# COMPUTING THE GENERATING FUNCTION OF A COINVARIANTS MAP

BY

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A thesis submitted in conformity with
the requirements for the degree of
Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

# ABSTRACT

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2023

A well-known source of knot invariants is representations of quasitriangular Hopf algebras (also known as quantum groups). These invariants require exponential time in the number of crossings to compute. Recent work has allowed for polynomial-time computations within the Hopf algebras themselves, using perturbed Gaußian differential operators. This thesis introduces and explores a partial expansion of the tangle invariant $Z$ introduced by Bar-Natan and van der Veen [BNvdVb]. We expand the use of the Hopf algebra $\mathfrak{U}(\mathfrak{sl}_{2+}^0)$ to include its space of coinvariants, providing an extension $Z^{\mathrm{tr}}$ of $Z$ from open tangles to links.

We compute a basis for the space of coinvariants of $\mathfrak{U}(\mathfrak{sl}_{2+}^0)$ and a closed-form expression for the exponential generating function of the corresponding trace map. The resulting function is not directly compatible with the previous research, so we also find a method of computing $Z^{\mathrm{tr}}$ on a subclass of links and write a program to compute $Z^{\mathrm{tr}}$ on two-component links. Contrary to expectations, we find that $Z^{\mathrm{tr}}$ is neither stronger nor weaker than the Multivariable Alexander polynomial. This unexplained behaviour warrants further study into $Z^{\mathrm{tr}}$ and its relationship with other invariants.

*To someone,*

*who did something nice.*

# ACKNOWLEDGEMENTS

Thank you:

To my advisor, Dror Bar-Natan, whose consistent meetings, guidance, and encouragement brought me to this point, and to my supervisory committee, Joel Kamnitzer and Eckhard Meinrenken, for their feedback and for helping me connect to more of the department and its research.

To Jason Siefken and the other teaching mentors in the math department: you have made me a better educator and communicator.

To the math department administration, especially Jemima Merisca and Sonja Injac, whose kindness made the department a comfortable place to be.

To Assaf Bar-Natan, Vincent Girard, Caleb Jonker, and Adriano Pacifico, whose friendship, bike trips, and cooking sessions provided much joy; to the people I met in the department, especially those who opened my thesis to confirm they were included here: thank you specifically.

To the care-taking staff at the St. George campus, especially Maria and Maria, whose diligent work and kind words have made this campus beautiful.

To the U of T Graduate Christian Fellowship for providing a refreshing breadth of conversation and companionship.

To Rosemary and Alan Johnstone, whose hospitality made me feel immediately welcome in a new city.

To my family, who encouraged me in my academic pursuits from the start: to my mom for embracing my atypical learning styles, for instilling my love of reading, and for teaching my how to think deeply; to my dad for teaching me to cycle, cook, and notice the invisible among us; and to my brother for all the laughter you bring to my life.

To the Perrins: you are my second family, and your support and love for me is precious.

To my fiancée, Emily Langridge, who helped me with the motivation to finish. I look forward to the next chapters of my life with you.

# CONTENTS

# 1

# EXECUTIVE SUMMARY

## 1.1 ALGEBRAIC TOOLS FOR UNDERSTANDING KNOTS

*Knotted objects*

In the field of knot theory, distinguishing between two knots or links has proven to be a difficult task. Computing strong invariants of knotted objects is a popular way to aid with the classification of these objects.

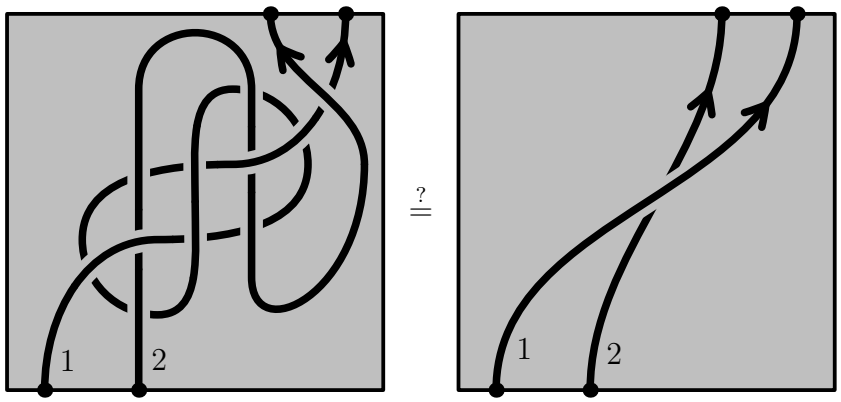



Figure 1.1: Two strings tangled together. Can they be disentangled?

Merely being able to distinguish between two knotted objects does not always provide us with enough information about these topological structures. For instance, one may ask if a link is a satellite (i.e. whether its complement has a non-trivial embedding of a torus), whether a knot is slice (i.e. it is the boundary of a disk in $\mathbb{R}^4$), or whether it is ribbon (i.e. it is the boundary of a disk in $\mathbb{R}^3$ with restricted types of singularities). (For more details see, for instance, Lickorish's [Lic].) Many interesting properties of knots can be phrased in terms of certain topological operations, such as strand doubling (taking a strand and replacing it with two copies of itself, as in figure 1.2) or strand stitching (joining two open components together to form one longer one).

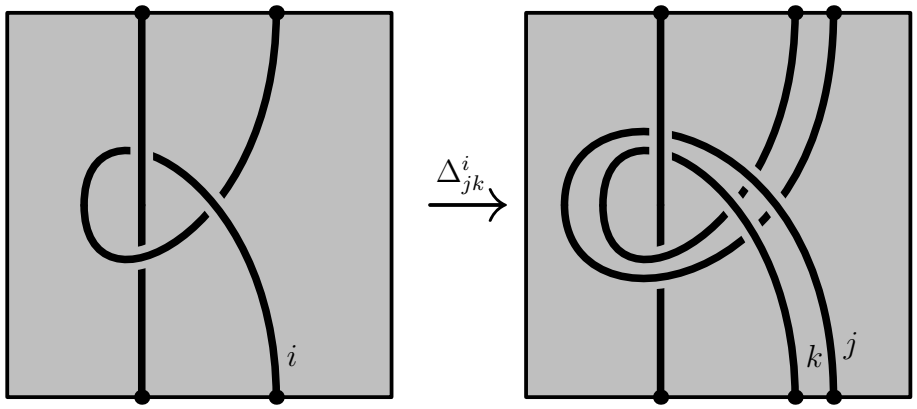



Figure 1.2: An example of strand-doubling.

Open problems such as the Ribbon Slice Conjecture (asking whether there exists a slice knot which is not ribbon, posed by R. H. Fox [Fox]) may be advanced by the development of "topologically aware" invariants—those which preserve topological data in a retrievable way.

*Quantum invariants*

One such class of topological invariants is derived from quasitriangular Hopf algebras, which are algebraic structures whose operations mimic those of knotted objects. In this approach, one takes a knotted object and decomposes it into a sequence of topological operations (such as stitching strands or doubling strands), then maps each of these operations to a corresponding algebraic operation. The composition of these algebraic operations is the value of the invariant.

More specifically, a Hopf algebra is an algebra $A$ together with several maps between various tensor powers thereof (for instance, a comultiplication map $\Delta\colon A \to A \otimes A$). Each crossing assigned an element of $A \otimes A$. Tensor factors which belong to the same strand are concatenated by multiplying the associated algebra elements. The value of the invariant is an element of a tensor power of $A$. We go over this in more detail in chapter 2.

While this formulation is elegant, it has a notable drawback: computing the invariant of a tangle with many components requires manipulating large tensor powers of $A$. One remedy is to instead perform the computation in a representation $V$ of $A$ with a low dimension, though the issue of exponential

growth in complexity remains. This limitation restricts the utility of quantum invariants to smaller knots.

## 1.2 COMPUTATIONAL IMPROVEMENTS USING THE UNIVERSAL INVARIANT

To avoid the issue of exponential computational complexity, one can instead investigate the set of all values of the universal invariant (using the algebra itself instead of a representation) as a subset of the algebra and its tensor powers. For a particular choice of algebra (namely $\widehat{\mathfrak{U}}(\mathfrak{sl}_{2+}^{\epsilon})$, as investigated by Dror Bar-Natan and Roland van der Veen in [BNvdVb]) the space of values the corresponding invariant $Z$ can take is significantly smaller than the whole space; the rank of the space of values grows only quadratically with the number of crossings in the knotted object. In particular, by looking at the generating functions of the algebra operations, one sees that the value of $Z$ on tangles always take the form of a (perturbed) Gaußian. Computationally, this means one need only keep track of a quadratic form and a small perturbation. The invariant $Z$ dominates the $\mathfrak{sl}_2$-coloured Jones polynomial. Here, we will focus on the case when $\epsilon = 0$, for which $Z$ becomes an efficient computation of the Alexander polynomial $\Delta$ on knots. This topic is described in chapter 3.

## 1.3 EXTENDING $Z$ TO LINKS

The research program outlined by Bar-Natan and van der Veen computes $Z$ only for (open) tangles–that is, collections of open strands whose endpoints are fixed to a boundary circle. (Note that this includes long knots, which are exactly the one-component tangles.) This thesis is focused on extending $Z$ and its computations to tangles with closed components, which includes links. Here we summarize chapter 4.

### *Computing the extended map*

The first task is to determine the space in which the extended invariant, which we will call $Z^{\mathrm{tr}}$, lives. One may observe that in a matrix algebra, one is able to contract two matrices together via matrix multiplication. When one

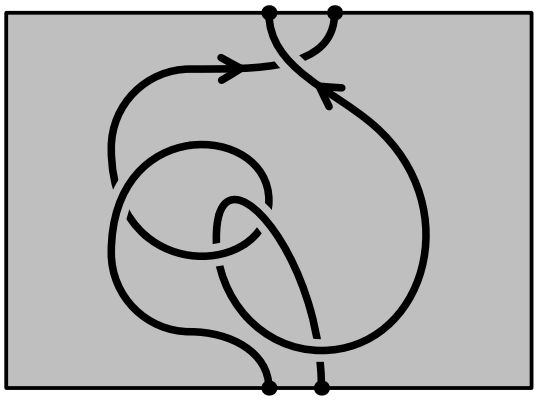


Figure 1.3: An open tangle. All components intersect the boundary.

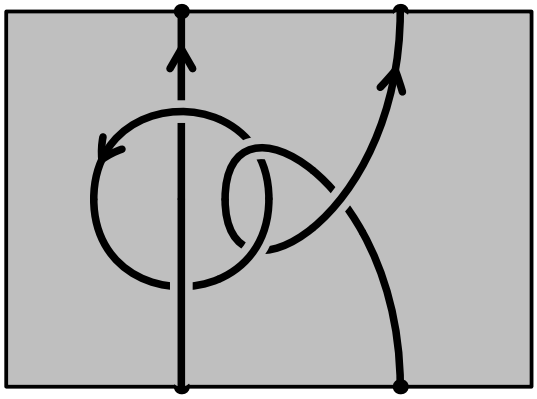


Figure 1.4: A tangle with a closed component.

wishes to contract a matrix along itself, one uses the trace map. Analogously, since stitching two strands in a tangle corresponds to multiplication, closing a strand into a loop should correspond algebraically to a trace map.

In a generic algebra $A$, the trace map is defined as the projection onto the set of coinvariants: $\mathrm{tr}\colon A \to A_A = A/[A,A]$.[†1] In order to extend $Z$ in this framework, we must first compute the space of coinvariants for the algebra $U = \widehat{\mathfrak{U}}(\mathfrak{sl}^0_{2+})$, then compute the coinvariants map, and write it as a generating function. (This is accomplished in section 4.2.)

†1 Here, $[A,A] = \mathrm{span}\{\,[x,y] \mid x,y \in A\,\}$ refers to the vector space of Lie brackets, not the ideal generated thereby. The space $A_A$ does not have an algebra structure in general.

*Performing computations*

Unfortunately, the resulting trace map does not take the form of a perturbed Gaußian in a way that readily connects to the existing framework. In order to determine whether further study of $Z^{\mathrm{tr}}$ is merited, we must find an alternative computation method to get a preliminary sense of the strength of $Z^{\mathrm{tr}}$.

For a subclass of links (which includes all two-component links), we compute an explicit closed form for the trace map, then implement a computer program to compute the value of $Z^{\mathrm{tr}}$ on all two-component links with up to 11 crossings. When applied to knots, $Z$ computes the Alexander polynomial. When applied to two-component tangles, one may expect that $Z^{\mathrm{tr}}$ would produce the natural generalization to multiple components: the Multivariable Alexander polynomial (MVA). Surprisingly, the MVA and $Z^{\mathrm{tr}}$ are incomparable, with each being able to distinguish pairs of links the other cannot. (See section 5.1 for more information.)

## 1.4 FURTHER STUDY

As $Z^{\mathrm{tr}}$ does not generalize $Z$ in the manner expected, several interesting avenues of further research are opened. Firstly, determining what the relationship between $Z^{\mathrm{tr}}$ and the MVA remains open. Second is the challenge of finding an efficient method for computing $Z^{\mathrm{tr}}$ on links with more than two components, which currently is mired in complications with the presence of non-elementary functions where quadratic forms normally appear. Third is the question of the existence of other viable trace candidates. In particular, it may be worth exploring whether a universal trace with respect to the perturbed Gaußian framework defines a sufficiently useful invariant. See chapter 5 for more information.

# 2

# TENSOR PRODUCTS AND META-OBJECTS

## 2.1 TENSOR PRODUCT NOTATION

In what follows, we will extensively use tensor products, tensor powers, and generalizations thereof. We begin by introducing the notation that will make working with these objects more straightforward, similar to the slot-naming index notation introduced by Penrose [Pen].

Let $\Bbbk$ be a field and $V$ a $\Bbbk$-vector space (for the moment assumed to be finite dimensional). When working with a large tensor power $V^{\otimes n}$ of $V$, it will often be more convenient to label tensor factors with elements of a finite set $S$ (with $|S| = n$) rather than by their position in a linear order.

For example, consider the vector $u \otimes v \otimes w \in V^{\otimes 3}$. Let us choose an index set $S = \{i, j, k\}$. We then may equivalently write this vector by labelling each tensor factor with one of the elements of $S$, say $u_i v_j w_k$. Since the labels serve to distinguish the separate factors, this vector may equivalently be written as $u_i v_j w_k = v_j u_i w_k = w_k u_j u_i \in V^{\otimes S}$, where $V^{\otimes S}$ denotes the tensor power of $V$. An additional notation which we will prefer is $V_S = V^{\otimes S}$. We formalize the idea below:

**Definition 2.1** (indexed tensor powers)**.** Let $V$ be a vector space and $S = \{s_1, \dots, s_n\}$ be a finite set. We define the indexed tensor power of $V$ to be the collection of formal linear combinations of functions from $S$ to $V$

$$V_S := V^{\otimes S} := \operatorname{span}\{f \colon S \to V\} / \sim \tag{2.1}$$

subject to the standard multilinear relations, namely multi-additivity and the factoring of scalars:

By multi-additivity, we mean that for each $i \in S$ and $f, g \in V_S$ satisfying $f(s) = g(s)$ for each $s \in S \setminus \{i\}$, we have:

$$f + g \sim \left( s \mapsto \begin{cases} f(s) = g(s) & \text{if } s \neq i \\ f(i) + g(i) & \text{if } s = i \end{cases} \right) \tag{2.2}$$

We will write such functions $f \colon S \to V$ with $f(s_i) = v_i$ with the following notation:

$$(v_1)_{s_1}(v_2)_{s_2} \cdots (v_n)_{s_n} := f \tag{2.3}$$

The factoring of scalars relation is:

$$(v_1)_{s_1}(v_2)_{s_2} \cdots (\lambda v_i)_{s_i} \cdots (v_n)_{s_n} = \lambda \cdot (v_1)_{s_1}(v_2)_{s_2} \cdots (v_n)_{s_n} \tag{2.4}$$

Similarly, equation (2.2) in the style of equation (2.3) becomes:

$$\begin{aligned} \left((v_1)_{s_1}(v_2)_{s_2} \cdots x_{s_i} \cdots (v_n)_{s_n}\right) + \left((v_1)_{s_1}(v_2)_{s_2} \cdots y_{s_i} \cdots (v_n)_{s_n}\right) \\ = (v_1)_{s_1}(v_2)_{s_2} \cdots (x + y)_{s_i} \cdots (v_n)_{s_n} \end{aligned} \tag{2.5}$$

Next, we introduce notation for maps between tensor powers so that we may unambiguously refer to appropriate tensor factors while defining morphisms. We accomplish this task by adding a convenient way of writing the domain and codomain of a map. Let $D$ and $C$ be finite sets, and $T \colon V_D \to V_C$. We will denote $T$ alternatively by $T^D_C$, so that its domain and codomain are easily read off. It is important to note that when $T$ is not symmetric in its arguments, the order of the indices in this notation matters.

**Example 2.2.** Let $V = \mathbb{R}^3$, and $T^{a,b}_c$ (equivalently, $T \colon V_{\{a,b\}} \to V_{\{c\}}$) defined by $T^{a,b}_c(\vec{v}_a \vec{w}_b) = (\vec{v} \times \vec{w})_c$ denote the cross product. Stating that the cross product is antisymmetric may be accomplished without referencing vectors by writing:

$$T^{a,b}_c = -T^{b,a}_c \tag{2.6}$$

*Remark* 2.3. There are three special cases with this notation:

- Given a (multi)linear functional $\phi \colon V_S \to \Bbbk \cong V_\emptyset$, we will write $\phi^S$ instead of $\phi^S_\emptyset$. The linear order on $S$ remains in this notation.
- Elements $v \in V_S$ will be interpreted as a map $v \colon \Bbbk = V_\emptyset \to V_S$ written $v_S$ instead of $v^\emptyset_S$.

- When only one index is present in a subscript or superscript, and its omission does not introduce an ambiguity in an expression, then it may be omitted to improve readability. For instance, a map $\phi\colon V_{\{1,2\}} \to V_{\{3\}}$ may be written as $\phi^{1,2}$ instead of $\phi_3^{1,2}$, with the canonical isomorphism $V \cong V_{\{3\}}$ being suppressed.

When taking the tensor product of two tensor powers, we follow [BNS] and use the notation "$\sqcup$" instead of "$\otimes$":

$$V_X \sqcup V_Y \coloneqq V_{X \sqcup Y} \tag{2.7}$$

Additionally, given $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ such that $D_1 \cap D_2 = \emptyset = C_1 \cap C_2$, we have a product morphism $\phi_{C_1}^{D_1}\psi_{C_2}^{D_2} \coloneqq \phi \otimes \psi\colon V_{D_1 \sqcup D_2} \to V_{C_1 \sqcup C_2}$, which we also write with concatenation.

Finally, we point out that any morphism $T_C^D$ may be extended to one with larger domain and codomain. We introduce the notation $T[S] \coloneqq T_S^D \operatorname{id}_S^S$ (recalling the concatenation means $T \otimes \operatorname{id}_S^S$) for this concept. When no ambiguity arises, we will also suppress the "$[S]$" so that $T_C^D$ represents more generally:

$$(T_C^D)(v_D w_S) \coloneqq (T_C^D v_D) w_S \tag{2.8}$$

for any $v_D \in V_D$ and $w_S \in V_S$.

## 2.2 META-OBJECTS

While the above notation is helpful when working with vector spaces, we are interested in also using the same notation to describe a tangle. Our formulation of tangles (introduced in section 2.4) is not a tensor product, though it shares many similarities. In particular, the domains and codomains of the maps we have discussed so far have only depended on the index set. With this observation, we replace the notation of tensor powers with that of a so-called meta-object. We introduce this concept by starting with monoids.

We now go through the process of defining a meta-monoid, which is a generalization of a monoid object. Traditionally, the data of a monoid object are the following:

- An object $M$ in a category $\mathcal{C}$.
- A morphism $m\colon M \times M \to M$ called the "multiplication" operation.

- A "unit" morphism $\eta\colon \{1\} \to M$.[†1]
- A collection of relations between the operations, written as equalities of morphisms between Cartesian powers of $M$. For example, associativity may be written:

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m\times\mathrm{id}} & M \times M \\ {\scriptstyle \mathrm{id}\times m}\downarrow & & \downarrow{\scriptstyle m} \\ M \times M & \xrightarrow[m]{} & M \end{array} \tag{2.9}$$

Further, the data of these relations is extended to higher powers of $M$ by acting on other components by the identity:

$$\begin{array}{ccc} M^{n+3} & \xrightarrow{m\times\mathrm{id}^{n+1}} & M^{n+2} \\ {\scriptstyle \mathrm{id}\times m\times\mathrm{id}^n}\downarrow & & \downarrow{\scriptstyle m\times\mathrm{id}^n} \\ M^{n+2} & \xrightarrow[m\times\mathrm{id}^n]{} & M^{n+1} \end{array} \tag{2.10}$$

Let us alter how we package these data so as to maximize the clarity of the meta-monoid structure:

1. Instead of linear orders of factors $M \times \cdots \times M$, we will index factors by a finite set $X$, writing it $M_X := \{f\colon X \to M\}$ in the style of equation (2.1).[†2]
2. The indexed factors will determine how the monoid operations act. For instance, multiplication of factor $i$ and $j$ together, with the result labelled in factor $k$ is to be written $m_k^{ij}\colon M_{\{i,j\}} \to M_{\{k\}}$.
3. Instead of implicitly including extensions of operations to higher powers by the identity, we will parametrize the extension by finite sets by $\phi_C^D[X] := \phi_C^D \times \mathrm{id}_X^X$. For example, multiplication $m_k^{ij}\colon M_{\{i,j\}} \to M_{\{k\}}$ generates a family of maps $m_k^{ij}[X]\colon M_{\{i,j\}\sqcup X} \to M_{\{k\}\sqcup X}$, each of which must satisfy the relations of the monoid object such as equation (2.10). (Again, the "$[X]$" is frequently omitted from writing.)

This way of packaging the data leads us to the following generalization:

†1 When $\mathcal{C} = \mathbf{Set}$, we usually write the unit as an element $1 = \eta(1) \in M$

†2 Indeed, when $\mathcal{C} = \mathbf{Vect}$, these definitions are identical when the monoidal product is $\otimes$. In this case, they are called algebras.

**Definition 2.4.** A meta-monoid in $\mathcal{C}$ is the following data:

- A family of objects $M_X \in \mathcal{C}$, indexed over finite sets $X$, with set bijections $\psi\colon X \xrightarrow{\sim} Y$ inducing isomorphisms $M_X \cong M_Y$.
- A family of morphisms $m_k^{ij}[X]\colon M_{\{i,j\}\sqcup X} \to M_{\{k\}\sqcup X}$ called "multiplication".
- A family of "unit" morphisms $\eta_i[X]\colon M_X \to M_{\{i\}\sqcup X}$.
- A collection of relations between the morphisms, written as equalities of morphisms between the $M_X$'s. In particular, associativity:

$$\begin{array}{ccc} M_{\{1,2,3\}\sqcup X} & \xrightarrow{m_1^{1,2}[X\sqcup\{3\}]} & M_{\{1,3\}\sqcup X} \\ {\scriptstyle m_2^{2,3}[X\sqcup\{1\}]}\downarrow & & \downarrow{\scriptstyle m_1^{1,3}[X]} \\ M_{\{1,2\}\sqcup X} & \xrightarrow[m_1^{1,2}[X]]{} & M_{\{1\}\sqcup X} \end{array} \tag{2.11}$$

  and the identity:

$$\begin{array}{ccc} M_{\{1\}\sqcup X} & \xrightarrow{\eta_2[X]} & M_{\{1,2\}\sqcup X} \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle m_1^{2,1}[X]}\downarrow{\scriptstyle m_1^{1,2}[X]} \\ & & M_{\{1\}\sqcup X} \end{array} \tag{2.12}$$

**Example 2.5** (monoid objects are meta-monoids)**.** Any monoid object $M$ in a strict, symmetric monoidal category $(\mathcal{C}, \otimes, \{1\})$ has the structure of a meta-monoid $\{M_X\}_X$ via $M_X := M^{\otimes X}$†3, $m_k^{ij}[X] := m_k^{ij} \otimes \mathrm{id}_X^X$, and $\eta_i[X](v) := 1_i \otimes v$ for any $v \in M^{\otimes X}$.

Consider the following structure, which satisfies the definition of a meta-monoid, but is not a monoid in the traditional sense:

**Example 2.6** (the meta-monoid of square matrices)**.** Let $\Bbbk$ be a field and $M_X := \mathrm{Mat}_{X\times X}(\Bbbk)$ be the set of square matrices whose rows and columns are indexed by the finite set $X = \{x_i\}_i$. Define $m_k^{ij}[X]\colon M_{X\sqcup\{i,j\}} \to M_{X\sqcup\{k\}}$ by $m_k^{ij}[X]((a_{rs})_{rs}) := \big(a_{rs} + \delta_{rk}(a_{is} + a_{js}) + \delta_{sk}(a_{si} + a_{sj})\big)_{rs}$. That is, the

†3 By $M^{\otimes X}$ we mean $M^{\otimes |X|}$ together with a choice of assigning each factor an index, analogous to definition 2.1.

multiplication of two indices corresponds to the summation of their respective rows and columns, the result of which is stored in row and column $k$:

$$\begin{bmatrix} a_{x_1,x_1} & \cdots & a_{x_1,i} & a_{x_1,j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i,x_1} & \cdots & a_{ii} & a_{ij} \\ a_{j,x_1} & \cdots & a_{ji} & a_{jj} \end{bmatrix} \xrightarrow{m_k^{ij}} \begin{bmatrix} a_{x_1,x_1} & \cdots & a_{x_1,i} + a_{x_1,j} \\ \vdots & \ddots & \vdots \\ a_{i,x_1} + a_{j,x_1} & \cdots & a_{ii} + a_{ji} + a_{ij} + a_{jj} \end{bmatrix} \tag{2.13}$$

Where the last column and row on the right-hand-side is indexed by $k$. The unit $\eta_i[X]((a_{rs})_{rs})$ extends $(a_{rs})_{rs}$ to include a row and column of 0's, each labelled by the index $i$.

**Example 2.7** (tangles form a meta-algebra)**.** Tangles are the main example of a meta-algebra which is not an algebra in the traditional sense. We go into more detail in section 2.4.

In order to define other meta-objects (such as a meta-colagebra or a meta-semigroup) we provide the following more general definition:

**Definition 2.8** (meta-object)**.** Let $\mathcal{C}$ be a category. A meta-object in $\mathcal{C}$ is four things:

1. A collection of objects $A_X$, one for each choice of finite set $X$. (This serves as the analogue to monoidal powers.)

2. For each bijection $\psi\colon X \xrightarrow{\sim} Y$ of finite sets $X$ and $Y$, a reindexing isomorphism $\iota_\psi\colon A_X \xrightarrow{\sim} A_Y$.

3. A collection of operations $\phi_1, \phi_2, \ldots, \phi_n$ each with a signature $|\phi_i| \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. For any pair of finite sets $(D, C)$ satisfying $(|D|, |C|) = |\phi|$, we have a morphism:
$$\phi_C^D\colon A_D \to A_C \tag{2.14}$$

4. For each operation $\phi_C^D$, there is a collection of morphisms $\phi_C^D[\cdot]$ indexed by finite sets such that for each finite set $S$, $T$:
   a) $\phi[S]\colon A_{C \sqcup S} \to A_{D \sqcup S}$
   b) $\phi[\emptyset] = \phi$
   c) $(\phi[S])[T] = \phi[S \sqcup T]$

When no ambiguity arises, we will omit the portion written in square brackets, so that $\phi$ will stand for $\phi[X]$, with the set $X$ determined from context.

Finally, we may define the product of two spaces $A_S$ and $A_T$ by $A_S A_T = A_{S \sqcup T}$. Given operations $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ such that $D_1 \cap D_2 = \emptyset = C_1 \cap C_2$, we have a product morphism $\phi_{C_1}^{D_1} \psi_{C_2}^{D_2} : \mathcal{C}_{D_1 \sqcup D_2} \to \mathcal{C}_{C_1 \sqcup C_2}$. This is visualized in figure 2.1.

Composition of operators $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ is defined when $C_1 = D_2$:

$$\psi_{C_2}^{D_2} \circ \phi_{C_1}^{D_1} : \mathcal{C}_{D_1} \to \mathcal{C}_{C_2} \tag{2.15}$$

*Remark* 2.9. In this text, we will denote left-to-right composition with the symbol "$/\!/$" (pronounced "then"): $f /\!/ g := g \circ f$. Writing function composition in this order assists with readability when there are many functions to apply.

*Remark* 2.10. To make expressions easier to read, we introduce the domain extension implicitly in the following context: given morphisms $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ such that $C_2 \cap (C_1 \setminus D_2) = \emptyset = D_1 \cap (D_2 \setminus C_1)$, we define:

$$\phi_{C_1}^{D_1} /\!/ \psi_{C_2}^{D_2} := \phi_{C_1}^{D_1}[D_2 \setminus C_1] /\!/ \psi_{C_2}^{D_2}[C_1 \setminus D_2] \tag{2.16}$$

Figure 2.2 visualizes this extension.

The two extreme cases of this definition are:

- When $C_1 \cap D_2 = \emptyset$, equation (2.16) becomes $\phi_{C_1}^{D_1} \psi_{C_2}^{D_2}$.
- When $C_1 = D_2$, equation (2.16) becomes the composition $\phi_{C_1}^{D_1} /\!/ \psi_{C_2}^{D_2}$ exactly.

## 2.3 ALGEBRAIC DEFINITIONS

We now introduce the algebraic structures which will be used to define the tangle invariant. These definitions follow those given by Majid in [Maj], presented in a way that their corresponding meta-structure are readily visible.

**Definition 2.11** (meta-algebra)**.** A meta-algebra (or meta-monoid[†4]) is a collection of objects $\{A_X\}_X$ in $\mathcal{C}$ together with an associative multiplication

†4 This is a repeat of definition 2.4. The only difference between an algebra object and a monoid object is the presence of a linear structure.

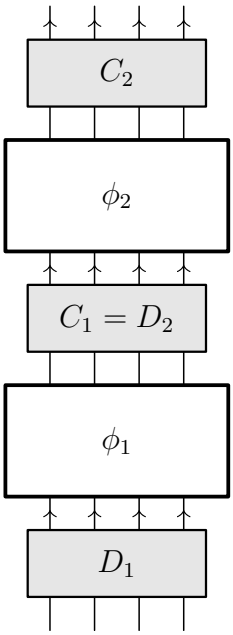



Figure 2.1: We may visualize a composition of morphisms with a graphical calculus. This graphic represents $(\phi_1)^{D_1}_{C_1} /\!\!/ (\phi_2)^{D_2}_{C_2}$ when $C_1 = D_2$. Each arrow represents one factor. Indices are written in grey boxes.

$m^{i,j}_k \colon A_{\{i,j\}} \to A_{\{k\}}$ (satisfying equation (2.17)), and a unit $\eta_i \colon A_\emptyset \to A_{\{i\}}$ satisfying equation (2.18).

*Remark* 2.12. When $\mathcal{C} = (\mathbf{Vect}, \otimes)$ and $A_X = V_X = V^{\otimes X}$ for some vector space $V$, definition 2.11 becomes the more familiar definition of an algebra. Then $A_\emptyset$ is a field. It is more common think of the unit as an element $\mathbf{1} \in V$. The unit map is then defined by linearly extending the assignment $\eta_i(1) = \mathbf{1}_i$.

$$\begin{array}{ccc} A_{\{1,2,3\}} & \xrightarrow{m_1^{1,2}} & A_{\{1,3\}} \\ {\scriptstyle m_2^{2,3}}\downarrow & & \downarrow{\scriptstyle m_1^{1,3}} \\ A_{\{1,2\}} & \xrightarrow[m_1^{1,2}]{} & A_{\{1\}} \end{array} \qquad (2.17)$$

$$\begin{array}{ccc} A_{\{1\}} & \xrightarrow{\eta_2} & A_{\{1,2\}} \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle m_1^{2,1}} \quad \downarrow{\scriptstyle m_1^{1,2}} \\ & & A_{\{1\}} \end{array} \qquad (2.18)$$

*Remark* 2.13. Associativity allows us to denote repeated multiplication by using extra indices. For instance: $m^{i,j,k}_\ell := m^{i,j}_r /\!\!/ m^{r,k}_\ell = m^{j,k}_s /\!\!/ m^{i,s}_\ell$.

There is also the dual notion of a coalgebra, which arises by reversing the arrows in equations (2.17) and (2.18):

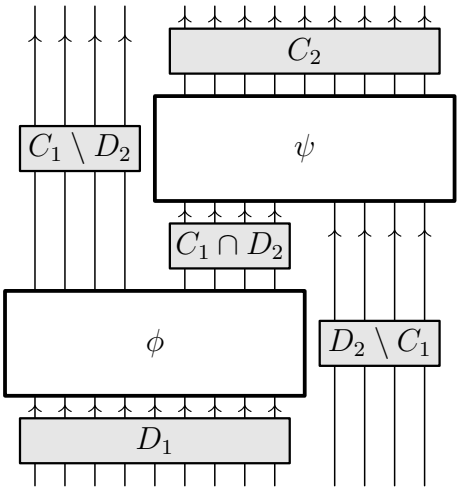



Figure 2.2: Visual mnemonic for extending morphisms. This graphic represents equation (2.16).

**Definition 2.14** (meta-coalgebra)**.** A meta-colagebra (or meta-comonoid) is a collection $\{C_X\}_X$ together with a comultiplication $\Delta^i_{jk} \colon C_{\{i\}} \to C_{\{j,k\}}$ which is coassociative (equation (2.19)) and a counit, which is a map $\epsilon^i \colon A_i \to A_\emptyset$ satisfying equation (2.20).

$$\begin{array}{ccc} C_{\{1,2,3\}} & \xleftarrow{\Delta^1_{1,2}} & C_{\{2,3\}} \\ {\scriptstyle \Delta^2_{2,3}}\uparrow & & \uparrow{\scriptstyle \Delta^1_{1,3}} \\ C_{\{1,2\}} & \xleftarrow[\Delta^1_{1,2}]{} & C_{\{1\}} \end{array} \tag{2.19}$$

$$\begin{array}{ccc} C_{\{1\}} & \xleftarrow{\epsilon^2} & C_{\{1,2\}} \\ & {\scriptstyle \mathrm{id}}\nwarrow \quad {\scriptstyle \Delta^1_{1,2}}\uparrow\uparrow{\scriptstyle \Delta^1_{2,1}} & \\ & C_{\{1\}} & \end{array} \tag{2.20}$$

*Remark* 2.15. Coassociativity allows us to denote repeated comultiplication by using extra indices. For instance: $\Delta^i_{j,k,\ell} := \Delta^i_{j,r} /\!\!/ \Delta^r_{k,\ell} = \Delta^i_{s,\ell} /\!\!/ \Delta^s_{j,j}$.

If a meta-object $\{B_X\}_x$ has the structure of both an algebra and a coalgebra, we introduce a definition for when the structures are compatible with each other:

**Definition 2.16** (meta-bialgebra)**.** A meta-bialgebra (or meta-bimonoid) is a meta-algebra $(B, m, \eta)$ and a meta-coalgebra $(B, \Delta, \epsilon)$, such that $\Delta$ and $\epsilon$ are meta-algebra morphisms. †5

†5 $B_X$ inherits a meta-(co)algebra structure from $B$, given by $(B_X)_Y := B_{X^Y}$ and componentwise operations. The bialgebra structure on $B_\emptyset$ is given by $m = \eta = \Delta = \epsilon = \mathrm{id}$.

$$\begin{array}{ccc} B_{\{1,2\}} & \xrightarrow{m_1^{1,2}} & B_{\{1\}} \\ \downarrow{\scriptstyle \Delta_{1,3}^1 /\!\!/ \Delta_{2,4}^2} & & \downarrow{\scriptstyle \Delta_{1,2}^1} \\ B_{\{1,2,3,4\}} & \xrightarrow[m_1^{1,2} /\!\!/ m_2^{3,4}]{} & B_{\{1,2\}} \end{array} \tag{2.21}$$

$$\begin{array}{ccc} & B_{\{1\}} & \\ {\scriptstyle \eta_1} \nearrow & & \downarrow{\scriptstyle \Delta_{1,2}^1} \\ B_\emptyset & & \\ {\scriptstyle \eta_1 /\!\!/ \eta_2} \searrow & & \\ & B_{\{1,2\}} & \end{array} \tag{2.22}$$

$$\begin{array}{ccc} B_{\{1,2\}} & \xrightarrow{m_1^{1,2}} & B_{\{1\}} \\ & {\scriptstyle \epsilon^1 /\!\!/ \epsilon^2} \searrow \quad \swarrow {\scriptstyle \epsilon^1} & \\ & B_\emptyset & \end{array} \tag{2.23}$$

$$\begin{array}{ccc} B_\emptyset & \xrightarrow{\eta_1} & B_{\{1\}} \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle \epsilon^1} \\ & & B_\emptyset \end{array} \tag{2.24}$$

*Remark* 2.17. The conditions for $\Delta$ being an algebra morphism are presented in equations (2.21) and (2.22), while those for $\epsilon$ are in equations (2.23) and (2.24). Observing invariance under arrow reversal, it may not come as a surprise that equations (2.21) and (2.23) also are the conditions for $m$ being a coalgebra morphism, and equations (2.22) and (2.24) tell us that $\eta$ is as well.

Next, we introduce a notion of invertibility which extends a (meta-)bialgebra to a (meta-)Hopf algebra:

**Definition 2.18** (meta-Hopf algebra)**.** A meta-Hopf algebra (or meta-Hopf monoid) is a meta-bialgebra $H$ together with a map $S\colon H \to H$ called the antipode, which satisfies $\Delta_{1,2}^1 /\!\!/ S_1^1 /\!\!/ m_1^{1,2} = \epsilon^1 /\!\!/ \eta_1 = \Delta_{1,2}^1 /\!\!/ S_2^2 /\!\!/ m_1^{1,2}$. As a commutative diagram, this looks like equation (2.25)

$$\begin{array}{ccccc} H_{\{1\}} & \xrightarrow{\epsilon^1} & H_\emptyset & \xrightarrow{\eta_1} & H_{\{1\}} \\ & {\scriptstyle \Delta_{1,2}^1} \searrow & & \nearrow {\scriptstyle m_1^{1,2}} & \\ & & H_{\{1,2\}} & & \end{array} \qquad H_{\{1,2\}} \underset{S_1^1}{\overset{S_2^2}{\rightrightarrows}} H_{\{1,2\}} \tag{2.25}$$

In order to do knot theory, we need an algebraic way to represent a crossing of two strands. This is accomplished by the $\mathcal{R}$-matrix:

**Definition 2.19** (quasitriangular meta-Hopf algebra)**.** A quasitriangular meta-Hopf algebra (or quasitriangular meta-Hopf monoid) is a meta-Hopf algebra $H$, together with an invertible element $\mathcal{R}_{i,j} \in H_{i,j}$, called the

$\mathcal{R}$-matrix, which satisfies the following properties: (we will denote the inverse by $\overline{\mathcal{R}}$)

$$\mathcal{R}_{13} /\!\!/ \Delta^1_{12} = \mathcal{R}_{13}\mathcal{R}_{24} /\!\!/ m^{34}_3 \tag{2.26}$$
$$\mathcal{R}_{13} /\!\!/ \Delta^3_{23} = \mathcal{R}_{13}\mathcal{R}_{42} /\!\!/ m^{14}_1 \tag{2.27}$$
$$\Delta^1_{21} = \Delta^1_{12}\mathcal{R}_{a_1,a_2}\overline{\mathcal{R}}_{p_1,p_2} /\!\!/ m_1^{a_1,1,p_1} /\!\!/ m_2^{a_2,2,p_2} \tag{2.28}$$

**Definition 2.20** (Drinfeld element)**.** In a quasitriangular meta-Hopf algebra $H$, the Drinfeld element, $\mathfrak{u} \in H$ is:

$$\mathfrak{u} \coloneqq \mathcal{R}_{21} /\!\!/ S^1_1 /\!\!/ m^{12} \tag{2.29}$$

**Definition 2.21** (monodromy)**.** Each quasitriangular meta-Hopf algebra has a monodromy $Q_{12} \coloneqq \mathcal{R}_{12}\mathcal{R}_{34} /\!\!/ m^{14}_1 /\!\!/ m^{23}_2$. Its inverse will be denoted $\overline{Q}_{12} = \overline{\mathcal{R}}_{12}\overline{\mathcal{R}}_{34} /\!\!/ m^{14}_1 /\!\!/ m^{23}_2$.

**Definition 2.22** (ribbon meta-Hopf algebra)**.** A quasitriangular meta-Hopf algebra $H$ is called ribbon if it has an element $\nu \in Z(H)$ such that:

$$\nu_1\nu_2 /\!\!/ m^{12} = \mathfrak{u}_1\mathfrak{u}_2 /\!\!/ S^2_2 /\!\!/ m^{12} \tag{2.30}$$
$$\nu_1 /\!\!/ \Delta^1_{12} = \nu_1\nu_2 /\!\!/ \overline{Q}_{34} /\!\!/ m^{13}_1 /\!\!/ m^{24}_2 \tag{2.31}$$
$$\nu /\!\!/ S = \nu \tag{2.32}$$
$$\nu /\!\!/ \epsilon = \eta /\!\!/ \epsilon = 1 \tag{2.33}$$

**Definition 2.23** (spinner)**.** A spinner[†6]) in a ribbon meta-Hopf algebra $H$ is an invertible element $C \in H$ (with inverse $\overline{C}$) such that for all $x \in H$:

$$C_1\nu_2C_3 /\!\!/ S^2_2 /\!\!/ m^{123} = \nu \tag{2.34}$$
$$C_1 /\!\!/ \Delta^1_{12} = C_1C_2 \tag{2.35}$$
$$C /\!\!/ S = \overline{C} \tag{2.36}$$
$$C_1x_2\overline{C}_3 /\!\!/ m^{123} = x /\!\!/ S /\!\!/ S \tag{2.37}$$
$$C /\!\!/ \epsilon = \eta /\!\!/ \epsilon = 1 \tag{2.38}$$

†6 These are more commonly referred to as distinguished grouplike elements. The term we use is inspired by **??**

**Lemma 2.24** (spinners and ribbon Hopf algebras)**.** *If a (meta-)Hopf algebra has either a ribbon element $\nu$ or a spinner $C$, then it must have the other as well, given by the formula: $C_1\nu_2 /\!\!/ m^{12} = \mathfrak{u}$.*

*Proof.* See Majid's work in [Maj] or Etingof and Schiffmann in [ES] for more details on this standard result. Note that the proof does not rely on the additive structure of the Hopf algebra, which allows us to extend this result to the realm of meta-Hopf algebras. □

## 2.4 THE META-ALGEBRA OF TANGLE DIAGRAMS

The particular structures introduced were chosen for their ability to represent the topological properties of knotted objects. We will now introduce the notion of a tangle and demonstrate its meta-algebraic structure.

### *Upright tangles*

For our purposes, a tangle will be visualised as follows: take a stiff (topologically) circular metal frame forming a Jordan curve (i.e. with a defined inside and outside), then attach a collection of strings to the wire, ensuring that the strings always remain inside the wire, and that each string is tied to the metal frame in two unique locations (that is, no two strings share an endpoint).

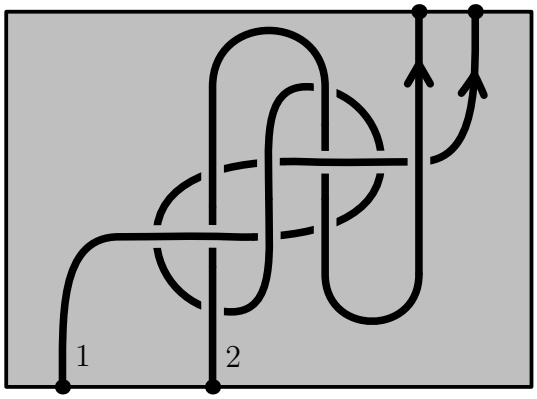



Figure 2.3: Example of a tangle with strands labelled 1 and 2.

**Definition 2.25** (open tangle)**.** An open tangle is an embedding of line segments (called components or strands) into the thickened unit disk $D \times$

$[-1,1]$ (or a disjoint union of such disks) such that the endpoints of the line segments are fixed along $\partial D \times \{0\}$. Each strand is labelled with elements of a set $X$. Two open tangles are considered equivalent if there exists an isotopy of the embedding which fixes the endpoints of the strands. The set of all tangles with strands indexed by $X$ will be denoted $\mathcal{T}_X$. (The term "open" refers to the absence of closed loops.)

The objects which are more natural for us to study are tangles with a framing, which one may thing of as open tangles with the strings replaced with thin ribbons.

**Definition 2.26** (framed tangle)**.** A framed tangle is an open tangle together with a choice of section of the normal bundle for each component, with endpoints of the section fixed pointing to the right of the tangent vector. This choice is taken up to endpoint-fixing homotopy. Unless otherwise mentioned, it will be assumed that all tangles are framed.

Observe that a generic projection of a tangle to its central core $D \times \{0\}$ results in the strands forming a graph, with each crossing of two strands in the tangle producing a vertex in the graph. By assigning to each vertex the sign of the corresponding crossing (either "positive" or "negative"), we end up with a combinatorial object which is equivalent to the original tangle.

**Definition 2.27** (open tangle diagram)**.** An open tangle diagram is a projection of a tangle onto its central core such that all the line segments are immersions which intersect both the boundary disk and the other strands transversally, together with an assignment of a sign to each strand intersection. Small open neighbourhoods of these intersections are called crossings, while the complement of the crossings is a collection of embedded line segments called arcs.

Two open tangle diagrams are considered equivalent if they differ by a finite sequence of Reidemeister moves, as outlined in figures 2.4 to 2.6

The rotation numbers of arcs will play a role in this thesis, so we will capture these data in the following way (as described in [BNvdVa]):

**Definition 2.28** (upright open tangle diagram)**.** An upright tangle diagrams is a tangle diagram with the further requirement that the endpoints of each arc must have a vertical tangent vector, and each crossing must involve only curves with tangent vectors that point (diagonally) upwards. Here, each

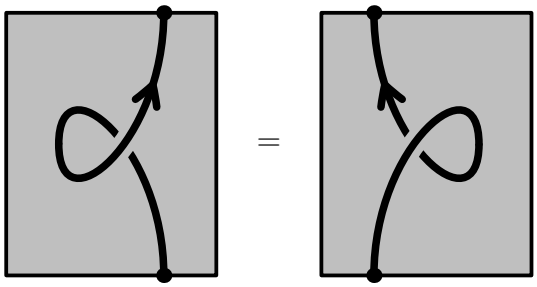


Figure 2.4: (Framed) Reidemeister move $R1'$

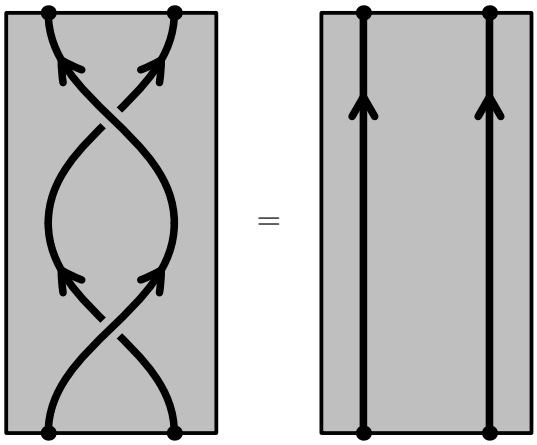


Figure 2.5: Reidemeister move $R2$

arc has well-defined integer rotation number. Two tangles are considered equivalent if they agree under the "rotational Reidemeister moves", which are figures 2.4 to 2.8. Given a finite set $X$, the set of $X$-indexed upright tangle diagrams will be denoted $\mathcal{T}_X^{\text{up}}$.

*Remark* 2.29. The concept of upright tangles was first introduced by Louis Kauffman in [Kau] under the name rotational virtual knot theory. In the formulation here, we insist that all strands end pointing upwards instead of merely requiring that endpoint vectors are vertical, so we will use the term "upright" to remind the reader of this difference.

Fortunately, ambient isotopy allows us to rotate any classical tangle into an upward-pointing form. Additionally, there is only one way to do this. We reproduce the proof of this fact by Bar-Natan and van der Veen in [BNvdVb] below:

**Lemma 2.30** (tangles inject into upright tangles)**.** *To each open tangle diagram $D$ there exists an upright open tangle diagram $D'$ obtained from*

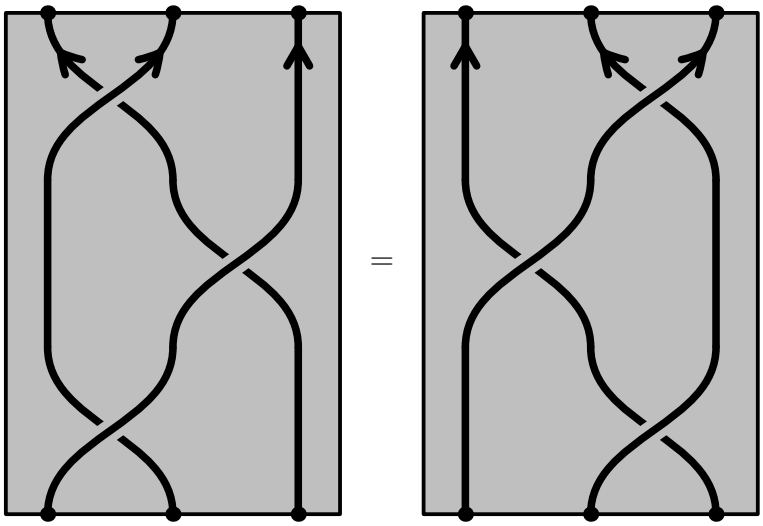


Figure 2.6: Reidemeister move $R3$

*$D$ by a planar isotopy. Further, if $D''$ is another such upright open tangle diagram obtained from $D$, then $D'$ and $D''$ differ by a finite sequence of rotational Reidemeister moves and a change of rotation number at the endpoints.*

*Proof.* Each arc and crossing in the diagram $D$ may be rotated so that its endpoints are pointing upwards, giving rise to a diagram $D'$. Two (nonupright) tangle diagrams are equivalent when they differ by a finite sequence of Reidemeister moves. Each of these Reidemeister may also be rotated to an equivalence of upright tangles, each of which is given as a rotational Reidemeister move figures 2.4 to 2.7. The last possibility is the rotation of an entire crossing, which is covered by figure 2.8. □

*The meta-algebra structure of upright tangle diagrams*

We now formally connect tangle diagrams with meta-algebras.

**Theorem 2.31** (tangles form a ribbon meta-Hopf algebra)**.** *The collection $\{\mathcal{T}_X^{up}\}_X$ forms a ribbon meta-Hopf algebra (in the category* **Set***) with the following operations:*

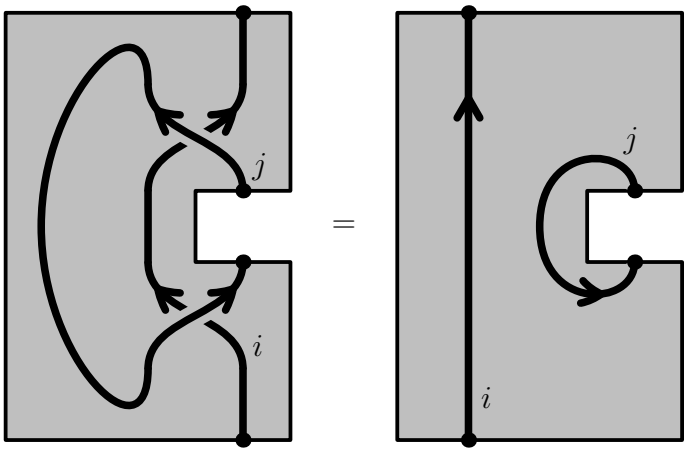



Figure 2.7: The (cyclic) rotational Reidemeister move $R2_{\text{rot}}$

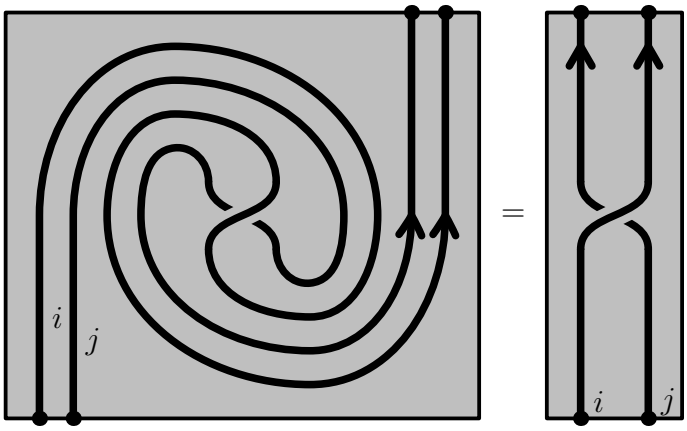



Figure 2.8: The whirling move

- *multiplication* $m_k^{ij}[X]$ *takes a tangle with strands* $X \sqcup \{i, j\}$ *and glues the end of strand* $i$ *to strand* $j$*, labelling the resulting strand* $k$ *(figure 2.9).*[†7]
- *the unit* $\eta_i[X]$ *takes a tangle diagram with strands* $X$ *and introduces a new strand* $i$ *which does not touch any of the other strands (figure 2.10).*
- *the comultiplication* $\Delta_{jk}^i[X]$ *takes a tangle with strands* $X \sqcup \{i\}$ *and doubles strand* $i$*, separating the two strands along the framing of strand* $i$*, calling the right strand* $j$ *and the left one* $k$ *(figure 2.12).*[†8]

†7 Strictly speaking, this operation is only defined when the end of strand $i$ is adjacent to strand $j$. See remark 2.32 for more details.

†8 While this convention appears unfortunate, we follow the notation laid out in [BNvdVb] so that the antipode and spinner have a more memorable representation, namely looking like the letters they are represented by (see figures 2.13 and 2.17 for the resemblance).

- *the counit $\epsilon^i[X]$ takes a tangle with strands indexed by $X \sqcup \{i\}$ and returns the tangle with strand labelled by $i$ deleted (figure 2.11).*
- *The antipode $S^i_j[X]$ takes a tangle with strands labelled by $X \sqcup \{i\}$ and reverses the direction of strand $i$, then adds a counter-clockwise cap to the new beginning, and a clockwise cup to the end. This new strand is called $j$. When applied to a single vertical strand, the resulting tangle looks like the letter "S" (figure 2.13).*
- *the $\mathcal{R}$-matrix $\mathcal{R}_{ij}$ is given by the two-strand tangle with a single positive crossing of strand $i$ over strand $j$. The inverse $\mathcal{R}$-matrix $\overline{\mathcal{R}}_{ij}$ is the two-strand tangle with a negative crossing of strand $i$ over strand $j$ (figure 2.14).*
- *The spinner $C_i[X]$ takes a tangle in $\mathcal{T}^{up}_X$ and adds a new strand with rotation number 1 which has no interactions with any other strands. This new strand looks like the letter "C" (figure 2.17).*

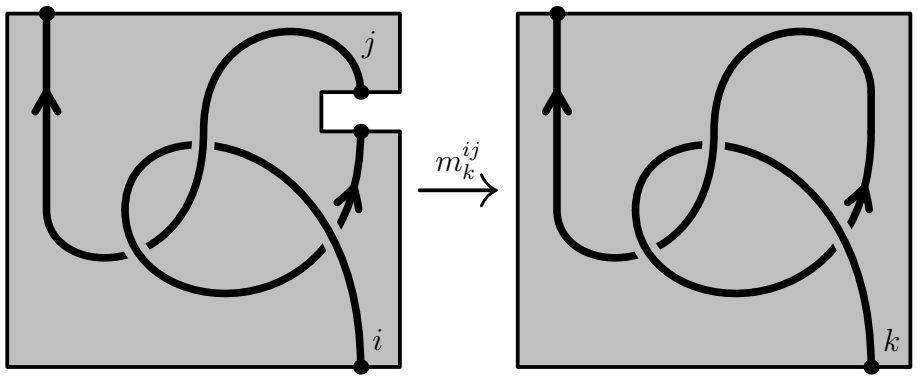



Figure 2.9: Multiplication $m^{ij}_k$ stitches two strands in a tangle together.

*Proof.* We go through a straightforward verification of the various axioms:

1. We show $\mathcal{T}^{\text{up}}_X$ is a meta-algebra:
   a) Associativity (equation (2.17)) holds, since stitching a sequence of strands together in a particular order, the order in which the individual stitches are made does not affect the resulting strand. The resulting strand has the same endpoints and follows the same path.

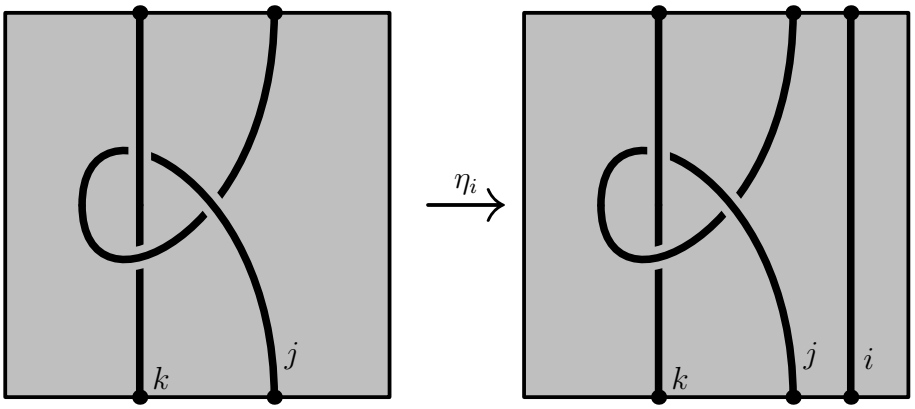



Figure 2.10: The unit $\eta_i$ introduces a new strand in a tangle.

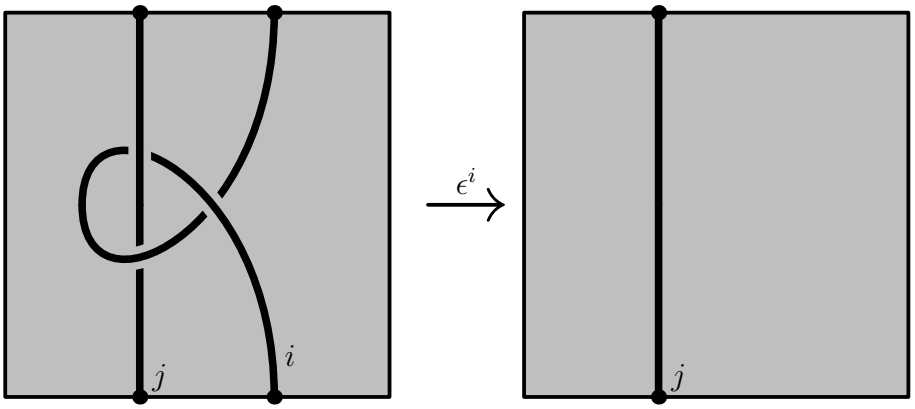



Figure 2.11: The counit $\epsilon^i$ deletes a strand in a tangle.

b) To see that the unit axiom (equation (2.18)) holds, observe that adding a non-interacting strand to a diagram, then stitching it to an existing strand does not change any of the combinatorial data, resulting in identical diagrams.

2. We show $\mathcal{T}_X^{\text{up}}$ is a meta-coalgebra:

   a) Establishing coassociativity (equation (2.19)) amounts to the same argument that cutting a strip of paper into three parallel strips does not depend on the order of cutting.

   b) The counit identity (equation (2.20)) states deleting a strand is the same operation as first doubling it, then deleting both resulting strands.

3. The meta-bialgebra axioms we verify next:

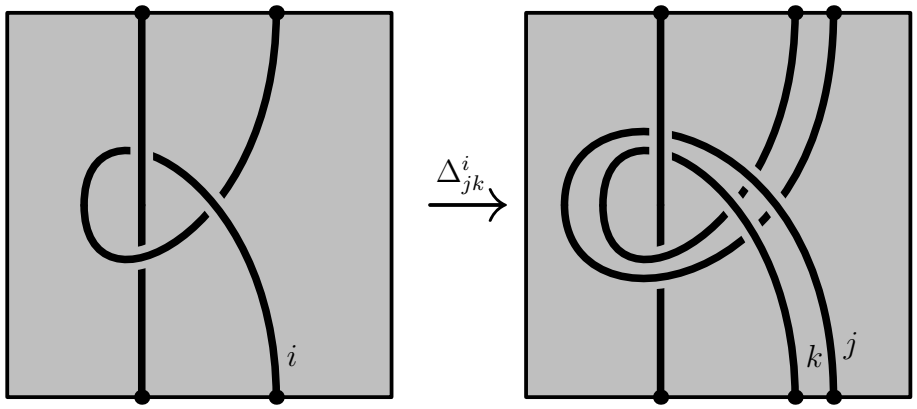



Figure 2.12: The comultiplication $\Delta^i_{jk}$ doubles a strand in a tangle along its framing. Notice the right-to-left strand labels.

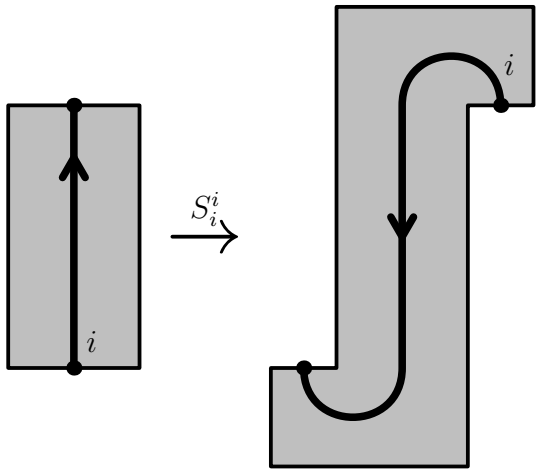
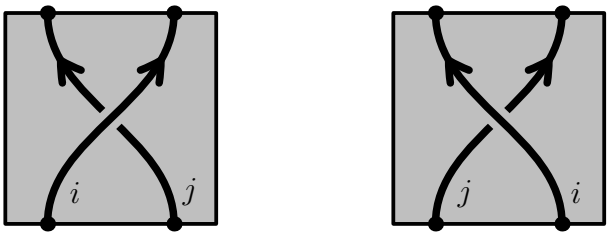



Figure 2.13: The antipode $S^i_i$ reverses a strand, rotating the endpoints to maintain an upright tangle.

a) Equation (2.21) states that if two strands are stitched together, then the resulting strand is doubled, this could have equivalently been achieved by doubling each of the original strands, then performing a stitching on both resulting pairs of strands.

b) Equation (2.23) simply states that stitching two strands together, then removing the resulting strand could have equally been achieved by removing both of the original strands without stitching them first.

c) Equation (2.24) states that introducing a strand, then immediately removing it is the identity operation.




(a) A positive crossing, represented by $\mathcal{R}_{ij}$

(b) A negative crossing, represented by $\overline{\mathcal{R}}_{ij}$

Figure 2.14: The $\mathcal{R}$-matrix and its inverse represent a tangle with a single crossing.

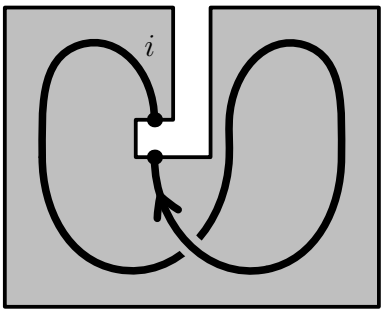



Figure 2.15: The Drinfeld element $\mathfrak{u}_i$ in the meta-Hopf algebra of tangles.

d) Equation (2.22) says that doubling a newly-introduced (and therefore free of crossings) strand is the same operation as introducing two strands separately. (For those worried that this equation depends on the location of the separately introduced strands, this is one place that the use of virtual tangles will be used, which does not heed the relative locations of disjoint strands.)

4. Next, we show $\mathcal{T}_X^{\text{up}}$ is a meta-Hopf algebra. Equation (2.25) states that when a strand is doubled, then one of the two strands is reversed, multiplying the two strands together results in a strand which can be rearranged to not interact with any of the other strands. This can be readily seen, as this newly-created strand looks like a snake weaving through the tangle diagram. One can remove the snake by applying a series of Reidemeister 2 moves, resulting in a strand disjoint from the rest of the diagram. This is the same as deleting the original strand, then introducing a new disjoint one.

5. The quasitriangular axioms are equalities of pairs of three-strand tangles:

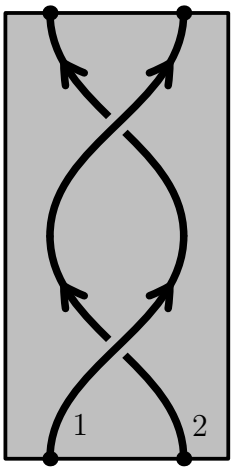
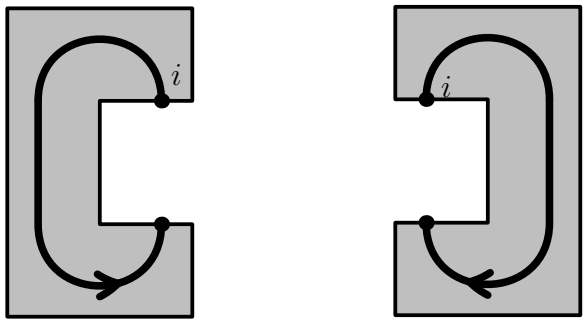



Figure 2.16: The monodromy in the meta-Hopf algebra of tangles.




(a) The spinner $C_i$ has rotation number 1.

(b) The inverse spinner $\overline{C}_i$ has rotation number $-1$.

Figure 2.17: The spinners represent strands with a unit rotation number.

a) Equations (2.26) and (2.27) tell us that doubling a strand involved in a single crossing can also be built by adjoining two crossings together. This is visualized in figures 2.19 and 2.20.

b) Equation (2.28) tells us that we can swap the order of a doubled strand by adding crossings to either end (reminiscent of a Reidemeister 2 move).

6. Next, we establish that $\mathcal{T}_X^{\text{up}}$ is ribbon. Using lemma 2.24, it is enough to verify the spinner axioms (equations (2.34) to (2.38)). All these axioms have corresponding pictures one can draw, keeping in mind the orientations in the definitions of the relevant operations.

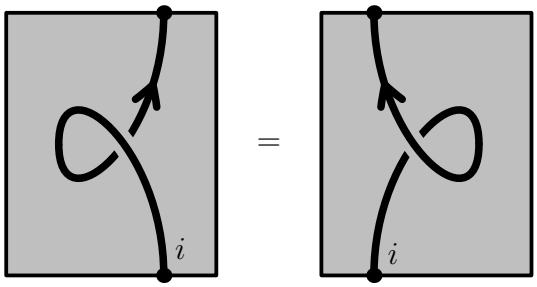



Figure 2.18: A ribbon element $\nu_i$ in the meta-Hopf algebra of tangles. One can use lemma 2.24 to verify this is compatible with the spinner.

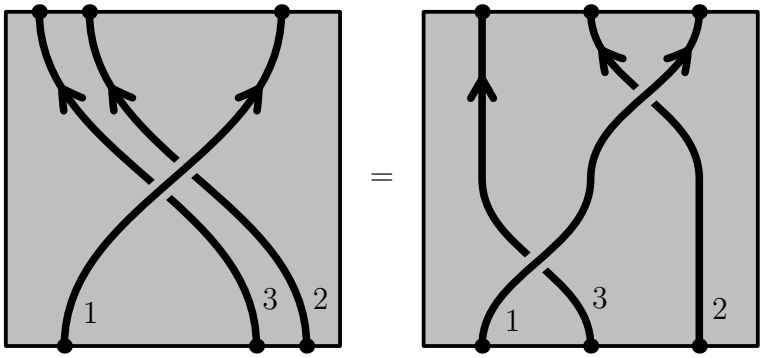



Figure 2.19: Example of a tangle satisfying equation (2.27)

Finally, we observe that the Reidemeister moves do not introduce any new relations. Reidemeister 2 follows from the invertibility of the $\mathcal{R}$-matrix. Next, it is readily seen that the quasitriangular relations governing the $\mathcal{R}$-matrix force it to solve the Yang-Baxter equation, which is one equivalent to the Reidemeister 3 in this case. □

*Remark* 2.32. One may object that strand-stitching $m_k^{ij}$ is not defined when the endpoint of strand $i$ is not adjacent to the starting point of strand $j$. This issue is resolved in multiple ways:

1. Extend the collection of tangles we work with to include virtual tangles. This generalization of tangles deals exactly with the issue that multiplication need not produce a planar tangle diagram. In fact, virtual tangles can be thought of as merely non-planar tangle diagrams.

2. Commit to only apply multiplication when doing so would result in a valid (classical) tangle. This is the approach we will take when performing computations on tangles.

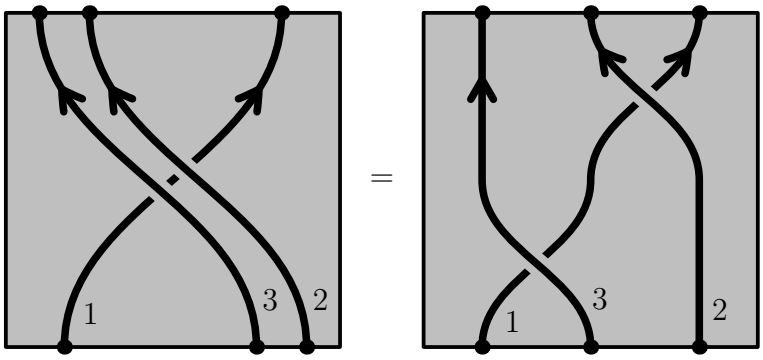



Figure 2.20: Example of a tangle satisfying equation (2.26)

## 2.5 THE META-ALGEBRA $U$

Here we define the ribbon Hopf algebra $U = \widehat{\mathfrak{U}}(\mathfrak{sl}_{2+}^0)$, and point out some of its properties.

**Definition 2.33** (The ribbon Hopf algebra $U$)**.** Define the Lie algebra

$$\mathfrak{g} := \operatorname{span}_{\mathbb{Q}}\left\{ \mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x} \,\middle|\, [\mathbf{a}, \mathbf{x}] = \mathbf{x}, [\mathbf{a}, \mathbf{y}] = -\mathbf{y}, [\mathbf{x}, \mathbf{y}] = \mathbf{b}, [\mathbf{b}, \cdot\,] = 0 \right\} \tag{2.39}$$

Then the algebra $U$ is defined to be the $b$-adic completion of the universal enveloping algebra $\widehat{\mathfrak{U}}(\mathfrak{g})$. With $\mathbf{B} := \mathbf{e}^{-\mathbf{b}}$, the bialgebra structure of $U$ is:

$$\begin{aligned}
\Delta_{i,j}(\mathbf{y}) &= \frac{\mathbf{b}_i + \mathbf{b}_j}{1 - \mathbf{B}_i \mathbf{B}_j} \left( \mathbf{B}_j \frac{1 - \mathbf{B}_i}{\mathbf{b}_i} \mathbf{y}_i + \frac{1 - \mathbf{B}_j}{\mathbf{b}_j} \mathbf{y}_j \right) \\
\Delta_{i,j}(\mathbf{b}) &= \mathbf{b}_i + \mathbf{b}_j \\
\Delta_{i,j}(\mathbf{a}) &= \mathbf{a}_i + \mathbf{a}_j \\
\Delta_{i,j}(\mathbf{x}) &= \mathbf{x}_i + \mathbf{x}_j
\end{aligned} \tag{2.40}$$

For any $\mathbf{z} \in \{\mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x}\}$, we have $\epsilon(\mathbf{z}) = 0$ (extended multiplicatively by equation (2.23)).

Next, we define the Hopf algebra structure by defining the antipode, which is defined as $S(\mathbf{z}) := -\mathbf{z}$ for each $\mathbf{z} \in \{\mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x}\}$, extended antimultiplicatively.

Next, we introduce the ribbon structure of $U$ with an $\mathcal{R}$-matrix and the spinner $C$:

$$\mathcal{R}_{i,j} := \exp(\mathbf{b}_i \mathbf{a}_j) \exp\left(\frac{1-\mathbf{B}_i}{\mathbf{b}_i} \mathbf{y}_i \mathbf{x}_j\right) \tag{2.41}$$

$$C := \sqrt{\mathbf{B}} \tag{2.42}$$

$$\nu := \overline{\mathcal{R}}_{31} \overline{C}_2 /\!\!/ m^{123} = \overline{\mathcal{R}}_{13} C_2 /\!\!/ m^{123} \tag{2.43}$$

**Lemma 2.34** (Commutation relations in $U$)**.** *Given $f \in \mathbb{Q}[\mathbf{a}]$, we have the following relations in $U$:*

$$f(\mathbf{a})\mathbf{y}^r = \mathbf{y}^r f(\mathbf{a}-r) \qquad\qquad \mathbf{x}^r f(\mathbf{a}) = f(\mathbf{a}-r)\mathbf{x}^r \tag{2.44}$$

*These relations also hold when $f(a)$ is replaced with $f(a,b) \in \mathbb{Q}[\mathbf{a}][\![\mathbf{b}]\!]$.*

*Proof.* We begin by commuting a single "$\mathbf{a}$" past a power of $\mathbf{y}$:

$$\mathbf{a}y^r = \mathbf{y}(\mathbf{a}-1)\mathbf{y}^{r-1} = \mathbf{y}^2(\mathbf{a}-2)\mathbf{y}^{r-2} = \cdots = \mathbf{y}^r(\mathbf{a}-r) \tag{2.45}$$

Repeating the above process for multiple copies of $a$ yields:

$$\mathbf{a}^k y^r = \mathbf{a}^{k-1}(^r) = \mathbf{a}^{k-1}\mathbf{y}^r(\mathbf{a}-r) = \mathbf{a}^{k-2}\mathbf{y}^r(\mathbf{a}-r)^2 = \cdots = \mathbf{y}^r(\mathbf{a}-r)^k \tag{2.46}$$

Equation (2.46) establishes a linear relation, so the result extends to any polynomial in $\mathbf{a}$, and thereby to any element of $\mathbb{Q}[\mathbf{a}][\![\mathbf{b}]\!]$.

A similar argument exists to show $\mathbf{x}^r f(\mathbf{a}) = f(\mathbf{a}-r)\mathbf{x}^r$. □

Next, we observe that since $[\mathbf{x}, \mathbf{y}] = \mathbf{b}$ is central, the Weyl canonical commutation relation holds:

**Lemma 2.35** (Weyl canonical commutation relation)**.** *Let $\xi, \eta \in \mathbb{Q}[\![\mathbf{b}]\!]$. Then:*

$$\mathbf{e}^{\xi\mathbf{x}}\mathbf{e}^{\eta\mathbf{y}} = \mathbf{e}^{\xi\eta\mathbf{b}}\mathbf{e}^{\eta\mathbf{y}}\mathbf{e}^{\xi\mathbf{x}} \tag{2.47}$$

*Proof.* See, Weyl's [Wey] or Hall's [Hal] for details on this result. □

We have further commutation relations with exponentials:

**Lemma 2.36** (Exponential commutation relations in $U$)**.** *Let $\alpha, \eta, \xi \in \mathbb{Q}[\![\mathbf{b}]\!]$, and write $\mathcal{A} := \mathbf{e}^{\alpha}$. Then we have*

$$\mathbf{e}^{\alpha \mathbf{a}} \mathbf{e}^{\eta \mathbf{y}} = \mathbf{e}^{\frac{\eta}{\mathcal{A}} \mathbf{y}} \mathbf{e}^{\alpha \mathbf{a}} \tag{2.48}$$

$$\mathbf{e}^{\xi \mathbf{x}} \mathbf{e}^{\alpha \mathbf{a}} = \mathbf{e}^{\alpha \mathbf{a}} \mathbf{e}^{\frac{\xi}{\mathcal{A}} \mathbf{x}} \tag{2.49}$$

*Proof.* Using equation (2.44), we notice

$$\mathbf{e}^{\alpha \mathbf{a}} \mathbf{e}^{\eta \mathbf{y}} = \mathbf{e}^{\alpha \mathbf{a}} \sum_n \frac{(\eta \mathbf{y})^n}{n!} = \sum_n \frac{(\eta \mathbf{y})^n}{n!} \mathbf{e}^{\alpha(\mathbf{a} - n)} = \mathbf{e}^{\frac{\eta}{\mathcal{A}} \mathbf{y}} \mathbf{e}^{\alpha \mathbf{a}} \tag{2.50}$$

similarly,

$$\mathbf{e}^{\xi \mathbf{x}} \mathbf{e}^{\alpha \mathbf{a}} = \sum_n \frac{(\xi \mathbf{x})^n}{n!} \mathbf{e}^{\alpha \mathbf{a}} = \sum_n \mathbf{e}^{\alpha(\mathbf{a} - n)} \frac{(\xi \mathbf{x})^n}{n!} = \mathbf{e}^{\alpha \mathbf{a}} \mathbf{e}^{\frac{\xi}{\mathcal{A}} \mathbf{x}} \tag{2.51}$$

□

**Lemma 2.37** (the algebra $U$ is ribbon)**.** *The algebra $U$ has a ribbon structure given by the above $\mathcal{R}$-matrix and spinner $C$.*

*Proof.* The Hopf algebra structure of $U$ is straightforward, and is left to the reader to verify. We will focus our attention on verifying quasitriangularity and the ribbon structure.

Let us verify equation (2.27) first. The left-hand side is:

$$\mathcal{R}_{12} /\!\!/ \Delta^2_{23} = \exp(\mathbf{b}_1(\mathbf{a}_2 + \mathbf{a}_3)) \exp\left( \frac{1 - \mathbf{B}_1}{\mathbf{b}_1} \mathbf{y}_1 (\mathbf{x}_2 + \mathbf{x}_3) \right) \tag{2.52}$$

Equality with the right-hand side follows by commutativity of $\mathbf{b}_1$ and $\mathbf{y}_1$:

$$\begin{aligned} \mathcal{R}_{13} \mathcal{R}_{42} /\!\!/ m_1^{14} &= \exp(\mathbf{b}_1 \mathbf{a}_3) \exp\left( \frac{1 - \mathbf{B}_1}{\mathbf{b}_1} \mathbf{y}_1 \mathbf{x}_3 \right) \exp(\mathbf{b}_1 \mathbf{a}_2) \exp\left( \frac{1 - \mathbf{B}_1}{\mathbf{b}_1} \mathbf{y}_1 \mathbf{x}_2 \right) \\ &= \exp(\mathbf{b}_1(\mathbf{a}_2 + \mathbf{a}_3)) \exp\left( \frac{1 - \mathbf{B}_1}{\mathbf{b}_1} \mathbf{y}_1 (\mathbf{x}_2 + \mathbf{x}_3) \right) \end{aligned} \tag{2.53}$$

Next we verify equation (2.26), whose left-hand side is:

$$\mathcal{R}_{13} /\!\!/ \Delta^1_{12} = \exp((\mathbf{b}_1 + \mathbf{b}_2)\mathbf{a}_3) \exp\left( \left( \mathbf{B}_2 \frac{1 - \mathbf{B}_1}{\mathbf{b}_1} y_1 + \frac{1 - \mathbf{B}_2}{\mathbf{b}_2} y_2 \right) \mathbf{x}_3 \right) \tag{2.54}$$

On the right-hand side, we have

$$\begin{aligned}\mathcal{R}_{13}\mathcal{R}_{24} /\!\!/ m_3^{34} &= \exp(\mathbf{b}_1\mathbf{a}_3)\exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{y}_1\mathbf{x}_3\right)\exp(\mathbf{b}_2\mathbf{a}_3)\exp\left(\frac{1-\mathbf{B}_2}{\mathbf{b}_2}\mathbf{y}_2\mathbf{x}_3\right)\\ &= \exp((\mathbf{b}_1+\mathbf{b}_2)\mathbf{a}_3)\exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{B}_2\mathbf{y}_1\mathbf{x}_3\right)\exp\left(\frac{1-\mathbf{B}_2}{\mathbf{b}_2}\mathbf{y}_2\mathbf{x}_3\right)\end{aligned} \tag{2.55}$$

We use lemma 2.36 to write the expression in a canonical order. Finally, the right two exponentials may be combined since each variable commutes with the others, either by belonging to separate tensor factors, or in the case of $\mathbf{b}$, being central. The verifications of equation (2.28) and equations (2.34) to (2.38) follow with similar computations. □

## 2.6 MORPHISMS BETWEEN META-OBJECTS

We can define a tangle invariant by considering a morphism between the meta-structure of tangles and that of an algebraic object. We now define a morphism between meta-objects:

**Definition 2.38** (morphism of meta-objects)**.** Let $\{A_X\}_X$ and $\{B_X\}_X$ be compatible meta-objects (i.e. ones with the same operations and relations between the operations). A morphism $\Phi$ between these meta-objects is map $\Phi\colon \{A_X\}_X \to \{B_X\}_X$ sending $A_X \mapsto B_X$ such that each operation $\varphi_Y^X$ in $A$ intertwines with that in $B$:

$$\Phi(\phi_Y^X) = \phi_Y^X \tag{2.56}$$

*Upright tangle invariants from a ribbon meta-Hopf algebra*

We define a $U$-valued tangle invariant in the following way:

1. Given a open tangle, disconnect each crossing from its neighbours, as well as each arc with a nonzero rotation number.
2. Replace each crossing with an $\mathcal{R}$-matrix $\mathcal{R}_{ij} \in U_{\{i,j\}}$, and each rotation of an arc with a spinner $C_i \in U_{\{i\}}$.
3. For each disconnection, there is a corresponding stitching operation required to bring the tangle back to its original state. Replace each stitching operation with a multiplication operation in $U$.

Figure 2.21 provides an example of this process. There, the meta-morphism $Z$ sends the right-hand tangle to $\mathcal{R}_{3,7}\mathcal{R}_{6,2}\mathcal{R}_{1,5}\overline{C}_4$. By equation (2.56), the image of the left-hand tangle (which is the knot $3_1$) under $Z$ is $\mathcal{R}_{3,7}\mathcal{R}_{6,2}\mathcal{R}_{1,5}\overline{C}_4 /\!\!/ m_i^{1,2,\dots,7}$. As an algebra element, this is

$$\exp(\mathbf{b}_3\mathbf{a}_7)\exp\left(\frac{1-\mathbf{B}_3}{\mathbf{b}_3}\mathbf{y}_3\mathbf{x}_7\right)\exp(\mathbf{b}_6\mathbf{a}_2)\exp\left(\frac{1-\mathbf{B}_6}{\mathbf{b}_6}\mathbf{y}_6\mathbf{x}_2\right) \cdot \exp(\mathbf{b}_1\mathbf{a}_5)\exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{y}_1\mathbf{x}_5\right) /\!\!/ m_i^{1,2,\dots,7} \tag{2.57}$$

In order to meaningfully compare such expressions, we must first write them in a canonical form. The form we use here is to order monomials so that they are of the form $\mathbf{y}^{n_1}\mathbf{b}^{n_2}\mathbf{a}^{n_3}\mathbf{x}^{n_4}$. This may always be done through the use of the relations with the Lie bracket. Doing so efficiently is a difficult task, and will be addressed in the following chapter.

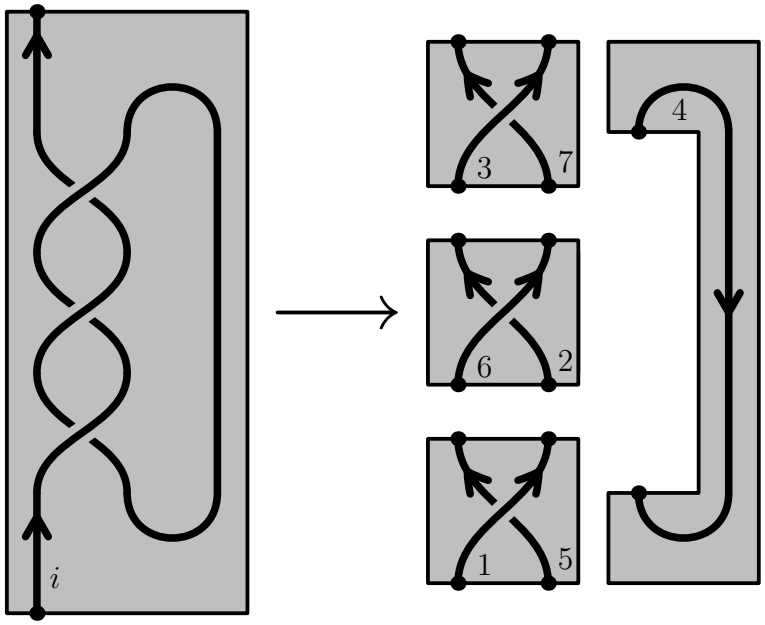



Figure 2.21: Breaking up a tangle into its constituent components.

# 3

# PERTURBED GAUSSIANS

We now summarize the work of Bar-Natan and van der Veen in [BNvdVb], which develops a universal knot invariant using perturbed Gaußians.

## 3.1 EXPRESSING MORPHISMS AS GENERATING FUNCTIONS

The invariant we have defined takes the form of building a knot out of several crossings and spinners, then replacing each crossing with an $\mathcal{R}$-matrix $\mathcal{R}_{ij}$ and spinners $C_k$. Finally, we apply a series of multiplication operations to join the various tensor factors into the final value. In order to be able to apply the operations efficiently, we need a compact way of encoding morphisms. In [BNvdVb], Bar-Natan and van der Veen achieve this through the use of generating functions, whose definition we reproduce below:

For $A$ and $B$ finite sets, consider the set $\hom(\mathbb{Q}[z_A], \mathbb{Q}[z_B])$ of linear maps between multivariate polynomial rings. Such a map is determined by its values on the monomials $z_A^{\mathbf{n}}$ for each multi-index $\mathbf{n} \in \mathbb{N}^A$.

**Definition 3.1** (Exponential generating function)**.** Given a linear map $\Phi\colon \mathbb{Q}[z_A] \to \mathbb{Q}[z_B]$ between polynomial spaces, its exponential generating function is:

$$\mathcal{G}(\Phi) := \sum_{\mathbf{n}\in\mathbb{N}^A} \frac{\Phi(z_A^{\mathbf{n}})}{\mathbf{n}!}\zeta_A^{\mathbf{n}} \in \mathbb{Q}[z_B][\![\zeta_A]\!] \tag{3.1}$$

*Remark* 3.2. We may extend the domain of $\Phi$ from $\mathbb{Q}[z_B]$ to $\mathbb{Q}[z_B][\![\zeta_A]\!]$ via extension of scalars $\mathbb{Q} \to \mathbb{Q}[\![\zeta_A]\!]$. This extension allows us to write the exponential generating function in a new way:

$$\mathcal{G}(\Phi) = \Phi\left(\sum_{\mathbf{n}\in\mathbb{N}^A} \frac{(z_A\zeta_A)^{\mathbf{n}}}{\mathbf{n}!}\right) = \Phi\Big(\mathcal{G}\big(\mathrm{id}_{\mathbb{Q}[z_A]}\big)\Big) \tag{3.2}$$

By the PBW theorem, we know that $U$ is isomorphic as a vector space to the polynomial ring $\mathbb{Q}[y, b, a, x]$ by choosing an ordering of the generators (following [BNvdVb], we use $(\mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x})$):

$$\begin{aligned} \mathbb{O} \colon \mathbb{Q}[y, b, a, x] &\xrightarrow{\sim} U \\ y^{n_1} b^{n_2} a^{n_3} x^{n_4} &\mapsto \mathbf{y}^{n_1} \mathbf{b}^{n_2} \mathbf{a}^{n_3} \mathbf{x}^{n_4} \end{aligned} \tag{3.3}$$

To efficiently describe $\Bbbk$-linear maps between tensor powers of the algebra $U$, we define categories $\mathcal{U}$, $\mathcal{H}$, and $\mathcal{C}$ with objects finite sets and morphisms:

$$\mathrm{Hom}_{\mathcal{U}}(J, K) \coloneqq \mathrm{Hom}_{\Bbbk}(U^{\otimes J}, U^{\otimes K}) \tag{3.4}$$

$$\mathrm{Hom}_{\mathcal{H}}(J, K) \coloneqq \mathrm{Hom}_{\Bbbk}(\mathbb{Q}[z_J], \mathbb{Q}[z_K]) \tag{3.5}$$

$$\mathrm{Hom}_{\mathcal{C}}(J, K) \coloneqq \mathbb{Q}[z_K][\![\zeta_J]\!] \tag{3.6}$$

Where equations (3.4) and (3.5) explicitly denote vector space maps, not just algebra or ring homomorphisms. There exist monoidal isomorphisms between these categories, namely $\mathbb{O} \colon \mathcal{H} \xrightarrow{\sim} \mathcal{U}$ and $\mathcal{G} \colon \mathcal{H} \xrightarrow{\sim} \mathcal{C}$ as introduced in equations (3.1) and (3.3).

We use this formulation because of the existence of a computationally amenable subcategory of $\mathcal{C}$ which contains the image of this invariant.

*Formulating composition in other categories*

Composing operations in $\mathcal{U}$ or $\mathcal{H}$ is straightforward to define, but lacks a closed form. However, on $\mathcal{C}$, the corresponding definition of composition takes the following form (quoted from [BNvdVb, Lemma 3]):

**Lemma 3.3** (Composition of generating functions)**.** *Suppose $J$, $K$, $L$ are finite sets and $\phi \in \mathrm{Hom}(\mathbb{Q}[z_J], \mathbb{Q}[z_K])$ and $\psi \in \mathrm{Hom}(\mathbb{Q}[z_K], \mathbb{Q}[z_L])$. We have*

$$\mathcal{G}(\phi /\!\!/ \psi) = \left( \mathcal{G}(\phi)|_{z_K \mapsto \partial_{\zeta_K}} \mathcal{G}(\psi) \right) \Big|_{\zeta_K = 0} \tag{3.7}$$

Since the above notation will occur several times, we will use the notion of contraction used by Bar-Natan and van der Veen (taken from [BNvdVb, Definition 4]):

**Definition 3.4** (Contraction)**.** Let $f \in \Bbbk[\![r, s]\!]$ be a powerseries. The contraction of $f = \sum_{k,l} c_{k,l} r^k s^l$ along the pair $(r, s)$ is:

$$\langle f \rangle_{(r,s)} := \sum_k c_{k,k} k! = \sum_{k,l} c_{k,l} \partial_s^k s^l \Big|_{s=0} \tag{3.8}$$

Further, this notation is to be extended to allow for multiple consecutive contractions for $f \in \Bbbk[\![r_i, s_i]\!]_{i \leq n}$:

$$\langle f \rangle_{((r_i)_i,(s_i)_i)} := \Big\langle \Big\langle \langle f \rangle_{(r_1,s_1)} \Big\rangle_{(r_2,s_2)} \cdots \Big\rangle_{(r_n,s_n)} \tag{3.9}$$

It is important to note that contraction does not always define a convergent expression. We will focus our attention on cases when convergence is well-defined, and especially those where the computation is accessible.

The theorem we will rely heavily on in this thesis is the following, taken from [BNvdVb, Theorem 6]:

**Theorem 3.5** (Contraction theorem)**.** *For any $n \in \mathbb{N}$, consider the ring $R_n = \mathbb{Q}[r_j, g_j][\![s_j, W_{ij}, f_j \mid 1 \leq i, j \leq n]\!]$. Then*

$$\langle \mathbf{e}^{gs+rf+rWs} \rangle_{r,s} = \det(\tilde{W}) \mathbf{e}^{g\tilde{W}f} \tag{3.10}$$

*where $\tilde{W} = (1 - W)^{-1}$.*

The main takeaway of this theorem is this: morphisms whose generating functions are Gaußians have a clean formula for composition. Furthermore, this formula is computationally reasonable, growing only polynomially in complexity with $n$. This is contrasted with the conventional approach of choosing a representation $V$ of $U$. When one considers morphisms between large tensor powers $V^{\otimes n}$, the computational complexity is exponential in $n$.

*Expressing Hopf algebra operations as perturbed Gaußians*

Using this vector space isomorphism, [BNvdVb] expresses all Hopf algebra operations as power series in a closed form, namely as perturbed Gaußians.

**Theorem 3.6** (The meta-Hopf structure of $U$ is Gaußian)**.** *Each of the meta-Hopf algebra operations for $U$ as defined in section 2.5 all have the form of a perturbed Gaußian. That is, when the generators $(y, b, a, x)$ are assigned*

*weights of* $(1,0,2,1)$ *respectively, and their dual variables are assigned complementary weights so that* $\operatorname{wt} z + \operatorname{wt} z^* = 2$*, we have the following expressions which are either Gaußian or central:*

$$\mathcal{G}\left(m_k^{ij}\right) = \exp\Bigg((\alpha_i+\alpha_j)a_k + (\beta_i+\beta_j+\xi_i\eta_j)b_k + \left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right)x_k + \left(\frac{\eta_j}{\mathcal{A}_i}+\eta_i\right)y_k\Bigg) \tag{3.11}$$

$$\mathcal{G}(\eta_i) = 1 \tag{3.12}$$

$$\mathcal{G}\left(\Delta_{jk}^i\right) = \exp\Bigg(\eta_i\frac{b_j+b_k}{1-B_jB_k}\left(B_k\frac{1-B_j}{b_j}y_j + \frac{1-B_k}{b_k}y_k\right) + \beta_i(b_j+b_k) + \alpha_i(a_j+a_k) + \xi_i(x_j+x_k)\Bigg) \tag{3.13}$$

$$\mathcal{G}(\epsilon^i) = 1 \tag{3.14}$$

$$\mathcal{G}(S_i^i) = \exp(-\eta_i\mathcal{A}_iy_i - \beta_ib_i + \eta_i\mathcal{A}_i\xi_ib_i - a_i\alpha_i - \mathcal{A}_i\xi_ix_i) \tag{3.15}$$

$$\mathcal{G}(\mathcal{R}_{ij}) = \exp\left(a_jb_i + \frac{1-B_i}{b_i}y_ix_j\right) \tag{3.16}$$

$$\mathcal{G}(C_i) = \sqrt{B_i} \tag{3.17}$$

$$\mathcal{G}(\nu_i) = \sqrt{B_i}\exp\left(a_ib_i + \frac{1-B_i}{b_i}x_iy_i\right) \tag{3.18}$$

*Proof.* To prove equation (3.11), we use equations (2.47) to (2.49), which allows us to commute exponentials past each other to bring expressions into *ybax*-order. Below we omit the index $k$ for readability:

$$\begin{aligned}
\mathcal{G}(m^{ij}) &= (\mathbb{O}^{-1}\circ m^{ij}\circ\mathbb{O})(\mathbf{e}^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i}\mathbf{e}^{\eta_jy_j+\beta_jb_j+\alpha_ja_j+\xi_jx_j}) \\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{\eta_i\mathbf{y}}\mathbf{e}^{\beta_i\mathbf{b}}\mathbf{e}^{\alpha_i\mathbf{a}}\mathbf{e}^{\xi_i\mathbf{x}}\mathbf{e}^{\eta_j\mathbf{y}}\mathbf{e}^{\beta_j\mathbf{b}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\xi_j\mathbf{x}}\right) \\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{\eta_i\mathbf{y}}\mathbf{e}^{\beta_i\mathbf{b}}\mathbf{e}^{\alpha_i\mathbf{a}}\left(\mathbf{e}^{-\xi_i\eta_j\mathbf{b}}\mathbf{e}^{\eta_j\mathbf{y}}\mathbf{e}^{\xi_i\mathbf{x}}\right)\mathbf{e}^{\beta_j\mathbf{b}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\xi_j\mathbf{x}}\right) \\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)\mathbf{b}}\mathbf{e}^{\eta_i\mathbf{y}}\mathbf{e}^{\frac{\eta_j}{\mathcal{A}_i}\mathbf{y}}\mathbf{e}^{\alpha_i\mathbf{a}}\mathbf{e}^{\xi_i\mathbf{x}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\xi_j\mathbf{x}}\right) \\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)\mathbf{b}}\mathbf{e}^{(\eta_i+\frac{\eta_j}{\mathcal{A}_i})\mathbf{y}}\mathbf{e}^{\alpha_i\mathbf{a}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\frac{\xi_i}{\mathcal{A}_j}\mathbf{x}}\mathbf{e}^{\xi_j\mathbf{x}}\right) \\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{\left(\eta_i+\frac{\eta_j}{\mathcal{A}_i}\right)\mathbf{y}}\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)\mathbf{b}}\mathbf{e}^{(\alpha_i+\alpha_j)\mathbf{a}}\mathbf{e}^{\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right)\mathbf{x}}\right) \\
&= \mathbf{e}^{\left(\eta_i+\frac{\eta_j}{\mathcal{A}_i}\right)y}\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)b}\mathbf{e}^{(\alpha_i+\alpha_j)a}\mathbf{e}^{\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right)x}
\end{aligned} \tag{3.19}$$

Since this expression is now written in the **ybax**-order, we conclude that the corresponding generating function is this same expression, but written with commuting variables.

The other computation we must verify is the antipode, which follows similarly:

$$\begin{aligned}
\mathcal{G}(S) &= (\mathbb{O}^{-1} \circ S \circ \mathbb{O})(\mathbf{e}^{\eta y+\beta b+\alpha a+\xi x}) \\
&= \mathbb{O}^{-1}\big(\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\alpha\mathbf{a}}\mathbf{e}^{-\beta\mathbf{b}}\mathbf{e}^{-\eta\mathbf{y}}\big) \\
&= \mathbb{O}^{-1}\big(\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{-\beta\mathbf{b}}\mathbf{e}^{-\alpha\mathbf{a}}\big) \\
&= \mathbb{O}^{-1}\big(\mathbf{e}^{\xi\mathcal{A}\eta\mathbf{b}}\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\alpha\mathbf{a}}\mathbf{e}^{-\beta\mathbf{b}}\big) \\
&= \mathbb{O}^{-1}\big(\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{\xi\mathcal{A}\eta\mathbf{b}}\mathbf{e}^{-\beta\mathbf{b}}\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\alpha\mathbf{a}}\big) \\
&= \mathbb{O}^{-1}\big(\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{(\eta\mathcal{A}\xi-\beta)\mathbf{b}}\mathbf{e}^{-\alpha\mathbf{a}}\mathbf{e}^{-\xi\mathcal{A}\mathbf{x}}\big) \\
&= \mathbf{e}^{-\mathcal{A}\eta y+(\eta\mathcal{A}\xi-\beta)b+-\alpha a+-\xi\mathcal{A}x}
\end{aligned} \tag{3.20}$$

Finally, equations (3.12), (3.14) and (3.16) to (3.18) follow immediately. □

*Notational conventions*

The generating function of a tangle is not the entirety of this definition, for the additional data is the domain and codomain of the corresponding morphism.

We will thereby write a morphism with domain $D$, codomain $C$, and generating function $f(\zeta_D, z_C)$ as $f(\zeta_D, z_C)^D_C$.

When applied to knots, this invariant computes the Alexander polynomial $\Delta$:

$$Z(K_{3,1}) = \left(\frac{1}{B_1^{-1}+1+B_1^1}\right)^{\emptyset}_{\{1\}} = \Delta(K_{3,1})^{-1} \tag{3.21}$$

Here is a second example, where we define $D_i^n := B_i^n + B_i^{-n}$ to emphasize the palindromic nature of the Alexander polynomial:

$$Z(K_{11a10}) = \left(\frac{1}{2D_1^3 - 11D_1^2 + 25D_1^1 - 31}\right)^{\emptyset}_{\{1\}} = \Delta(K_{11a10})^{-1} \tag{3.22}$$

Since each tangle is expressed as an object, the domains in these examples are empty.

# 4

# CONSTRUCTING THE TRACE

## 4.1 EXTENDING AN OPEN TANGLE INVARIANT TO LINKS AND GENERAL TANGLES

Thus far, the algebraic setting we have defined allows us to describe invariants of tangles with no closed components. We now extend the notion of a meta-Hopf algebra to include closed components.

**Definition 4.1** (traced meta-algebra)**.** A traced meta-algebra is a family of meta-algebras: for each finite set $L$, we assign one meta-algebra $\{A_{L,S}\}_S$.[†1] The multiplication maps $m_k^{i,j}[L]$ then take the form:

$$m_k^{i,j}[L][S] \colon A_{\{i,j\}\sqcup S,L} \to A_{\{k\}\sqcup S,L} \tag{4.1}$$

for $i$, $j$, $k$ disjoint from both $S$ and $L$.

There is an additional structure, called a trace. The compatibility of the families of meta-algebras is governed this trace in the following way: $\mathrm{tr}^i \colon A_{\{i\}\sqcup S,L} \to A_{S,\{i\}\sqcup L}$ which is universal with respect to the cyclic property:

$$m_k^{i,j} /\!\!/ \mathrm{tr}^k = m_k^{j,i} /\!\!/ \mathrm{tr}^k \tag{4.2}$$

Furthermore, $\mathrm{tr}^i$ is a morphism of meta-coalgebras. That is:

$$\Delta_{jk}^i /\!\!/ \mathrm{tr}^j /\!\!/ \mathrm{tr}^k = \mathrm{tr}^i /\!\!/ \Delta_{jk}^i \tag{4.3}$$

$$\mathrm{tr}^i /\!\!/ \epsilon^i = \epsilon^i \tag{4.4}$$

The first example we give is that of mixed tangles.

**Definition 4.2** (mixed upright tangles)**.** Let $\overline{\mathcal{T}}^{\mathrm{up}}_{L,S}$ be the set of upright tangles with open strands indexed by $S$ and closed strands indexed by $L$. The

†1 These sets index the "strands" $S$ and the "loops" $L$.

operations $\phi[L][S]$ are defined analogously to the $\phi[S]$ given in theorem 2.31. (Here $\phi$ varies over $m$, $\eta$, $\Delta$, $\epsilon$, $S$, $\mathcal{R}$, and $C$.)

**Lemma 4.3** (tangles as a traced algebra)**.** *The collection of all $\overline{\mathcal{T}}^{up}_{L,S}$ is a traced ribbon meta-Hopf algebra, with trace map given by closing a strand into a loop.*

*Proof.* When $L = \emptyset$, the situation is exactly the case of theorem 2.31, so $\overline{\mathcal{T}}^{\text{up}}_{\emptyset,S} = \mathcal{T}^{\text{up}}_{S}$ is a meta-Hopf algebra. Furthermore, since the Reidemeister moves are local operations, the presence of closed components does not affect our ability to verify the identities on the Hopf-algebra operations.

The last point to verify is that closing a strand into a loop is a cyclic operation. Given two strands, we must verify that stitching one end together, then tracing the other yields the same diagram as stitching the other ends together, then taking the trace. However, by definition of trace, these two actions yield identical diagrams, the two strands replaced by the same closed loop. □

**Lemma 4.4** (coinvariants as a trace map)**.** *Let $A$ be an algebra, and denote by $A_A := A/[A, A]$ its set of coinvariants. Then define $A_{S,L} := A^{\otimes S} \otimes A_A^{\otimes L}$. Then $A$ defines a traced meta-algebra with trace map given by $\text{tr}^i_j \colon A_i \to (A_A)_j$.*

*Proof.* Observe that for any choice of $L$, extending morphisms by the identity yield an isomorphism of traced meta-Hopf algebras:

$$\begin{aligned} \phi_L \colon \{A^{\otimes S}\}_S &\xrightarrow{\sim} \{A^{\otimes S} \otimes A_A^{\otimes L}\}_S \\ A^{\otimes S} &\mapsto A^{\otimes S} \otimes A_A^{\otimes L} \\ f &\mapsto f \otimes \text{id}_{A_A}^{\otimes L} \end{aligned} \tag{4.5}$$

Next, we must show that $m^{ij}_k /\!\!/ \text{tr}^k = m^{ji}_k /\!\!/ \text{tr}^k$. This amounts to showing that, given $u, v \in A$, that $\overline{uv} = \overline{vu} \in A_A$. However, by the construction of the coinvariants, $\overline{uv} - \overline{vu} = \overline{uv - vu} = \overline{0} \in A$, and we are done. □

## 4.2 THE COINVARIANTS OF $U$

We start with a result which simplifies working with coinvariants:

**Lemma 4.5** (Coinvariant simplification)**.** *Let $\mathfrak{h}$ be a Lie algebra. Then $\mathfrak{U}(\mathfrak{h})_{\mathfrak{U}(\mathfrak{h})} = \mathfrak{U}(\mathfrak{h})_{\mathfrak{h}}$.*

*Proof.* First, observe that for any $u$, $v$, $f \in \mathfrak{U}(\mathfrak{h})$, $\mathrm{ad}_{uv}(f) = \mathrm{ad}_u(vf) + \mathrm{ad}_v(fu)$. Proceeding inductively, for any monomial $\mu \in \mathfrak{U}(\mathfrak{h})$, $\mathrm{ad}_\mu(u)$ is a linear combination of elements of $[\mathfrak{h}, \mathfrak{U}(\mathfrak{h})]$. By linearity of ad, we conclude $[\mathfrak{U}(\mathfrak{h}), \mathfrak{U}(\mathfrak{h})] = [\mathfrak{h}, \mathfrak{U}(\mathfrak{h})]$. □

**Theorem 4.6.** *The coinvariants of $U$, $U_U$, has basis $\{y^n a^k x^n\}_{n,k\geq 0}$.*

*Proof.* By lemma 4.5, we need only compute $[\mathfrak{g}, U]$ to determine $U_U$. Using lemma 2.34, we compute the adjoint actions of $y$, $a$, and $x$. (Recall $b$ is central.)

$$\mathrm{ad}_a f(x) = xf'(x) \qquad \mathrm{ad}_a f(y) = -yf'(y) \tag{4.6}$$

$$\mathrm{ad}_x f(y) = bf'(y) \qquad \mathrm{ad}_x f(a) = -\nabla[f](a)x \tag{4.7}$$

$$\mathrm{ad}_y f(x) = -bf'(x) \qquad \mathrm{ad}_y f(a) = y\nabla[f](a) \tag{4.8}$$

(Here $\nabla$ is the backwards finite difference operator $\nabla[f](x) := f(x) - f(x-1)$.) Observe for any $n$, $m$, $k$, and polynomials $f$ and $g$:

$$\mathrm{ad}_a(y^m g(b,a)x^n) = (n-m)y^m g(b,a)x^n \tag{4.9}$$

$$\mathrm{ad}_x(y^{n+1}b^{m-1}f(a)x^k) = (n+1)y^n b^m f(a)x^k - y^{n+1}b^{m-1}\nabla[f](a)x^{k+1} \tag{4.10}$$

$$\mathrm{ad}_y(y^n b^{m-1}f(a)x^{k+1}) = -(k+1)y^n b^m f(a)x^k + y^{n+1}b^{m-1}\nabla[f](a)x^{k+1} \tag{4.11}$$

By equation (4.9), any monomial whose powers of $y$ and $x$ differ vanish in $U_{\mathfrak{g}}$. As a consequence, in equations (4.10) and (4.11), the only nontrivial case is when $n = k$, resulting in the same relation. By induction on $n$, we conclude that:

$$y^n b^m f(a)x^k \sim \delta_{nk}\frac{n!}{(n+m)!}y^{n+m}\nabla^m[f](a)x^{n+m} \tag{4.12}$$

where $\sim$ refers to equivalence in the set of coinvariants. Observing when $f$ is a monomial in equation (4.12), we see $U_{\mathfrak{g}}$ is spanned by $\{y^n a^k x^n\}_{n,k\geq 0}$.

Finally, all that remains to show is this set is linearly independent. This is equivalent to no two $y^n a^k x^n$'s with distinct choices of exponents being

related under a sequence of relations. Since lemma 4.5 allows us to only consider sequences of relations of the form $\mathrm{ad}_z$ for $z$ a one-letter word in $\{y, b, a, x\}$, inspection of the above comprehensive summary of all one-letter relations (in particular, equation (4.12) with $m = 0$) allows us to conclude that this set is indeed linearly independent. □

*A generating function for the coinvariants*

In order to define a generating function, we need to choose an appropriate basis for the space of coinvariants. We define an isomorphism from the space of coinvariants to a polynomial space, tweaking the earlier-defined basis by scalar multiples. Since it plays the role of the ordering map, we also name it $\mathbb{O}$.

$$\begin{aligned} \mathbb{O} \colon \mathbb{Q}[a, z] &\xrightarrow{\sim} U_U \\ a^n z^k &\mapsto \frac{1}{k!} y^k a^n x^k \\ k! \nabla^m[f](a) z^{k+m} &\mapsfrom y^k b^m f(a) x^k \end{aligned} \tag{4.13}$$

This defines a commutative square upon whose bottom edge $\tau = \mathbb{O} /\!\!/ \mathrm{tr} /\!\!/ \mathbb{O}^{-1}$ we compute the generating function:

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{tr}} & U_U \\ {\scriptstyle\mathbb{O}}\uparrow & & \uparrow{\scriptstyle\mathbb{O}} \\ \mathbb{Q}[y, b, a, x] & \xrightarrow{\tau} & \mathbb{Q}[a, z] \end{array} \tag{4.14}$$

We begin with a result on finite differences:

**Lemma 4.7** (finite differences of exponentials)**.** *The finite difference operator acts in the following way on exponentials:*

$$\nabla^n[\mathbf{e}^{\alpha a}](a) = (1 - \mathbf{e}^{-\alpha})^n \mathbf{e}^{\alpha a} \tag{4.15}$$

*Proof.* Using the fact that $\nabla^n[f](x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x - k)$, we see that $\nabla^n[\mathbf{e}^{\alpha a}](a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbf{e}^{\alpha a - \alpha k} = (1 - \mathbf{e}^{-\alpha})^n \mathbf{e}^{\alpha a}$. □

We now are ready to compute the generating function for the trace:

**Theorem 4.8** (Generating function for the trace of $U$)**.**

$$\mathcal{G}(\mathrm{tr}) = \exp\Big(\alpha a + (\eta\xi + \beta(1 - \mathbf{e}^{-\alpha}))z\Big) \tag{4.16}$$

*Proof.* Using lemma 4.7 and the extension of scalars of tr to $\mathbb{Q}[\![\eta, \beta, \alpha, \xi]\!]$, we see

$$\begin{aligned}
&\mathcal{G}(\mathbb{O} /\!\!/ \operatorname{tr} /\!\!/ \mathbb{O}^{-1}) = (\mathbf{e}^{\eta y}\mathbf{e}^{\beta b}\mathbf{e}^{\alpha a}\mathbf{e}^{\xi x}) /\!\!/ \operatorname{tr} /\!\!/ \mathbb{O}^{-1} \\
&= \mathbb{O}^{-1} \sum_{i,j,k} \operatorname{tr}\left( \frac{(\eta y)^i}{i!} \frac{(\beta b)^j}{j!} \mathbf{e}^{\alpha a} \frac{(\xi x)^k}{k!} \right) \\
&= \sum_{i,j} \frac{\eta^i \beta^j \xi^i}{i!j!} (1 - \mathbf{e}^{-\alpha})^j \mathbf{e}^{\alpha a} z^{i+j} = \mathbf{e}^{\alpha a + (\eta\xi + \beta(1-\mathbf{e}^{-\alpha}))z} \qquad \square
\end{aligned} \tag{4.17}$$

*Evaluation of the trace on a generic element*

Here we will outline a computation involving the trace by using Bar-Natan and van der Veen's Contraction Theorem.

A typical value for a tangle invariant that arises is of the form:

$$P\mathbf{e}^{c+\alpha a_i + \beta b_i + \xi(b_i)x_i + \eta(b_i)y_i + \lambda(b_i)x_i y_i} \tag{4.18}$$

Here, $c$, $\alpha$, and $\beta$ denote constants with respect to the variables $y_i$, $b_i$, $a_i$, and $x_i$ (collectively referred to as "$v_i$"s), while $\xi$, $\eta$, and $\lambda$ are potentially $b_i$-dependent, and $P$ is a (rational) function in (the square root of) $B_i$.

**Theorem 4.9** (The trace of a Gaußian)**.** *With symbols as defined above, let* $f(y_i, b_i, a_i, x_i) = P(B_i)\mathbf{e}^{c+\alpha a_i + \beta b_i + \xi(b_i)x_i + \eta(b_i)y_i + \lambda(b_i)x_i y_i}$*. Then*

$$\left\langle f(y_i, b_i, a_i, x_i) \operatorname{tr}^i \right\rangle_{v_i} = \frac{P(\mathbf{e}^{-\mu})}{1 - \lambda(\mu)\bar{z}_i} \mathbf{e}^{c + \alpha\bar{a}_i + \beta\mu + \frac{\eta(\mu)\xi(\mu)\bar{z}_i}{1-\lambda(\mu)\bar{z}_i}} \tag{4.19}$$

*where* $\mu := (1 - \mathbf{e}^{-\alpha})\bar{z}_i$.

*Proof.* Let us compute the trace of equation (4.18). For clarity, we will put bars over the coinvariants variables $a_i$ and $z_i$, as they do not play a role in the contraction.

$$\begin{aligned}
&\langle P(B_i)\mathbf{e}^{c+\alpha a_i+\beta b_i+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i}\operatorname{tr}^i\rangle_{v_i}\\
&=\langle P(B_i)\mathbf{e}^{c+\beta b_i+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i+\eta_i\xi_i\bar{z}_i+\beta_i(1-\mathbf{e}^{-\alpha_i})\bar{z}_i}\mathbf{e}^{\alpha a_i+\alpha_i\bar{a}_i}\rangle_{v_i}\\
&=\mathbf{e}^{\alpha\bar{a}_i}\langle P(B_i)\mathbf{e}^{c+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i+\eta_i\xi_i\bar{z}_i}\mathbf{e}^{\beta b_i+\beta_i(1-\mathbf{e}^{-\alpha})\bar{z}_i}\rangle_{b_i,x_i,y_i}
\end{aligned}$$

In what follows, we let $\mu := (1-\mathbf{e}^{-\alpha})\bar{z}_i$:

$$\begin{aligned}
&=\mathbf{e}^{c+\alpha\bar{a}_i+\beta\mu}P(\mathbf{e}^{-\mu})\langle\mathbf{e}^{\eta(\mu)y_i}\mathbf{e}^{(\xi(\mu)+\lambda(\mu)y_i)x_i+\xi_i\eta_i\bar{z}_i}\rangle_{x_i,y_i}\\
&=\mathbf{e}^{c+\alpha\bar{a}_i+\beta\mu}P(\mathbf{e}^{-\mu})\langle\mathbf{e}^{\eta(\mu)y_i+\xi(\mu)\bar{z}_i\eta_i+\lambda(\mu)\bar{z}_i\eta_iy_i}\rangle_{y_i}\\
&=\frac{P(\mathbf{e}^{-\mu})}{1-\lambda(\mu)\bar{z}_i}\mathbf{e}^{c+\alpha\bar{a}_i+\beta\mu+\frac{\eta(\mu)\xi(\mu)\bar{z}_i}{1-\lambda(\mu)\bar{z}_i}}
\end{aligned}\tag{4.20}$$

□

We point out that the outcome of this computation is not guaranteed to be a Gaußian. This puts a limitation on the applicability of this formula to links with more than two components, explored in chapter 5.

*Computational examples*

Using the formula given in equation (4.19), let us do some preliminary examples:

$$\operatorname{tr}^i(R_{ij}) = 1 \tag{4.21}$$

$$\operatorname{tr}^j(R_{ij}) = \mathbf{e}^{b_i\bar{a}_j} \tag{4.22}$$

$$\begin{aligned}
&\operatorname{tr}^2\left(\sqrt{B_2}\mathbf{e}^{-a_2b_1-a_1b_2+\frac{(B_1^{-1})x_2y_1}{b_1B_1}+\frac{(B_2^{-1})x_1y_2}{b_2B_2}}\right)=\\
&\mathbf{e}^{\frac{a_1(\bar{z}_2-B_1\bar{z}_2)}{B_1}-b_1\bar{a}_2+\frac{e^{B_1\bar{z}_2}(x_1y_1e^{B_1^{-1}\bar{z}_2}-x_1y_1e^{\bar{z}_2})}{b_1}+\frac{1}{2}B_1^{-1}\bar{z}_2-\frac{\bar{z}_2}{2}}
\end{aligned}\tag{4.23}$$

Equations (4.21) and (4.22) are the values one obtains for the two (virtual) one-crossing, two-component link, while equation (4.23) is the value of the invariant on the Hopf link.

When computing this on a link, however, it is important to keep track of which strands are open, and which are closed. We will extend the notation

from the previous section to differentiate between open and closed indices. We write a morphism with domain $D = D_{\mathrm{o}} \sqcup D_{\mathrm{c}}$, codomain $C = C_{\mathrm{o}} \sqcup C_{\mathrm{c}}$ (here $D_{\mathrm{o}}$ denotes domain indices which are open, while $D_{\mathrm{c}}$ those which are closed, with the same convention for $C$) and generating function $f(\zeta_D, z_C)$ as $f(\zeta_D, z_C)^{(D_{\mathrm{o}}, D_{\mathrm{c}})}_{(C_{\mathrm{o}}, C_{\mathrm{c}})}$.

# 5

# CONCLUSIONS

*Limitations of this definition*

For some inputs to the trace, expressions involving the Lambert $W$-function appear, which complicates attempts to keep the invariant valued in the space of perturbed Gaußians.

## 5.1 COMPARISON WITH THE MULTIVARIABLE ALEXANDER POLYNOMIAL

Given that the long knot (i.e. one-component) case of this invariant encodes the Alexander Polynomial, it was suspected that the invariant on long links (i.e. multiple components, one of which is long) formed by adding the trace would encode the MVA. However, there are links which the MVA separates which this invariant does not.

On all two-component links with at most 11 crossings (a collection of size 914), the trace map attains 878 distinct values, while the MVA attains only 778. However, the two invariants are incomparable in terms of their strength.

The links $L_{5a1}$ and $L_{10a43}$ are not distinguished by their partial traces, with both returning a value of:

$$\left(\left(\frac{B_1}{B_1^2 t_2 - 2B_1 t_2 + B_1 + t_2}\right)_{(\{1\},\{2\})}, \left(\frac{B_2^{3/2}}{B_2^2 t_1 - 2B_2 t_1 + B_2 + t_1}\right)_{(\{2\},\{1\})}\right) \tag{5.1}$$

The values of these links under the MVA are, however

$$\frac{(B_1-1)(B_2-1)}{\sqrt{B_1}\sqrt{B_2}} \text{ and } -\frac{(B_1-1)(B_2-1)(B_1+B_2-1)(B_2B_1-B_1-B_2)}{B_1^{3/2}B_2^{3/2}} \tag{5.2}$$

respectively.

In the other direction, there are also pairs of links in the same fibre of the MVA which this traced invariant can distinguish. In particular $L_{5a1}$ and $L_{7n2}$ both have the same value under the MVA:

$$\frac{(B_1-1)(B_2-1)}{\sqrt{B_1}\sqrt{B_2}} \tag{5.3}$$

The trace yields the following values (respectively):

$$\left(\left(\frac{B_1}{B_1^2t_2-2B_1t_2+B_1+t_2}\right)_{(\{1\},\{2\})},\left(\frac{B_2^{3/2}}{B_2^2t_1-2B_2t_1+B_2+t_1}\right)_{(\{2\},\{1\})}\right) \tag{5.4}$$

$$\left(\left(\frac{B_1}{B_1^2t_2-2B_1t_2+B_1+t_2}\right)_{(\{1\},\{2\})},\left(\frac{B_2^{5/2}}{B_2^2t_1-2B_2t_1+B_2^2-B_2+t_1+1}\right)_{(\{2\},\{1\})}\right) \tag{5.5}$$

This example also serves to highlight that the information provided by leaving one strand open is not enough to recover the value of a different strand being left open.

## 5.2 FURTHER WORK

While all other Hopf algebra operations in $U$ are expressed by [BNvdVb] as perturbed Gaußians, the form in equation (4.16) does not to conform to the same structure. Further work is needed to either implement this operation into the established framework, or to suitably extend the framework (perhaps with the use of Lambert $W$-functions).

# A

# CODE

## A.1 IMPLEMENTATION OF THE INVARIANT $Z$

This is a Mathematica™ implementation by Bar-Natan and van der Veen in [BNvdVb], modified by the author. The full source code is available at https://github.com/phro/GDO. We begin by setting some variables, as well as a method for modifying associations.

```
γ = 1; ħ = 1; $k = 0;
setValue[value_,obj_,coord_]:=Module[
        {b=Association@@obj},
        b[coord] = value; Head[obj]@@Normal@b
]
```

We introduce notation `PG[L, Q, P]` to be interpreted as the Perturbed Gaußian $P\mathbf{e}^{L+Q}$. The function `fromE` serves as a compatibility layer between a former version of the code.

```
toPG[L_, Q_, P_] := PG["L"->L, "Q"->Q, "P"->P]
fromE[e_\[DoubleStruckCapitalE]] := toPG@@e/.
        Subscript[(v:y|b|t|a|x|B|T|η|β|τ|α|ξ|A), i_] -> v[i]
```

We define the Kronecker-$\delta$ function next.

```
δ[i_,j_] := If[SameQ[i,j],1,0]
```

Next we introduce helper functions for the reading and manipulating of `PG`-objects:

```
getL[pg_PG] := Lookup[Association@@pg,"L",0]
getQ[pg_PG] := Lookup[Association@@pg,"Q",0]
getP[pg_PG] := Lookup[Association@@pg,"P",1]

setL[L_][pg_PG] := setValue[L, pg, "L"];
setQ[Q_][pg_PG] := setValue[Q, pg, "Q"];
setP[P_][pg_PG] := setValue[P, pg, "P"];

applyToL[f_][pg_PG] := pg//setL[pg//getL//f]
applyToQ[f_][pg_PG] := pg//setQ[pg//getQ//f]
applyToP[f_][pg_PG] := pg//setP[pg//getP//f]
```

Next is a function `CF`, which bring objects into canonical form allows us to compare for equality effectively. This is defined by Bar-Natan and van der Veen.

```
CCF[e_] := ExpandDenominator@ExpandNumerator@Together[
        Expand[e] //. E^x_ E^y_ :> E^(x + y) /. E^x_ :> E^CCF[x]
];
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[e_] := Module[
        {vs = Union[
                Cases[e, (y|b|t|a|x|η|β|τ|α|ξ)[_], ∞],
                {y, b, t, a, x, η, β, τ, α, ξ}
        ]},
        Total[CoefficientRules[Expand[e], vs] /.
                (ps_ -> c_) :> CCF[c] (Times @@ (vs^ps))
        ]
];
CF[e_PG] := e//applyToL[CF]//applyToQ[CF]//applyToP[CF]
```

We must also define the notion of equality for `PG`-objects, as well as what it means to multiply them.

```
Congruent[x_, y_, z__] := And[Congruent[x, y], Congruent[y, z]]
PG /: Congruent[pg1_PG, pg2_PG] := And[
        CF[getL@pg1 == getL@pg2],
        CF[getQ@pg1 == getQ@pg2],
        CF[Normal[getP@pg1-getP@pg2] == 0]
]
```

```
41
42 PG /: pg1_PG * pg2_PG := toPG[
43         getL@pg1 + getL@pg2,
44         getQ@pg1 + getQ@pg2,
45         getP@pg1 * getP@pg2
46 ]
47
48 setEpsilonDegree[k_Integer][pg_PG] := setP[Series[Normal@getP@pg,{ϵ,
   ↪  0, k}]][pg]
```

The variables $y$, $b$, $t$, $a$, and $x$ are paired with their dual variables $\eta$, $\beta$, $\tau$, $\alpha$, and $\xi$. This applies as well when they have subscripts.

```
49 ddsl2vars = {y, b, t, a, x, z};
50 ddsl2varsDual = {η, β, τ, α, ξ, ζ};
51
52 Evaluate[Dual/@ddsl2vars] = ddsl2varsDual;
53 Evaluate[Dual/@ddsl2varsDual] = ddsl2vars;
54 Dual@z = ζ;
55 Dual@ζ = z;
56 Dual[u_[i_]]:=Dual[u][i]
```

Since various exponentials of the lowercase variables frequently appear, we introduce capital variable names to handle various exponentiated forms.

```
57 U2l = {
58         B[i_]^p_. :> E^(-p ħ γ b[i]), B^p_. :> E^(-p ħ γ b),
59         W[i_]^p_. :> E^(w[i])       , W^p_. :> E^(p w),
60         T[i_]^p_. :> E^(-p ħ t[i])  , T^p_. :> E^(-p ħ t),
61         A[i_]^p_. :> E^(p γ α[i])   , A^p_. :> E^(-p γ α)
62 };
63 l2U = {
64         E^(c_. b[i_] + d_.) :> B[i]^(-c/(ħ γ))E^d,
65         E^(c_. b + d_.)     :> B^(-c/(ħ γ))E^d,
66         E^(c_. t[i_] + d_.) :> T[i]^(-c/ħ)E^d,
67         E^(c_. t + d_.)     :> T^(-c/ħ)E^d,
68         E^(c_. α[i_] + d_.) :> A[i]^(c/γ)E^d,
69         E^(c_. α + d_.)     :> A^(c/γ)E^d,
70         E^(c_. w[i_] + d_.) :> W[i]^(c)E^d,
71         E^(c_. w + d_.)     :> W^(c)E^d,
```

```
        E^expr_             :> E^Expand@expr
};
```

Below the notion of differentiation is defined for expressions which involve both upper- and lower-case variables.

```
DD[f_, b]     := D[f, b   ] - ℏ γ B    D[f, B   ];
DD[f_, b[i_]] := D[f, b[i]] - ℏ γ B[i] D[f, B[i]];

DD[f_, t    ] := D[f, t   ] - ℏ T    D[f, T   ];
DD[f_, t[i_]] := D[f, t[i]] - ℏ T[i] D[f, T[i]];

DD[f_, α    ] := D[f, α   ] + γ A    D[f, A   ];
DD[f_, α[i_]] := D[f, α[i]] + γ A[i] D[f, A[i]];

DD[f_, v_] := D[f, v];
DD[f_, {v_,0}] := f;
DD[f_, {}] := f;
DD[f_, {v_,n_Integer}] := DD[DD[f,v],{v,n-1}];
DD[f_, {l_List, ls___}] := DD[DD[f, l], {ls}];
```

What follows now is the implementation of contraction as introduced in definition 3.4. We begin with the introduction of contractions of (finite) polynomials.

```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[expr_, ζ_] := Collect[expr, ζ];

Zip[{}][P_] := P;
Zip[ζs_List][Ps_List] := Zip[ζs]/@Ps;
Zip[{ζ_,ζs___}][P_] := (collect[P // Zip[{ζs}],ζ] /.
        f_. ζ^d_. :> DD[f,{Dual[ζ], d}]) /.
        Dual[ζ] -> 0 /.
        ((Dual[ζ] /. {b->B, t->T, α -> A}) -> 1)
```

We define contraction along the variables $x$ and $y$ (here packaged into the matrix `Q`).

```
QZip[ζs_List][pg_PG] := Module[{Q, P, ζ, z, zs, c, ys, ηs, qt, zrule,
↪  ζrule},
        zs = Dual/@ζs;
        Q  = pg//getQ;
        P  = pg//getP;
        c  = CF[Q/.Alternatives@@Union[ζs, zs]->0];
        ys = CF/@Table[D[Q,ζ]/.Alternatives@@zs->0,{ζ,ζs}];
        ηs = CF/@Table[D[Q,z]/.Alternatives@@ζs->0,{z,zs}];
        qt = CF/@#&/@(Inverse@Table[
                δ[z, Dual[ζ]] - D[Q,z,ζ],
                {ζ,ζs},{z,zs}
        ]);
        zrule = Thread[zs -> CF/@(qt . (zs + ys))];
        ζrule = Thread[ζs -> ζs + ηs . qt];
        CF@setQ[c + ηs.qt.ys]@setP[Det[qt] Zip[ζs][P /. Union[zrule,
        ↪  ζrule]]]@pg
]
```

We define contraction along the variables $a$ and $b$ (here packaged into the matrix L).

```
LZip[ζs_List][pg_PG] := Module[
        {
                L, Q, P, ζ, z, zs, Zs, c, ys, ηs, lt,
                zrule, Zrule, ζrule, Q1, EEQ, EQ, U
        },
        zs = Dual/@ζs;
        {L, Q, P} = Through[{getL, getQ, getP}@pg];
        Zs = zs /. {b -> B, t -> T, α -> A};
        c  = CF[L/.Alternatives@@Union[ζs, zs]->0/.Alternatives@@Zs ->
        ↪  1];
        ys = CF/@Table[D[L,ζ]/.Alternatives@@zs->0,{ζ,ζs}];
        ηs = CF/@Table[D[L,z]/.Alternatives@@ζs->0,{z,zs}];
        lt = CF/@#&/@Inverse@Table[
                δ[z, Dual[ζ]] - D[L,z,ζ],
                {ζ,ζs},{z,zs}
        ];
        zrule = Thread[zs -> CF/@(lt . (zs + ys))];
        Zrule = Join[zrule, zrule /.
                r_Rule :> ( (U = r[[1]] /. {b -> B, t -> T, α -> A})
                ↪  ->
                (U /. U2l /. r //. l2U))
```

```
        ];
        \[Zeta]rule = Thread[\[Zeta]s -> \[Zeta]s + \[Eta]s . lt];
        Q1 = Q /. Union[Zrule, ζrule];
        EEQ[ps___] :=
                EEQ[ps] = (
                        CF[E^-Q1 DD[E^Q1,Thread[{zs,{ps}}]] /.
                                {Alternatives@@zs -> 0, Alternatives
                                 ↪  @@Zs -> 1}]
                );
        CF@toPG[
                c + ηs.lt.ys,
                Q1 /. {Alternatives@@zs -> 0, Alternatives@@Zs -> 1},
                Det[lt] (Zip[ζs][(EQ@@zs) (P /. Union[Zrule,ζrule])]
                 ↪  /.
                        Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ
                         ↪  ->1
                )
        ]

]
```

The function `Pair` combines the above zipping functions into the final contraction map.

```
Pair[{}][L_PG,R_PG] := L R;
Pair[is_List][L_PG,R_PG] := Module[{n},
        Times[
                L /. ((v: b|B|t|T|a|x|y)[#] -> v[n@#]&/@is),
                R /. ((v: β|τ|α|A|ξ|η)[#] -> v[n@#]&/@is)
        ] // LZip[Join@@Table[Through[{β, τ, a}[n@i]],{i, is}]] //
        QZip[Join@@Table[Through[{ξ, y}[n@i]],{i, is}]]
]
```

Our next task is to provide domain and codomain information for the `PG`-objects. These will be packaged inside a `GDO`, (Gaußian Differential Operator). The four lists' names refer to whether it is a domain or a codomain, and whether the index corresponds to an open strand or a closed one.

```
toGDO[do_List,dc_List,co_List,cc_List,L_,Q_,P_] := GDO[
        "do" -> do,
```

```
        "dc" -> dc,
        "co" -> co,
        "cc" -> cc,
        "PG" -> toPG[L, Q, P]
]

toGDO[do_List,dc_List,co_List,cc_List,pg_PG] := GDO[
        "do" -> do,
        "dc" -> dc,
        "co" -> co,
        "cc" -> cc,
        "PG" -> pg
]
```

Next are defined functions for accessing and modifying sub-parts of `GDO`-objects. The last argument of `Lookup` is the default value if nothing is specified. This means that a morphism with empty domain or codomain may be specified as such by omitting that portion of the definition.

```
getDO[gdo_GDO] := Lookup[Association@@gdo, "do", {}]
getDC[gdo_GDO] := Lookup[Association@@gdo, "dc", {}]
getCO[gdo_GDO] := Lookup[Association@@gdo, "co", {}]
getCC[gdo_GDO] := Lookup[Association@@gdo, "cc", {}]

getPG[gdo_GDO] := Lookup[Association@@gdo, "PG", PG[]]

getL[gdo_GDO] := gdo//getPG//getL
getQ[gdo_GDO] := gdo//getPG//getQ
getP[gdo_GDO] := gdo//getPG//getP

setPG[pg_PG][gdo_GDO] := setValue[pg, gdo, "PG"]

setL[L_][gdo_GDO] := setValue[setL[L][gdo//getPG], gdo, "PG"]
setQ[Q_][gdo_GDO] := setValue[setQ[Q][gdo//getPG], gdo, "PG"]
setP[P_][gdo_GDO] := setValue[setP[P][gdo//getPG], gdo, "PG"]

setDO[do_][gdo_GDO] := setValue[do, gdo, "do"]
setDC[dc_][gdo_GDO] := setValue[dc, gdo, "dc"]
setCO[co_][gdo_GDO] := setValue[co, gdo, "co"]
setCC[cc_][gdo_GDO] := setValue[cc, gdo, "cc"]

```

```
applyToDO[f_][gdo_GDO] := gdo//setDO[gdo//getDO//f]
applyToDC[f_][gdo_GDO] := gdo//setDC[gdo//getDC//f]
applyToCO[f_][gdo_GDO] := gdo//setCO[gdo//getCO//f]
applyToCC[f_][gdo_GDO] := gdo//setCC[gdo//getCC//f]

applyToPG[f_][gdo_GDO] := gdo//setPG[gdo//getPG//f]

applyToL[f_][gdo_GDO] := gdo//setL[gdo//getL//f]
applyToQ[f_][gdo_GDO] := gdo//setQ[gdo//getQ//f]
applyToP[f_][gdo_GDO] := gdo//setP[gdo//getP//f]
```

The canonical form function (`CF`) and the contraction mapping (`Pair`) we extend to include `GDO`-objects. Furthermore, on the level of `GDO`-objects we can compose morphisms and keep track of the corresponding domains and codomains, using the left-to-right composition operator "//".

```
CF[e_GDO] := e//
        applyToDO[Union]//
        applyToDC[Union]//
        applyToCO[Union]//
        applyToCC[Union]//
        applyToPG[CF]

Pair[is_List][gdo1_GDO, gdo2_GDO] := GDO[
        "do" -> Union[gdo1//getDO, Complement[gdo2//getDO, is]],
        "dc" -> Union[gdo1//getDC, gdo2//getDC],
        "co" -> Union[gdo2//getCO, Complement[gdo1//getCO, is]],
        "cc" -> Union[gdo1//getCC, gdo2//getCC],
        "PG" -> Pair[is][gdo1//getPG, gdo2//getPG]
]

gdo1_GDO // gdo2_GDO :=
 ↪  Pair[Intersection[gdo1//getCO,gdo2//getDO]][gdo1,gdo2];
```

We also define notions of equality and multiplication (by concatenation) for `GDO`'s.

```
GDO /: Congruent[gdo1_GDO, gdo2_GDO] := And[
        Sort@*getDO/@Equal[gdo1, gdo2],
        Sort@*getDC/@Equal[gdo1, gdo2],
```

```
        Sort@*getCO/@Equal[gdo1, gdo2],
        Sort@*getCC/@Equal[gdo1, gdo2],
        Congruent[gdo1//getPG, gdo2//getPG]
]

GDO /: gdo1_GDO gdo2_GDO := GDO[
        "do" -> Union[gdo1//getDO, gdo2//getDO],
        "dc" -> Union[gdo1//getDC, gdo2//getDC],
        "co" -> Union[gdo1//getCO, gdo2//getCO],
        "cc" -> Union[gdo1//getCC, gdo2//getCC],
        "PG" -> (gdo1//getPG)*(gdo2//getPG)
]
```

For the sake of compatibility with Bar-Natan and van der Veen's program, we introduce several conversion functions between the two notations.

```
setEpsilonDegree[k_Integer][gdo_GDO] :=
        setP[Series[Normal@getP@gdo,{ϵ,0,k}]][gdo]

fromE[Subscript[\[DoubleStruckCapitalE],{do_List, dc_List}->{co_List,
 ↪  cc_List}][
        L_, Q_, P_
]] := toGDO[do, dc, co, cc, fromE[\[DoubleStruckCapitalE][L, Q, P]]]

fromE[Subscript[\[DoubleStruckCapitalE], dom_List->cod_List][
        L_, Q_, P_
]] := GDO["do" -> dom, "co" -> cod,
        "PG" -> fromE[\[DoubleStruckCapitalE][L, Q, P]]
]
```

It is at this point that we implement the morphisms of the algebra $U$. Each operation is prepended with a "c" to emphasize that this is a classical algebra, not a quantum deformation.

```
fromLog[l_] := CF@Module[
        {L, l0 = Limit[l, ϵ->0]},
        L = l0 /. (η|y|ξ|x)[_]->0;
        PG[
                "L" -> L,
                "Q" -> l0 - L
```

```
        ]/.l2U
]

cΛ = (η[i] + E^(-γ α[i] - ϵ β[i]) η[j]/(1+γ ϵ η[j]ξ[i])) y[k] +
     (β[i] + β[j] + Log[1 + γ ϵ η[j]ξ[i]]/ϵ            ) b[k] +
     (α[i] + α[j] + Log[1 + γ ϵ η[j]ξ[i]]/γ            ) a[k] +
     (ξ[j] + E^(-γ α[j] - ϵ β[j]) ξ[i]/(1+γ ϵ η[j]ξ[i])) x[k];

cm[i_, j_, k_] = GDO["do" -> {i,j}, "co" -> {k}, "PG" -> fromLog[cΛ]];

cη[i_] = GDO["co" -> {i}];
cσ[i_,j_] = GDO["do"->{i},"co"->{j},
        "PG"->fromLog[β[i] b[j] + α[i] a[j] + η[i] y[j] + ξ[i] x[j]]
];
cϵ[i_] = GDO["do" -> {i}];
cΔ[i_, j_, k_] = GDO["do"->{i}, "co"->{j, k},
        "PG" -> fromLog[
                β[i](b[j] + b[k]) +
                α[i](a[j] + a[k]) +
                η[i]
                        ((b[j]+b[k])/(1-B[j]B[k]))
                        (
                                B[k]((1-B[j])/b[j])y[j]+
                                    ((1-B[k])/b[k])y[k]
                        ) +
                ξ[i](x[j] + x[k])
        ]
];

sY[i_, j_, k_, l_, m_] = GDO["do"->{i}, "co"->{j, k, l, m},
        "PG" -> fromLog[β[i]b[k] + α[i]a[l] + η[i]y[j] + ξ[i]x[m]]
];

sS[i_] = GDO["do"->{i},"co"->{i},
        "PG"->fromLog[-(β[i] b[i] + α[i] a[i] + η[i] y[i] + ξ[i]
         ↪  x[i])]
];

cS[i_] = sS[i] // sY[i, 1, 2, 3, 4] // cm[4,3, i] // cm[i, 2, i] //
 ↪  cm[i, 1, i];

cR[i_, j_] = GDO[
        "co" -> {i,j},
```

```
        "PG" -> toPG[ħ a[j] b[i], (B[i]-1)/(-b[i]) x[j] y[i], 1]
]

cRi[i_, j_] = GDO[
        "co" -> {i,j},
        "PG" -> toPG[-ħ a[j] b[i], (B[i]-1)/(B[i] b[i]) x[j] y[i], 1]
]

CC[i_] :=  GDO["co"->{i},"PG"->PG["P"->B[i]^( 1/2)]]
CCi[i_] := GDO["co"->{i},"PG"->PG["P"->B[i]^(-1/2)]]

cKink[i_]  = Module[{k}, cR[i,k] CCi[k] // cm[i, k, i]]
cKinki[i_] = Module[{k}, cRi[i,k] CC[k] // cm[i, k, i]]

cKinkn[0][i_]  = cη[i]
cKinkn[1][i_]  = cKink[i]
cKinkn[-1][i_] = cKinki[i]
cKinkn[n_Integer][i_] :=
  ↪  Module[{j},cKinkn[n-1][i]cKink[j]//cm[i,j,i]]/; n > 1
cKinkn[n_Integer][i_] :=
  ↪  Module[{j},cKinkn[n+1][i]cKinki[j]//cm[i,j,i]]/; n < -1

uR[i_, j_]  = Module[{k}, cR[i,j] cKinki[k] // cm[i, k, i]]
uRi[i_, j_] = Module[{k}, cRi[i,j] cKink[k] // cm[i, k, i]]
```

## A.2 IMPLEMENTATION OF THE TRACE

Now we implement the trace. We introduce several functions which extract the various coefficients of a `GDO`, so that we may apply equation (4.19). Coefficients are extracted based on whether they belong to the matrix `L` or the matrix `Q`.

```
getConstLCoef::usage = "getConstLCoef[i][gdo] returns the terms in the
  ↪  L-portion of a GDO expression which are not a function of y[i],
  ↪  b[i], a[i], nor x[i]."
getConstLCoef[i_][gdo_] :=
        (SeriesCoefficient[#, {b[i],0,0}]&) @*
        (Coefficient[#, y[i], 0]&) @*
        (Coefficient[#, a[i], 0]&) @*
        (Coefficient[#, x[i], 0]&) @*
```

```
321         ReplaceAll[U2l] @*
322         getL@
323         gdo
324
325 getConstQCoef::usage = "getConstQCoef[i][gdo] returns the terms in the
    ↪  Q-portion of a GDO expression which are not a function of y[i],
    ↪  b[i], a[i], nor x[i]."
326 getConstQCoef[i_][gdo_][bb_] :=
327         ReplaceAll[{b[i]->bb}] @*
328         (Coefficient[#, y[i], 0]&) @*
329         (Coefficient[#, a[i], 0]&) @*
330         (Coefficient[#, x[i], 0]&) @*
331         ReplaceAll[U2l] @*
332         getQ@
333         gdo
334
335 getyCoef::usage = "getyCoef[i][gdo][b[i]] returns the linear
    ↪  coefficient of y[i] as a function of b[i]."
336 getyCoef[i_][gdo_][bb_] :=
337         ReplaceAll[{b[i]->bb}] @*
338         ReplaceAll[U2l] @*
339         (Coefficient[#, x[i],0]&) @*
340         (Coefficient[#, y[i],1]&) @*
341         getQ@
342         gdo
343
344 getbCoef::usage = "getbCoef[i][gdo] returns the linear coefficient of
    ↪  b[i]."
345 getbCoef[i_][gdo_] :=
346         (SeriesCoefficient[#, {b[i],0,1}]&) @*
347         (Coefficient[#, a[i],0]&) @*
348         (Coefficient[#, x[i],0]&) @*
349         (Coefficient[#, y[i],0]&) @*
350         ReplaceAll[U2l] @*
351         getL@
352         gdo
353
354 getPCoef::usage = "getPCoef[i][gdo] returns the perturbation P of a
    ↪  GDO as a function of b[i]."
355 getPCoef[i_][gdo_][bb_] :=
356         ReplaceAll[{b[i]->bb}] @*
357         (Coefficient[#, a[i],0]&) @*
358         (Coefficient[#, x[i],0]&) @*
```

```
359         (Coefficient[#, y[i],0]&) @*
360         ReplaceAll[U2l] @*
361         getP@
362         gdo
363
364 getaCoef::usage = "getaCoef[i][gdo] returns the linear coefficient of
    ↪  a[i]."
365 getaCoef[i_][gdo_] :=
366         (SeriesCoefficient[#, {b[i],0,0}]&) @*
367         (Coefficient[#, a[i],1]&) @*
368         ReplaceAll[U2l] @*
369         getL@
370         gdo
371
372 getxCoef::usage = "getxCoef[i][gdo][b[i]] returns the linear
    ↪  coefficient of x[i] as a function of b[i]."
373 getxCoef[i_][gdo_][bb_] :=
374         ReplaceAll[{b[i]->bb}] @*
375         ReplaceAll[U2l] @*
376         (Coefficient[#, y[i],0]&) @*
377         (Coefficient[#, x[i],1]&) @*
378         getQ@
379         gdo
380
381 getabCoef::usage = "getabCoef[i][gdo] returns the linear coefficient
    ↪  of a[i]b[i]."
382 getabCoef[i_][gdo_] :=
383         (SeriesCoefficient[#,{b[i],0,1}]&) @*
384         (Coefficient[#,a[i],1]&) @*
385         ReplaceAll[U2l] @*
386         getL@
387         gdo
388
389 getxyCoef::usage = "getxyCoef[i][gdo][b[i]] returns the linear
    ↪  coefficient of x[i]y[i] as a function of b[i]."
390 getxyCoef[i_][gdo_][bb_] :=
391         ReplaceAll[{b[i]->bb}] @*
392         ReplaceAll[U2l] @*
393         (Coefficient[#,x[i],1]&) @*
394         (Coefficient[#,y[i],1]&) @*
395         getQ@
396         gdo
```

In order to run more efficiently, limits are first computed by direct evaluation, unless such an operation is ill-defined. In such a case, the corresponding series is computed and evaluated at the limit point.

```
safeEval[f_][x_] := Module[{fx, x0},
        If[(fx=Quiet[f[x]]) === Indeterminate,
                Series[f[x0],{x0,x,0}]//Normal,
                fx
        ]
]

closeComponent[i_][gdo_GDO]:=gdo//
        setCO[Complement[gdo//getCO,{i}]]//
        setCC[Union[gdo//getCC,{i}]]
```

Now we come to the implementation of the trace map. The current implementation requires that the coefficient of $a_i b_i$ be zero. (See chapter 5 for how this restriction limits computability.)

```
tr::usage = "tr[i] computes the trace of a GDO element on component i.
  ↪ Current implementation assumes the Subscript[a, i] Subscript[b, i]
  ↪ term vanishes and $k=0."
tr::nonzeroSigma = "tr[`1`]: Component `1` has writhe: `2`, expected:
  ↪ 0."
tr[i_][gdo_GDO] := Module[
        {
                cL = getConstLCoef[i][gdo],
                cQ = getConstQCoef[i][gdo],
                βP = getPCoef[i][gdo],
                ηη = getyCoef[i][gdo],
                ββ = getbCoef[i][gdo],
                αα = getaCoef[i][gdo],
                ξξ = getxCoef[i][gdo],
                λ = getxyCoef[i][gdo],
                ta
        },
        ta = (1-Exp[-αα]) z[i];
        expL = cL + αα w[i] + ββ ta;
        expQ = safeEval[cQ[#] + z[i]ηη[#]ξξ[#]/(1-z[i] λ[#])&][ta];
        expP = safeEval[βP[#]/(1-z[i] λ[#])&][ta];
```

```
        CF[(gdo//closeComponent[i]//
         ↪  setL[expL]//setQ[expQ]//setP[expP])//.l2U]
] /; Module[
        {σ = getabCoef[i][gdo]},
        If[σ == 0,
                True,
                Message[tr::nonzeroSigma, i, ToString[σ]]; False
        ]
]
```

Here we introduce some formatting to display the output more aesthetically.

```
Format[gdo_GDO] := Subsuperscript[\[DoubleStruckCapitalE],
        Row[{gdo//getCO, ",", gdo//getCC}],
        Row[{gdo//getDO, ",", gdo//getDC}]
][gdo//getL, gdo//getQ, gdo//getP];
Format[pg_PG] := \[DoubleStruckCapitalE][pg//getL, pg//getQ,
 ↪  pg//getP];

SubscriptFormat[v_] := (Format[v[i_]] := Subscript[v, i]);

SubscriptFormat/@{y,b,t,a,x,z,w,η,β,α,ξ,A,B,T,W};
```

*Implementing the full invariant*

Now we are in a position to implement the $Z$ invariant to tangles with a closed component. We begin by defining an object representing an isolated strand with arbitrary integer rotation number, `CCn`:

```
CCn[i_][n_Integer]:=Module[{j},
  If[n==0,
    GDO["co"->{i}],
    If[n>0,
      If[n==1,
        CC[i],
        CC[j]//CCn[i][n-1]//cm[i,j,i]
      ],
      If[n==-1,
```

```
        CCi[i],
        CCi[j]//CCn[i][n+1]//cm[i,j,i]
      ]
    ]
  ]
]
```

Since multiplication is associative, we may implement a generalized multiplication which can take any number of arguments. It is also named `cm`, with a first argument given as an ordered list of indices to be concatenated.

```
cm[{}, j_] := cη[j]
cm[{i_}, j_] := cσ[i,j]
cm[{i_, j_}, k_] := cm[i,j,k]
cm[ii_List, k_] := Module[
        {
                i  = First[ii],
                is = Rest[ii],
                j  ,
                js ,
                l
        },
        j  = First[is];
        js = Rest[is];
        cm[i,j,l] // cm[Prepend[js, l], k]
]
```

The function `toGDO` serves as the invariant for the generators of the tangles. We define its value on crossings and on concatenations of elements.

```
toGDO[Xp[i_,j_]] := cR[i,j]
toGDO[Xm[i_,j_]] := cRi[i,j]
toGDO[xs_Strand] := cm[List@@xs, First[xs]]
toGDO[xs_Loop]   := Module[{x = First[xs]}, cm[List@@xs, x]//tr[x]]

getIndices[RVT[cs_List, _List, _List]] := Sort@Catenate@(List@@@cs)

TerminalQ[cs_List][i_] := MemberQ[Last/@cs,i];
next[cs_List][i_]:=If[TerminalQ[cs][i],
        Nothing,
```

```
        Extract[cs, ((#/.{c_,j_}->{c,j+1}&)@FirstPosition[i]@cs)]
]

InitialQ[cs_List][i_] := MemberQ[First/@cs,i];
prev[cs_List][i_]:=If[InitialQ[cs][i],
        Nothing,
        Extract[cs, ((#/.{c_,j_}->{c,j-1}&)@FirstPosition[i]@cs)]
]
```

To minimize the size of computations, whenever adjacent indices are present in the partial computation, they are to be concatenated before more crossings are introduced.

```
MultiplyAdjacentIndices[{cs_List,calc_GDO}]:=Module[
        { is=getCO[calc]
        , i
        , i2
        },
        i = SelectFirst[is,MemberQ[is,next[cs][#]]&];
        If[Head[i]===Missing,
                {cs,calc},
                i2 = next[cs][i];
                {DeleteCases[cs,i2,2], calc//cm[i,i2,i]}
        ]
]

MultiplyAllAdjacentIndices[{cs_List, calc_GDO}] :=
        FixedPoint[MultiplyAdjacentIndices, {cs, calc}]

generateGDOFromXing[x:_Xp|_Xm,rs_Association]:=Module[
        {p, i,j, in, jn},
        {i,j} = List@@x;
        {in,jn} = Lookup[rs,{i,j},0];
        toGDO[x]*CCn[p[i]][in]*CCn[p[j]][jn]
        ↪  //cm[p[i],i,i]//cm[p[j],j,j]
]

addRotsToXingFreeStrands[rvt_RVT] := GDO[] * Times @@ (
        CCn[#][Lookup[rvt[[3]], #, 0]] & /@
        First /@ Select[rvt[[1]], Length@# == 1 &]
)
```

Next we implement the framed link invariant `ZFramed`.

```
ZFramedStep[{_List,{},_Association,calc_GDO}]:={{},{},<||>,calc};
ZFramedStep[{cs_List,xs_List,rs_Association,calc_GDO}]:=Module[
        { x=First[xs], xss=Rest[xs]
        , csOut, calcOut
        , new
        },
        new=calc*generateGDOFromXing[x,rs];
        {csOut,calcOut} = MultiplyAllAdjacentIndices[{cs,new}];
        {csOut,xss,rs,calcOut}
]

ZFramed[rvt_RVT] := Last@FixedPoint[ZFramedStep, {Sequence @@ rvt,
        addRotsToXingFreeStrands[rvt]}]
ZFramed[L_] := ZFramed[toRVT@L]
```

Finally, when we wish to consider the unframed invariant, we apply the function `Unwrithe`, defined below.

```
Z[rvt_RVT] := Unwrithe@Last@FixedPoint[ZFramedStep, {Sequence @@ rvt,
 ↪  GDO[]}]
Z[L_] := Z[toRVT@L]

combineBySecond[l_List] := mergeWith[Total,#]& /@ GatherBy[l, First];
combineBySecond[lis___] := combineBySecond[Join[lis]]

mergeWith[f_, l_] := {l[[1, 1]], f@(#[[2]] & /@ l)}

Reindex[RVT[cs_, xs_, rs_]] := Module[
  {
    sf,
    cs2, xs2, rs2,
    repl, repl2
  },
   sf = Flatten[List@@#&/@cs];
   repl = (Thread[sf -> Range[Length[sf]]]);
   repl2 = repl /. {(a_ -> b_) -> ({a, i_} -> {b, i})};
   cs2 = cs /. repl;
   xs2 = xs /. repl;
   rs2 = rs /. repl2;
   RVT[cs2, xs2, rs2]
```

```
]

UnwritheComp[i_][gdo_GDO] := Module[
        {n = gdo//getL//SeriesCoefficient[#,{a[i]b[i],0,1}]&, j},
        gdo//(cKinkn[-n][j])//cm[i,j,i]
]

Unwrithe[gdo_GDO]:=(Composition@@(UnwritheComp/@(gdo//getCO)))@gdo

toRVT[L_RVT] := L
```

The partial trace is what we use to close a subset of the strands in a tangle. It takes the trace of all but one component, then returns the collection of all such ways of leaving one component open. (As described in ??).

```
ptr[L_RVT] := Module[
        {
                ZL = Z[L],
                cod
        },
        cod = getCO@ZL;
        Table[(Composition@@Table[tr[j],
         ↪ {j,Complement[cod,{i}]}])[ZL], {i,cod}]
]
ptr[L_] := ptr[toRVT[L]]
```

In order to be able to compare GDO's properly, we require a way to canonically represent them. This is achieved by reindexing the strands of the link and selecting one who's resulting invariant comes first in an (arbitrarily-selected) order, in this case the built-in ordering of expressions as defined by Mathematica™.

```
getGDOIndices[gdo_GDO]:=Sort@Catenate@Through[{getDO, getDC, getCO,
 ↪  getCC}@gdo]

isolateVarIndices[i_ -> j_] :=
 ↪  (v:y|b|t|a|x|η|β|α|ξ|A|B|T|w|z|W)[i]->v[j];

ReindexBy[f_][gdo_GDO] := Module[
        {
```

```
577         replacementRules,
578         varIndexFunc,
579         repFunc,
580         indices = getGDOIndices[gdo]
581         },
582         replacementRules = Thread[indices->(f/@indices)];
583         repFunc = ReplaceAll[replacementRules];
584         varIndexFunc =
            ↪  ReplaceAll[Thread[isolateVarIndices[replacementRules]]];
585         gdo//applyToPG[varIndexFunc]//
586                 applyToCO[repFunc]//
587                 applyToDO[repFunc]//
588                 applyToDC[repFunc]//
589                 applyToCC[repFunc]
590 ]
591
592 fromAssoc[ass_] := Association[ass][#] &
593
594 ReindexToInteger[gdos_List] := Module[
595         {is = getGDOIndices@gdos[[1]], f},
596         f = fromAssoc@Thread[is -> Range[Length[is]]];
597         ReindexBy[f]/@gdos
598 ]
599
600 getReindications[gdos_List] := Module[
601         {
602                 gdosInt = ReindexToInteger[gdos],
603                 is,
604                 fs,
605                 ls
606         },
607         is = getGDOIndices[gdosInt[[1]]];
608         fs = (fromAssoc@*Association@*Thread)/@(is -> # & /@
            ↪  Permutations[is]);
609         ls = CF@ReindexBy[#]/@gdosInt&/@fs;
610         Sort[Sort/@ls]
611 ]
612
613 getCanonicalIndex[gdo_] := First@getReindications@gdo
614
615 deleteIndex[i_][expr_] := SeriesCoefficient[expr/.U2l, Sequence @@
    ↪  ({#[i], 0, 0} & /@ {
```

```
        y, b, t, a, x, z, w
})]/.l2U
```

Here we introduce functions to further verify the co-algebra structure of a traced ribbon meta-Hopf algebra. In particular, the counit is responsible for deleting a strand. This has further applications in determining whether the invariants of individual components are contained in those of more complex links.

```
deleteIndexPG[i_][pg_PG] := pg//
        applyToL[deleteIndex[i]]//
        applyToQ[deleteIndex[i]]//
        applyToP[deleteIndex[i]]

deleteLoop[i_][gdo_] := gdo//
        applyToCC[Complement[#,{i}]&]//
        applyToPG[deleteIndexPG[i]]
```

## A.3 IMPLEMENTATION OF ROTATION NUMBER ALGORITHM

*Description of algorithm for knots*

Bar-Natan and van der Veen develop an algorithm to convert a classical long knot into an upright tangle. It involves passing a line segment, called the front, over the knot, requiring that everything behind the front is in upright form. For example, consider the link: By pulling the crossings along

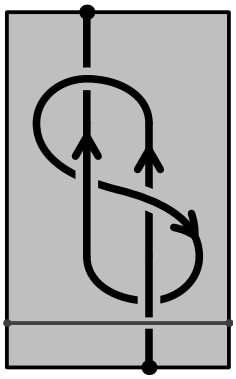


Figure A.1: A knot which is not in upright form. The front is written in grey.

the arc which touches the front, we can bring the knot to upright form.

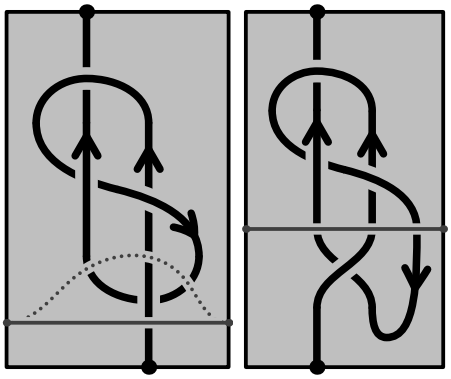


Figure A.2: By advancing the front over a crossing, we bring a crossing into upright form. A dashed front indicates where the front is advancing to.

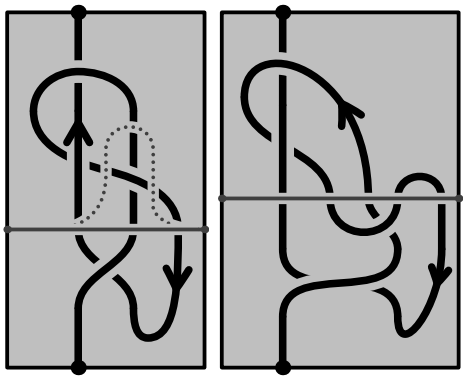


Figure A.3: By advancing the front over a crossing, we bring a crossing into upright form. A dashed front indicates where the front is advancing to.

*Extending the algorithm to multiple components*

An algorithm to convert a classical knot diagram into an upright knot diagram was implemented by Bar-Natan and van der Veen. Here we generalize the algorithm to convert a classical tangle with one open component to an upright tangle diagram. This generalization allows us to consider tangle diagrams with multiple components.

**Lemma A.1** ([BNvdVb], Lemma 43)**.** *For each classical tangle with one open component, there exists a unique upright tangle whose unbounded arcs have rotation numbers* 0*.*

This is a Haskell implementation[†1] of the algorithm `toRVT`,[†2] which takes a classical tangle and produces an upright tangle by computing a compatible choice of rotation numbers for each arc. This follows largely the same logic as above, except the leftmost strand is always prioritized for absorption, regardless of which component it belongs to.

*Use case*

For example, to query the **SX**-form of a link (i.e. its skeleton-crossing form), one writes:

```
>>> link 4 True 1
```

to obtain

```
SX [Loop [1,2,3,4],Loop [5,6,7,8]]
   [Xm 1 6,Xm 3 8,Xm 5 2,Xm 7 4]
```

To convert from the **SX** form to an upright tangle form (here written **RVT**), we must first replace one of the closed **Loop**s with an open **Strand** (accomplished by `openFirstStrand`):

```
>>> toRVT . openFirstStrand $ link 4 True 1
```

to obtain

```
RVT [Strand [1,2,3,4],Loop [5,6,7,8]]
    [Xm 1 6,Xm 3 8,Xm 5 2,Xm 7 4]
    [(5,-1),(6,1),(8,1)]
```

Reading off the final line, we see that arc 5 has rotation number $-1$, arcs 6 and 8 have rotation number 1, and the rest of the arcs have rotation number 0.

†1 The full source code is available at https://github.com/phro/KnotTheory.
†2 Here, the acronym RVT stands for "Rotational Virtual Tangle", which is another term for "Upright Tangle".

*Implementation*

We begin with a series of imports of common functions, relating to list manipulations and type-wrangling. The exact details are not too important.

```haskell
{-# LANGUAGE DeriveFunctor #-}
module KnotTheory.PD where
import Data.Maybe (listToMaybe, catMaybes, mapMaybe, fromMaybe,
 ↪  fromJust)
import Data.List (find, groupBy, sortOn, partition, intersect, union,
 ↪  (\\))
import Data.Tuple (swap)
import Data.Function (on)
import Control.Monad ((>=>))
import Control.Arrow ((>>>))
```

Next, we introduce the crossing type, which can be either positive `Xp` or negative `Xm` (using the mnemonic "plus" and "minus"):

```haskell
type Index = Int
data Xing i = Xp i i | Xm i i -- | Xv i i
  deriving (Eq, Show, Functor)
```

We define several functions which extract basic data from a crossing.

```haskell
sign :: (Integral b) => Xing Index -> b
sign (Xp _ _) = 1
sign (Xm _ _) = -1

isPositive :: Xing i -> Bool
isPositive (Xp _ _) = True
isPositive (Xm _ _) = False

isNegative :: Xing i -> Bool
isNegative (Xp _ _) = False
isNegative (Xm _ _) = True

overStrand :: Xing i -> i
overStrand (Xp i _) = i
overStrand (Xm i _) = i
```

```haskell

underStrand :: Xing i -> i
underStrand (Xp _ i) = i
underStrand (Xm _ i) = i
```

Next, we introduce the notion of a planar diagram, whose data is comprised of a collection of **Strand**s and **Loop**s (indexed by some type `i`, typically an integer). The **Skeleton** of a planar diagram is defined to be the collection of **Component**s, each of which is either an open **Strand** or a closed **Loop**.

```haskell
type Strand i = [i]
type Loop i = [i]
data Component i = Strand (Strand i) | Loop (Loop i)
  deriving (Eq, Show, Functor)

type Skeleton i = [Component i]
```

Next, we introduce the notion of a **KnotObject**, which has its components labelled by the same type `i`. We further define a function `toRVT` which converts a generic **KnotObject** into an upright tangle (in this codebase, the term **RVT!** (**RVT!**) is frequently used for the notion of an upright tangle). We call an object a planar diagram (or **PD**) if it has a notion of **Skeleton** and a collection of crossings.

```haskell
class KnotObject k where
  toSX  :: (Ord i) => k i -> SX i
  toRVT :: (Ord i) => k i -> RVT i
  toRVT = toRVT . toSX

class PD k where
  skeleton :: k i -> Skeleton i
  xings :: k i -> [Xing i]

```

The **SX** form of a diagram just contains the **Skeleton** and the **Xing**s (crossings), while the **RVT** form also assigns each arc an integral rotation number.

```haskell
data SX  i = SX  (Skeleton i) [Xing i] deriving (Show, Eq, Functor)
data RVT i = RVT (Skeleton i) [Xing i] [(i,Int)] deriving (Show, Eq,
  ↪ Functor)
```

Given any labelling of the arcs in a diagram, we can re-label the arcs using consecutive whole numbers. This is accomplised with `reindex`:

```haskell
reindex :: (PD k, Functor k, Eq i) => k i -> k Int
reindex k = fmap (fromJust . flip lookup table) k
  where
    table = zip (skeletonIndices s) [1..]
    s = skeleton k
```

Most importantly, we now declare that a diagram expressed in **SX** form (that is, without any rotation data) may be assigned rotation numbers to each of its arcs in a meaningful way. The bulk of the work is done by `getRotNums`, which is defined farther below. We handle the case where the entire tangle is a single crossingless strand separately.

```haskell
instance KnotObject SX where
  toSX = id
  toRVT k@(SX cs xs) = RVT cs xs rs where
    rs = filter ((/=0) . snd) . mergeBy sum $ getRotNums k f1
    i1 = head . toList $ s
    Just s = find isStrand cs
    f1 = case next i1 (toList s) of
           Just _  -> [(Out,i1)]
           Nothing -> []

instance KnotObject RVT where
  toRVT = id
  toSX (RVT s xs _) = SX s xs

instance PD SX where
  skeleton (SX s _) = s
  xings (SX _ xs)   = xs

instance PD RVT where
  skeleton (RVT s _ _) = s
  xings (RVT _ xs _)   = xs
```

Next, we include a series of functions which answer basic questions about planar diagrams. Note in rotnum, if a rotation number is not present in the table of values, it is assumed to be 0.

```
rotnums :: RVT i -> [(i,Int)]
rotnums (RVT _ _ rs) = rs

rotnum :: (Eq i) => RVT i -> i -> Int
rotnum k i = fromMaybe 0 . lookup i . rotnums $ k

isStrand :: Component i -> Bool
isStrand (Strand _) = True
isStrand _          = False

isLoop :: Component i -> Bool
isLoop (Loop _) = True
isLoop _        = False

toList :: Component i -> [i]
toList (Strand is) = is
toList (Loop is)   = is

skeletonIndices :: Skeleton i -> [i]
skeletonIndices = concatMap toList

involves :: (Eq i) => Xing i -> i -> Bool
x `involves` k = k `elem` [underStrand x, overStrand x]

otherArc :: (Eq i) => Xing i -> i -> Maybe i
otherArc x i
  | i == o     = Just u
  | i == u     = Just o
  | otherwise = Nothing
  where o = overStrand x
        u = underStrand x

next :: (Eq i) => i -> Strand i -> Maybe i
next e = listToMaybe . drop 1 . dropWhile (/= e)

prev :: (Eq i) => i -> Strand i -> Maybe i
prev e = next e . reverse

nextCyc :: (Eq i) => i -> Loop i -> Maybe i
```

```haskell
nextCyc e xs = next e . take (length xs + 1). cycle $ xs

prevCyc :: (Eq i) => i -> Loop i -> Maybe i
prevCyc e xs = prev e . take (length xs + 1). cycle $ xs

isHeadOf :: (Eq i) => i -> [i] -> Bool
x `isHeadOf` ys = x == head ys

isLastOf:: (Eq i) => i -> [i] -> Bool
x `isLastOf` ys = x == last ys

nextComponentIndex :: (Eq i) => i -> Component i -> Maybe i
nextComponentIndex i (Strand is) = next i is
nextComponentIndex i (Loop is) = nextCyc i is

prevComponentIndex :: (Eq i) => i -> Component i -> Maybe i
prevComponentIndex i (Strand is) = prev i is
prevComponentIndex i (Loop is) = prevCyc i is

isHeadOfComponent :: (Eq i) => i -> Component i -> Bool
isHeadOfComponent _ (Loop   _ ) = False
isHeadOfComponent i (Strand is) = i `isHeadOf` is

isLastOfComponent :: (Eq i) => i -> Component i -> Bool
isLastOfComponent _ (Loop   _ ) = False
isLastOfComponent i (Strand is) = i `isLastOf` is

isTerminalOfComponent :: (Eq i) => Component i -> i ->  Bool
isTerminalOfComponent c i = i `isHeadOfComponent` c || i
  ↪  `isLastOfComponent` c

isTerminalIndex :: (Eq i) => Skeleton i -> i ->  Bool
isTerminalIndex cs i = any (`isTerminalOfComponent` i) cs

nextSkeletonIndex :: (Eq i) => Skeleton i -> i -> Maybe i
nextSkeletonIndex s i = listToMaybe . mapMaybe (nextComponentIndex i)
  ↪  $ s

prevSkeletonIndex :: (Eq i) => Skeleton i -> i -> Maybe i
prevSkeletonIndex s i = listToMaybe . mapMaybe (prevComponentIndex i)
  ↪  $ s
```

In order to obtain all the crossing indices, we must take every combination of the under- and over-strands and their following indices:

```haskell
getXingIndices :: (Eq i) => Skeleton i -> Xing i -> [i]
getXingIndices s x = catMaybes
        [ f a | f <- [id, (>>= nextSkeletonIndex s)], a <- [o, u] ]
        where o = return (overStrand x)
              u = return (underStrand x)

δ :: (Eq a) => a -> a -> Int
δ x y
  | x == y    = 1
  | otherwise = 0

mergeBy :: (Ord i) => ([a] -> b) -> [(i,a)] -> [(i,b)]
mergeBy f = map (wrapIndex f) . groupBy ((==) `on` fst) . sortOn fst
  where
    wrapIndex :: ([a] -> b) -> [(i,a)] -> (i,b)
    wrapIndex g xs@(x:_) = (fst x, g . map snd $ xs)
```

Here we come to the main function, `getRotNums`, for which we have the following requirements (not expressed in the code):

1. The diagram `k` is a $(1, n)$-tangle (a tangle with only one open component)
2. The underlying graph of `k` is a planar.
3. The diagram `k` is a connected.

Only in this case will the function `toRVT` will then output a planar $(1, n)$-upright tangle which corresponds to a classical (i.e. planar) diagram.

This function involves taking a simple open curve (a Jordan curve passing through infinity) called the **Front**, and passing it over arcs in the diagram. This curve is characterized by the arcs it passes through, together with their orientations. Each intersection of the **Front** with the diagram provides a different **View**, either **In** or **Out** of the **Front** when following the orientation of the intersecting arc.

```haskell
type Front i = [View i]
type View  i = (Dir, i)
```

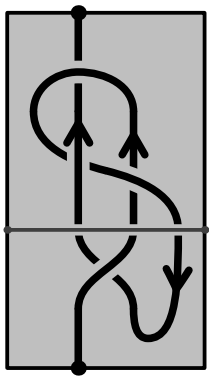


Figure A.4: A tangle with a front passing over it. The portion of the tangle below the front has all crossings in upright form. All arcs fully below the front have an integer rotation number.

We obtain the rotation numbers by successively passing the front across new crossings (achieved by `advanceFront`), keeping track of the rotation numbers of arcs which have already passed by the front. Once the front has passed across every crossing, all the rotation numbers have been computed.

Next, we define `converge`, which iterates a function until a fixed point is achieved.

```
176 converge :: (Eq a) => (a -> a) -> a -> a
177 converge f x
178         | x == x'   = x
179         | otherwise = converge f x'
180         where x' = f x
```

The function `convergeT` wraps `converge` in monadic transformations. In our context, the monad will be used to keep track of rotation numbers of the arcs.

```
181 convergeT :: (Monad m, Eq (m a)) => (a -> m a) -> a -> m a
182 convergeT f = return >>> converge (>>= f)
```

The implementation of `getRotNums` takes a front and advances it along a diagram until no more changes occur.

```haskell
getRotNums :: (Eq i) => SX i -> Front i -> [(i,Int)]
getRotNums k = convergeT (advanceFront k) >>> fst
```

When advancing the **Front**, we start by absorbing arcs that intersect with the front twice until the leftmost **View** no longer connects directly back to the **Front**. At this point, we can absorb a crossing into the front.

```haskell
advanceFront :: (Eq i) => SX i -> Front i -> ([(i,Int)], Front i)
advanceFront k = convergeT (absorbArc k) >=> absorbXing k
```

We next check for the case where the leftmost arc connects back to the **Front**. If it is pointing **Out** (and therefore connects back **In** further to the right), we adjust the rotation number of the arc by $-1$. Otherwise, we leave both the **Front** and the rotation numbers unchanged.

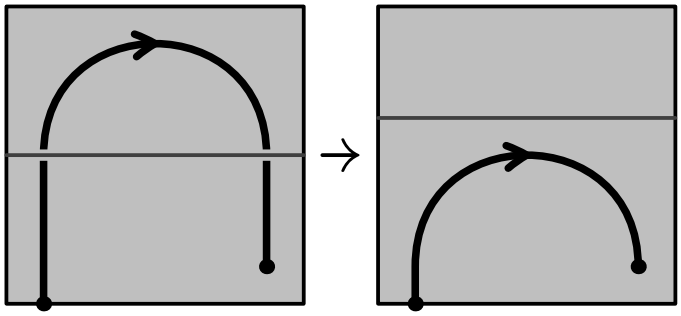


Figure A.5: Example of absorbing an arc which intersects the front multiple times. If the horizontal tangent vector points to to right, as in this picture, then the rotation number of the arc is decreased by 1. Otherwise, no change in the rotation number is recorded.

```haskell
absorbArc :: (Eq i) => SX i -> Front i -> ([(i,Int)],Front i)
absorbArc k []     = return []
absorbArc k f@(f1:fs) = case fs1 of
        ( In,i):_ -> (return (i,-1), fss)
        (Out,i):_ -> return fss           -- No new rotation numbers
        []        -> return f
        where (fs1,fss) = partition (((==) `on` snd) f1) fs
```

Our goal is to repeat this operation until we get a fixed point, which is encoded in `absorbArcs`:

```
absorbArcs :: (Eq i) => SX i -> Front i -> ([(i,Int)],Front i)
absorbArcs k = convergeT (absorbArc k)
```

Absorb a crossing involves expanding one's view at an arc from looking at a crossing to all the views one gets when looking in every direction at the crossing (namely, to the left, along the arc, and to the right). The function `absorbXing` performs this task on the leftmost **`View`** on the **`Front`**. The transverse strand receives a positive rotation number if it moves from left to right. The arc receiving the rotation depends on how the crossing is oriented.

```
absorbXing :: (Eq i) => SX i -> Front i -> ([(i,Int)],Front i)
absorbXing _ [] = return []
absorbXing k (f:fs) = (rs,newFront++fs) where
        newFront = catMaybes [l, a, r]
        l = lookLeft k f
        a = lookAlong k f
        r = lookRight k f
        rs = case (l,f,r) of
                (Just (In,i), (Out,_),_           ) -> [(i,1)]
                (_          , (In ,_),Just (Out, j)) -> [(j,1)]
                _                                    -> [     ]

data Dir = In | Out
  deriving (Eq, Show)
```

The following functions take a **`View`**, returning the **`View`** one has when looking in the corresponding direction. Since it is possible for the resulting gaze to be merely the boundary, it is possible for these functions to return **`Nothing`**.

```
lookAlong :: (Eq i, PD k) => k i -> View i -> Maybe (View i)
lookAlong k (d, i) = case d of
        Out -> sequence (Out, nextSkeletonIndex s i)
        In  -> sequence (In , prevSkeletonIndex s i)
        where s = skeleton k

lookSide :: (Eq i, PD k) => Bool -> k i -> View i -> Maybe (View i)
lookSide isLeft k di@(Out,i) = do
```

```
218         x <- findNextXing k di
219         j <- otherArc x i
220         if isLeft == ((underStrand x == i) == isPositive x)
221         then return (In, j)
222         else sequence (Out, nextSkeletonIndex (skeleton k) j)
223 lookSide isLeft k (In,i) =
224         sequence (Out, prevSkeletonIndex (skeleton k) i) >>=
225         lookSide (not isLeft) k
226
227 lookLeft  :: (Eq i, PD k) => k i -> View i -> Maybe (View i)
228 lookLeft = lookSide True
229
230 lookRight :: (Eq i, PD k) => k i -> View i -> Maybe (View i)
231 lookRight = lookSide False
232
233 findNextXing :: (Eq i, PD k) => k i -> View i -> Maybe (Xing i)
234 findNextXing k (Out,i) = find (`involves` i) $ xings k
235 findNextXing k (In ,i) = do
236   i' <- prevSkeletonIndex (skeleton k) i
237   find (`involves` i') $ xings k
```

# BIBLIOGRAPHY


[BNS] Dror Bar-Natan and Sam Selmani, Meta-monoids, meta-bicrossed products, and the Alexander polynomial, no. 10, 1350058, Publisher: World Scientific Publishing Co.

[BNvdVa] Dror Bar-Natan and Roland van der Veen, A perturbed-Alexander invariant, no. arXiv:2206.12298.

[BNvdVb] _____, Perturbed Gaußian generating functions for universal knot invariants, no. arXiv:2109.02057.

[ES] Pavel I. Etingof and Olivier Schiffmann, Lectures on quantum groups, International Press.

[Fox] R. H. Fox, Some problems in knot theory, Publisher: Prentice-Hall.

[Hal] Brian C. Hall, Quantum theory for mathematicians, Graduate Texts in Mathematics, vol. 267, Springer.

[Kau] Louis H. Kauffman, Rotational virtual knots and quantum link invariants.

[Lic] W. B. Raymond Lickorish, An introduction to knot theory, Springer Science & Business Media.

[Maj] Shahn Majid, A quantum groups primer, London Mathematical Society Lecture Note Series, Cambridge University Press.

[Pen] Roger Penrose, Spinors and space-time. volume 1, two-spinor calculus and relativistic fields, Cambridge monographs on mathematical physics, University Press.

[Wey] H. Weyl, Quantenmechanik und gruppentheorie, no. 1, 1–46.

## COLOPHON

Version 0.2.1

This thesis was typeset using the typographical look-and-feel `classicthesis` developed by André Miede and Ivo Pletikosić.

The style was inspired by Robert Bringhurst's seminal book on typography The Elements of Typographic Style.