CS 376 Hybrid Systems : HW 8

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1 Hybrid Simulation of a Tank Sytem

Consider the two-tank systems shown in Fig. 1. The system consists of two identical cylindrical tanks, of unlimited height, that are connected by a pipe at level h. We denote by h_1 and h_2 the water levels in tanks 1 and 2 respectively. The input flow Q_{in} is provided by a pump and it is described by

$$Q_{in} = V_{in} k_{in} u(t), \tag{1}$$

where $V_{in} \in \{0, 1\}$ represents a valve that can be used to turn on or off the pump (no partially open valve), k_{in} is a linear gain, and $u(t)$ is the input signal representing the flow at the pump. The flow Q_a between the two tanks is controlled by a valve V_a. An outlet value V_{out} located at the bottom of tank 2 is used to empty the tank. Tank 2 is equipped with a sensor that measures the output flow which is described by

$$Q_{out} = V_{out} k_{out} \sqrt{\rho g h_2} \tag{2}$$

where $V_{out} \in \{0, 1\}$ represents the outlet valve, k_{out} is a linear gain, ρ is the density of the water, and g is the gravitational constant.

The dynamic evolution of the systems is described by

$$\dot{h_1} = \frac{1}{A}(Q_{in} - Q_a) \tag{3}$$

$$\dot{h_2} = \frac{1}{A}(Q_a - Q_a) \tag{4}$$

where A is the section of the identical cylindrical tanks. Following Toricelli's law, the flow Q_a depends on the water levels h_1 and h_2 as follows:

$$Q_a = \begin{cases} 0, & \text{if } h_1 < h \wedge h_2 < h \\ V_a k_a \sqrt{\rho g (h_1 - h)}, & \text{if } h_1 > h \wedge h_2 < h \\ V_a k_a \sqrt{\rho g (h - h_2)}, & \text{if } h_1 < h \wedge h_2 > h \\ sign(h_1 - h_2) V_a k_a \sqrt{\rho g |h_1 - h_2|}, & \text{if } h_1 > h \wedge h_2 > h \end{cases} \tag{5}$$

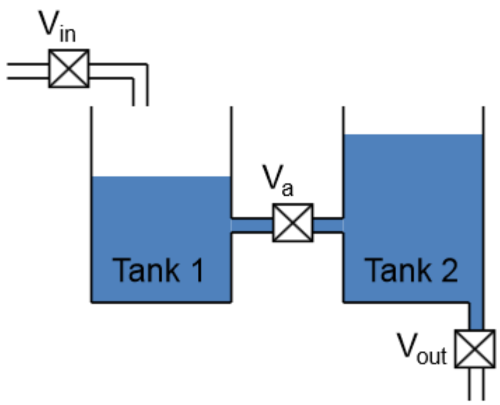

Figure 1: Two-tank system

where $V_a \in \{0, 1\}$ and k_a is a linear gain. The evolution of the continuous state can be described by

$$x = [h_1, h_2]^T \tag{6}$$
$$\dot{x} = f_q(x(t), u(t)) \tag{7}$$

where q is the discrete mode of the sytem. The mode transitions are based on guard conditions indicated in Equation 5.

Assume the following values for the system parameters and initial conditions:

$$V_{in} = V_a = V_{out} = 1, \text{ all valves are open} \tag{8}$$
$$\tag{9}$$

$u(t)$ is a pulse with amplitude 1 m^3 and frequency 1 Hz resulting in an average rate of 1 m^3/sec.

$$h = 0.3 \text{ m} \tag{10}$$
$$k_{in} = 0.06, \tag{11}$$
$$k_a = 0.001, \tag{12}$$
$$k_{out} = 0.001, \tag{13}$$
$$g = 9.81\ m/sec^2\ , \tag{14}$$
$$\rho = 1000\ kg/m^3\ , \tag{15}$$
$$A = 0.0154\ m^2 \tag{16}$$

2 A hybrid automaton model

The formal model for the two-tank system hybrid automaton.

$$H = (q, X, Init, f, Inv, E, G, R) \tag{17}$$

The set of discrete modes.

$$\text{if } h_1 < h \wedge h_2 < h, : separated \tag{18}$$
$$\text{if } h_1 > h \wedge h_2 < h, : from_1 \tag{19}$$
$$\text{if } h_1 < h \wedge h_2 > h, : from_2 \tag{20}$$
$$\text{if } h_1 > h \wedge h_2 > h, : balancing \tag{21}$$

$$q = \{separated, from_1, from_2, balancing\} \tag{22}$$

The set of continuous variables. $X = \mathbb{R}$

$$X = \{u(t), x, Q_a, Q_{out}\} \tag{23}$$

The set of initial conditions. $Init \subseteq Q \times X$

$$Init = \{\} \tag{24}$$

The vector field. $f : Q \times X$

$$f = f_{separated} \cup f_{from_1} \cup f_{from_2} \cup f_{balancing} \tag{25}$$

The invariant set. (all empty) $Q \mapsto 2^X$

$$Inv = Inv_{separated} \cup Inv_{from_1} \cup Inv_{from_2} \cup Inv_{balancing} \tag{26}$$

The invariants are the same for each mode. $Inv_i =$

$$h_1 \geq 0 \tag{27}$$
$$h_2 \geq 0 \tag{28}$$

$f_{separated} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{29}$$
$$Q_a = 0, \tag{30}$$
$$\dot{h_1} = \frac{Q_{in}}{A} \tag{31}$$
$$\dot{h_2} = -\frac{Q_{out}}{A} \tag{32}$$
$$Q_out = k_{out}\sqrt{\rho g h_2} \tag{33}$$

$f_{from_1} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{34}$$
$$Q_a = k_a\sqrt{\rho g(h_1 - h)}, \tag{35}$$
$$\dot{h_1} = \frac{Q_{in} - Q_a}{A} \tag{36}$$
$$\dot{h_2} = \frac{Q_a - Q_{out}}{A} \tag{37}$$
$$Q_out = k_{out}\sqrt{\rho g h_2} \tag{38}$$

$f_{from_2} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{39}$$
$$Q_a = k_a\sqrt{\rho g(h - h_2)}, \tag{40}$$
$$\dot{h_1} = \frac{Q_{in} - Q_a}{A} \tag{41}$$
$$\dot{h_2} = \frac{Q_a - Q_{out}}{A} \tag{42}$$
$$Q_out = k_{out}\sqrt{\rho g h_2} \tag{43}$$

$f_{balancing} =$

$$Q_{in} = k_{in}\ m/sec^2 \tag{44}$$
$$Q_a = sign(h_1 - h_2)k_a\sqrt{\rho g|h_1 - h_2|} \tag{45}$$
$$\dot{h_1} = \frac{Q_{in} - Q_a}{A} \tag{46}$$
$$\dot{h_2} = \frac{Q_a - Q_{out}}{A} \tag{47}$$
$$Q_out = k_{out}\sqrt{\rho g h_2} \tag{48}$$

The collection of discrete transitions. $E \subset Q \times Q$

$$E = \{(separated, from_1) \tag{49}$$
$$(separated, from_2) \tag{50}$$
$$(separated, balanced) \tag{51}$$
$$(from_1, separated) \tag{52}$$
$$(from_1, from_2) \tag{53}$$
$$(from_1, balanced) \tag{54}$$
$$(from_2, from_1) \tag{55}$$
$$(from_2, separated) \tag{56}$$
$$(from_2, balanced) \tag{57}$$
$$(balanced, from_1) \tag{58}$$
$$(balanced, from_2) \tag{59}$$
$$(balanced, separated)\} \tag{60}$$

The guards on the transitions. $G : E \mapsto 2^X$

$$
\begin{aligned}
E = \{&(separated, from_1), h_1 > h \wedge h_2 < h, &(61)\\
&(separated, from_2), h_1 < h \wedge h2 > h, &(62)\\
&(separated, balanced), h_1 > h \wedge h_2 > h, &(63)\\
&(from_1, separated), h_1 < h \wedge h_2 < h, &(64)\\
&(from_1, from_2), h_1 < h \wedge h2 > h, &(65)\\
&(from_1, balanced), h_1 > h \wedge h_2 > h, &(66)\\
&(from_2, from_1), h_1 > h \wedge h_2 < h, &(67)\\
&(from_2, separated), h_1 < h \wedge h_2 < h, &(68)\\
&(from_2, balanced), h_1 > h \wedge h_2 > h, &(69)\\
&(balanced, from_1), h_1 > h \wedge h_2 < h, &(70)\\
&(balanced, from_2), h_1 < h \wedge h2 > h, &(71)\\
&(balanced, separated), h_1 < h \wedge h_2 < h, \} &(72)
\end{aligned}
$$

The reset relation on the transitions. $R : E \times X \mapsto 2^X$

$$R = \emptyset \tag{73}$$

3 Simulink Simulation

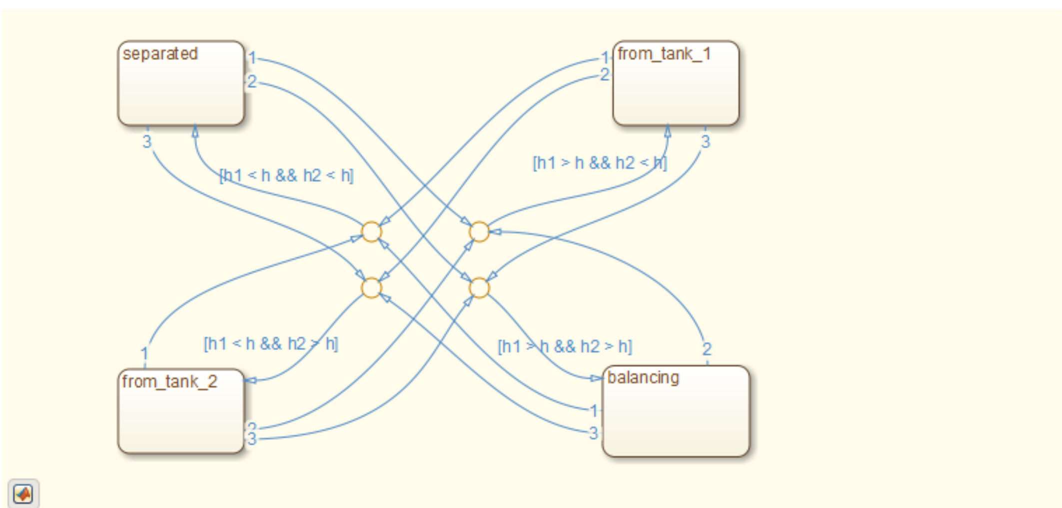

Figure 2: Two-tank SimuLink Model

The model is simulated for several initial conditions $x_0 = [h_1, h_2]^T$. The continuous state Q_a, x, discrete state q, and output Q_{out} of the system are plotted.

3.1 $x_0 = [0.20, 0.75]^T$

hw8_20_75.png

Figure 3: Two-tank SimuLink Model, $x_0 = [0.2, 0.75]^T$

3.2 $x_0 = [0.50, 0.20]^T$

hw8_50_20.png

Figure 4: Two-tank SimuLink Model, $x_0 = [0.50, 0.20]^T$

3.3 $x_0 = [0.50, 0.50]^T$

hw8_50_50.png

Figure 5: Two-tank SimuLink Model, $x_0 = [0.50, 0.50]^T$

3.4 $x_0 = [0.20, 0.20]^T$

hw8_20_20.png

Figure 6: Two-tank SimuLink Model, $x_0 = [0.20, 0.20]^T$