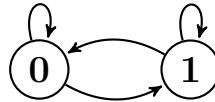
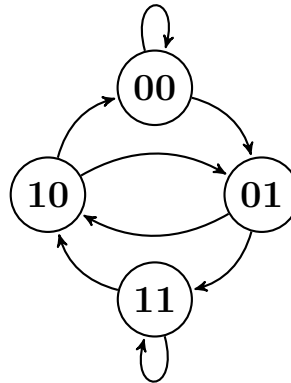


1. For a dynamical system  $(T, X)$ , a point  $x \in X$  is called *eventually periodic* if there exists  $m > m'$  so that  $T^m x = T^{m'} x$ .
  - (a) Let  $(T, [0, 1))$  be the doubling map and let  $(\sigma, \Omega)$  be the full two shift. For each system, give an example of (i) a point that is eventually periodic, and (ii) a point that is eventually periodic but *not* periodic.
  - (b) Let  $\mathbb{C}$  be the coding function for the doubling map with the usual partition  $(\mathcal{P}_0 = [0, 1/2)$  and  $\mathcal{P}_1 = [1/2, 1)$ .  
Prove  $\mathbb{C}(x)$  is eventually periodic in  $(\sigma, \Omega)$  if and only if  $x$  is eventually periodic in  $(T, [0, 1))$ .
  - (c) Prove that  $\mathbb{C}(x)$  is eventually periodic if and only if  $x$  is a rational number.
  - (d) Prove that a binary expansion of a number in  $[0, 1)$  (viewed as an element of  $(\sigma, \Omega)$ ) is eventually periodic if and only if that number is rational. (*Hint:  $\mathbb{C}(x)$  is always a binary expansion, but not all binary expansions come from codings*).
  - (e) Prove that the base  $n$  expansion of a number in  $[0, 1)$  is eventually periodic if and only if that number is rational.
2. Let  $(\sigma, \Omega)$  be the full two-shift. Let  $G \subseteq \Omega$  be the set of sequences without two ones in a row. Let  $X \subseteq \Omega$  be the set of sequences without *three* ones in a row.
  - (a) Show that  $(\sigma, X)$  is a subshift.
  - (b) A *two-step* Markov chain is a Markov chain where the transition probabilities between states depend on the current state *and* the previous state.  
A *two-step* Markov chain on a state space  $S$  can be thought of as a one-step (regular) Markov chain on the state space  $S \times S$ . For example, let  $\mathcal{M}$  be the Markov chain on states 0 and 1 with equal transition probabilities. Normally,  $\mathcal{M}$  has a graph like this:



However, we can also model  $\mathcal{M}$  as a Markov chain  $\mathcal{M}'$  with state space 00, 01, 10, and 11 and transition graph like this:



Explain how  $\mathcal{M}'$  models  $\mathcal{M}$ . In particular, explain why the graph for  $\mathcal{M}'$  has no edge between 00 and 11.

- (c) Let  $\mathcal{M}_G$  be a Markov chain whose set of realizations is  $G$ .
  - i. Draw a graph for  $\mathcal{M}_G$  and find its associated transition matrix.
  - ii. Use the transition matrix for  $\mathcal{M}_G$  to compute the entropy of  $(\sigma, G)$ .

- iii. Model  $\mathcal{M}_G$  as a two-step Markov chain  $\mathcal{M}'_G$ , which can be viewed as a Markov chain with state space  $\{00, 01, 10, 11\}$ .  
 Draw the graph associated with  $\mathcal{M}'_G$  and find its transition matrix.
- iv. Using the transition matrix for  $\mathcal{M}'_G$ , compute the entropy of  $(\sigma, G)$ .
- (d) Can  $X$  be modeled by a one-step Markov chain? Why or why not?
- (e) Find the entropy of  $(\sigma, X)$ .
3. Let  $(\sigma, \Omega)$  be the full two shift.
- (a) Prove that  $(\sigma, \Omega)$  is *expansive*.
- (b) Prove that  $(\sigma, \Omega)$  is *transitive*.
- (c) Show that  $(\sigma, \Omega)$  is *chaotic*.
- (d) Let  $(T, X)$  be a dynamical system and let  $x, y \in X$ . We say  $x$  and  $y$  are  $\delta$ -correlated for  $n$  steps if the distance between  $T^i x$  and  $T^i y$  is at most  $\delta$  for  $i = 0, \dots, n$ .  
 Let  $x, y \in \Omega$  be points that are at distance  $2^{-k}$  of each other. For how many steps will  $x$  and  $y$  be  $\delta$ -correlated when  $\delta = 1/4$ ?
- (e) If  $(T, X)$  is a chaotic dynamical system, can two points be  $\delta$ -correlated indefinitely? What implications does this have for measurement error?
4. Two dynamical systems  $(T, X)$  and  $(S, Y)$  are called *conjugate* if there exists a continuous, invertible function  $\Phi : Y \rightarrow X$  so that  $T = \Phi^{-1} \circ S \circ \Phi$ . In this case,  $\Phi$  is called a *conjugacy*.  
 The systems are called *semi-conjugate* if there exist continuous, onto function  $\Phi : Y \rightarrow X$  so that  $T \circ \Phi = \Phi \circ S$ . In this case,  $\Phi$  is called a *semi-conjugacy*.
- (a) Let  $(T, X)$  and  $(S, Y)$  be conjugate dynamical systems.
- i. Show that if there exists a point of period  $k$  in  $Y$ , there exists a point of period  $k$  in  $X$ .
- ii. Show that if  $(S, Y)$  is transitive, then  $(T, X)$  is transitive.
- iii. Is it true that if  $(S, Y)$  is expansive, then  $(T, X)$  is necessarily expansive?
- (b) Prove that if  $(T, X)$  and  $(S, Y)$  are conjugate dynamical systems, that they are also semi-conjugate dynamical systems.
- (c) Let  $(T, X)$  and  $(S, Y)$  be semi-conjugate dynamical systems.
- i. Show that if there exists a point of period  $k$  in  $Y$ , there exists a point of period  $\leq k$  in  $X$ .
- ii. Show that if  $(S, Y)$  is transitive, then  $(T, X)$  is transitive.
- iii. Is it true that if  $(S, Y)$  is expansive, then  $(T, X)$  is necessarily expansive?
- (d) Let  $(T, [0, 1])$  be the doubling map and let  $(\sigma, \Omega)$  be the full two shift.
- i. Show that  $(T, [0, 1])$  and  $(\sigma, \Omega)$  are semi-conjugate.
- ii. Show that  $(T, [0, 1])$  is chaotic.
- (e) Let  $(T, [0, 1])$  be the doubling map and let  $(L, [0, 1])$  be the *logistic map* defined by  $L(x) = rx(1 - x)$  for  $r = 4$ .
- i. Define  $f : [0, 1] \rightarrow [0, 1]$  by  $f(x) = \sin^2(2\pi x)$ . Show that  $f$  is a semi-conjugacy between  $(L, [0, 1])$  and  $(T, [0, 1])$ .
- ii. Show that  $(L, [0, 1])$  contains points of every period.
- iii. Show that  $(L, [0, 1])$  is chaotic.

## Programming Problems

For the programming problems, please use the Jupyter notebook available at

<https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2023-MAT-335-webpage&subPath=homework/homework4-exercises.ipynb>

Make sure to comment your code and use “Markdown” style cells to explain your answers.

1. The *logistic map with parameter  $r$*  is the function  $f : [0, 1] \rightarrow [0, 1]$  defined by  $f(x) = rx(1 - x)$  (for  $r \in [0, 4]$ ). It was invented as a simple model for population in biology. The idea is that if  $x$  is your original population, there will be some birth rate  $r$  governing population growth; however, when the population outstretches its resources, its growth will be constrained. The  $1 - x$  parameter models this constraint on growth.

Despite being so simple, the logistic map can have amazingly complex behaviour!

- (a) A two-parameter logistic function `f` has been predefined. Create a function `orbit_f` which inputs a starting value, a parameter  $r$ , and an orbit length  $n$ , and returns a list with the first  $n$  points along the orbit of `f`.
  - (b) Given a dynamical system  $(T, X)$ , the  $n$ -orbit of  $x$  is  $\mathcal{O}^n(x) = \{x, Tx, \dots, T^{n-1}x\}$ . Using  $r = 2$ , plot the 20-orbits of  $x$  for at least 10 different  $x$ 's (plot time vs. value). What do you notice?
  - (c) Does the logistic map with  $r = 2$  have a *basin of attraction*<sup>1</sup>? Justify your conclusion.
  - (d) Repeat part 1b using  $r = 3$ . What do you notice? Is there a basin of attraction consisting of a single point?
  - (e) In general, a *basin of attraction* for a dynamical system is a set which all points limit to. Does the logistic map have a basin of attraction when  $r = 3$ ? If so, describe it.
  - (f) Repeat part 1b with  $r = 4$ . Is there a basin of attraction? Why or why not?
2. We are going to plot the basins of attraction for the logistic map as a function of  $r$ . We can approximate a basin of attraction by taking the 1000-orbit of several points, and then taking the set consisting of the last 100 points in each orbit.
    - (a) Create a function `approximate_basin` which takes in a parameter  $r$  and returns a list of points that approximate the basin of attraction of the logistic map with parameter  $r$ .  
*Hint:* It may be useful to round your results to 3 or 4 decimal places and then use `np.unique` to get a list of manageable length.
    - (b) Plot the basins of attraction of the logistic map vs.  $r$  for at 1000 different values of  $r$  between 0 and 4.
    - (c) What do you notice about the basins of attraction? Did you expect this?
    - (d) In the *proofs* part of this homework set, you proved that the logistic map is chaotic when  $r = 4$ . What does that imply about the basin of attraction?
    - (e) If a population is well-modeled by the logistic map with a parameter of  $r = 3.82$ , what can you say about the population after 1000 days? What if it is modeled by a logistic map with parameter  $r = 3.83$ ? Is the population more or less predictable?

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<sup>1</sup>Look back in the notes if you've forgotten this term.