

# CS 376 Hybrid Systems : HW 8

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19 November 2014

## 1 Hybrid Simulation of a Tank Sytem

Consider the two-tank systems shown in Fig. 1. The system consists of two identical cylindrical tanks, of unlimited height, that are connected by a pipe at level  $h$ . We denote by  $h_1$  and  $h_2$  the water levels in tanks 1 and 2 respectively. The input flow  $Q_{in}$  is provided by a pump and it is described by

$$Q_{in} = V_{in}k_{in}u(t), \quad (1)$$

where  $V_{in} \in \{0, 1\}$  represents a valve that can be used to turn on or off the pump (no partially open valve),  $k_{in}$  is a linear gain, and  $u(t)$  is the input signal representing the flow at the pump. The flow  $Q_a$  between the two tanks is controlled by a valve  $V_a$ . An outlet valve  $V_{out}$  located at the bottom of tank 2 is used to empty the tank. Tank 2 is equipped with a sensor that measures the output flow which is described by

$$Q_{out} = V_{out}k_{out}\sqrt{\rho gh_2} \quad (2)$$

where  $V_{out} \in \{0, 1\}$  represents the outlet valve,  $k_{out}$  is a linear gain,  $\rho$  is the density of the water, and  $g$  is the gravitational constant.

The dynamic evolution of the systems is described by

$$\dot{h}_1 = \frac{1}{A}(Q_{in} - Q_a) \quad (3)$$

$$\dot{h}_2 = \frac{1}{A}(Q_a - Q_{out}) \quad (4)$$

where  $A$  is the section of the identical cylindrical tanks. Following Toricelli's law, the flow  $Q_a$  depends on the water levels  $h_1$  and  $h_2$  as follows:

$$Q_a = \begin{cases} 0, & \text{if } h_1 < h \wedge h_2 < h \\ V_a k_a \sqrt{\rho g(h_1 - h)}, & \text{if } h_1 > h \wedge h_2 < h \\ V_a k_a \sqrt{\rho g(h - h_2)}, & \text{if } h_1 < h \wedge h_2 > h \\ \text{sign}(h_1 - h_2) V_a k_a \sqrt{\rho g|h_1 - h_2|}, & \text{if } h_1 > h \wedge h_2 > h \end{cases} \quad (5)$$

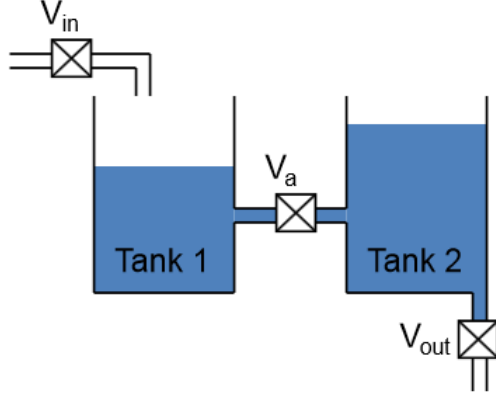


Figure 1: Two-tank system

where  $V_a \in \{0, 1\}$  and  $k_a$  is a linear gain. The evolution of the continuous state can be described by

$$x = [h_1, h_2]^T \quad (6)$$

$$\dot{x} = f_q(x(t), u(t)) \quad (7)$$

where  $q$  is the discrete mode of the system. The mode transitions are based on guard conditions indicated in Equation 5.

Assume the following values for the system parameters and initial conditions:

$$V_{in} = V_a = V_{out} = 1, \text{ all valves are open} \quad (8)$$

$$(9)$$

$u(t)$  is a pulse with amplitude  $1 \text{ m}^3$  and frequency  $1 \text{ Hz}$  resulting in an average rate of  $1 \text{ m}^3/\text{sec}$ .

$$h = 0.3 \text{ m} \quad (10)$$

$$k_{in} = 0.06, \quad (11)$$

$$k_a = 0.001, \quad (12)$$

$$k_{out} = 0.001, \quad (13)$$

$$g = 9.81 \text{ m/sec}^2, \quad (14)$$

$$\rho = 1000 \text{ kg/m}^3, \quad (15)$$

$$A = 0.0154 \text{ m}^2 \quad (16)$$

## 2 A hybrid automaton model

The formal model for the two-tank system hybrid automaton.

$$H = (q, X, Init, f, Inv, E, G, R) \quad (17)$$

The set of discrete modes.

$$\text{if } h_1 < h \wedge h_2 < h, : \textit{separated} \quad (18)$$

$$\text{if } h_1 > h \wedge h_2 < h, : \textit{from}_1 \quad (19)$$

$$\text{if } h_1 < h \wedge h_2 > h, : \textit{from}_2 \quad (20)$$

$$\text{if } h_1 > h \wedge h_2 > h, : \textit{balancing} \quad (21)$$

$$q = \{\textit{separated}, \textit{from}_1, \textit{from}_2, \textit{balancing}\} \quad (22)$$

The set of continuous variables.  $X = \mathbb{R}$

$$X = \{u(t), x, Q_a, Q_{out}\} \quad (23)$$

The set of initial conditions.  $Init \subseteq Q \times X$

$$Init = \{\} \quad (24)$$

The vector field.  $f : Q \times X$

$$f = f_{\textit{separated}} \cup f_{\textit{from}_1} \cup f_{\textit{from}_2} \cup f_{\textit{balancing}} \quad (25)$$

The invariant set. (all empty)  $Q \mapsto 2^X$

$$Inv = Inv_{\textit{separated}} \cup Inv_{\textit{from}_1} \cup Inv_{\textit{from}_2} \cup Inv_{\textit{balancing}} \quad (26)$$

The invariants are the same for each mode.  $Inv_i =$

$$h_1 \geq 0 \quad (27)$$

$$h_2 \geq 0 \quad (28)$$

$f_{\textit{separated}} =$

$$Q_{in} = k_{in} \text{ m/sec}^2 \quad (29)$$

$$Q_a = 0, \quad (30)$$

$$\dot{h}_1 = \frac{Q_{in}}{A} \quad (31)$$

$$\dot{h}_2 = -\frac{Q_{out}}{A} \quad (32)$$

$$Q_{out} = k_{out} \sqrt{\rho g h_2} \quad (33)$$

$$f_{from_1} =$$

$$Q_{in} = k_{in} \text{ m/sec}^2 \quad (34)$$

$$Q_a = k_a \sqrt{\rho g (h_1 - h)}, \quad (35)$$

$$\dot{h}_1 = \frac{Q_{in} - Q_a}{A} \quad (36)$$

$$\dot{h}_2 = \frac{Q_a - Q_{out}}{A} \quad (37)$$

$$Q_{out} = k_{out} \sqrt{\rho g h_2} \quad (38)$$

$$f_{from_2} =$$

$$Q_{in} = k_{in} \text{ m/sec}^2 \quad (39)$$

$$Q_a = k_a \sqrt{\rho g (h - h_2)}, \quad (40)$$

$$\dot{h}_1 = \frac{Q_{in} - Q_a}{A} \quad (41)$$

$$\dot{h}_2 = \frac{Q_a - Q_{out}}{A} \quad (42)$$

$$Q_{out} = k_{out} \sqrt{\rho g h_2} \quad (43)$$

$$f_{balancing} =$$

$$Q_{in} = k_{in} \text{ m/sec}^2 \quad (44)$$

$$Q_a = \text{sign}(h_1 - h_2) k_a \sqrt{\rho g |h_1 - h_2|} \quad (45)$$

$$\dot{h}_1 = \frac{Q_{in} - Q_a}{A} \quad (46)$$

$$\dot{h}_2 = \frac{Q_a - Q_{out}}{A} \quad (47)$$

$$Q_{out} = k_{out} \sqrt{\rho g h_2} \quad (48)$$

The collection of discrete transitions.  $E \subset Q \times Q$

$$E = \{(\text{separated}, \text{from}_1) \quad (49)$$

$$(\text{separated}, \text{from}_2) \quad (50)$$

$$(\text{separated}, \text{balanced}) \quad (51)$$

$$(\text{from}_1, \text{separated}) \quad (52)$$

$$(\text{from}_1, \text{from}_2) \quad (53)$$

$$(\text{from}_1, \text{balanced}) \quad (54)$$

$$(\text{from}_2, \text{from}_1) \quad (55)$$

$$(\text{from}_2, \text{separated}) \quad (56)$$

$$(\text{from}_2, \text{balanced}) \quad (57)$$

$$(\text{balanced}, \text{from}_1) \quad (58)$$

$$(\text{balanced}, \text{from}_2) \quad (59)$$

$$(\text{balanced}, \text{separated})\} \quad (60)$$

The guards on the transitions.  $G : E \mapsto 2^X$

$$E = \{(separated, from_1), h_1 > h \wedge h_2 < h, \quad (61)$$

$$(separated, from_2), h_1 < h \wedge h_2 > h, \quad (62)$$

$$(separated, balanced), h_1 > h \wedge h_2 > h, \quad (63)$$

$$(from_1, separated), h_1 < h \wedge h_2 < h, \quad (64)$$

$$(from_1, from_2), h_1 < h \wedge h_2 > h, \quad (65)$$

$$(from_1, balanced), h_1 > h \wedge h_2 > h, \quad (66)$$

$$(from_2, from_1), h_1 > h \wedge h_2 < h, \quad (67)$$

$$(from_2, separated), h_1 < h \wedge h_2 < h, \quad (68)$$

$$(from_2, balanced), h_1 > h \wedge h_2 > h, \quad (69)$$

$$(balanced, from_1), h_1 > h \wedge h_2 < h, \quad (70)$$

$$(balanced, from_2), h_1 < h \wedge h_2 > h, \quad (71)$$

$$(balanced, separated), h_1 < h \wedge h_2 < h, \} \quad (72)$$

The reset relation on the transitions.  $R : E \times X \mapsto 2^X$

$$R = \emptyset \quad (73)$$

### 3 Simulink Simulation

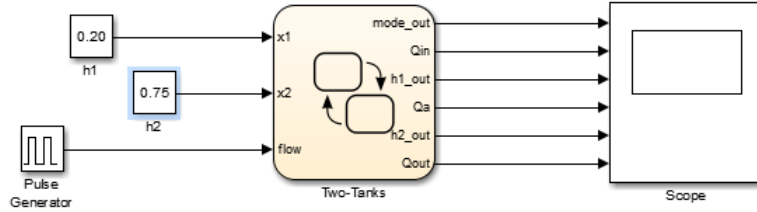


Figure 2: Two-tank SimuLink Model

The model is simulated for several initial conditions  $x_0 = [h_1, h_2]^T$ . The continuous state  $Q_a, x$ , discrete state  $q$ , and output  $Q_{out}$  of the system are plotted.

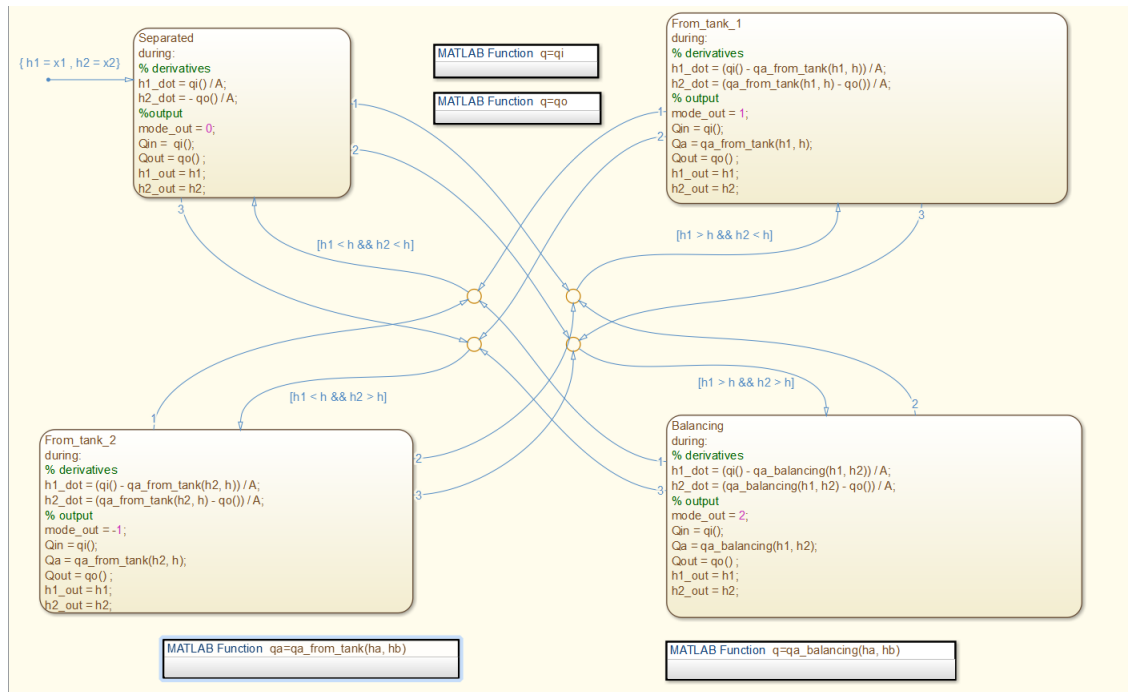


Figure 3: Two-tank StateFlow Model

```

1  function q=qi()
2  %%QO Compute output flow rate from tank 2
3  % A negative flow value should be avoided.
4  % It would be non-physical.
5  q = Kin * flow;
6

```

Figure 4: Input Flow Function

```

1  function q=qo()
2  %%QO Compute output flow rate from tank 2
3  % A negative flow rate is allowed so the
4  % solver can detect zero-crossing.
5  q = sign(h2) * Kout * sqrt(rho * g * abs(h2));
6

```

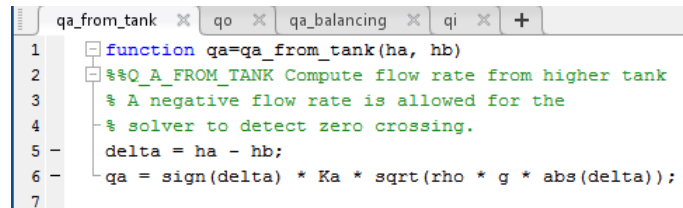
Figure 5: Output Flow Function

3.1  $x_0 = [0.20, 0.75]^T$

3.2  $x_0 = [0.50, 0.20]^T$

3.3  $x_0 = [0.50, 0.50]^T$

3.4  $x_0 = [0.20, 0.20]^T$



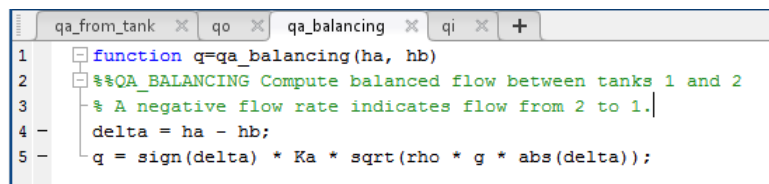
The image shows a MATLAB editor window with the following tabs: qa\_from\_tank, qo, qa\_balancing, qi, and a plus sign. The 'qa\_from\_tank' tab is active, displaying the following code:

```

1 function qa=qa_from_tank(ha, hb)
2 %%Q_A_FROM_TANK Compute flow rate from higher tank
3 % A negative flow rate is allowed for the
4 % solver to detect zero crossing.
5 - delta = ha - hb;
6 - qa = sign(delta) * Ka * sqrt(rho * g * abs(delta));
7

```

Figure 6: From Tank Flow Function



The image shows a MATLAB editor window with the following tabs: qa\_from\_tank, qo, qa\_balancing, qi, and a plus sign. The 'qa\_balancing' tab is active, displaying the following code:

```

1 function q=qa_balancing(ha, hb)
2 %%QA_BALANCING Compute balanced flow between tanks 1 and 2
3 % A negative flow rate indicates flow from 2 to 1.
4 - delta = ha - hb;
5 - q = sign(delta) * Ka * sqrt(rho * g * abs(delta));

```

Figure 7: Balancing Flow Function

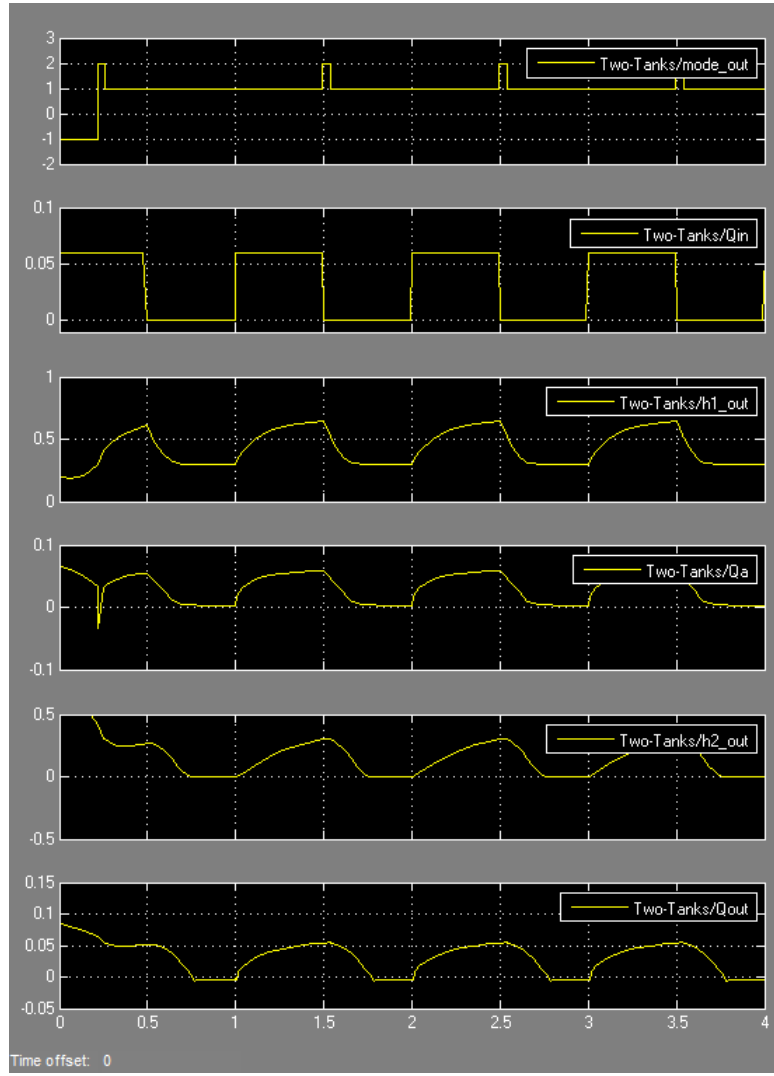


Figure 8: Two-tank SimuLink Model,  $x_0 = [0.2, 0.75]^T$



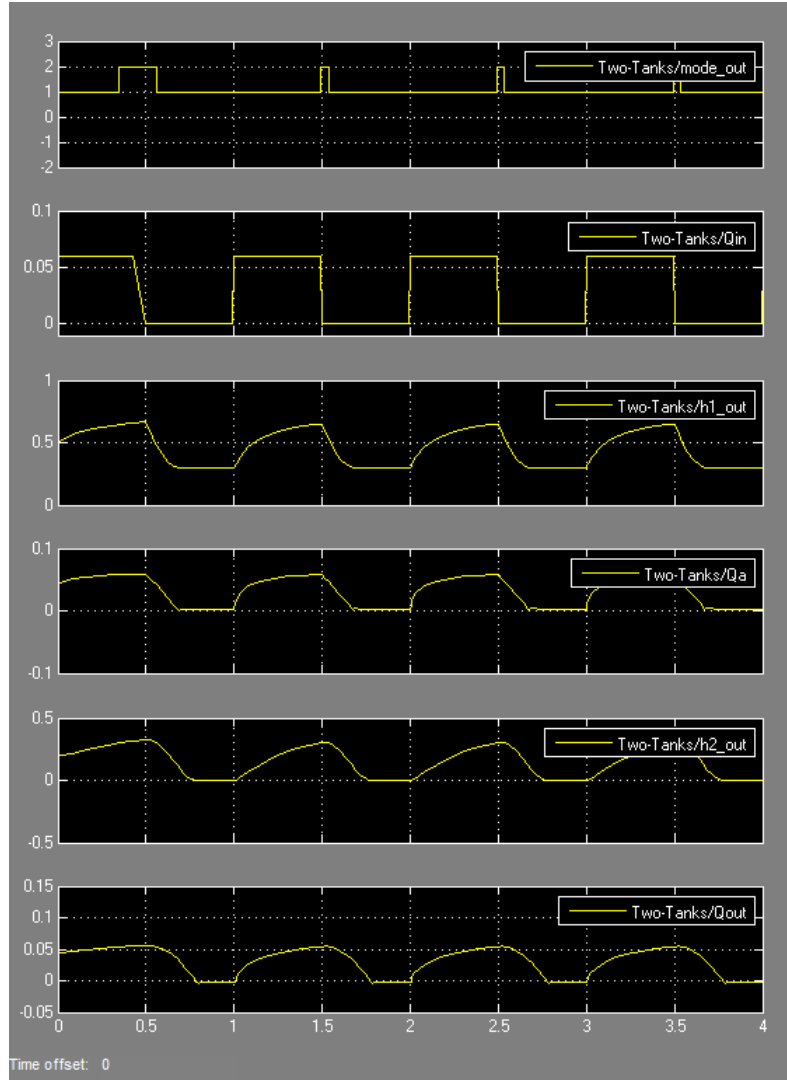


Figure 9: Two-tank SimuLink Model,  $x_0 = [0.50, 0.20]^T$

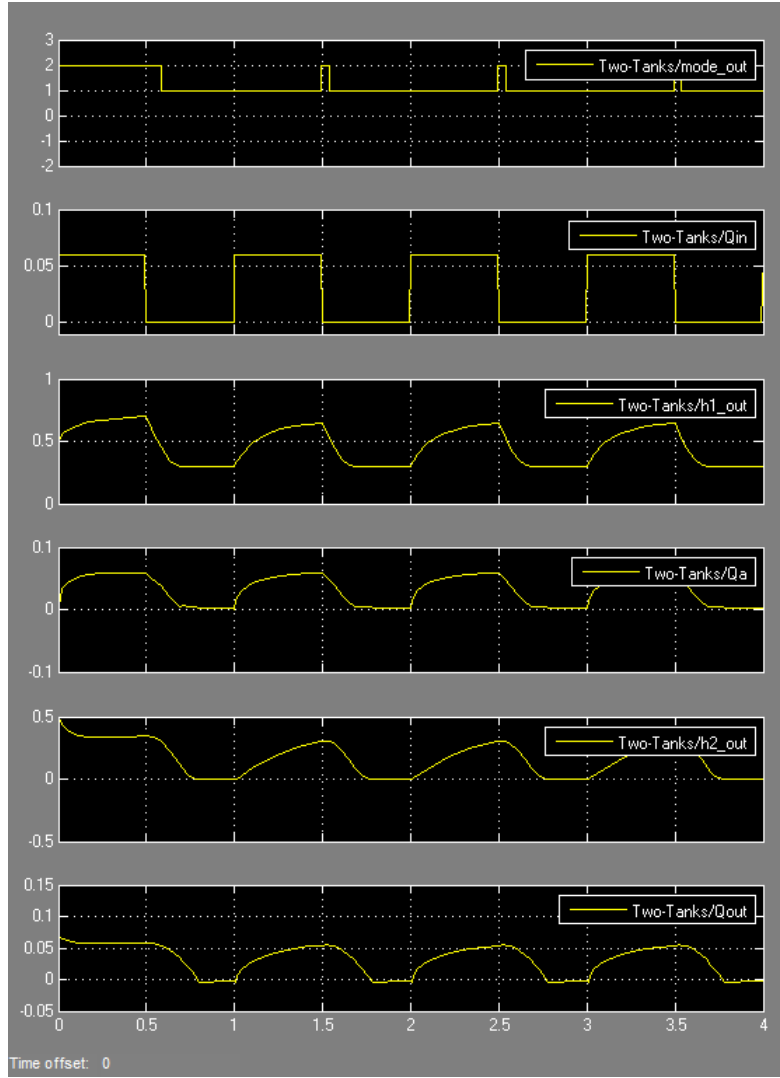


Figure 10: Two-tank SimuLink Model,  $x_0 = [0.50, 0.50]^T$

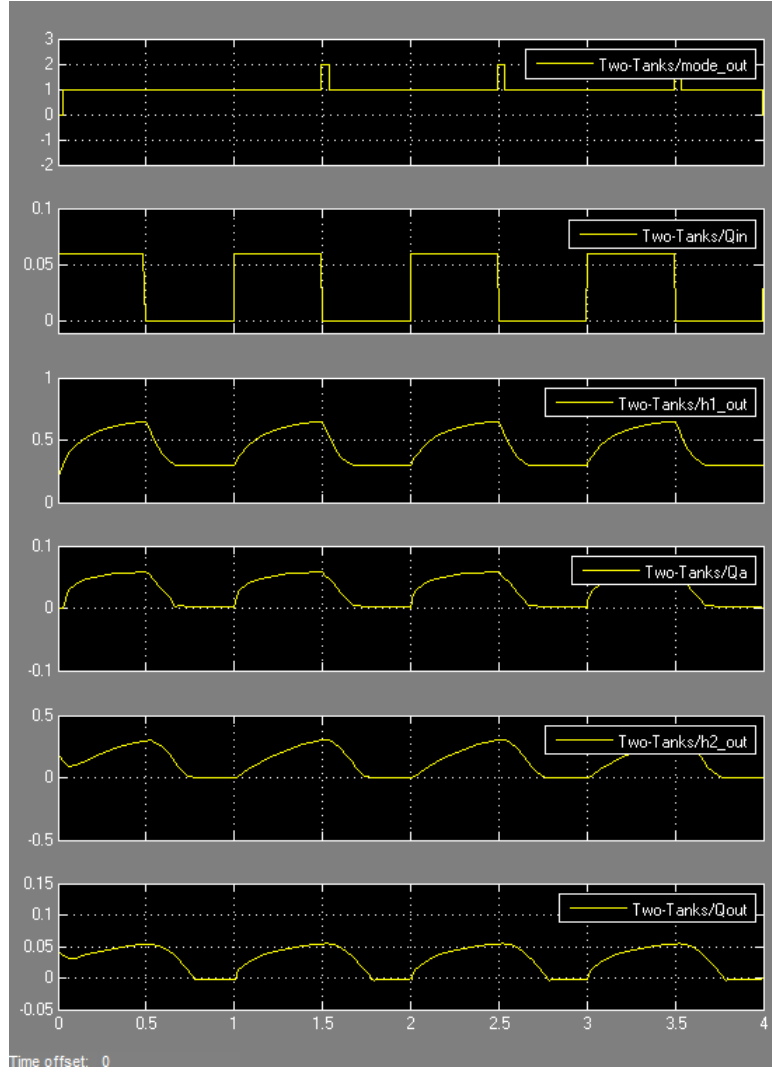


Figure 11: Two-tank SimuLink Model,  $x_0 = [0.20, 0.20]^T$