CS 376 Hybrid Systems : HW 8

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19 November 2014

1 Hybrid Simulation of a Tank Sytem

Consider the two-tank systems shown in Fig. 1. The system consists of two identical cylindrical tanks, of unlimited height, that are connected by a pipe at level h. We denote by h_1 and h_2 the water levels in tanks 1 and 2 respectively. The input flow Q_{in} is provided by a pump and it is described by

$$Q_{in} = V_{in} k_{in} u(t), \tag{1}$$

where $V_{in} \in \{0, 1\}$ represents a valve that can be used to turn on or off the pump (no partially open valve), k_{in} is a linear gain, and $u(t)$ is the input signal representing the flow at the pump. The flow Q_a between the two tanks is controlled by a valve V_a. An outlet value V_{out} located at the bottom of tank 2 is used to empty the tank. Tank 2 is equipped with a sensor that measures the output flow which is described by

$$Q_{out} = V_{out} k_{out} \sqrt{\rho g h_2} \tag{2}$$

where $V_{out} \in \{0, 1\}$ represents the outlet valve, k_{out} is a linear gain, ρ is the density of the water, and g is the gravitational constant.

The dynamic evolution of the systems is described by

$$\dot{h_1} = \frac{1}{A}(Q_{in} - Q_a) \tag{3}$$

$$\dot{h_2} = \frac{1}{A}(Q_a - Q_a) \tag{4}$$

where A is the section of the identical cylindrical tanks. Following Toricelli's law, the flow Q_a depends on the water levels h_1 and h_2 as follows:

$$Q_a = \begin{cases} 0, & \text{if } h_1 < h \wedge h_2 < h \\ V_a k_a \sqrt{\rho g (h_1 - h)}, & \text{if } h_1 > h \wedge h_2 < h \\ V_a k_a \sqrt{\rho g (h - h_2)}, & \text{if } h_1 < h \wedge h_2 > h \\ sign(h_1 - h_2) V_a k_a \sqrt{\rho g |h_1 - h_2|}, & \text{if } h_1 > h \wedge h_2 > h \end{cases} \tag{5}$$

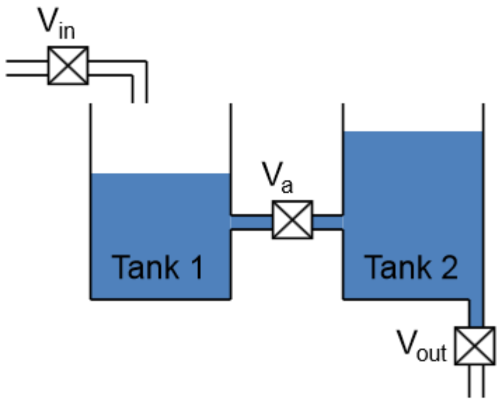


Figure 1: Two-tank system

where $V_a \in \{0, 1\}$ and k_a is a linear gain. The evolution of the continuous state can be described by

$$x = [h_1, h_2]^T \tag{6}$$
$$\dot{x} = f_q(x(t), u(t)) \tag{7}$$

where q is the discrete mode of the sytem. The mode transitions are based on guard conditions indicated in Equation 5.

Assume the following values for the system parameters and initial conditions:

$$V_{in} = V_a = V_{out} = 1, \text{ all valves are open} \tag{8}$$
$$\tag{9}$$

$u(t)$ is a pulse with amplitude 1 m^3 and frequency 1 Hz resulting in an average rate of 1 m^3/sec.

$$h = 0.3 \text{ m} \tag{10}$$
$$k_{in} = 0.06, \tag{11}$$
$$k_a = 0.001, \tag{12}$$
$$k_{out} = 0.001, \tag{13}$$
$$g = 9.81\ m/sec^2\ , \tag{14}$$
$$\rho = 1000\ kg/m^3\ , \tag{15}$$
$$A = 0.0154\ m^2 \tag{16}$$

2 A hybrid automaton model

The formal model for the two-tank system hybrid automaton. There were a number of simplifications which could have been made. Rather than make these simplifications I tried to stay as close to the specification as possible.

All the V_i variables are 1 so they have no net effect on the model so they have been ommitted.

There is a further assumption on this problem in that the input flow is allways positive (or zero).

$$H = (q, X, Init, f, Inv, E, G, R) \tag{17}$$

The set of discrete modes.

$$\text{if } h_1 < h \wedge h_2 < h, : separated \tag{18}$$
$$\text{if } h_1 > h \wedge h_2 < h, : from_1 \tag{19}$$
$$\text{if } h_1 < h \wedge h_2 > h, : from_2 \tag{20}$$
$$\text{if } h_1 > h \wedge h_2 > h, : balancing \tag{21}$$

$$q = \{separated, from_1, from_2, balancing\} \tag{22}$$

The set of continuous variables. $X = \mathbb{R}$

$$X = \{u(t), x, Q_a, Q_{out}\} \tag{23}$$

The set of initial conditions. $Init \subseteq Q \times X$

$$Init = \{h_1 = h_{1_0}, h_2 = h_{2_0}\} \tag{24}$$

The vector field. $f : Q \times X$

$$f = f_{separated} \cup f_{from_1} \cup f_{from_2} \cup f_{balancing} \tag{25}$$

The invariant set. (all empty) $Q \mapsto 2^X$

$$Inv = Inv_{separated} \cup Inv_{from_1} \cup Inv_{from_2} \cup Inv_{balancing} \tag{26}$$

The invariants are the same for each mode. $Inv_i =$

$$h_1 \geq 0 \tag{27}$$
$$h_2 \geq 0 \tag{28}$$

$f_{separated} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{29}$$
$$Q_a = 0, \tag{30}$$
$$\dot{h_1} = \frac{Q_{in}}{A} \tag{31}$$
$$\dot{h_2} = -\frac{Q_{out}}{A} \tag{32}$$
$$Q_{out} = k_{out}\sqrt{\rho g h_2} \tag{33}$$

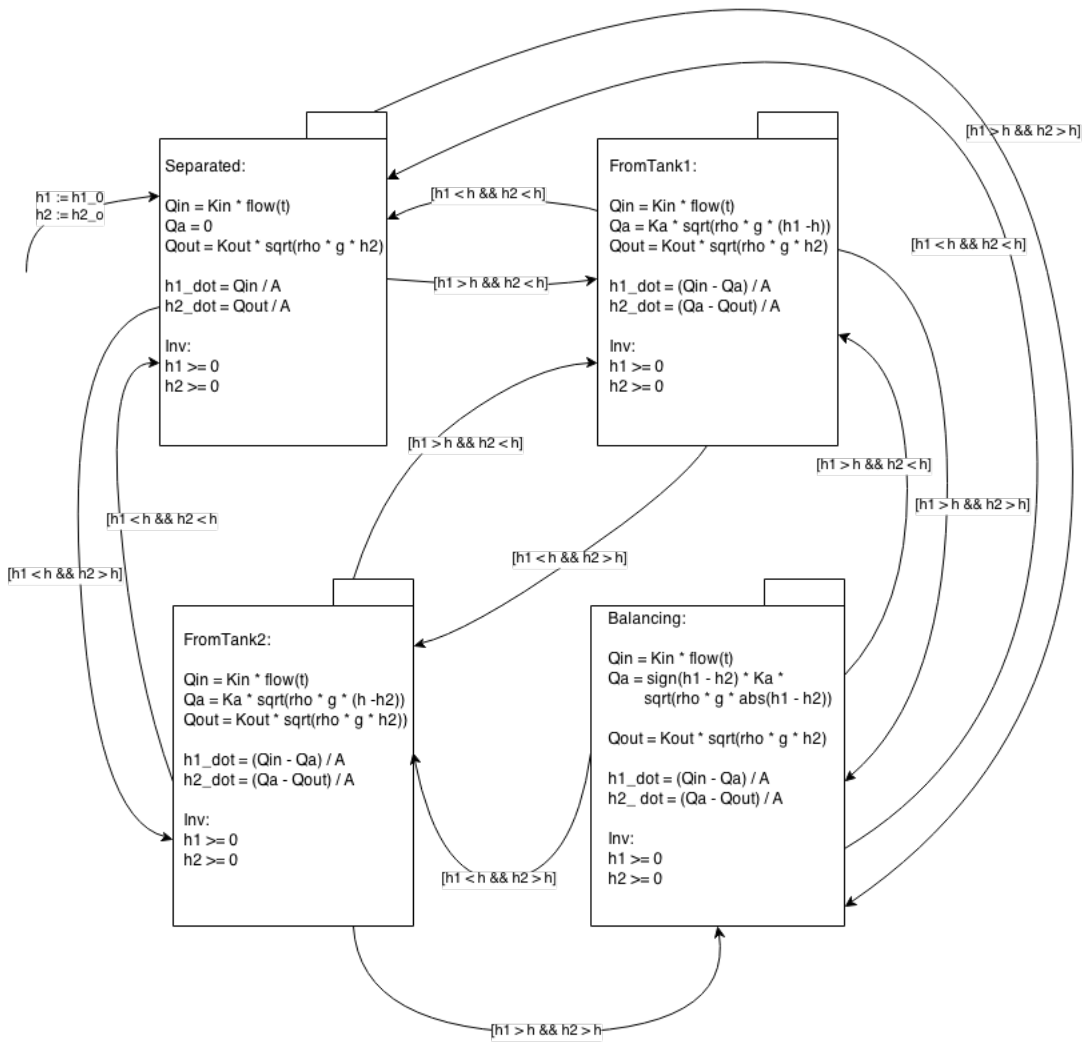


Figure 2: Two-tank Hybrid Model

$f_{from_1} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{34}$$

$$Q_a = k_a\sqrt{\rho g(h_1 - h)}, \tag{35}$$

$$\dot{h_1} = \frac{Q_{in} - Q_a}{A} \tag{36}$$

$$\dot{h_2} = \frac{Q_a - Q_{out}}{A} \tag{37}$$

$$Q_{out} = k_{out}\sqrt{\rho g h_2} \tag{38}$$

$f_{from_2} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{39}$$

$$Q_a = k_a\sqrt{\rho g(h - h_2)}, \tag{40}$$

$$\dot{h_1} = \frac{Q_{in} - Q_a}{A} \tag{41}$$

$$\dot{h_2} = \frac{Q_a - Q_{out}}{A} \tag{42}$$

$$Q_{out} = k_{out}\sqrt{\rho g h_2} \tag{43}$$

$f_{balancing} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{44}$$

$$Q_a = sign(h_1 - h_2)k_a\sqrt{\rho g|h_1 - h_2|} \tag{45}$$

$$\dot{h_1} = \frac{Q_{in} - Q_a}{A} \tag{46}$$

$$\dot{h_2} = \frac{Q_a - Q_{out}}{A} \tag{47}$$

$$Q_{out} = k_{out}\sqrt{\rho g h_2} \tag{48}$$

The collection of discrete transitions. $E \subset Q \times Q$

$$E = \{(separated, from_1) \tag{49}$$

$$(separated, from_2) \tag{50}$$

$$(separated, balanced) \tag{51}$$

$$(from_1, separated) \tag{52}$$

$$(from_1, from_2) \tag{53}$$

$$(from_1, balanced) \tag{54}$$

$$(from_2, from_1) \tag{55}$$

$$(from_2, separated) \tag{56}$$

$$(from_2, balanced) \tag{57}$$

$$(balanced, from_1) \tag{58}$$

$$(balanced, from_2) \tag{59}$$

$$(balanced, separated)\} \tag{60}$$

The guards on the transitions. $G : E \mapsto 2^X$

$$
\begin{aligned}
E = \{&(separated, from_1), h_1 > h \wedge h_2 < h, &(61)\\
&(separated, from_2), h_1 < h \wedge h2 > h, &(62)\\
&(separated, balanced), h_1 > h \wedge h_2 > h, &(63)\\
&(from_1, separated), h_1 < h \wedge h_2 < h, &(64)\\
&(from_1, from_2), h_1 < h \wedge h2 > h, &(65)\\
&(from_1, balanced), h_1 > h \wedge h_2 > h, &(66)\\
&(from_2, from_1), h_1 > h \wedge h_2 < h, &(67)\\
&(from_2, separated), h_1 < h \wedge h_2 < h, &(68)\\
&(from_2, balanced), h_1 > h \wedge h_2 > h, &(69)\\
&(balanced, from_1), h_1 > h \wedge h_2 < h, &(70)\\
&(balanced, from_2), h_1 < h \wedge h2 > h, &(71)\\
&(balanced, separated), h_1 < h \wedge h_2 < h, \} &(72)
\end{aligned}
$$

The reset relation on the transitions. $R : E \times X \mapsto 2^X$

$$R = \emptyset \tag{73}$$

3 Simulink Simulation

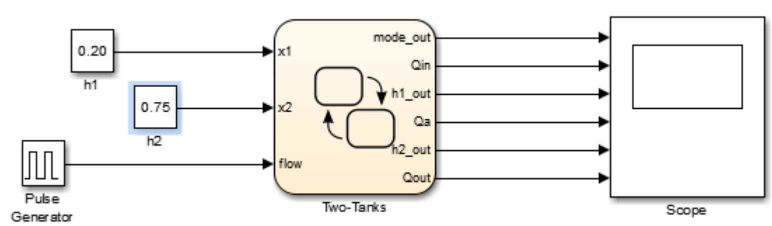


Figure 3: Two-tank SimuLink Model

I choose to implement this model using continuous StateFlow. The tricky part of this simulation was constructing functions on continuous variables. The difficulty was the limitations imposed on the continuous variables. In particular relational checks were not allowed. I used the $sign()$ and $abs()$ functions to accomdate this limitation.

It was likely that the functions would require taking the $sqrt()$ of a negative value. Also, the $sqrt()$ function does not actually cross zero. The functions have been modified to handle the invariant conditions by using a modified square root that is monotinic increasing, continuous, and differentiable.

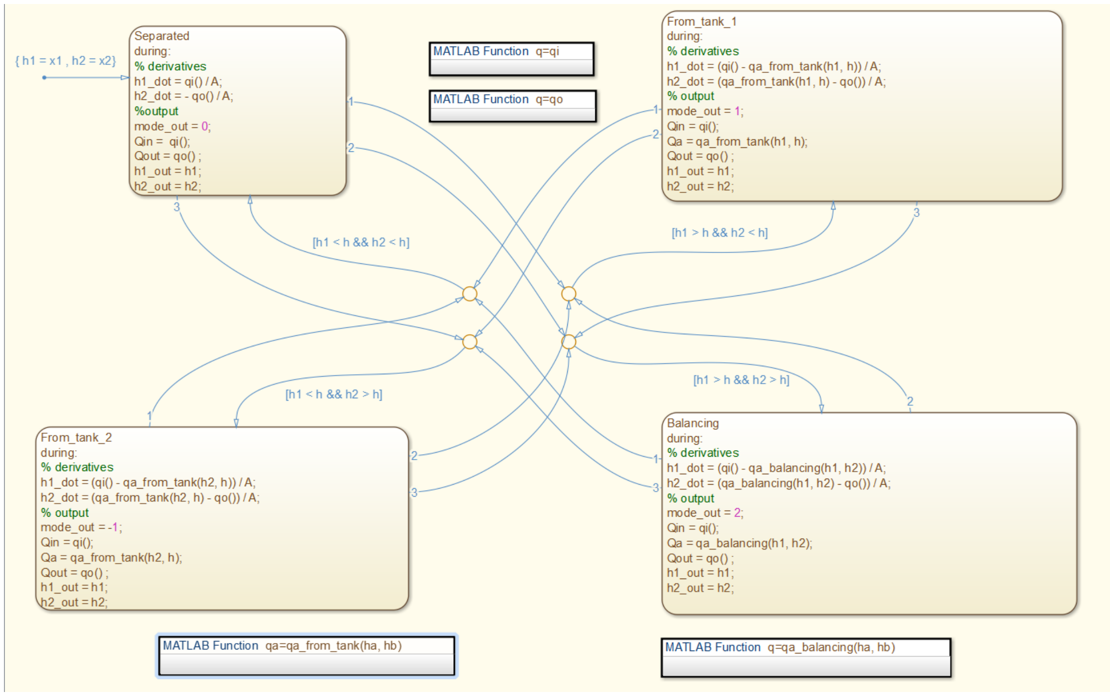
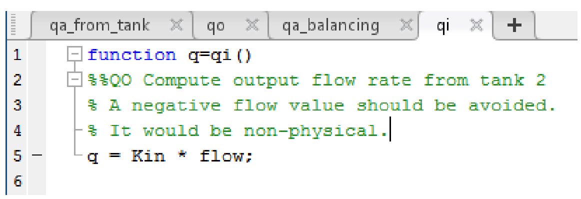
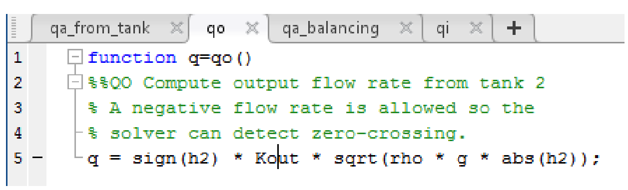


Figure 4: Two-tank StateFlow Model




```
function q=qi()
%%QO Compute output flow rate from tank 2
% A negative flow value should be avoided.
% It would be non-physical.
q = Kin * flow;
```

Figure 5: Input Flow Function

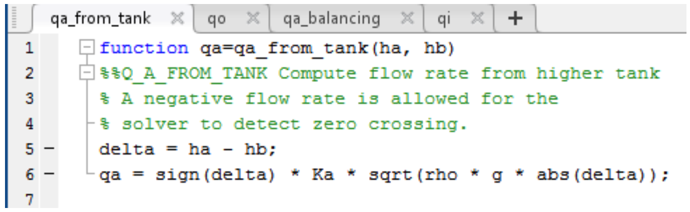



```
function q=qo()
%%QO Compute output flow rate from tank 2
% A negative flow rate is allowed so the
% solver can detect zero-crossing.
q = sign(h2) * Kout * sqrt(rho * g * abs(h2));
```

Figure 6: Output Flow Function

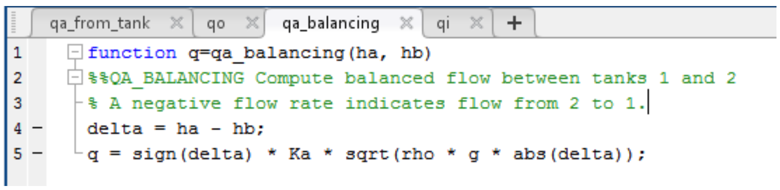
$$qa = K_a * V_a * \sqrt{\rho * g * |delta|} * sign(delta) \tag{74}$$

The model is simulated for several initial conditions $x_0 = [h_1, h_2]^T$. The



```
function qa=qa_from_tank(ha, hb)
%%Q_A_FROM_TANK Compute flow rate from higher tank
% A negative flow rate is allowed for the
% solver to detect zero crossing.
delta = ha - hb;
qa = sign(delta) * Ka * sqrt(rho * g * abs(delta));
```

Figure 7: From Tank Flow Function



```
function q=qa_balancing(ha, hb)
%%QA_BALANCING Compute balanced flow between tanks 1 and 2
% A negative flow rate indicates flow from 2 to 1.
delta = ha - hb;
q = sign(delta) * Ka * sqrt(rho * g * abs(delta));
```

Figure 8: Balancing Flow Function

continuous state Q_a, x, discrete state q, and output Q_{out} of the system are plotted.

3.1 $x_0 = [0.20, 0.75]^T$

For the most part the outcome is unremarkable. Regardless of the initial state the system quickly reaches a quasi-steady-state condition alternating between brief periods where the $h_1 > h \wedge h_2 > h$ immediately before the end of a pulse and the condition where $h_1 > 1 \wedge h_2 < h$. The $Q_in = 0.06$ when the pulse is active. The h_1 level rises during the pulse approaching ≈ 0.65 and between pulses it falls to 0.3 and holds. The Q_a generally follows the h_1 except near the end of the pulse.

There is an initial shift, Figure **??**, of water from Tank 2 back to Tank 1 but this is short lived, resolving itself before the end of the first pulse.

3.2 $x_0 = [0.50, 0.20]^T$

There is a slightly lengthened duration in $h1 > h \wedge h2 > h$, Figure **??** but this is short lived, resolving itself before the end of the first pulse.

3.3 $x_0 = [0.50, 0.50]^T$

There is a immediate and lengthened duration in $h1 > h \wedge h2 > h$, Figure **??** and the initial output flow is increased but this is short lived, resolving itself before the end of the first pulse.

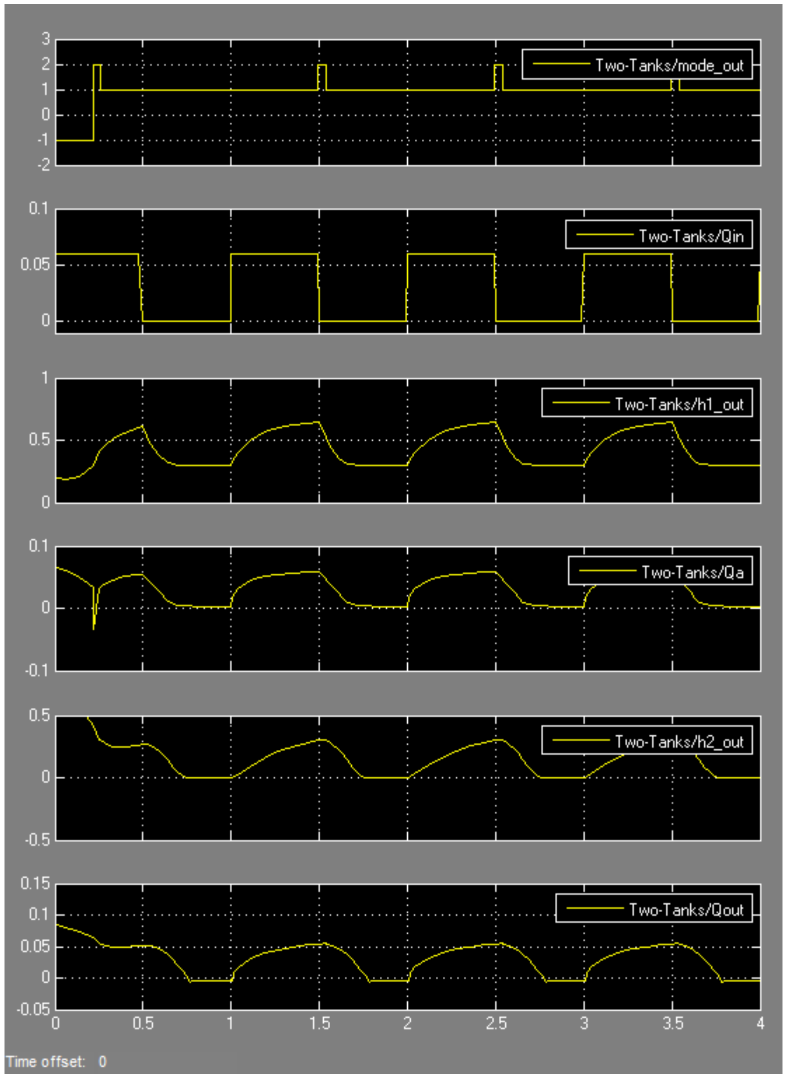


Figure 9: Two-tank SimuLink Model, $x_0 = [0.2, 0.75]^T$

3.4 $x_0 = [0.20, 0.20]^T$

There no time spent in $h1 > h \wedge h2 > h$, Figure **??** and the initial output flow is decreased but this is short lived, resolving itself before the end of the first pulse.

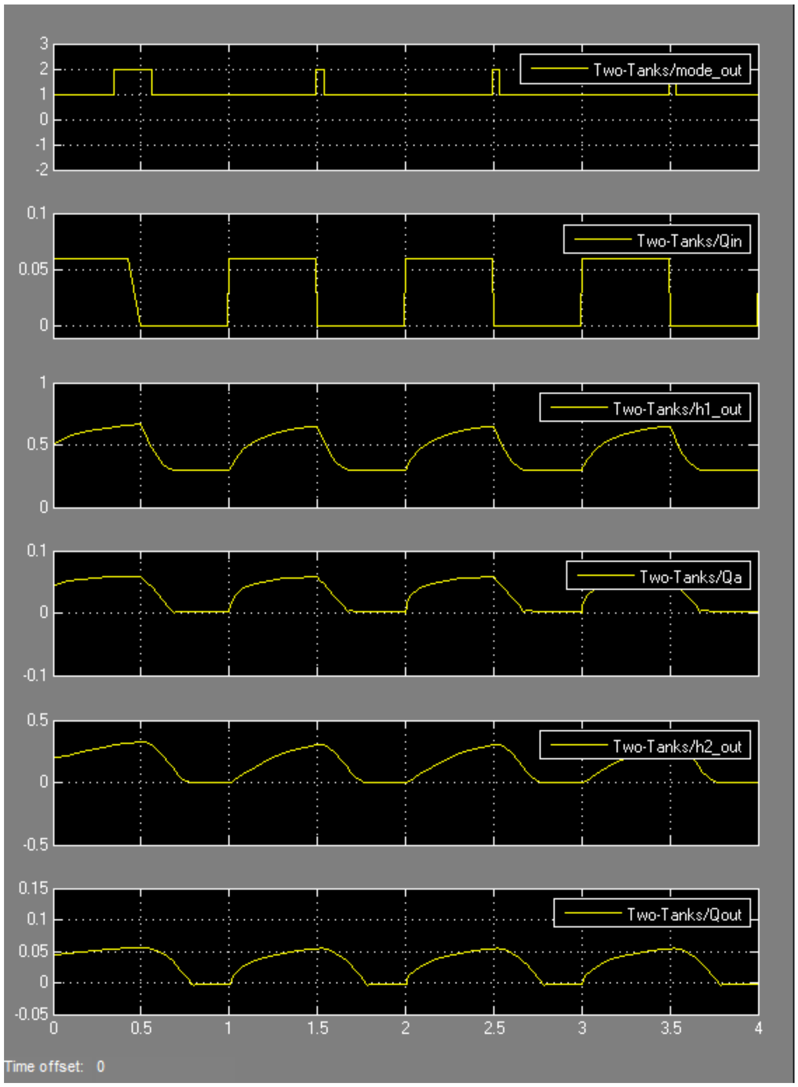


Figure 10: Two-tank SimuLink Model, $x_0 = [0.50, 0.20]^T$

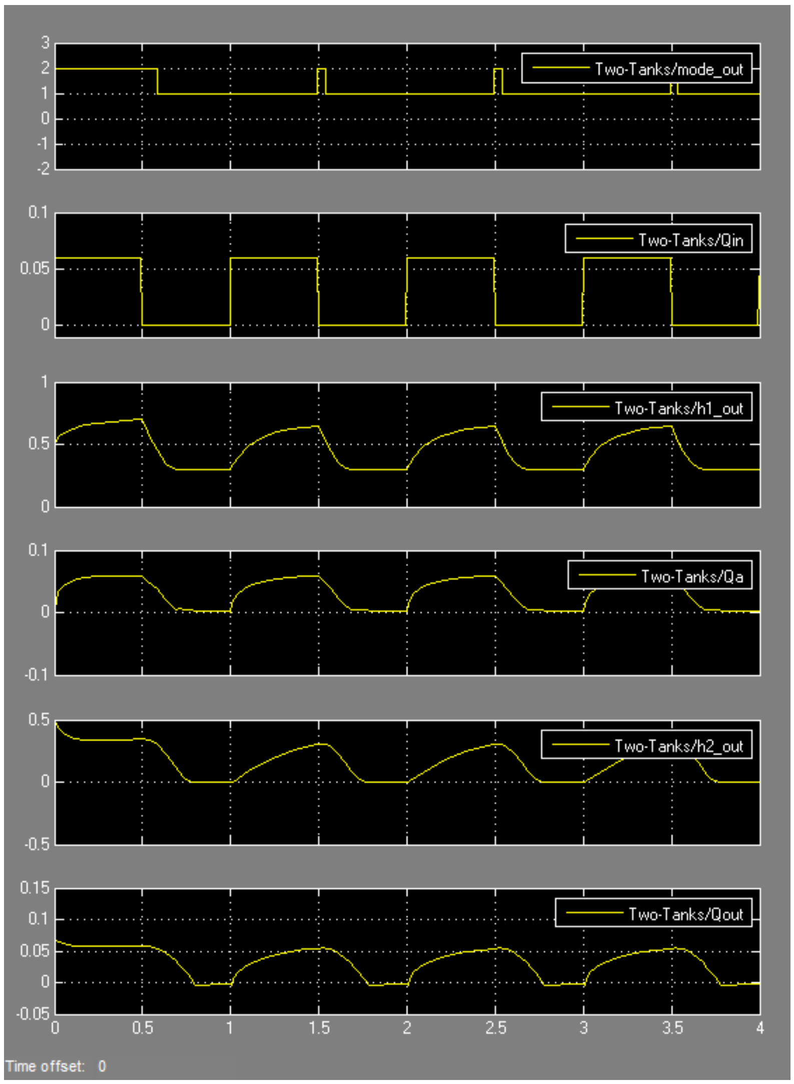


Figure 11: Two-tank SimuLink Model, $x_0 = [0.50, 0.50]^T$

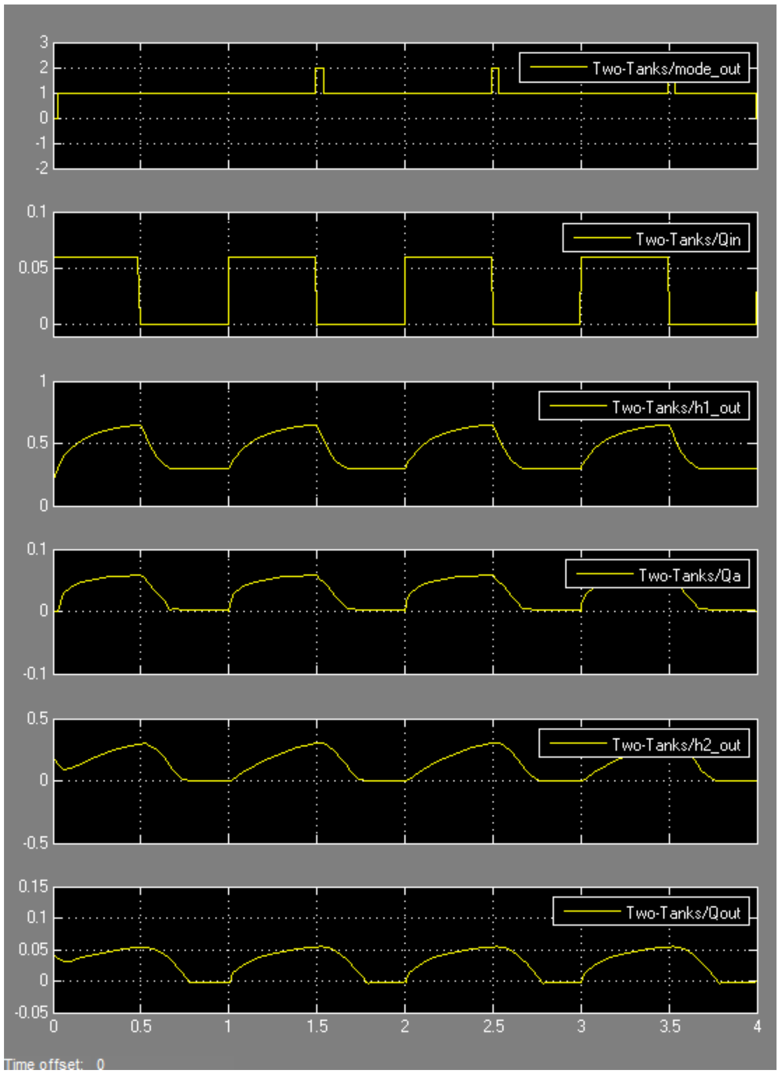


Figure 12: Two-tank SimuLink Model, $x_0 = [0.20, 0.20]^T$