function of density.

$$t = \frac{\rho_{snow} B}{A - \rho_{snow}} \tag{14}$$

$$\frac{d\rho_{snow}(t)}{dt} = \frac{A}{B(1 + \frac{\rho_{snow}(t)}{A - \rho_{snow}(t)})^2} \tag{15}$$

Compaction occurs in all modes except in the absence of a snowpack.

2.5 Adsorbtion

Snow acts as a sponge, adsorbing water into its matrix. This fact distinguishes the thawing and melting modes. The thawing mode is characterized by the snow's ability to adsorb more water, while the melting mode is supersaturated and additional conversion of ice into water results in run-off. This idea is expressed in the volumetric water content of the snow, θ_{snow}.

$$\theta_{snow} = \frac{V_{water}}{V_{total}} \tag{16}$$

$$= \frac{M_{water}/\rho_{water}}{M_{snow}/\rho_{snow}} \tag{17}$$

The maximum θ_{snow} signals the transition between modes, designated θ_r. The mass of water in the saturated snowpack follows directly. And, differentiating by parts once gives the rate that the water in the snowpack is changing.

$$M_{water} = \theta_r \rho_{water} \frac{M_{ice}}{\rho_{snow} - \theta_r \rho_{water}} \tag{18}$$

$$\frac{dM_{water}}{dt} = \theta_r \rho_{water} \left[\frac{\frac{dM_{water}}{dt}(\rho_{snow} - \theta_r \rho_{water}) - M_{ice} \frac{d\rho_{snow}}{dt}}{(\rho_{snow} - \theta_r \rho_{water})^2} \right] \tag{19}$$

Intuitively this relation makes sense. A higher density in the snowpack results is a lower adsorbtivity. These provide a sufficient physical model to construct the hybrid model. The output of most interest in the report is the snow-water-equivalent (SWE). This denotes the height of water that would result if the remaining snow were converted to water.

$$swe(t) = \frac{M_{ice}(t) + M_{water}(t)}{\rho_{water}} \tag{20}$$

3 Hybrid Model

The hybrid model is a bulk model for an arbitrary core sample of the snowpack.

We start with the formal definition of the model.

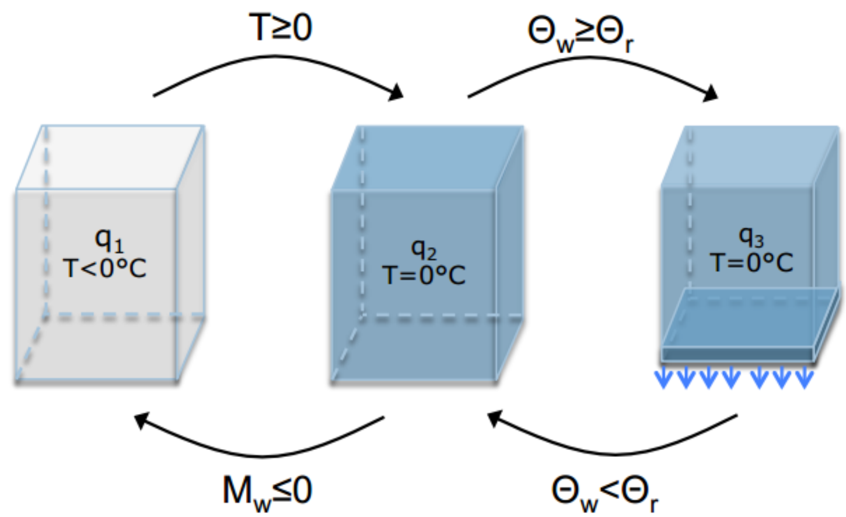

Figure 1: The snowpack modes

$$H = (Q, X, Init, U, f, Dom, R, Y) \tag{21}$$
$$X = \{x \in \mathbb{R}^4 | x = [M_{ice}, M_{water}, \rho_{snow}, T]^T\} \tag{22}$$

3.1 Discrete Model

There are four discrete modes in this model. It has two top level modes, one representing the absence of snow, and the other representing the presence of a snowpack. The snowpack mode is further discriminated into three modes: frozen, thawing and melting. Frozen indicates the absence of water in the snow matrix, only ice and air are present. Thawing indicates that the ice and water are in a phase change condition; all the water present is adsorbed by the snow matrix. Melting indicates a supersaturated state where the volumetric water content of the snowpack is at a maximum.

$$Q = \{q_0, q_1, q_2, q_3\} \tag{23}$$
$$= \{\text{empty}, \text{frozen}, \text{thawing}, \text{melting}\} \tag{24}$$

$$Dom_{empty} = \{x \in X \big| M_{ice} <= 0\} \tag{25}$$
$$Dom_{frozen} = \{x \in X \big| T <= 0\} \tag{26}$$
$$Dom_{thawing} = \{x \in X \big| T >= 0 \wedge 0 < M_{water} < \theta_r \rho_{water} \frac{M_{ice} + M_{water}}{\rho_{snow}}\} \tag{27}$$
$$Dom_{melting} = \{x \in X \big| M_{water} >= \theta_r \rho_{water} \frac{M_{ice} + M_{water}}{\rho_{snow}}\} \tag{28}$$

The initial states are arbitrarily chosen but must meet the following criteria regardless of the initial mode.

$$M_{ice} >= 0 \tag{29}$$
$$M_{water} >= 0 \tag{30}$$
$$M_{water} <= \theta_r \rho_{water} \frac{M_{ice} + M_{water}}{\rho_{snow}} \tag{31}$$
$$\rho_{snow} >= 0 \tag{32}$$
$$\rho_{snow} <= A \text{ (the maximum saturation level)} \tag{33}$$
$$T <= 0 \tag{34}$$

The domain also suggests the reset relations.

$$R(empty, x) = (frozen, x) \qquad \text{if } 0 < M_{ice} \tag{35}$$
$$R(frozen, x) = (thawing, x) \qquad \text{if } 0 \leq T \tag{36}$$
$$R(thawing, x) = (frozen, x) \qquad \text{if } T \leq 0 \tag{37}$$
$$R(thawing, x) = (melting, x) \qquad \text{if } \theta_r \leq \frac{M_{water}/\rho_{water}}{(M_{ice} + M_{water})/\rho_{snow}} \tag{38}$$
$$R(melting, x) = (thawing, x) \qquad \text{if } \frac{M_{water}/\rho_{water}}{(M_{ice} + M_{water})/\rho_{snow}} < \theta_r \tag{39}$$
$$R(q, x) = (empty, x) \qquad \text{otherwise} \tag{40}$$

Notice the implementation issue with the $R(frozen, x) \rightarrow (thawing, x)$ and $R(thawing, x) \rightarrow (frozen, x)$ the implementation will need to force at least one time step before leaving.

3.2 Dynamic Model

When there is no snowpack some of the values in x are undefined and are arbitrarily set to zero. The mass of ice remaining is clearly $M_{ice} = 0$.

In each case the power is given by Equation **??**.

$$f_{empty}(t, x(t), u(t)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

In each of the remaining cases the snowpack density, p_{snow} increases due to compaction, Equation: 15.

When in the sub-zero, frozen, mode there is no change in mass composition. The snow does compact and the temperature is free to change.

$$f_{frozen}(t, x(t), u(t)) = \begin{bmatrix} 0 \\ 0 \\ Equation : 15 \\ Equation : 8 \end{bmatrix} \tag{42}$$

In the thawing mode the temperature is locked to the material phase transition temperature ($0C$).

$$f_{thawing}(t, x(t), u(t)) = \begin{bmatrix} -Equation : 10 \\ Equation : 10 \\ Equation : 15 \\ 0 \end{bmatrix} \tag{43}$$

$$f_{melting}(t, x(t), u(t)) = \begin{bmatrix} -Equation : 10 \\ Equation : 19 \\ Equation : 15 \\ 0 \end{bmatrix} \tag{44}$$

The output of the system, Y is given by Equation 20.

4 Work Performed

Using the hybrid model and based on the recommended Stateflow®Simulink®design pattern [1] I constructed a hybrid model.

Several variations on inputs were examined, two of which follow. In each of these cases the mass is in kg, the time in in days, the density is in kg/m^3, energy (insolation) is in kJ/m^2/day, and height (SWE) is in meters.

In this example, Figure 2, the energy is diurnal. Notice that the temperature increases more rapidly than the melt. This is due to the fact that the latent heat of fusion is much greater than the heat capacity for water. Note that the heat lost at night is not sufficiently large to move the mode back into thawing or freezing. With this in mind, a simulation was performed with a much higher nightly radiation.

In Figure 3 the general behavior looks as expected with the water in the snowpack refreezing each night until the system enters the melting mode.

5 Modeling Issues

There are several ways to model systems such as this, each with their adavantages and drawbacks. The model using Simulink®models supervised by a Stateflow®model and driven by a Simulink®model was choosen as it closely

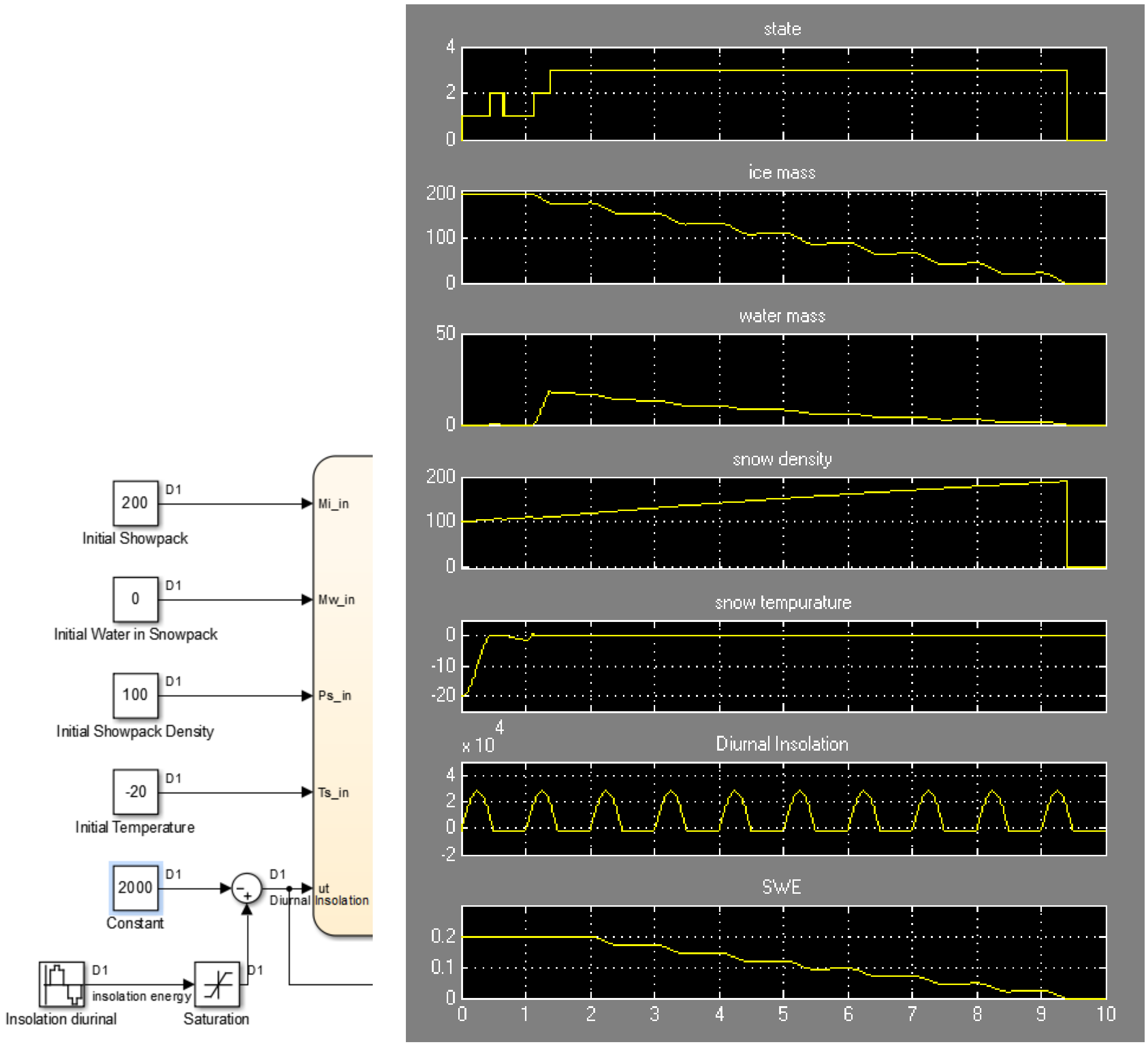

Figure 2: Snowpack consistently melting

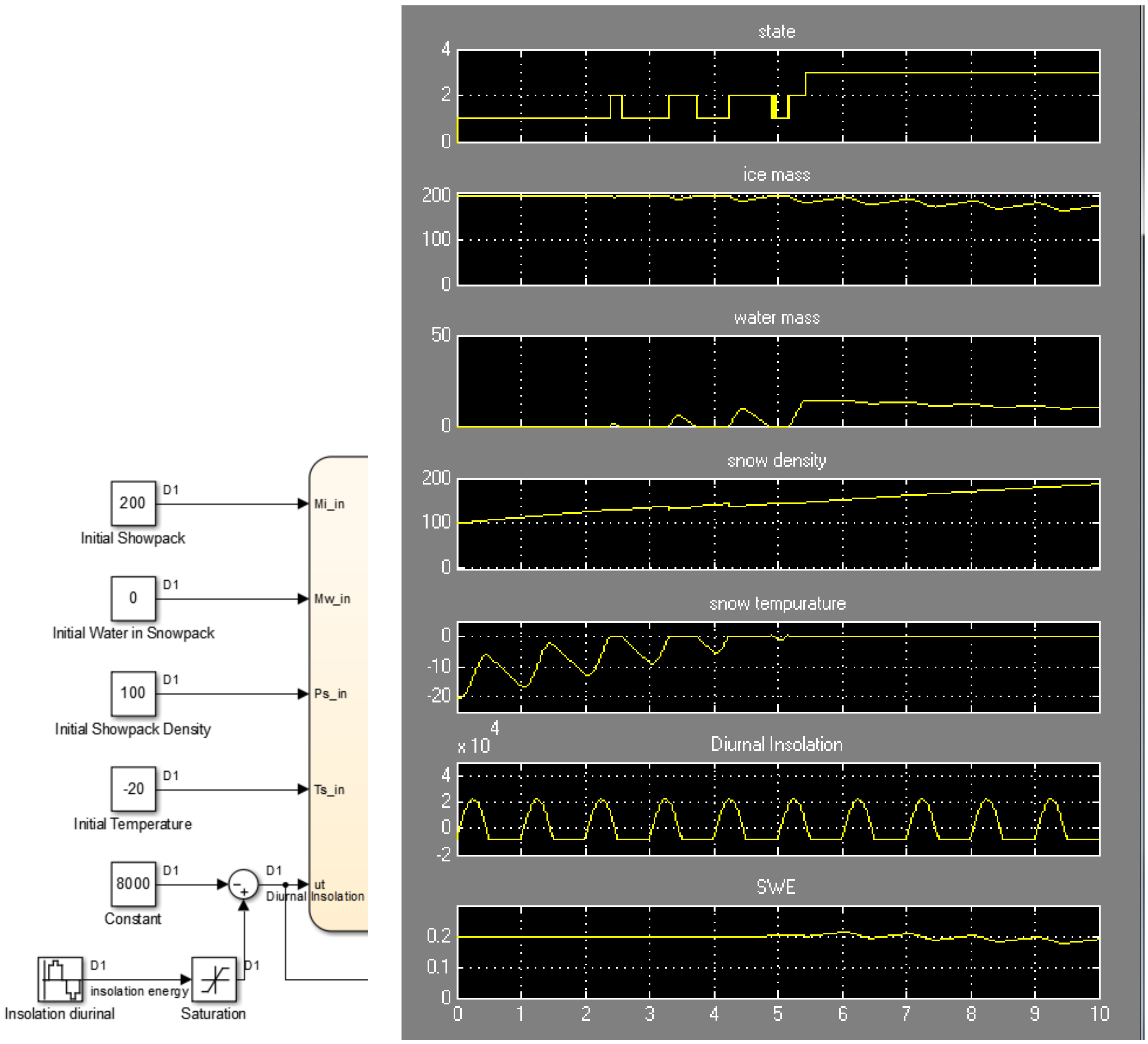

Figure 3: Snowpack thawing and refreezing at night

imitates the formal hybrid model. It has the drawback of requiring the transfer of state information between integrators which makes use of the reset with its added complexity.

The other significant issue is the proper calibration of the model. Typically the models produced with Stateflow®Simulink®are working with time frames in the one second region. This problem uses time frames in the hour or day region.

A minor issue is that the paper does not describe the input in much detail. The data they use is that recorded in various surveys. This will be approximated in my model with a sine wave that immitates the diurnal insolation with the trough giving up energy.

There were a few minor errors in the equations in paper that were corrected.

6 Physical Constants and Conditions

$$u(t) = a \sin 2\pi t - c_{rad} \tag{45}$$
$$= -c_{rad} \tag{46}$$
$$U(t) = \frac{a \cos 2\pi t}{2\pi}\Big|_0^{1/2} - c_{rad} t\Big|_0^1 \tag{47}$$
$$= 3 \text{ kW-hr}/m^2 \tag{48}$$
$$= 10,800 \text{ kJ}/m^2 \tag{49}$$
$$c_{rad} = 800 \text{ kJ}/m^2\text{-day} \tag{50}$$
$$a = (10,800 - c_{rad}) \times 2\pi \text{ kJ}/m^2\text{-day} \tag{51}$$
$$= 20,000\pi \text{ kJ}/m^2\text{-day} \tag{52}$$
$$\approx 63,000 \text{ kJ}/m^2\text{-day} \tag{53}$$

Where t is measured in days. A is the average daily insolation. A reasonable, Springtime, value for A is near 3kW-hr/m^2.

$$C_{ice} = 2.05 \text{ kJ/kg-K} \tag{54}$$
$$C_s = C_{ice} \tag{55}$$
$$\theta_r = 0.01 \tag{56}$$
$$L_f = 334 \text{ kJ/kg} \tag{57}$$
$$\rho_w = 1000 \text{ kg}/m^3 \tag{58}$$
$$A = 450 \text{ kg}/m^3 \tag{59}$$
$$B = 20 \text{ days} \tag{60}$$

7 Conclusion

This problem shows that natural systems may be effectively modelled using hybrid-automata. Some areas that could be extended would include models for simultaneous snow accumulation.

References

[1] Michael Carone. Design patterns for integrating simulink with stateflow. pages 175–191, 2009.

[2] Brando Kerkez, Steven D. Glaser, John A. Dracup, and Roger C. Bales. A hybrid system model of seasonal snowpack water balance. 2010.