A Hybrid System Model of Seasonal Snowpack Water Balance

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ABSTRACT

It is estimated that seasonal snow cover is the primary source of water supply for over 60 million people in the western United States. Informed decision making, which ensures reliable and equitable distribution of this limited water resource, thus needs to be motivated by an understanding of the physical snowmelt process. We present a direct application of hybrid systems for the modeling of the seasonal snowmelt cycle, and show that through the hybrid systems framework it is possible to significantly reduce the complexity offered by conventional PDE modeling methods. Our approach shows how currently existing heuristics can be embedded into a coherent mathematical framework to allow for powerful analytical techniques while preserving physical intuition about the problem. Snowmelt is modeled as a three state hybrid automaton, representing the sub-freezing, subsaturated, and fully saturated physical states that an actual snowpack experiences. We show that the model accurately reproduces melt patterns, by simulating over actual data sets collected in the Sierra Nevada mountains. We further explore the possibility of merging this model with a currently existing wireless sensing infrastructure to create reliable prediction techniques that will feed into large scale control schemes of dams in mountain areas.

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Categories and Subject Descriptors

C.1.m [**Hybrid systems**]: Miscellaneous

General Terms

Measurement, Management

Keywords

hydrology, snowmelt, hybrid systems, water resources

1. INTRODUCTION

This paper presents a direct application of hybrid systems for the modeling of a natural phenomenon, and shows that through the hybrid systems framework it is possible to model extremely complex system dynamics, such as hydrologic process in mountains basins, as set of much simpler switched dynamics. We derive a mathematical model for the dynamics of snow, and show how hybrid systems can be used to blend heuristics and physical laws within a coherent mathematical framework. Our model is motivated by water supply shortages in the western United States, and provides a powerful new tool to aid in critical water management decisions.

As the following sections will lay out, the physical snowpack is known to be a highly nonlinear system which, to be modeled properly, requires solutions to nonlinear partial differential equations (PDEs). This often leads to comprehensive, but highly parametrized models with many state variables. We show, however, that it is possible to cast the problem in the framework of hybrid systems, allowing us to describe the evolution of the snowmelt process as a decoupled system of three regimes. We will highlight how a hybrid system model can simplify the complex dynamics of offered by similar PDE approaches without sacrificing model accuracy. Casting our model in this framework brings a new sense of intuition to the modeling process, something which is lost in more complex PDE or heuristic approaches. In the past, successful applications of hybrid systems theory have ranged from studies of bio-molecular networks, control of electronic vehicle management systems, and cognitive radio networks [31, 1, 17, 8]. Our model not only brings further mathematical rigor, and new analytical tools to hydrologic sciences, but validates the appropriateness of hybrid systems for the modeling of real-world phenomena. This paper is the first stage of a larger objective to couple snow melt models, and sensing infrastructure with already existing water management tools (e.g., dams, locks, and canals), to optimally distribute and control water throughout the western United States.

2. MOTIVATION

It is estimated that seasonal snow cover is the primary source of water supply for over 60 million people in the western United States, and that melting snow is responsible for 80% or more of soil moisture and streamflow in semi-arid mountain basins in this region [3, 22]. Recent droughts, and the general water crisis in the state of California, are a motivating example of why informed decision making, which ensures reliable and equitable distribution of this limited water resource, needs to be motivated by an understanding of the physical snowmelt process. Factors such as the quantity of available snow, and rate of snowmelt are important indicators to water resource officials who wish to forecast water availability. While the study of hydrologic phenomena is a relatively mature field, the hydrologic processes pertaining to water and energy fluxes in mountainous regions, where basins are largely dominated by snowmelt, are yet not well captured by hydrologic models [3]. It is especially not known to what extent, if any, recent trends in climate change will affect mountain hydrology and, specifically, future water supplies.

Water resource management in snowmelt-dominated regions depends on sparsely distributed snow surveys to estimate available quantities of snow [30]. Methods entail physically sending hydrographers into the mountains every month to measure snow depth, density, and temperature distributions of the snow pack. These measurements are then converted into what is known as the snow-water-equivalent (SWE), an important measure which denotes the height of water that would theoretically result if the entire snowpack melted instantaneously [16]. The importance of these surveys can not be understated, as utility companies, and the agricultural industry, to name a few, depend heavily on SWE estimates for operational purposes. Additionally, these estimates can be used to forecast potential catastrophic flooding induced by rapid melting of snow during warm weather periods, or rain-on-snow events [27].

Hydrologic processes are inherently heterogeneous, guided by varying topographies, soil characteristics, geological inhomogeneities, and vegetative cover [3]. This is especially evident in mountainous regions, where these conditions are coupled with yet not well understood natural feedback cycles that guide water and energy balances. A natural step towards improving the general process-understanding of such systems is the implementation of more sophisticated and distributed sensing techniques. Past sensor deployments, and the previously described snow surveying methods, are limited to few point-scale measurements and, due to high operating costs (both monetary and energy related), representative dense spatial and temporal data are not available. Recent advances in wireless sensing technologies have the potential to allow for the sampling of hydrologic processes at unprecedented resolutions. High sampling frequencies, along with dense distributions of sensing networks, will expose the variability of the parameters guiding hydrologic systems, and will allow for the confident decoupling of underlying natural processes. The potential of these low-power wireless networks to span large areas also permits data to be available in real time, improving the ability of planners to make informed decisions. Recent deployments of such technologies in the Sierra Nevada mountain range are currently part of the effort conducted by the Critical Zone Observatories [2, 21]. Data from this observatory, collected at spatial and temporal resolutions much finer than previous campaigns, allow us to develop robust estimates of environmental conditions through our models. We will focus on deriving a mathematical model for snowmelt, and will highlight those process that guide water-, and heat-transfer through the snowpack. The main aim of the model is to reliably forecast weekly snow water equivalent, given real-time data acquired by wireless sensor networks. Special attention will be paid to construct a model that relies only on parameters that can readily be sensed in real-time by currently available technology.

3. THE PHYSICS OF SNOW

3.1 Physical Behavior

The physical snowpack is an inherently unstable system, continuously subject to fluxes in mass, energy, and momentum. Mass and energy are introduced into the system in the form of water, in either a a solid (snow), liquid (rain), or vapor form. Additionally, energy is exchanged with the snowpack through evaporation, soil heat flux, as well as solar and thermal radiation. As energy is introduced onto the snowpack, the surface melts, and the subsequently produced liquid water, compelled by gravitational forces, begins to flow through the snow matrix. Depending on the temperature and physical characteristics of the snow below the melt surface, this water may either refreeze onto the snow grains below, or flow into or onto the underlying soil surface. As water melts on the surface and moves into the underlying snow matrix, the latent heat generated by the refreezing of the water is used to heat the snow below the melt surface. It is thus possible to classify the bulk equations governing snowmelt using standard conservation laws, along with constitutive relations for water flux through porous media [7, 36].

At a basic level, snowmelt is thus governed by a coupled interaction between energy and mass balance components:

$$\frac{dE}{dt} = f(M(t), E(t)) \tag{1}$$

$$\frac{dM}{dt} = g(M(t), E(t))) \tag{2}$$

where M and E reflect specific mass[1] and energy com-

[1]Mass per unit area. It should be noted that from here on, any references or calculations involving mass are to be taken as references to specific mass. This will allow us to model snowpack with regard to its evolution over an arbitrarily sized area.

ponents. It is also customary to introduce a momentum conservation relation, which tracks the movement of water and deformation of the snow matrix. Since our main goal is to model snow water equivalent, rather than track front propagation within the snow matrix, we take the approach of Marks et al. [26], and model compaction in a prescribed, rather than explicit, manner. An energy balance governs the flux of energy into the system, usually in the form of solar and thermal radiation, latent heat transfer, and soil and evaporative fluxes. General conduction theory along with the heat equation can be used to model the temperature evolution through the snowpack. Realistically, temperature distribution in the snowpack is non-uniform, but due to the inability to sense these distributions, most authors assume an isothermal snowpack (e.g., [10, 14]).

Water flow through snow is a non-linear process, complicated not only by microscopic characteristics of the snow pack, but also by the tendency of the water to refreeze as it flows through the snow matrix. Water flow through porous media, described in detail by [7], can be broadly classified into two regimes: saturated flow, and unsaturated flow. Water flow near the surface of a melting snowpack, where a layer of wet snow exists, is governed by Darcy's law [7]. The flow is proportional to the gravitational pressure gradient, as well as microscopic properties of the snow grains. Once water leaves this region, flow in the dry region can be modeled by the nonlinear Richard's PDE [7], which is highly dependent on microscopic properties, and for which a closed form solution does not exist. It can thus be seen how the coupling of all these interactions has the potential to make the proper modeling of snowpack a non-trivial process. Prior to engaging the inner workings our proposed model, it will be advantageous to discuss previous efforts from the snowmelt modeling literature.

3.2 Previous Work

A number of authors have explored the empirical and mathematical modeling of snowmelt processes. A series of notable works was published by Colbeck, in which the author explored both the thermodynamic phenomena guiding snowmelt, as well as water flow through the snow matrix [9, 14, 13, 11, 12]. In these papers, snow was treated as a porous medium consisting of ice, water and air. A mass balance approach along with a model for water flow through porous media, was used to arrive at a PDE that guides the evolution of the pack, and whose solution depended upon a finite difference approach. Various other authors extended this work by realizing that snow is a medium in which the ice matrix and the water phases are, depending in the energy input, interchangeable. Sellers showed, by building upon [20, 29], that snow can be modeled as a phase-changing porous medium, where the water resulting from melt, is interconvertible with the rigid ice matrix [32, 33]. Additionally, the author developed a method by which to track the propagation of the melting snow surface and the percolating interior boundary, by noting that the problem can be formulated as a generalized Stefan condition (or, a free moving boundary) [15]. Tseng et al., guided by a model for water flow through subfreezing snow by Illangasekare [18], arrived at a similar solution, but formulated the problem using volumetric supply terms [35]. More recent work on the subject was conducted in a comprehensive three part paper by Bartelt and Lehing, in which the authors modeled all the phenomena described in the previous section by making simplifying assumptions about snow micro-structure. The complexity of their extensive model, however, warranted a solution only by a finite element approach [4, 5, 6]. Currently, two popular point based modeling packages exist: SNOBAL, and SNTHERM [23, 19]. The former is a two layer heuristic model that tracks mass and energy fluxes well, while the latter is a highly parametrized 20 state PDE model that tracks the compaction and formation of layers within the snow-matrix in detail. Our approach, however, offers an elegant hybrid solution, preserving the motivating heuristics, while retaining mathematical rigor.

4. MODELING THE SNOWPACK

The complexity of the previously described PDE approaches has the tendency to remove general intuition regarding the physics of the snowmelt process. The procedures usually entail lengthy mathematical derivations along with a high degree of parameters, arriving at a solution that is difficult to intuitively comprehend. Additionally, many of the inputs into these models are not readily available (e.g. snow grain diameter), and must be estimated. Our proposed model addresses these concerns by splitting the snowmelt process into stages, each of which is governed by relatively simple dynamics. The evolution of the entire pack in time can then be modeled as a switched system between these stages. A number of reasonable assumptions will justify simplified dynamics in each of the three proposed regimes. The snowpack is at all times assumed to have a uniform bulk temperature T (see section 4.1 for details). Furthermore, we assume that the snowpack can be modeled as a single-layer homogeneous mixture of ice, water and air with bulk density ρ_S. Most authors have taken a multilayer approach to model melt over long periods of time [19, 23], but since this model aims to forecast weekly SWE values rather than track long-term structural metamorphosis, a single layer was deemed to be adequate. Model validation will show this assumption to be appropriate. The following sections will decouple the snowmelt process into three regimes, and will derive the necessary equations to describe the dynamics of the snowpack in each regime.

4.1 The Sub-freezing Snowpack

During this stage no liquid water is present within the snow matrix. The snowpack is completely frozen and the volume occupied by the system is strictly filled with ice and air. We denote E as the energy introduced into the system. A positive energy input will heat the surface of the pack until it reaches a temperature of 0^oC, upon which the ice at the surface will begin to melt and percolate into the snow matrix. The water will then refreeze, and the latent heat released by refreezing will in turn raise the temperature of the surrounding snow. This process steadily progresses until the water front eventually reaches the bottom of the snow pack. At this point, the entire snowpack is assumed to be at a bulk temperature $T = 0^oC$. Since the primary purpose of our analysis is to model the water content of the snowpack, it is not necessary to track the evolution of this interior saturation boundary, but rather to identify the point at which the snowpack reaches a bulk temperature of $T = 0^oC$. Following the approach of [18] and using general conduction theory [36], it is possible to identify the point in time at which the snowpack will reach an isothermal state at $T = 0^oC$. The relation giving the change in temperature of the snowpack during this regime can be given by

$$\frac{dE}{dt} = M_s C_s \frac{dT}{dt} \tag{3}$$

where E is the energy input into the system, M_s is the mass of snow, and C_s is the specific heat of snow. We take a proscribed approach to snow pack metamorphosis, merging approaches of [23, 28] and note that during this stage of snowmelt, change in snowpack height is governed by an exponential decay model. Analysis of field data shows this to be a reasonable assumption.

The change in density of the snowpack is given by:

$$\frac{d\rho_s}{dt} = \frac{A}{B\left(1 + \frac{\rho_s(t)}{A-\rho_s(t)}\right)^2} \tag{4}$$

where ρ_s is the bulk density of the snowpack, and A and B are easily obtained model parameters. We derive (4) in Appendix 4.1. When the bulk temperature of the snowpack reaches $T = 0^oC$, the system jumps into the second regime of snowmelt described next.

4.2 The isothermal Sub-saturated Snowpack

At this stage the snowpack has reached an isothermal state, where the bulk temperature is 0^oC throughout. It is physically impossible for the bulk temperature to climb beyond this value. Therefore, any energy input into the system at this point is used to convert (melt) snow at 0^oC to water at 0^oC. Two more equations can now be added to reflect that ice is being converted to water at a rate proportional to the energy input:

$$L_f \frac{dM_i}{dt} = -\frac{dE}{dt} \tag{5}$$

$$L_f \frac{dM_w}{dt} = \frac{dE}{dt} \tag{6}$$

where M_w and M_i are the mass constituents of ice and water of the pack, and the mass of snow is given by $M_s = M_i + M_w$. L_f is the latent heat of fusion of ice. Water is still not exiting the snowpack because it is held in place by capillary action within the ice matrix [4]. The volumetric water content of snow θ_w, the volume of water over the total volume of snow, is given by

$$\theta_w = \frac{V_w}{V_{total}} = \frac{M_w/\rho_w}{M_s/\rho_s} \tag{7}$$

where ρ_w and ρ_s are the density of water, and snow, respectively. θ_w has to climb above the threshold value θ_r (typically 1%) for water to begin flowing out of the snowpack [24]. Thus, in our model the snow pack begins to drain once the volumetric water content condition has been satisfied, or

$$\theta_w \geq \theta_r \tag{8}$$

or, equivalently, when

$$\frac{M_w/\rho_w}{(M_i + M_w)/\rho_s} \geq \theta_r. \tag{9}$$

4.3 The Isothermal Saturated Snowpack

At this stage, the volumetric water content has reached the critical threshold value, and water can begin to flow out of the snowpack. Similarly, if the volumetric water content drops below θ_r the system relapses back into the second regime. Snowpack density and change in the ice mass are still evolving according to the relations in the previous regime. The amount of water that can now leave the system is equal to the amount of ice being melted along with the amount being discharged due to any excess volumetric water content. The relation giving the change in water content of the snowpack is given by

$$\begin{aligned} &(\rho_s - \rho_w)^2 \frac{dM_w}{dt} \\ &= -\theta_r \rho_w \left(\frac{1}{L_f} \frac{dE}{dt} (\rho_s - \theta_r \rho_w) + M_i \frac{d\rho_s}{dt} \right). \end{aligned} \tag{10}$$

We show in Appendix 4.3 that eq. (10) can easily be derived using the volumetric water requirement of regime three. Intuitively, the above relation accounts for the fact that as density of the snowpack is changing, the volumetric water content changes accordingly, and water must leave the system to ensure $\theta_w = \theta_r$.

5. HYBRID SYSTEMS MODELING

The previous section showed that is it possible to separate the natural process of snowmelt into three regimes, each reflecting physical states of the snowpack. As such, a hybrid systems model of snowmelt is an appropriate means by which to formalize the interaction between the above dynamics. Formulating the problem in this framework will allow for a model whose solution is dependent on a series of switched ordinary differential equations (ODEs). Our hybrid model captures the behaviors of more complex models, while ensuring that a sense of intuition regarding the snowmelt process is preserved. More information on hybrid systems can be found in [34].

5.1 Preliminaries

We base our definition of hybrid systems on a class of hybrid systems given in [34].

Definition 1. An autonomous hybrid automaton H is a collection

$$H = (Q, X, Init, U, f, Dom, R, Y)$$

where
$Q = \{q_1, q_2, ...\}$ is a set of discrete states;
$X = \mathbb{R}^n$ is the continuous state space;
$Init \subseteq Q \times X$ is a set of initial states;
U is a set of continuous input variables;
$f : Q \times X \times U \to \mathbb{R}^n$ is a vector field;
$Dom : Q \to 2^{X \times U}$ is a domain;
$R : Q \times X \times U \to 2^{Q \times X}$ is a reset relation;
Y is a collection of continuous output variables.

Definition 2. A hybrid time set is a finite or infinite sequence of intervals $\tau = \{I_i\}_{i=0}^N$ such that
$I_i = [\tau_i, \tau_i']$ for all $i < N$; and $\tau_i \leq \tau_i' = \tau_{i+1} \forall i$.
if $N < \infty$ then either $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$;

Definition 3. A hybrid trajectory (τ, q, x) consists of a hybrid time set $\tau = \{I_i\}_0^N$ and two sequences of functions $q = \{q_i(\cdot)\}_0^N$ and $x = \{x_i(\cdot)\}_0^N$ with $q_i(\cdot) : I_i \to Q$ and $x(\cdot)I_i \to \mathbb{R}^n$.

Definition 4. An execution of a hybrid automaton H is a hybrid trajectory, (τ, q, x) which satisfies:
Initial Condition: $(q_0, x_0) \in Init$;
Discrete Evolution: $(q_{i+1}(\tau_{i+1}), x_{i+1}(\tau_{i+1}) \in R(q_i(\tau_i'), x(\tau_i'))$;
Continuous evolution: for all i,
1. $q_i(\cdot) : I_i \to Q$ is constant over $t \in I_i$, that is, $q_i(t) = q_i(\tau_i)$ $\forall t \in I_i$;
2. $x_i(\cdot) : I_i \to X$ is the solution to the differential equation $\dot{x}_i = f(q_i(t), x_i(t), u(t))$ over I_i, starting at $x_i(\tau_i)$; and,
3. $\forall t \in [\tau_i, \tau_i')$, $x_i(t) \in Dom(q_i(t))$.

5.2 A Hybrid System Model of Snow

Using the above definition, it is now possible to model the physical snowmelt process as a hybrid system:

- $Q = \{q_1, q_2, q_3\}$ is the set of discrete states, corresponding respectively to each regime of snowmelt.

- $X = \mathbb{R}^4$ is the set of continuous states, where for $x \in X$ we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} M_i \\ M_w \\ \rho_s \\ T \end{bmatrix} \tag{11}$$

  where x_1 and x_2 are the mass of ice and water in the snowpack, x_3 is the density of the snowpack, and x_4 is the bulk temperature of the snowpack.

- The set of initial states $Init = Q \times \{x \in X : x_1 \geq 0 \wedge 0 \leq x_2 \leq \theta_r \rho_w \frac{x_1+x_2}{x_3} \wedge 0 \leq x_3 \leq A \wedge x_4 \leq 0\}$ where A is the same as in (4).

- The domains are given by $Dom(q_1) = \{x \in \mathbb{R}^4 | x_4 < 0\}$, $Dom(q_2) = \{x \in \mathbb{R}^4 | x_4 \geq 0 \wedge 0 < x_2 < \theta_r \rho_w \frac{x_1+x_2}{x_3}\}$, $Dom(q_3) = \{x \in \mathbb{R}^4 | x_2 \geq \theta_r \rho_w \frac{x_1+x_2}{x_3}\}$

- The system dynamics for each state are given by

$$f_1(t, x(t), u(t)) = \begin{bmatrix} 0 \\ 0 \\ h(x_3(t)) \\ \frac{1}{C_s} \frac{u(t)}{x_1(t)} \end{bmatrix} \tag{12}$$

$$f_2(t, x(t), u(t)) = \begin{bmatrix} -\frac{1}{L_f} u(t) \\ \frac{1}{L_f} u(t) \\ h(x_3(t)) \\ 0 \end{bmatrix} \tag{13}$$

$$f_3(t, x(t), u(t)) = \begin{bmatrix} -\frac{1}{L_f} u(t) \\ g(x_1(t), x_3(t), u(t)) \\ h(x_3(t)) \\ 0 \end{bmatrix} \tag{14}$$

  where

$$g(v, w, z) = \frac{-\theta_r \rho_w \left(\frac{1}{L_f} z \left(w - \theta_r \rho_w\right) + v h(w) \right)}{\left(w - \rho_w\right)^2} \tag{15}$$

  and

$$h(z) = \frac{A}{B\left(1 + \frac{z}{A - z}\right)^2} \tag{16}$$

  and all other terms are constants as defined previously. The continuous input $u(t) \in U \in \mathbb{R}$ represents the amount of energy input into the system (given in W/m^2). The continuous output of the hybrid automaton is, at all times, given by $y(t) = (x_1(t) + x_2(t))/\rho_w$, reflecting the SWE of the snowpack.

- $R(q_1, x) = (q_2, x)$ if $x_4 \geq 0$, $R(q_2, x) = (q_1, x)$ if $x_2 \leq 0$, $R(q_2, x) = (q_3, x)$ if $\frac{x_2/\rho_w}{(x_1+x_2)/x_3} \geq \theta_r$, and $R(q_3, x) = (q_2, x)$ if $\frac{x_2/\rho_w}{(x_1+x_2)/x_3} < \theta_r$ and $R(q, x) = \emptyset$ otherwise.

A graphical representation of the hybrid automaton is given in Figure 1.

5.3 Discrete Mode Dynamics

Due to the relatively straightforward nature of the discrete mode dynamics, a detailed analysis of the system would not offer more information than is provided by a qualitative overview:
State q_1: The steady state behavior of the hybrid automaton during this discrete mode reflects only densification of the snow pack, as an energy input is required for the temperature to be affected. During this state, the density of the snowpack, x_3, is evolving towards a terminal density value A, given in (4). Introduction of an energy flux $|u(t)| > 0$ evolves the temperature of the snowpack proportional to the sign and magnitude of the energy input.
State q_2: The evolution of x_1 and x_2 is equal in magnitude, and opposite in sign, and is proportional to the energy input into the system. The continuous state x_4, the temperature of the snowpack, does not evolve and remains at a steady $T = 0^o C$, since ice and snow can not exist in solid form beyond this region.
State q_3: With a steady positive energy flux, the snowpack will eventually melt and the dynamics of state q_3 will ensure that the mass of ice and water go to zero. In q_3, the steady state behavior does affect mass balance, since densification of the snowpack x_3, changes the system's volumetric water content. As the snowpack density approaches the terminal value A, the steady state contribution of the x_3 dynamics to overall mass balance becomes negligible. Given a positive energy influx, the dynamics of x_2 evolve to ensure that the volumetric water content of the snowpack θ_w is at all times equal to the threshold value θ_r.

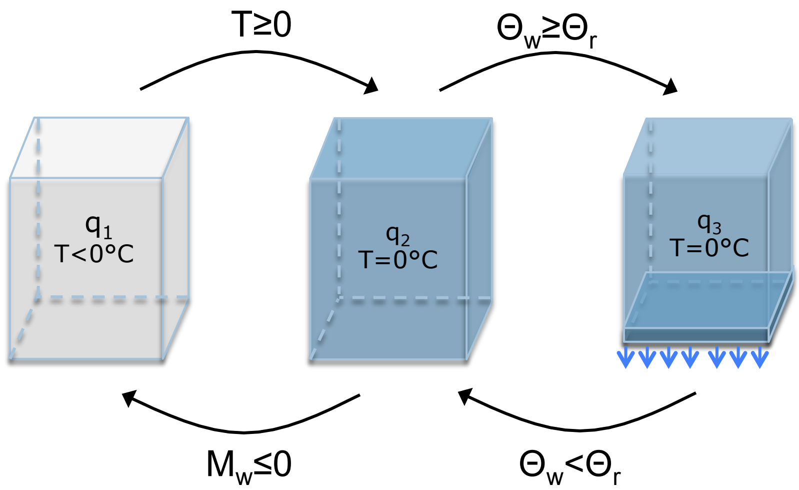


Figure 1: A visual representation of the snowmelt hybrid automaton. State q_1 is the first stage of snowmelt, where the entire snowpack exists at below freezing temperatures, and no liquid water is present. State q_2 represents the isothermal snowpack at $T = 0^oC$. Ice begins to melt, but water can not leave the snow matrix because it is held in place by capillary forces. Once the water content θ_w of the snowpack reaches a value greater than or equal to a threshold θ_r, water begins to exit the system and flow into the soil below. A negative energy input in q_2 causes existing water to refreeze. If all the water in state q_2 freezes, the system relapses into the sub-freezing state q_1.

6. MODEL VALIDATION

A set of real-world input and validation data was obtained from a research initiative at the Kings River Experimental Watershed (KREW) in the Southern Sierra Nevada, California. Two detailed data sets were analyzed, each collected at two different locations over the 2004 and 2006 winter seasons. The sites were located at an approximate elevation of 2100 meters above sea level. Pre-formatted hourly records of temperature, relative humidity, snow depth, SWE, wind direction, wind speed, precipitation, and solar radiation were available. In this data set, sensor readings were taken at fifteen-minute intervals, and combined into an hourly average, a practice commonly undertaken in hydrologic sciences. An ultrasonic depth sensor was deployed in both studies to measure snow depth. This device measures the time of flight of an acoustic ping and calculates the depth to the snow surface, adjusting automatically for the speed of sound as a function of air temperature. A low-pass filter was also applied to the raw sensor readings to remove commonly experienced high-frequency sensor noise. The sites experienced significant snow accumulation over the winter period. We regard these data to be of extremely good quality due to their temporal and spatial density.

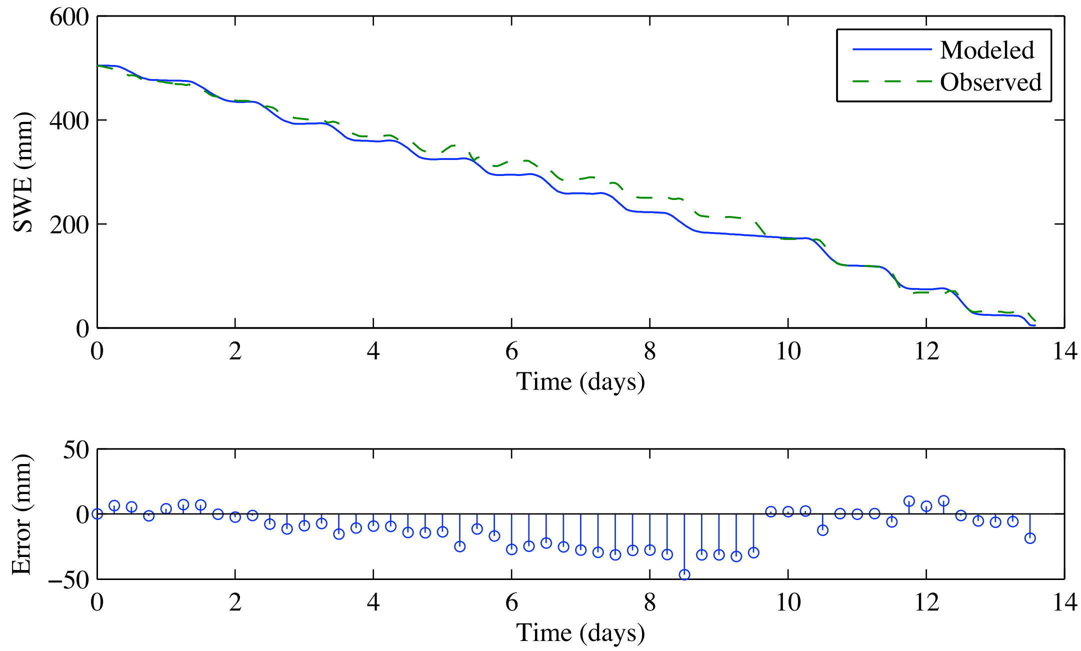
Energy input into the snowpack was calculated at every time step using methods developed by [27, 25, 24, 23]. The method takes a set of the measured hydrologic variables as an input, and outputs an energy flux into the snowpack (in W/m^2), which we subsequently employ as the input $u(t)$ to our hybrid model. The resulting energy input exhibits diurnal features. Peak energy input into the snowpack occurred during daytime, when solar radiation was highest, while nighttime energy contributions remained relatively low. As such, our system is deterministic, since all the energy inputs $u(t)$ are determined prior to system execution. It should be noted that this method requires soil temperature as a parameter to estimate net conductive energy fluxes on the snowpack. Such measurements were not available, but [14] points out that contribution to melt due to soil heat flux can be considered negligible compared to net radiative, latent, and advective energy inputs.

6.1 Initial Conditions and Parametrization

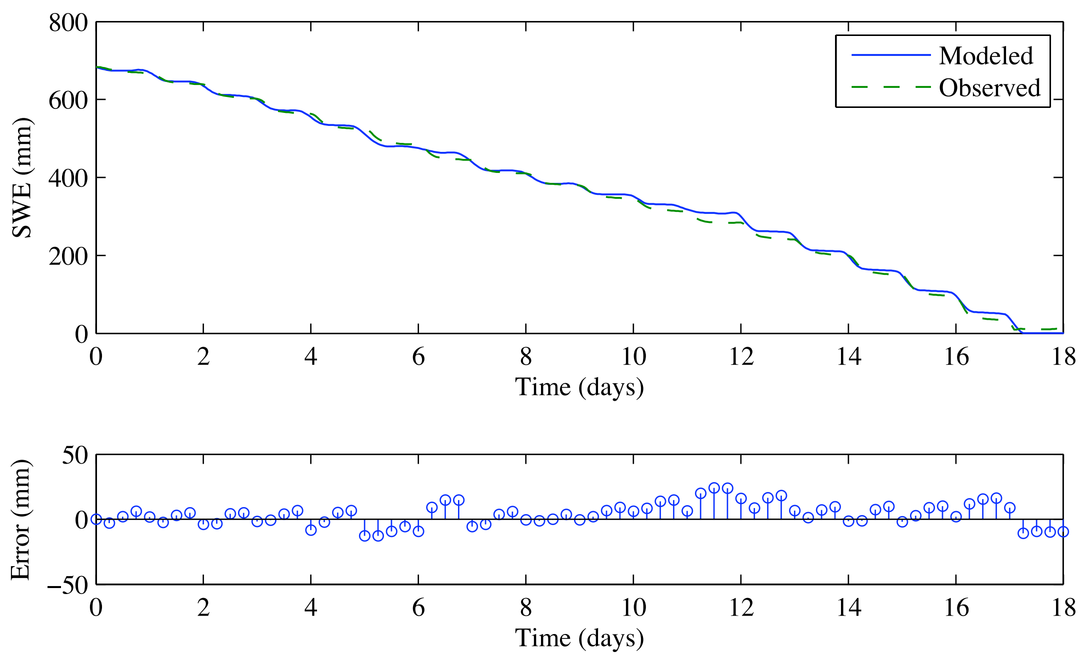
Values, such as the amount of liquid water, and the temperature of the snowpack, are difficult to measure in the field. Since these values are embedded in our state space, it becomes important to develop a proper approach to confidently estimate their initial conditions. The hybrid automaton was initialized with night-time values following a fresh snow-fall event. This maximized the likelihood of capturing the system during state q_1, as we assume the snowpack temperatures to be below freezing during such events. We believe that this method allows us to estimate initial states confidently, by assuming that at this stage, the temperature of the snowpack is equal to the air temperature, and the mass of liquid water in the system is zero. The density was initialized using physical measurements, and the mass of ice was calculated using density and snow height measurements. The free parameters, or physical constants, of the model are given in Appendix 6.1, and are based on commonly available physical constants, as well as known properties of snow in the Sierra Nevada. The model shows no significant sensitivity to initial conditions and parameters when they are constrained to physically realistic values.

6.2 Validation Results and Analysis

By inspection of figure 2, very close correspondence between modeled and observed SWE values becomes evident. Maximum error for a two week prediction of SWE, for both years, did not exceed 50mm of SWE. A measurement error in the range of 20mm is known to exist for the snow-depth sensor that was deployed in the study. Given the week long time horizon requirement imposed on this model, we regard these results to be indicative of very good model perfor-



(a) Observed SWE vs. Modeled SWE for a site in the Sierra Nevada during the 2004 snow season.



(b) Observed SWE vs. Modeled SWE for a different site in the Sierra Nevada during the 2006 snow season.

Figure 2: Plots of Snow Water Equivalent (SWE) at two different sites during 2004 and 2006. Snow water equivalent is the height of water that would result if the whole snowpack were to melt instantaneously. Step-like features in the time series are indications of discrete mode switches of the hybrid automaton.

mance. The accuracy of the model is further validated by the fact that it is able to correctly identify values and trends in SWE for two different sites, and snow seasons. The point at which complete melt occurs is identified correctly as well. Overall, the model performs extremely well within the desired seven day prediction window, and is able to capture high-frequency daily variations in SWE. Figure 2a shows a slight model undershoot at the five day mark. At this time the observed SWE value displays an actual increase, indicating a snow fall event has occurred. This is an acceptable behavior, since our main aim is to forecast SWE given only current snowpack conditions. It is assumed that during the operational phase, the model will be used as a moving window to incorporate changes in incoming meteorological forecasts. The step like response of the SWE plot in figure 2, indicates switching of the discrete system dynamics, and further intuitively validates the switched dynamics approach of our model.

6.3 Comparison to Existing Simulators

To fully identify the advantages of the proposed approach, a rigorous quantitative analysis and comparison with existing simulators is warranted. Given the currently available validation data, as well as the high dimensionality and parameterization requirements of both the SNOBAL [23] and SNTHERM [19] models, such an analysis was deemed beyond the scope of this particular paper. A series of test runs using SNOBAL found the model predictions of SWE to be inconsistent with the ground truth data. SNOBAL performed along the same error margin as the hybrid automaton simulation for the first four days of the melt cycle, only to significantly diverge through the remainder of the simulation period, predicting total melt with an error of seven days. This behavior is largely attributed to the model's apparent sensitivity to parameterization and snow layer initialization. It should also be noted that the source code of the SNOBAL model had to be modified to allow for inputs of solar radiation that were physically measured, but deemed unreasonable in the original model design. The SNTHERM model is, to our knowledge, one of the most comprehensive snowmelt simulator available. As such, it requires more inputs and parameters that were available to this study. Assumptions regarding these parameters could not be justified given the available data sets. A data set currently being collected in the Sierra Nevada, however, will permit for a more comprehensive error analysis of the proposed hybrid system model, along with providing confident estimates of model parameters for SNTHERM and SNOBAL.

Compared to existing simulators, computational tractability along with model analyzability stand out as the two main benefits gained by casting the snowmelt process in a hybrid systems framework. As mentioned previously, the primary aim of the proposed model is to reliably forecast weekly SWE given real-time data acquired by densely deployed wireless sensor networks. Spatial variation in snow cover will be accounted for by the distribution of network nodes, and separate snowmelt simulations will need to be carried out for each measurement location. The relatively simple structure of the proposed model allows it to run extremely fast on a standard personal computer, and offers promise in terms of scalability. Compared to existing simulators, the proposed model also lends itself to a number of potential hybrid systems analysis techniques. The model originated with the aspiration to control dams in the California mountains based on the input of real-time meteorological and snow cover data. Casting the snowmelt process in hybrid systems framework will allow for investigation of controller synthesis methods for such purposes. Additionally, hybrid system estimation, and reachability analyses can be carried out to provide water management officials with valuable information regarding water storage and snowmelt runoff.

7. CONCLUSIONS AND FUTURE WORK

In this paper, we derived a hybrid system model of the seasonal snowmelt water balance. This model is part of a larger project scope to control water management infrastructure and forecast available water supplies in the state of California. The main contribution of our paper hinges not only on its performance, but rather also on showing that it is possible to accurately model an extremely complex natural phenomenon using a system of simple switched dynamics. Compared to other approaches, our model significantly reduces the number of required state variables while delivering a high level of intuition to the modeling process. The approach also makes it possible to incorporate well known heuristics into a coherent mathematical framework. Dimensionality reduction is one of the main appealing features of the approach, when compared to traditional PDE models. It is believed that the dimensionality of the system could be further reduced in the future by taking into account that in each discrete mode, certain continuous states experience zero change. As (12), (13) and (14) show, the system could be modeled as a hybrid automaton with switching between state spaces of different dimensions. Future work will explore the choice of a new coordinate system which takes further advantage of known physical behaviors of snowmelt.

From a hydrologic perspective, future work will focus on investigating to which extent the model can accurately predict spatial distribution of SWE. Our current analysis was limited to point-wise forecasting of a monotonically decreasing SWE based on energy input measured at the exact location of the SWE estimate. To broaden the impact to flood and agricultural planning, our approach will be expanded to treat the spatial variation of the snowpack throughout a drainage basin. This approach will entail the interaction of multiple automata, each governing snowmelt in a specific region of the basin. Minor modifications to the current snowmelt automaton will account for thermodynamic interaction between the automata, and will permit water to flow into soil as well as into neighboring automata. New data sets are currently being collected, which will be used to validate the model over long-term freeze-thaw snowmelt cycles. The variability of energy fluxes on the model area, as well as appropriate terrain gridding techniques will be investigated. This model was designed to utilize real-time data provided by wireless sensor networks, which are currently being deployed. Estimation techniques will also be developed to ensure that the incoming input data is a valid representation of actual environmental conditions.

8. ACKNOWLEDGMENTS

This research is supported by an NSF graduate fellowship, and NSF EAR 0725097, Roger Bales, P.I., CZO: Critical Zone Observatory - Snowline Processes in the Southern

Sierra Nevada. The authors would also like to thank Danny Marks and Samuel Burden for their input.

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APPENDIX

A 4.1 Derivation of Bulk Density Evolution Equation

We are given the empirical observation [23] that the density of aging snow can be determined through the half saturation equation:

$$\rho_s(t) = \frac{A}{1 + B/t} \tag{17}$$

where A is the maximum saturation level (in kg/m^3) and B is the time at which half-saturation is achieved (days). We would like to derive a relation for the change in snow density, given the current density, from this relation. Given a density, we solve (17) for the corresponding time

$$t = \frac{\rho_s B}{A - \rho_s} \tag{18}$$

and then take the derivative with respect to t of (17) to get

$$\frac{d\rho_s}{dt}(t) = \frac{AB}{(B + t)^2} \tag{19}$$

after substituting (18) into (19) we get the desired relation in (4).

A 4.3 Derivation of Change in Water Mass Relation

When the system is in the saturated isothermal regime the mass of water in the snowpack at any time can be given by solving (7) for M_w to yield

$$M_w = \theta_r \rho_w \frac{M_i}{\rho_s - \theta_r \rho_w}. \tag{20}$$

Taking the derivative of (20) with respect to t gives

$$\frac{dM_w}{dt} = \theta_r \rho_w \left[\frac{\frac{dM_i}{dt}\left(\rho_s - \theta_r \rho_w\right) - M_i \frac{d\rho_s}{dt}}{\left(\rho_s - \theta_r \rho_w\right)^2} \right]. \tag{21}$$

Substituting with the relation in (5) and rearranging gives (10).

6.1 Model Parametrization

$C_s = C_{ice} = 2.05 \times 10^3 J/kg^o K$, $\theta_r = 0.01$, $L_f = 334 \times 10^3 J/kg$, $\rho_w = 1000kg/m^3$, $A = 450kg/m^3$, $B = 20$ days.