COMPUTING THE GENERATING FUNCTION OF A COINVARIANTS MAP

BY

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ABSTRACT

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A well-known source of strong link invariants comes from quantum groups. Typically, one uses a representation of a quantum group to build a computable invariant, though these computations require exponential time in the number of crossings. Recent work has allowed for direct and efficient computations within the quantum groups themselves, through the use of perturbed Gaußian differential operators. This thesis introduces and explores a partial expansion of the tangle invariant Z introduced by Bar-Natan and van der Veen [BNvdVb]. We expand the use of the quantum group $\mathfrak{U}(\mathfrak{sl}^0_{2+})$ to include its space of coinvariants, providing an extension Z^{tr} of Z from open tangles to links.

We compute a basis for the space of coinvariants, then compute a closed-form expression for the corresponding trace map in the form of an exponential generating function. The resulting function is not a compatible perturbed Gaußian with respect to the previous research. To respond to this limitation, we find a method of computing the link invariant for a subclass of links and write a program to compute the invariant Z^{tr} on two-component links. Contrary to expectations, we find that Z^{tr} is neither stronger nor weaker than the Multivariable Alexander polynomial. This unexplained behaviour

warrants further study into the invariant Z^{tr} and its relationship with other invariants.

To someone,

who did something nice.

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ACRONYMS

RVT Rotational Virtual Tangle

RVK Rotational Virtual Knot

RVL Rotational Virtual Link

MVA Multivariable Alexander polynomial

CONTENTS

1

EXECUTIVE SUMMARY

1.1 CONTEXT

Understanding Knotted Objects

In the field of knot theory, distinguishing between two knots or links has proven to be a difficult task. Computing strong invariants of knotted objects is a popular way to aid with the classification of these objects.

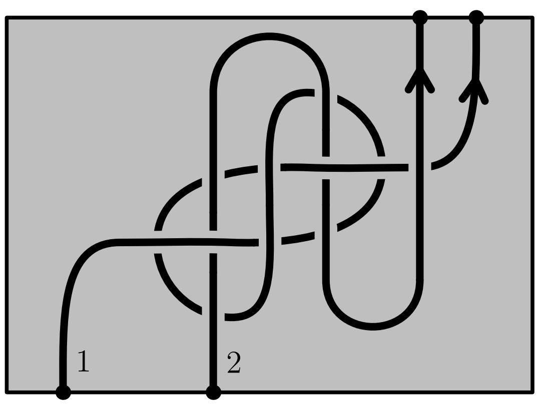

Figure 1.1: Two strings knotted together. Can they be disentangled?

Merely being able to distinguish between two knotted objects does not always provide us with enough information about these topological structures. For instance, one may ask if a particular link is a satellite of another (roughly: where one knot is embedded into a link by following one of its components), whether a knot is slice (i.e. it is the boundary of a disk in $\mathbb{R}^4$), or whether it is ribbon (i.e. the boundary of a disk in $\mathbb{R}^3$ with restricted types of singularities). Many interesting properties of knots can be phrased in terms of certain topological properties, such as strand doubling (taking a strand and replacing it with two copies of itself, as in figure 1.2) or strand stitching (joining two open components together to form one longer one).

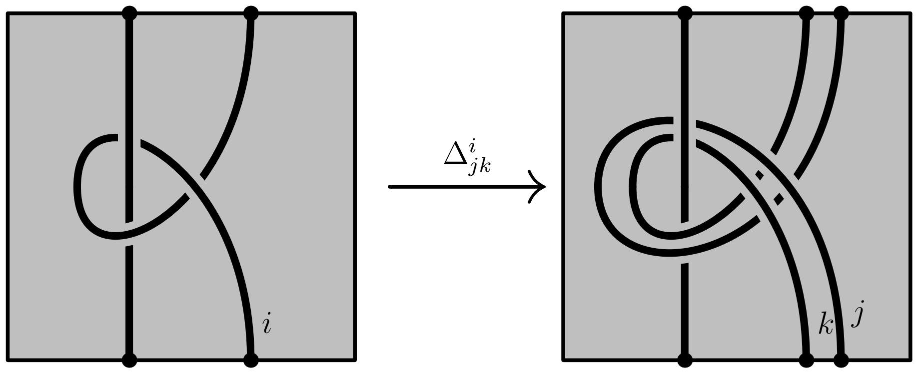

Figure 1.2: An example of strand-doubling.

Open problems such as the Ribbon Slice Conjecture (asking whether there exists a slice knot which is not ribbon) may be advanced by the development of "topologically aware" invariants—those which preserve topological data in a retrievable way.

Quantum Invariants

One such class of topological invariants is derived from quasitriangular Hopf algebras, which are algebraic structures whose operations mimic those of knotted objects. With this approach, one takes a knotted object and constructs it into a series of topological operations (such as stitching strands or doubling strands), then maps each of these operations to a corresponding algebraic operation. The composition of these algebraic operations is the value of the invariant.

More specifically, a Hopf algebra is a vector space A together with several maps between various tensor powers thereof (for instance, a multiplication map $m\colon A \otimes A \to A$). The value of the invariant is a vector in a tensor power of A, with each crossing assigned an element of $A \otimes A$. Tensor factors which belong to the same strand are concatenated by multiplying the associated algebra elements. While this formulation is elegant, it has a notable drawback: computing the value of the invariant of an object with many components requires manipulating large tensor powers of A. One remedy is to instead perform the computation in a representation of A, say a small vector space V, though the issue of exponential growth in complexity remains.

Images of the invariant

To avoid the issue of exponential computational complexity, one can instead investigate the nature of the set of all values of the invariant as a subset of the algebra and its tensor powers. For a particular choice of algebra (namely $\hat{\mathfrak{U}}(\mathfrak{sl}_{2+}^{\epsilon})$, as investigated by Dror Bar-Natan and Roland van der Veen in [BNvdVb]) the space of values the corresponding invariant Z can take is significantly smaller than the whole space; the rank of the space of values grows only quadratically with the number of crossings in the knotted object. In particular, by looking at the generating functions of the associated maps, instead of a generic power series, the value of Z on tangles always take the form of a (perturbed) Gaußian. Computationally, this means one needs only to keep track of the quadratic form and the perturbation. The invariant Z dominates the $\mathfrak{sl}_2$-coloured Jones polynomial. We will focus on the case when $\epsilon = 0$, for which Z becomes an efficient computation of the Alexander polynomial Δ on knots.

1.2 EXTENDING THE INVARIANT TO MORE KNOTTED OBJECTS

The research program outlined by Bar-Natan and van der Veen computes Z only for (open) tangles–that is, collections of open strands whose endpoints are fixed to a boundary circle. (Note that this includes long knots, which are exactly the one-component tangles.) This thesis is focused on extending Z and its computations to tangles with closed components.

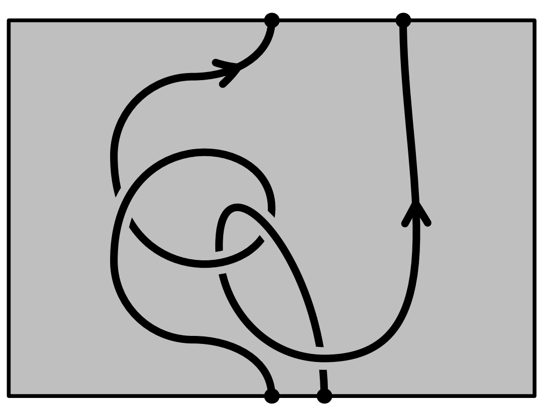

Figure 1.3: A open tangle. All components intersect the boundary.

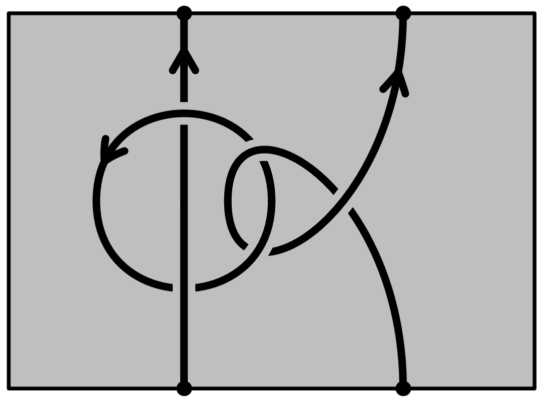

Figure 1.4: A tangle with a closed component.

Computing the extended map

The first task is to determine the space in which the extended invariant, which we will call Z^{tr}, lives. One may observe that in a matrix algebra, one is able to contract two matrices together via matrix multiplication. When one wishes to contract a matrix along itself, one uses the trace map. Analogously, since two strands in a tangle corresponds to multiplication, closing a strand into a loop should correspond algebraically to a trace map.

In a generic algebra A, the trace map is defined as the projection onto the set of coinvariants: $\mathrm{tr}\colon A \to A_A = A/[A, A]$.†1 In order to extend the invariant in this framework, we must first compute the space of coinvariants for $\hat{\mathfrak{U}}(\mathfrak{sl}^{\epsilon}_{2+})$, then compute the coinvariants map, and write it as a generating function. (This is accomplished in section 4.2.)

Performing computations

Unfortunately, the resulting trace map does not take the form of a perturbed Gaußian in a way that readily connects to the existing framework. In order to determine whether further study in this direction is merited, we must find an alternative computation method to get a preliminary sense of the strength of Z^{tr}.

For a subclass of links (which includes all two-component links), we compute an explicit closed form for the trace map, then implement a

†1 Here, $[A, A] = \mathrm{span}\{\, [x, y] \mid x, y \in A \,\}$ refers to the vector space of Lie brackets, not the ideal generated thereby. The space A_A does not have an algebra structure in general.

computer program to compute the value of Z^{tr} on all two-component links with up to 11 crossings. When applied to knots, Z computes the Alexander polynomial. When applied to two-component tangles, one may expect that Z^{tr} would produce the natural generalization to multiple components: the Multivariable Alexander polynomial (MVA). Surprisingly, the MVA and Z^{tr} are incomparable, with each being able to distinguish pairs of links the other cannot. (See section 5.1 for more information.)

Further study

As Z^{tr} does not generalize Z in the manner expected, several interesting avenues of further research are opened. Firstly, determining what the relationship between Z^{tr} and the MVA remains open. Second is the challenge of finding an efficient method for computing Z^{tr} on links with more than two components, which currently is mired in complications with the presence of non-elementary functions where quadratic forms normally appear. Third is the question of the existence of other viable trace candidates. In particular, it may be worth exploring whether a universal trace with respect to the perturbed Gaußian framework defines a sufficiently useful invariant.

2

TENSOR PRODUCTS AND META-OBJECTS

2.1 TENSOR PRODUCT NOTATION

In what follows, we will extensively use tensor products, tensor powers, and generalizations thereof. We begin by introducing the notation that will be used first for traditional tensor products, then for their generalizations.

Let V be a $\Bbbk$-vector space (for the moment assumed to be finite dimensional). When working with a large tensor power $V^{\otimes n}$ of V, it will often be more convenient to label tensor factors with elements of a finite set S (with $|S| = n$) rather than by their position in a linear order.

For example, consider the vector $u \otimes v \otimes w \in V^{\otimes 3}$. Let us choose an index set $S = \{1, 2, 3\}$. We then may equivalently write this vector by labelling each tensor factor with one of the elements of S, say $u_1 v_2 w_3$. Since the labels serve to distinguish the separate factors, this vector may equivalently be written as $u_1 v_2 w_3 = v_2 u_1 w_3 = w_3 u_2 u_1 \in V^{\otimes S}$. We will write the set $V^{\otimes S}$ with a subscript: V_S. We formalize the idea below:

Definition 2.1 (indexed tensor powers)**.** Let V be a vector space and $S = \{s_1, \dots, s_n\}$ be a finite set. We define the indexed tensor power of V to be the collection of formal linear combinations of functions from S to V

$$V_S \coloneqq \operatorname{span}\{f\colon S \to V\}/\sim \tag{2.1}$$

subject to the standard multilinear relations, namely multi-additivity and the factoring of scalars.

By multi-additivity, we mean that for each $i \in S$ and $f, g \in V_S$ satisfying $f(s) = g(s)$ for each $s \in S \setminus \{i\}$, we have:

$$(f+g)(s) = \begin{cases} f(s) = g(s) & \text{if } s \neq i \\ f(i) + g(i) & \text{if } s = i \end{cases} \tag{2.2}$$

In practice, we will write such functions $f\colon S \to V$ with $f(s_i) = v_i$ in the following notation:

$$(v_1)_{s_1}(v_2)_{s_2} \cdots (v_n)_{s_n} := f \tag{2.3}$$

With this notation, we may easily express the factoring of scalars as:

$$(v_1)_{s_1}(v_2)_{s_2} \cdots (\lambda v_i)_{s_i} \cdots (v_n)_{s_n} = \lambda \cdot (v_1)_{s_1}(v_2)_{s_2} \cdots (v_n)_{s_n} \tag{2.4}$$

Likewise, equation (2.2) becomes

$$\begin{aligned}\Big((v_1)_{s_1}(v_2)_{s_2} \cdots x_{s_i} \cdots (v_n)_{s_n}\Big) + \Big((v_1)_{s_1}(v_2)_{s_2} \cdots y_{s_i} \cdots (v_n)_{s_n}\Big) \\ = (v_1)_{s_1}(v_2)_{s_2} \cdots (x+y)_{s_i} \cdots (v_n)_{s_n}\end{aligned} \tag{2.5}$$

Next, to compactify notation, we introduce notation for maps between tensor powers so that we may unambiguously refer to appropriate tensor factors while defining morphisms. We accomplish this task by adding a convenient way of writing the domain and codomain of a map. Let D and C be finite sets, and $T\colon V_D \to V_C$. We will denote T alternatively by T_C^D, so that its domain and codomain are easily read off. It is important to note that when T is not symmetric in its arguments, the order of the indices in this notation matters.

Example 2.2. Let $V = \mathbb{R}^2$, and $T_c^{a,b}$ (equivalently, $T\colon V_{\{a,b\}} \to V_{\{c\}}$) defined by

$$\begin{aligned} T_c^{a,b}(\vec{v}_a(\vec{e}_1)_b) &= \vec{0}_c \\ T_c^{a,b}(\vec{v}_a(\vec{e}_2)_b) &= \vec{v}_c \end{aligned} \tag{2.6}$$

This function zeros out vectors whose b-component is $\vec{e}_1$. If we wish to define an analogous function for the a-component, we may simply reverse the order of the superscript: $T_c^{b,a}$, which sends $\vec{v}_b(\vec{e}_1)_a = (\vec{e}_1)_a\vec{v}_b$ to $\vec{0}_c$ and $\vec{v}_b(\vec{e}_2)_a$ to $\vec{v}_c$.

Finally, we point out that any morphism T_C^D may be extended to one with larger domain and codomain. We introduce the notation $T[S] := T \otimes \mathrm{id}_S^S$

for this concept. When no ambiguity arises, we will also suppress the "$[S]$" so that T_C^D becomes more generally:

$$(T_C^D)(v_D \otimes w_S) := (T_C^D v_D) \otimes w_S \tag{2.7}$$

for any $v_D \in V_D$ and $w_S \in V_S$.

Remark 2.3. There are three special cases with this notation:

- Given a (multi)linear functional $\phi\colon V_S \to \Bbbk \cong V_\emptyset$, we will write ϕ^S instead of $\phi_\emptyset^S$. The linear order on S remains in this notation.
- Elements $v \in V_S$ will be interpreted as a map $v\colon \Bbbk = V_\emptyset \to V_S$ written v_S instead of $v_S^\emptyset$.
- When only one index is present in a subscript or superscript, and its omission does not introduce an ambiguity in an expression, then it may be omitted to improve readability. For instance, a map $\phi\colon V_{\{1,2\}} \to V_{\{3\}}$ may be written as $\phi^{1,2}$ instead of $\phi_3^{1,2}$, with the canonical isomorphism $V \cong V_{\{3\}}$ being suppressed.

When taking the tensor product of two such tensor powers, we follow [BNS] and use the notation "$\sqcup$" instead of "$\otimes$":

$$V_X \sqcup V_Y := V_{X \sqcup Y} \tag{2.8}$$

Additionally, given $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ such that $D_1 \cap D_2 = \emptyset = C_1 \cap C_2$, we have a product morphism $\phi_{C_1}^{D_1}\psi_{C_2}^{D_2} := \phi \otimes \psi\colon V_{D_1 \sqcup D_2} \to V_{C_1 \sqcup C_2}$, which we also write with concatenation.

2.2 META-OBJECTS

Notation extension beyond vector spaces

While the above notation is helpful when working with vector spaces, we are interested in also using the same notation to describe a tangle. Our formulation of tangles (introduced in section 2.4) is neither a tensor product nor a monoidal category, though it shares many similarities with both concepts. In particular, the domains and codomains of the maps we have discussed so far have only depended on the index set. With this observation, we replace the notation of tensor powers with that of a so-called meta-object. We introduce this concept by starting with monoids.

Defining a meta-monoid

To make the above definition more concrete, we will go through the process of defining a meta-monoid, which is a generalization of a monoid object. Traditionally, the data of a monoid object are the following:

- An object M in a category $\mathcal{C}$.
- A morphism $m\colon M \times M \to M$ called the “multiplication” operation.
- A “unit” morphism $\eta\colon \{1\} \to M$.[†1]
- A collection of relations between the operations, written as equalities of morphisms between Cartesian powers of M. For example, associativity may be written:

$$\begin{tikzcd} M \times M \times M \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"'] & M \times M \arrow[d, "m"] \\ M \times M \arrow[r, "m"'] & M \end{tikzcd} \tag{2.9}$$

Further, the data of these relations is extended to higher powers of M by acting on other components by the identity:

$$\begin{tikzcd} M^{n+3} \arrow[r, "m\times\mathrm{id}^{n+1}"] \arrow[d, "\mathrm{id}\times m\times\mathrm{id}^{n}"'] & M^{n+2} \arrow[d, "m\times\mathrm{id}^{n}"] \\ M^{n+2} \arrow[r, "m\times\mathrm{id}^{n}"'] & M^{n+1} \end{tikzcd} \tag{2.10}$$

Let us alter how we package these data so as to maximize the clarity of the meta-monoid structure:

1. Instead of linear orders of factors $M \times \cdots \times M$, we will index factors by a finite set X, writing it $M_X := \{f\colon X \to M\}$ in the style of equation (2.1).

†1 When $\mathcal{C} = \mathbf{Set}$, we usually write the unit as an element $1 = \eta(1) \in M$

2. The indexed factors will determine how the monoid operations act. For instance, multiplication of factor i and j together, with the result labelled in factor k is to be written $m_k^{ij}: M_{\{i,j\}} \to M_{\{k\}}$.

3. Instead of implicitly including extensions of operations to higher powers by the identity, we will parametrize the extension by finite sets by $\phi_C^D[X] := \phi_C^D \times \mathrm{id}_X^X$. For example, multiplication $m_k^{ij}: M_{\{i,j\}} \to M_{\{k\}}$ generates a family of maps $m_k^{ij}[X]: M_{\{i,j\}\sqcup X} \to M_{\{k\}\sqcup X}$, each of which must satisfy the relations of the monoid object such as equation (2.10).

This way of packaging the data leads us to the following generalization:

Definition 2.4. A meta-monoid in $\mathcal{C}$ is the following data:

- A family of objects $M_X \in \mathcal{C}$, indexed over finite sets X, with set bijections $f: X \xrightarrow{\sim} Y$ inducing isomorphisms $M_X \cong M_Y$.
- A family of morphisms $m_k^{ij}[X]: M_{\{i,j\}\sqcup X} \to M_{\{k\}\sqcup X}$ called "multiplication".
- A family of "unit" morphisms $\eta_i[X]: M_X \to M_{\{i\}\sqcup X}$.
- A collection of relations between the morphisms, written as equalities of morphisms between the M_X's. In particular, associativity:

$$\begin{tikzcd} M_{\{1,2,3\}\sqcup X} \arrow[r, "m_1^{1,2}[X\sqcup\{3\}]"] \arrow[d, "m_2^{2,3}[X\sqcup\{1\}]"'] & M_{\{1,3\}\sqcup X} \arrow[d, "m_1^{1,3}[X]"] \\ M_{\{1,2\}\sqcup X} \arrow[r, "m_1^{1,2}[X]"'] & M_{\{1\}\sqcup X} \end{tikzcd} \tag{2.11}$$

and the identity:

$$\begin{tikzcd} M_{\{1\}\sqcup X} \arrow[r, "\eta_2[X]"] \arrow[dr, bend right, "\mathrm{id}"'] & M_{\{1,2\}\sqcup X} \arrow[d, shift right, "m_1^{2,1}[X]"'] \arrow[d, shift left, "m_1^{1,2}[X]"] \\ & M_{\{1\}\sqcup X} \end{tikzcd} \tag{2.12}$$

Example 2.5 (monoid objects are meta-monoids)**.** Any monoid object M in a monoidal category $(\mathcal{C}, \otimes, \{1\})$ has the structure of a meta-monoid $\{M_X\}_X$ via $M_X := M^{\otimes X}$, $m_k^{ij}[X] := m_k^{ij} \otimes \mathrm{id}_X^X$, and $\eta_i[X](v) := 1_i \otimes v$ for any $v \in M^{\otimes X}$.

Consider the following structure, which satisfies the definition of a meta-monoid, but is not a monoid in the traditional sense:

Example 2.6 (the meta-monoid of square matrices)**.** Let $\Bbbk$ be a field and $M_X := \mathrm{Mat}_{X\times X}(\Bbbk)$ be the set of square matrices whose rows and columns are indexed by the finite set X. Define $m_k^{ij}[X]\colon M_{X\sqcup\{i,j\}} \to M_{X\sqcup\{k\}}$ by $m_k^{ij}[X]((a_{rs})_{rs}) := \big(a_{rs} + \delta_{rk}(a_{is} + a_{js}) + \delta_{sk}(a_{si} + a_{sj})\big)_{rs}$. That is, the multiplication of two indices corresponds to the summation of their respective rows and columns, the result of which is stored in row and column k. The unit $\eta_i[X]((a_{rs})_{rs})$ extends $(a_{rs})_{rs}$ to include a row and column of 0's, each labelled by the index i.

Example 2.7 (tangles form a meta-algebra)**.** Tangles are the main example of a meta-algebra which is not an algebra in the traditional sense. We go into more detail in section 2.4.

Defining meta-objects

In order to define other meta-objects, such as a meta-colagebra or a meta-semigroup, we provide the following more general definition:

Definition 2.8 (meta-object)**.** Let $\mathcal{C}$ be a category. A meta-object in $\mathcal{C}$ is two things:

1. A collection of objects A_X, one for each choice of finite set X. (This serves as the analogue to monoidal powers.)

2. A collection of operations $\phi_1, \phi_2, \dots, \phi_n$ each with a signature $|\phi_i| \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. For any pair of finite sets (D, C) satisfying $(|D|, |C|) = |\phi|$, we have a morphism:
$$\phi_C^D\colon A_D \to A_C \tag{2.13}$$

Further, for each bijection $f\colon X \xrightarrow{\sim} Y$ of finite sets X and Y, a reindexing isomorphism $\iota_f\colon A_X \xrightarrow{\sim} A_Y$.

Next, for each operation ϕ_C^D, there is a collection of morphisms $\phi_C^D[\,\cdot\,]$ indexed by finite sets such that for each finite set S, T:

1. $\phi[S]\colon A_{C\sqcup S} \to A_{D\sqcup S}$

2. $\phi[\emptyset] = \phi$

3. $(\phi[S])[T] = \phi[S \sqcup T]$

When no ambiguity arises, we will omit the portion written in square brackets, so that ϕ will stand for $\phi[X]$, with the set X determined from context.

Finally, we may define the product of two spaces A_S and A_T by $A_S A_T = A_{S \sqcup T}$. Given morphisms $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ such that $D_1 \cap D_2 = \emptyset = C_1 \cap C_2$, we have a product morphism $\phi_{C_1}^{D_1}\psi_{C_2}^{D_2} \colon \mathcal{C}_{D_1 \sqcup D_2} \to \mathcal{C}_{C_1 \sqcup C_2}$.

Composition of operators $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ is defined when $C_1 = D_2$:

$$\psi_{C_2}^{D_2} \circ \phi_{C_1}^{D_1} \colon \mathcal{C}_{D_1} \to \mathcal{C}_{C_2} \tag{2.14}$$

Remark 2.9. In this text, we will denote left-to-right composition with the "$/\!\!/$" symbol: $f /\!\!/ g := g \circ f$. Writing function composition in this order assists with readability when there are many functions to apply.

Remark 2.10. To make expressions easier to read, we introduce the domain extension implicitly in the following context: given morphisms $\phi_{C_1}^{D_1}$ and $\psi_{C_2}^{D_2}$ such that $C_2 \cap (C_1 \setminus D_2) = \emptyset = D_1 \cap (D_2 \setminus C_1)$, we define:

$$\phi_{C_1}^{D_1} /\!\!/ \psi_{C_2}^{D_2} := \phi_{C_1}^{D_1}[D_2 \setminus C_1] /\!\!/ \psi_{C_2}^{D_2}[C_1 \setminus D_2] \tag{2.15}$$

The two extreme cases of this definition are:

- When $C_1 \cap D_2 = \emptyset$, equation (2.15) becomes $\phi_{C_1}^{D_1}\psi_{C_2}^{D_2}$.
- When $C_1 = D_2$, equation (2.15) becomes the composition $\phi_{C_1}^{D_1} /\!\!/ \psi_{C_2}^{D_2}$ exactly.

Remark 2.11. While the $/\!\!/$ operator is associative, care must be taken that the compositions are well-defined in the presence of duplicated indices. While it is sufficient for all the finite sets in a composition to be pairwise disjoint, this condition will prove too restrictive for clear communication of formulae.

2.3 ALGEBRAIC DEFINITIONS

We now introduce the algebraic structures which will be used to define the tangle invariant. These definitions follow those given by Majid in [Maj], although the ones presented below are given in a way that their corresponding meta-structure is readily visible.

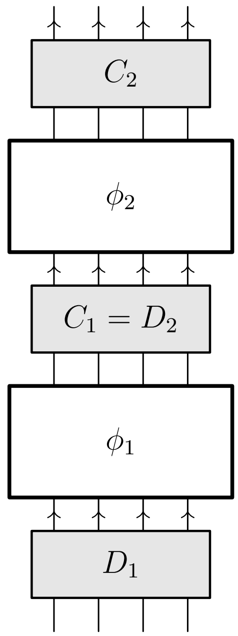

Figure 2.1: We may visualize a composition of morphisms with a graphical calculus. This graphic represents $(\phi_1)^{D_1}_{C_1} /\!\!/ (\phi_2)^{D_2}_{C_2}$ when $C_1 = D_2$.

Definition 2.12 (meta-algebra)**.** A meta-algebra (or meta-monoid[†2]) is a collection of objects $\{A_X\}_X$ in $\mathcal{C}$ together with an associative multiplication $m^{i,j}_k \colon A_{\{i,j\}} \to A_{\{k\}}$ (satisfying equation (2.16)), and a unit $\eta_i \colon A_\emptyset \to A_{\{i\}}$ satisfying equation (2.17).

Remark 2.13. When $\mathcal{C} = \mathbf{Vect}$ and $A_X = V^{\otimes X}$ for some vector space V, definition 2.12 becomes the more familiar definition of an algebra. When $A_\emptyset$ is a field, it is more common think of the unit as an element $\mathbf{1} \in V$. The unit map is then defined by linearly extending the assignment $\eta_i(1) = \mathbf{1}_i$.

$$\begin{tikzcd} A_{\{1,2,3\}} \arrow[r, "m_1^{1,2}"] \arrow[d, "m_2^{2,3}"'] & A_{\{1,3\}} \arrow[d, "m_1^{1,3}"] \\ A_{\{1,2\}} \arrow[r, "m_1^{1,2}"'] & A_{\{1\}} \end{tikzcd} \tag{2.16}$$

$$\begin{tikzcd} A_{\{1\}} \arrow[r, "\eta_2"] \arrow[dr, "\mathrm{id}"'] & A_{\{1,2\}} \arrow[d, shift right, "m_1^{2,1}"'] \arrow[d, shift left, "m_1^{1,2}"] \\ & A_{\{1\}} \end{tikzcd} \tag{2.17}$$

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†2 This is a repeat of definition 2.4. The only difference between an algebra object and a monoid object is the presence of a linear structure. Since meta-monoids are defined for any category, this distinction disappears.

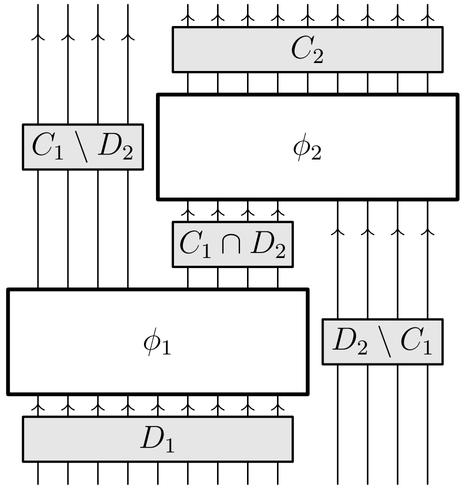

Figure 2.2: Visual mnemonic for extending morphisms. This graphic represents equation (2.15).

Remark 2.14. From now on, we will denote repeated multiplication as in equation (2.16) by using extra indices. For instance: $m_\ell^{i,j,k} := m_r^{i,j} /\!\!/ m_\ell^{r,k} = m_s^{j,k} /\!\!/ m_\ell^{i,s}$.

There is also the dual notion of a coalgebra, which arises by reversing the arrows in equations (2.16) and (2.17):

Definition 2.15 (meta-coalgebra). A meta-colagebra (or meta-comonoid) is a collection $\{C_X\}_X$ together with a comultiplication $\Delta^i_{jk} \colon C_{\{i\}} \to C_{\{j,k\}}$ which is coassociative (equation (2.18)) and a counit, which is a map $\epsilon^i \colon A_i \to A_\emptyset$ satisfying equation (2.19).

$$\begin{tikzcd} C_{\{1,2,3\}} & C_{\{2,3\}} \arrow[l, "\Delta^1_{1,2}"'] \\ C_{\{1,2\}} \arrow[u, "\Delta^2_{2,3}"] & C_{\{1\}} \arrow[l, "\Delta^1_{1,2}"] \arrow[u, "\Delta^1_{1,3}"'] \end{tikzcd} \tag{2.18}$$

$$\begin{tikzcd} C_{\{1\}} & C_{\{1,2\}} \arrow[l, "\epsilon^2"'] \\ & C_{\{1\}} \arrow[ul, "\mathrm{id}"] \arrow[u, shift left, "\Delta^1_{2,1}"'] \arrow[u, shift right, "\Delta^1_{1,2}"] \end{tikzcd} \tag{2.19}$$

Remark 2.16. From now on, we will denote repeated comultiplication as in equation (2.18) by using extra indices. For instance: $\Delta^i_{j,k,\ell} := \Delta^i_{j,r} /\!\!/ \Delta^r_{k,\ell} = \Delta^i_{s,\ell} /\!\!/ \Delta^s_{j,j}$.

If a meta-object $\{B_X\}_x$ satisfies both definitions of an algebra and a coalgebra, we introduce a definition for when the structures are compatible with each other in the following way:

Definition 2.17 (meta-bialgebra)**.** A meta-bialgebra (or meta-bimonoid) is a meta-algebra (B, m, η) and a meta-coalgebra (B, Δ, ϵ), such that Δ and ϵ are meta-algebra morphisms. [†3]

$$
\begin{tikzcd}
B_{\{1,2\}} \arrow[r, "m_1^{1,2}"] \arrow[d, "\Delta^1_{1,3} /\!\!/ \Delta^2_{2,4}"] & B_{\{1\}} \arrow[d, "\Delta^1_{1,2}"] \\
B_{\{1,2,3,4\}} \arrow[r, "m_1^{1,2} /\!\!/ m_2^{3,4}"'] & B_{\{1,2\}}
\end{tikzcd}
\tag{2.20}
$$

$$
\begin{tikzcd}
 & B_{\{1\}} \arrow[d, "\Delta^1_{1,2}"] \\
B_\emptyset \arrow[ur, "\eta_1"] \arrow[r, "\eta_1 /\!\!/ \eta_2"'] & B_{\{1,2\}}
\end{tikzcd}
\tag{2.21}
$$

$$
\begin{tikzcd}
B_{\{1,2\}} \arrow[rr, "m_1^{1,2}"] \arrow[dr, "\epsilon^1 /\!\!/ \epsilon^2"'] & & B_{\{1\}} \arrow[dl, "\epsilon^1"] \\
 & B_\emptyset &
\end{tikzcd}
\tag{2.22}
$$

$$
\begin{tikzcd}
B_\emptyset \arrow[r, "\eta_1"] \arrow[dr, "\mathrm{id}"'] & B_{\{1\}} \arrow[d, "\epsilon^1"] \\
 & B_\emptyset
\end{tikzcd}
\tag{2.23}
$$

Remark 2.18. The conditions for Δ being an algebra morphism are presented in equations (2.20) and (2.21), while those for ϵ are in equations (2.22) and (2.23).[†4] Observing invariance under arrow reversal, it may not come as a surprise that equations (2.20) and (2.22) also are the conditions for m being a coalgebra morphism, and equations (2.21) and (2.23) tell us that η is as well.

Next, we introduce a notion of invertibility which extends a bialgebra to a Hopf algebra.

Definition 2.19 (meta-Hopf algebra)**.** A meta-Hopf algebra (or meta-Hopf monoid) is a bialgebra H together with a map $S\colon H \to H$ called the antipode, which satisfies $\Delta^1_{1,2} /\!\!/ S^1_1 /\!\!/ m_1^{1,2} = \epsilon^1 /\!\!/ \eta_1 = \Delta^1_{1,2} /\!\!/ S^2_2 /\!\!/ m_1^{1,2}$. As a commutative diagram, this looks like equation (2.24)

$$
\begin{tikzcd}
H_{\{1\}} \arrow[rr, "\epsilon^1"] \arrow[dr, "\Delta^1_{1,2}"'] & & H_\emptyset \arrow[rr, "\eta_1"] & & H_{\{1\}} \\
 & H_{\{1,2\}} \arrow[rr, shift left, "S^2_2"] \arrow[rr, shift right, "S^1_1"'] & & H_{\{1,2\}} \arrow[ur, "m_1^{1,2}"'] &
\end{tikzcd}
\tag{2.24}
$$

†3 B_X inherits a (co)algebra structure from B, given by $(B_X)_Y := B_{X^Y}$ and component-wise operations. The bialgebra structure on $B_\emptyset$ is given by $m = \eta = \Delta = \epsilon = \mathrm{id}$.

†4 While notation explicitly naming each tensor factor appears cumbersome in these diagrams, it will prove invaluable later when used on tangle diagrams, so we leave it as is for the sake of consistency.

In order to do knot theory, we need an algebraic way to represent a crossing of two strands. This is accomplished by the so-called $\mathcal{R}$-matrix:

Definition 2.20 (quasitriangular meta-Hopf algebra)**.** A quasitriangular meta-Hopf algebra (or quasitriangular meta-Hopf monoid) is a Hopf algebra H, together with an invertible element $\mathcal{R}_{i,j} \in H_{i,j}$, called the $\mathcal{R}$-matrix, which satisfies the following properties: (we will denote the inverse by $\overline{\mathcal{R}}$)

$$\mathcal{R}_{13} \mathbin{/\!\!/} \Delta^1_{12} = \mathcal{R}_{13}\mathcal{R}_{24} \mathbin{/\!\!/} m^{34}_3 \tag{2.25}$$

$$\mathcal{R}_{13} \mathbin{/\!\!/} \Delta^3_{23} = \mathcal{R}_{13}\mathcal{R}_{42} \mathbin{/\!\!/} m^{14}_1 \tag{2.26}$$

$$\Delta^1_{21} = \Delta^1_{12}\mathcal{R}_{1',2'}\overline{\mathcal{R}}_{1'',2''} \mathbin{/\!\!/} m_1^{1',1,1''} \mathbin{/\!\!/} m_2^{2',2,2''} \tag{2.27}$$

Definition 2.21 (Drinfeld element)**.** In a quasitriangular meta-Hopf algebra H, the Drinfeld element, $\mathfrak{u} \in H$ is given by:

$$\mathfrak{u} := \mathcal{R}_{21} \mathbin{/\!\!/} S^1_1 \mathbin{/\!\!/} m^{12} \tag{2.28}$$

Definition 2.22 (monodromy)**.** Each quasitriangular meta-Hopf algebra has a monodromy $Q_{12} := \mathcal{R}_{12}\mathcal{R}_{34} \mathbin{/\!\!/} m^{14}_1 \mathbin{/\!\!/} m^{23}_2$. Its inverse will be denoted $\overline{Q}_{12} = \overline{\mathcal{R}}_{12}\overline{\mathcal{R}}_{34} \mathbin{/\!\!/} m^{14}_1 \mathbin{/\!\!/} m^{23}_2$.

Proof. See Majid's work in [Maj] or Etingof and Schiffmann in [ES] for more details on this standard result. Note that the proof does not rely on the additive structure of the Hopf algebra, which allows us to extend this result to the realm of meta-Hopf algebras. □

Definition 2.23 (ribbon meta-Hopf algebra)**.** A quasitriangular meta-Hopf algebra H is called ribbon if it has an element $\nu \in Z(H)$ such that:

$$\nu_1\nu_2 \mathbin{/\!\!/} m^{12} = \mathfrak{u}_1\mathfrak{u}_2 \mathbin{/\!\!/} S^2_2 \mathbin{/\!\!/} m^{12} \tag{2.29}$$

$$\nu_1 \mathbin{/\!\!/} \Delta^1_{12} = \nu_1\nu_2 \mathbin{/\!\!/} \overline{Q}_{34} \mathbin{/\!\!/} m^{13}_1 \mathbin{/\!\!/} m^{24}_2 \tag{2.30}$$

$$\nu \mathbin{/\!\!/} S = \nu \tag{2.31}$$

$$\nu \mathbin{/\!\!/} \epsilon = \eta \mathbin{/\!\!/} \epsilon = 1 \tag{2.32}$$

Definition 2.24 (distinguished grouplike element (spinner))**.** A distinguished grouplike element (or spinner) in a quasitriangular meta-Hopf algebra H is an invertible element $C \in H$ (with inverse $\overline{C}$) such that for all $x \in H$:

$$C_1 \nu_2 C_3 \mathbin{/\!/} S_2^2 \mathbin{/\!/} m^{123} = \nu \tag{2.33}$$

$$C_1 \mathbin{/\!/} \Delta^1_{12} = C_1 C_2 \tag{2.34}$$

$$C \mathbin{/\!/} S = \overline{C} \tag{2.35}$$

$$C_1 x_2 \overline{C}_3 \mathbin{/\!/} m^{123} = x \mathbin{/\!/} S \mathbin{/\!/} S \tag{2.36}$$

$$C \mathbin{/\!/} \epsilon = \eta \mathbin{/\!/} \epsilon = 1 \tag{2.37}$$

Lemma 2.25 (spinners and ribbon Hopf algebras)**.** *If a Hopf algebra has either a ribbon element ν or a spinner C, then it must have the other as well, given by the formula: $C_1 \nu_2 \mathbin{/\!/} m^{12} = \mathfrak{u}$.*

2.4 THE META-ALGEBRA OF TANGLE DIAGRAMS

The particular structures introduced were chosen for their ability to represent the topological properties of knotted objects. We will now introduce the notion of a tangle and demonstrate its meta-algebraic structure.

Upright tangles

For our purposes, a tangle will be visualised as follows: take a stiff circular metal frame and attach a collection of strings to the wire, ensuring that the strings always remain inside the circle, and that each string is tied to the metal frame in two unique locations (that is, no two strings share an endpoint).

Definition 2.26 (open tangle)**.** An open tangle is an embedding of line segments (called components or strands) into the thickened unit disk $D \times [-1, 1]$ (or a disjoint union of such disks) such that the endpoints of the line segments are fixed along $\partial D \times \{0\}$. Each strand is labelled with elements of a set X. Two open tangles are considered equivalent if there exists an isotopy of the embedding which fixes the endpoints of the strands. The term "open" refers to the absence of closed loops. The set of all tangles with strands indexed by X will be denoted $\mathcal{T}_X$.

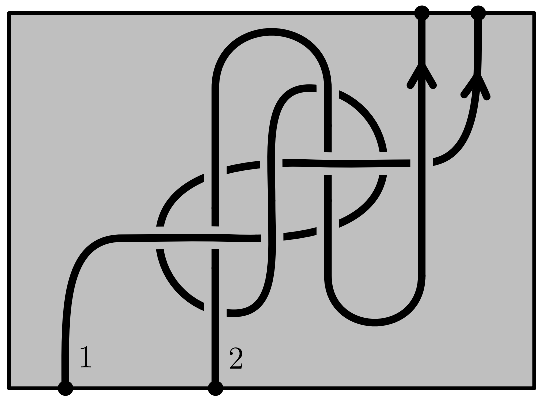

Figure 2.3: Example of a tangle with strands labelled 1 and 2.

Definition 2.27 (framed tangle)**.** The object which is more natural for us to deal with is the framed tangle, which is an open tangle together with a choice of section of the normal bundle for each component, with endpoints of the section fixed pointing to the right of the tangent vector. This choice is taken up to endpoint-fixing homotopy. One may equivalently think of replacing the strings in an open tangle with thin ribbons. Unless otherwise mentioned, it will be assumed that all tangles are framed.

In order to best capture the combinatorial properties of a tangle, observe that a generic projection of a tangle to its central core $D \times \{0\}$ will result in the strands forming a graph, with each crossing of two strands in the tangle producing a vertex in the graph. By assigning to each vertex the sign of the corresponding crossing (either “positive” or “negative”), we end up with a combinatorial object which is equivalent to the original concept of a tangle.

Definition 2.28 (open tangle diagram)**.** An open tangle diagram is a projection of a tangle onto its central core such that all the line segments are immersions which intersect both the boundary disk and the other strands transversally, together with an assignment of a sign to each strand intersection. Small open neighbourhoods of these intersections are called crossings, while the complement of the crossings is a collection of embedded line segments called arcs.

Two open tangle diagrams are considered equivalent if they differ by a finite sequence of Reidemeister moves, as outlined in figures 2.4 to 2.6

The rotation numbers of arcs will play a role in this thesis, so we will capture these data in the following way (as described in [BNvdVa]):

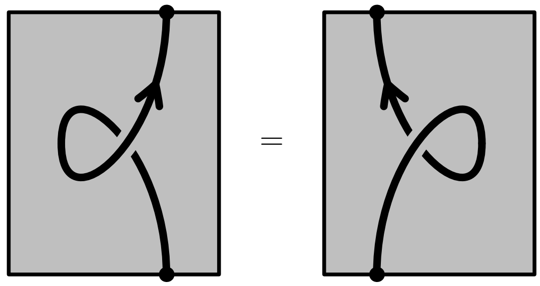

Figure 2.4: Reidemeister move $R1'$

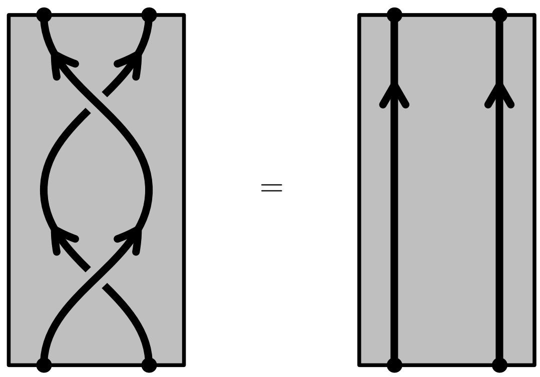

Figure 2.5: Reidemeister move $R2$

Definition 2.29 (upright open tangle diagram)**.** We will put a further requirement on our tangles: the endpoints of each arc must have a vertical tangent vector, and each crossing must involve only curves with tangent vectors that point (diagonally) upwards. With this requirement, each arc has well-defined integer rotation number. Two tangles are considered equivalent if they agree under the "rotational Reidemeister moves". Such tangle diagrams will be called upright tangle diagrams. Given a finite set X, the set of X-indexed upright tangle diagrams will be denoted $\mathcal{T}_X^{\mathrm{up}}$.

Remark 2.30*.* This concept was first introduced by Louis Kauffman in [Kau] under the name rotational virtual knot theory. In this formulation, we insist that all strands end pointing upwards instead of merely requiring that endpoint vectors are vertical, so we will use the term "upright" to remind the reader of this difference.

Fortunately, ambient isotopy allows us to rotate any classical tangle into an upward-pointing form. Additionally, there is only one way to do this. We reproduce the proof of this fact by Bar-Natan and van der Veen in [BNvdVb] below:

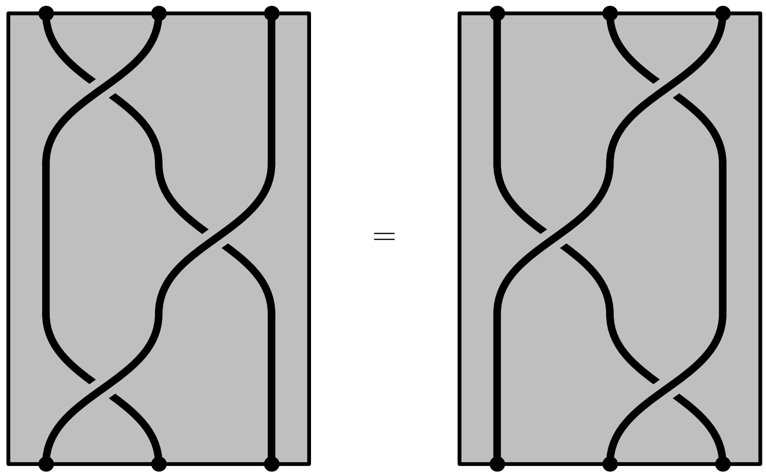

Figure 2.6: Reidemeister move $R3$

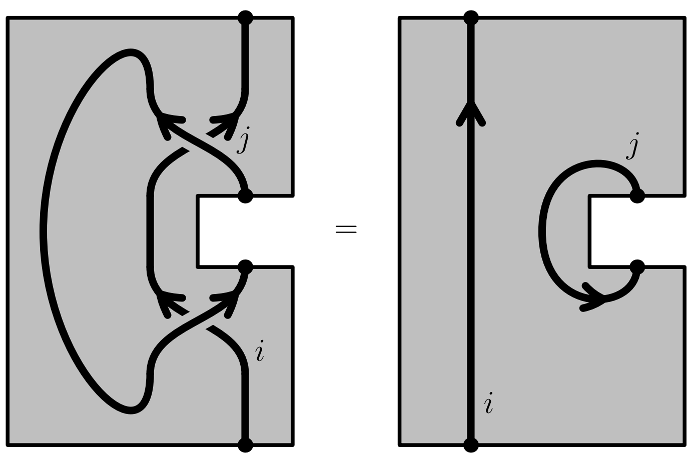

Figure 2.7: The (cyclic) rotational Reidemeister move $R2_{\mathrm{rot}}$

Lemma 2.31 (tangles inject into upright tangles)**.** *To each open tangle diagram D there exists an upright open tangle diagram D' obtained from D by a planar isotopy. Further, if D'' is another such upright open tangle diagram obtained from D, then D' and D'' differ by a finite sequence of rotational Reidemeister moves and a change of rotation number at the endpoints.*

Proof. Each arc and crossing in the diagram D may be rotated so that its endpoints are pointing upwards, giving rise to a diagram D'. Two (nonupright) tangle diagrams are equivalent when they differ by a finite sequence of Reidemeister moves. Each of these Reidemeister may also be rotated to an equivalence of upright tangles, each of which is given as a

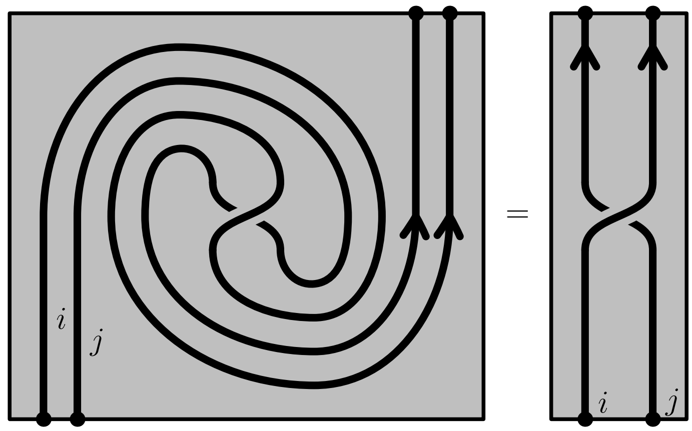

Figure 2.8: The whirling move

rotational Reidemeister move figures 2.4 to 2.7. The last possibility is the rotation of an entire crossing, which is covered by figure 2.8. □

The meta-algebra structure of upright tangle diagrams

Let us now formally connect tangle diagrams with meta-algebras.

Theorem 2.32 (tangles form a ribbon meta-Hopf algebra)**.** *The collection* $\{\mathcal{T}^{up}_X\}_X$ *forms a quasitriangular ribbon meta-Hopf algebra (in the category* **Set***) with the following operations:*

- *multiplication* $m^{ij}_k[X]$ *takes a tangle with strands* $X \sqcup \{i, j\}$ *and glues the end of strand* i *to strand* j*, labelling the resulting strand* k*.*[†5]
- *the unit* $\eta_i[X]$ *takes a tangle diagram with strands* X *and introduces a new strand* i *which does not touch any of the other strands.*
- *the comultiplication* $\Delta^i_{jk}[X]$ *takes a tangle with strands* $X \sqcup \{i\}$ *and doubles strand* i*, separating the two strands along the framing of strand* i*, calling the right strand* j *and the left one* k*.*[†6]
- *the counit* $\epsilon^i[X]$ *takes a tangle with strands indexed by* $X \sqcup \{i\}$ *and returns the tangle with strand labelled by* i *deleted.*

†5 Strictly speaking, this operation is only defined when the end of strand i is adjacent to strand j. See remark 2.33 for more details.

†6 While this convention appears unfortunate, we follow the notation laid out in [BNvdVb] so that the antipode and spinner have a more memorable representation, namely looking like the letters they are represented by (see ?? for more details).

- *The antipode $S^i_j[X]$ takes a tangle with strands labelled by $X \sqcup \{i\}$ and reverses the direction of strand i, then adds a counter-clockwise cap to the new beginning, and a clockwise cup to the end. This new strand is called j. When applied to a single vertical strand, the resulting tangle looks like the letter "S".*

- *the $\mathcal{R}$-matrix $\mathcal{R}_{ij}$ is given by the two-strand tangle with a single positive crossing of strand i over strand j. The inverse $\mathcal{R}$-matrix $\overline{\mathcal{R}}_{ij}$ is the two-strand tangle with a negative crossing of strand i over strand j.*

- *The spinner $C_i[X]$ takes a tangle in $\mathcal{T}^{up}_X$ and adds a new strand with rotation number 1 which has no interactions with any other strands. This new strand looks like the letter "C".*

Remark 2.33. One may object that strand-stitching m^{ij}_k is not defined when the endpoint of strand i is not adjacent to the starting point of strand j. This issue is resolved in multiple ways:

1. Extend the collection of tangles we work with to include virtual tangles. This generalization of tangles deals exactly with the issue that multiplication need not produce a planar tangle diagram. In fact, virtual tangles can be thought of as merely non-planar tangle diagrams.

2. Commit to only apply multiplication when doing so would result in a valid (classical) tangle. This is the approach we will take when performing computations on tangles.

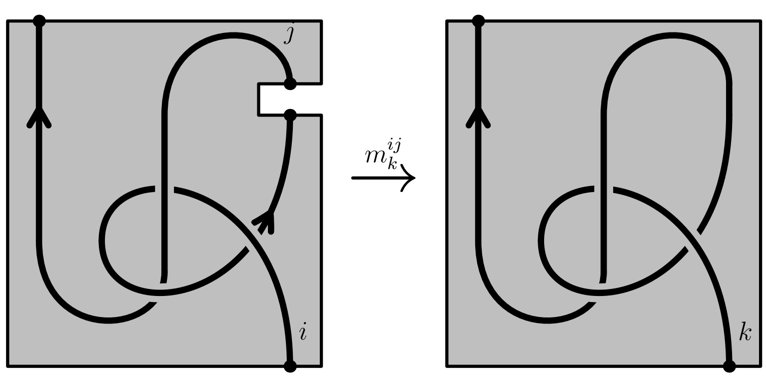

Figure 2.9: Multiplication m^{ij}_k stitches two strands in a tangle together.

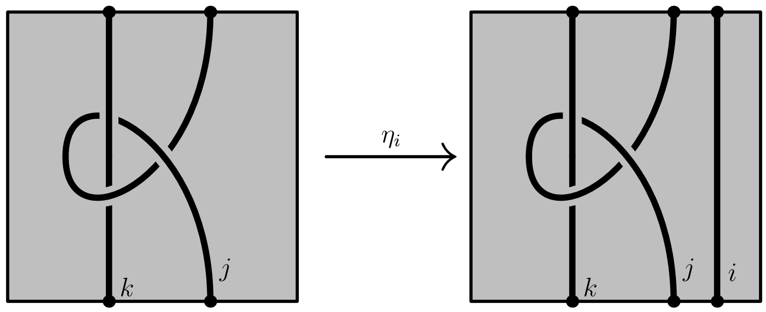

Figure 2.10: The unit η_i introduces a new strand in a tangle.

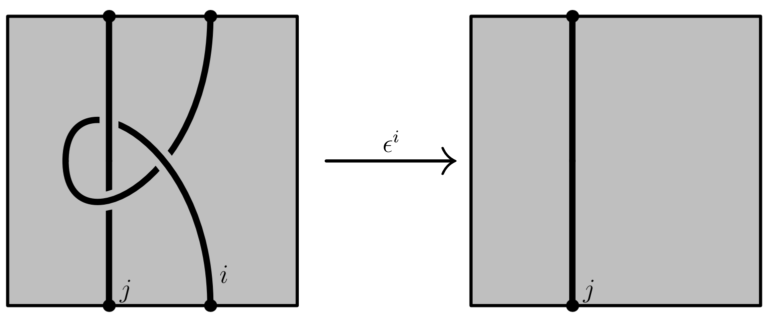

Figure 2.11: The counit ϵ^i deletes a strand in a tangle.

Proof. Associativity of multiplication (equation (2.16)) follows from the fact that stitching strands together amounts to concatenating the order of the crossings each strand interacts with. Since list concatenation is an associative operation, associativity follows in this case as well.

Adding a non-interacting strand to a diagram, then stitching it to an existing strand (equation (2.17)) does not change any of the combinatorial data in the diagram, and results in identical diagrams.

Establishing coassociativity (equation (2.18)) amount to the same argument that cutting a piece of paper into three strips does not depend on the order of cutting.

The counit identity (equation (2.19)) states deleting a strand is the same operation as first doubling it, then deleting both resulting strands.

The meta-bialgebra axioms we verify next:

Equation (2.20) states that if two strands are stitched together, then the resulting strand is doubled, this could have equivalently been achieved by doubling each of the original strands, then performing a stitching on both resulting pairs of strands.

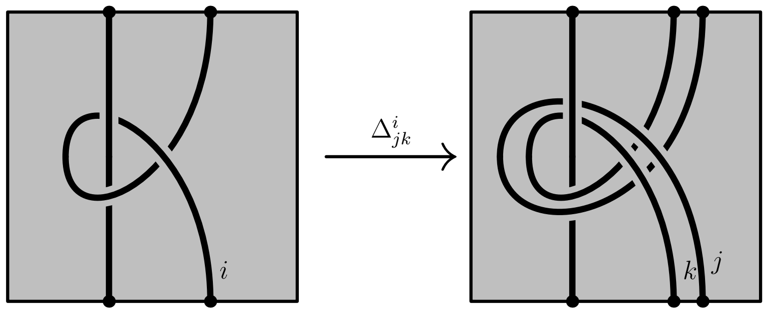

Figure 2.12: The comultiplication Δ^i_{jk} doubles a strand in a tangle along its framing. Notice the right-to-left strand labels.

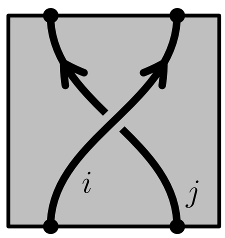

Figure 2.13: The $\mathcal{R}$-matrix $\mathcal{R}_{ij}$ represents a tangles with a single positive crossing.

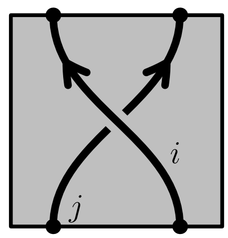

Figure 2.14: The inverse $\mathcal{R}$-matrix $\overline{\mathcal{R}}_{ij}$ represents a tangle with a single negative crossing.

Equation (2.22) simply states that stitching two strands together, then removing the resulting strand could have equally been achieved by removing both of the original strands without stitching them first.

Equation (2.23) states that introducing a strand, then immediately removing it is the identity operation.

Equation (2.21) says that doubling a newly-introduced (and therefore free of crossings) strand is the same operation as introducing two strands separately. (For those worried that this equation depends on the location of the separately introduced strands, this is one place that the use of virtual tangles will be used, which does not heed the relative locations of disjoint strands.)

Equation (2.24) states that when a strand is doubled, then one of the two strands is reversed, multiplying the two strands together results in a strand which can be rearranged to not interact with any of the other strands. This can be readily seen, as this newly-created strand looks like a snake weaving through the tangle diagram. One can remove the snake by applying a series

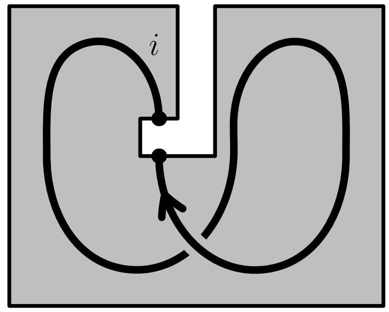

Figure 2.15: The Drinfeld element $\mathfrak{u}_i$ in the meta-Hopf algebra of tangles.

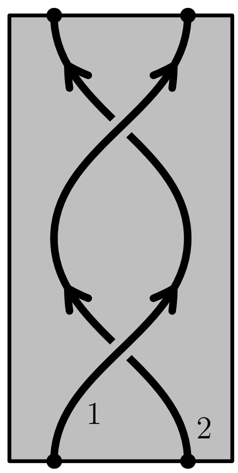

Figure 2.16: The monodromy in the meta-Hopf algebra of tangles.

of Reidemeister 2 moves, resulting in a strand disjoint from the rest of the diagram. This is the same as deleting the original strand, then introducing a new disjoint one.

The quasitriangular axioms are equalities of pairs of three-strand tangles:

- Equations (2.25) and (2.26) tell us that doubling a strand involved in a single crossing can also be built by adjoining two crossings together.
- Equation (2.27) tells us that we can swap the order of a doubled strand by adding crossings to either end (reminiscent of a Reidemeister 2 move)

Finally, we observe that the quotient we introduce to tangle diagrams by the Reidemeister moves does not introduce any new relations. Reidemeister 2 follows from the invertibility of the $\mathcal{R}$-matrix. Next, it is readily seen that the quasitriangular relations governing the $\mathcal{R}$-matrix force it to solve the

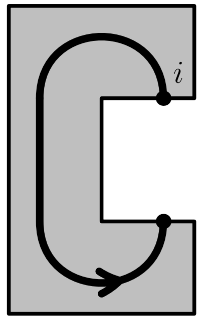

Figure 2.17: The spinner C_i represents a strand with rotation number 1.

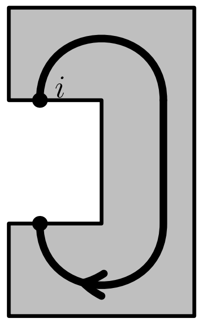

Figure 2.18: The inverse spinner $\overline{C}_i$ represents a strand with rotation number -1.

Yang-Baxter equation, which is one equivalent to the Reidemeister 3 in this case.

Using lemma 2.25, it is enough to verify the spinner axioms (equations (2.33) to (2.37)). All these axioms have corresponding pictures one can draw, keeping in mind the orientations in the definitions of the relevant operations. □

2.5 THE $ybax$ META-ALGEBRA

Here we define the ribbon Hopf algebra U.

Define the Lie algebra

$$\mathfrak{g} := \operatorname{span}\left\{ \mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x} \,\middle|\, [\mathbf{a}, \mathbf{x}] = \mathbf{x}, [\mathbf{a}, \mathbf{y}] = -\mathbf{y}, [\mathbf{x}, \mathbf{y}] = \mathbf{b}, [\mathbf{b}, \,\cdot\,] = 0 \right\} \quad (2.38)$$

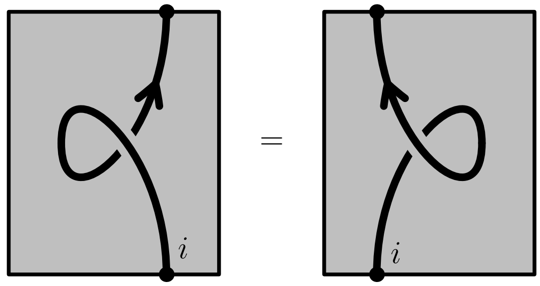

Figure 2.19: A ribbon element ν_i in the meta-Hopf algebra of tangles. One can use lemma 2.25 to verify this is compatible with the spinner.

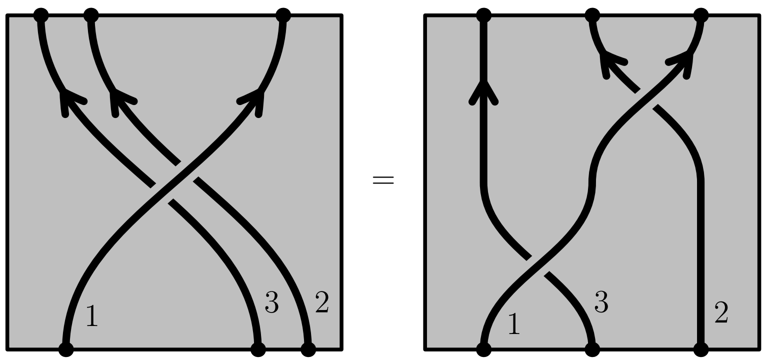

Figure 2.20: Example of a tangle satisfying equation (2.26)

Then the algebra U is defined to be the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. The bialgebra structure of U is:

$$\begin{aligned}
\Delta_{i,j}(\mathbf{y}) &= \frac{\mathbf{b}_i + \mathbf{b}_j}{1 - \mathbf{B}_i \mathbf{B}_j} \left(\mathbf{B}_j \frac{1 - \mathbf{B}_i}{\mathbf{b}_i} \mathbf{y}_i + \frac{1 - \mathbf{B}_j}{\mathbf{b}_j} \mathbf{y}_j \right) \\
\Delta_{i,j}(\mathbf{b}) &= \mathbf{b}_i + \mathbf{b}_j \\
\Delta_{i,j}(\mathbf{a}) &= \mathbf{a}_i + \mathbf{a}_j \\
\Delta_{i,j}(\mathbf{x}) &= \mathbf{x}_i + \mathbf{x}_j
\end{aligned} \tag{2.39}$$

For any $\mathbf{z} \in \{\mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x}\}$, we have $\epsilon(\mathbf{z}) = 0$ (extended multiplicatively by equation (2.22)).

Next, we define the Hopf algebra structure by defining the antipode, which is defined as $S(\mathbf{z}) \coloneqq -\mathbf{z}$ for each $\mathbf{z} \in \{\mathbf{y}, \mathbf{b}, \mathbf{a}, \mathbf{x}\}$, extended antimultiplicatively.

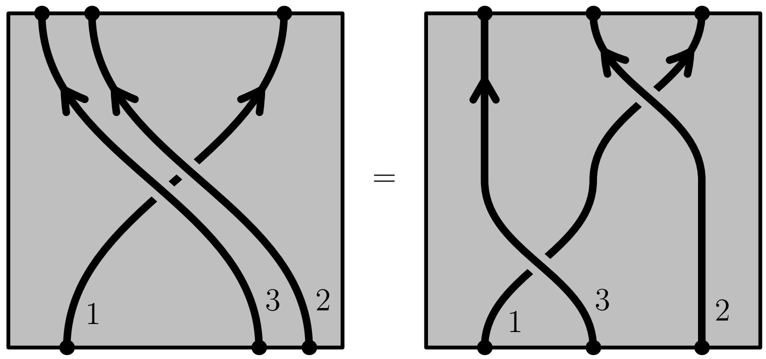

Figure 2.21: Example of a tangle satisfying equation (2.25)

Next, we introduce the ribbon structure of U with an $\mathcal{R}$-matrix and the spinner C:

$$\mathcal{R}_{i,j} := \exp\big(\mathbf{b}_i \mathbf{a}_j\big) \exp\left(\frac{1-\mathbf{B}_i}{\mathbf{b}_i} \mathbf{y}_i \mathbf{x}_j\right) \tag{2.40}$$

$$C := \sqrt{\mathbf{B}} \tag{2.41}$$

$$\nu := \overline{\mathcal{R}}_{31} \overline{C}_2 \,/\!\!/\, m^{123} \tag{2.42}$$

We point out the following commutation relations in U, beginning with the Weyl canonical commutation relation:

$$\mathbf{e}^{\xi \mathbf{x}} \mathbf{e}^{\eta \mathbf{y}} = \mathbf{e}^{-\xi \eta \mathbf{b}} \mathbf{e}^{\eta \mathbf{y}} \mathbf{e}^{\xi \mathbf{x}} \tag{2.43}$$

Secondly, using equation (4.6) and $\mathcal{A} := \mathbf{e}^{\alpha}$, we notice

$$\mathbf{e}^{\alpha \mathbf{a}} \mathbf{e}^{\eta \mathbf{y}} = \mathbf{e}^{\alpha \mathbf{a}} \sum_n \frac{(\eta \mathbf{y})^n}{n!} = \sum_n \frac{(\eta \mathbf{y})^n}{n!} \mathbf{e}^{\alpha(\mathbf{a}-n)} = \sum_n \frac{(\eta \mathbf{y})^n}{n!} \mathbf{e}^{\alpha \mathbf{a}} \mathcal{A}^{-n} = \mathbf{e}^{\frac{\eta}{\mathcal{A}} \mathbf{y}} \mathbf{e}^{\alpha \mathbf{a}} \tag{2.44}$$

similarly,

$$\mathbf{e}^{\xi \mathbf{x}} \mathbf{e}^{\alpha \mathbf{a}} = \sum_n \frac{(\xi \mathbf{x})^n}{n!} \mathbf{e}^{\alpha \mathbf{a}} = \sum_n \mathbf{e}^{\alpha(\mathbf{a}-n)} \frac{(\xi \mathbf{x})^n}{n!} = \sum_n \mathbf{e}^{\alpha \mathbf{a}} \frac{(\frac{\xi \mathbf{x}}{\mathcal{A}})^n}{n!} = \mathbf{e}^{\alpha \mathbf{a}} \mathbf{e}^{\frac{\xi}{\mathcal{A}} \mathbf{x}} \tag{2.45}$$

Lemma 2.34 (the algebra U is ribbon)**.** *The algebra U has a ribbon structure given by the above $\mathcal{R}$-matrix and spinner C.*

Proof. The Hopf algebra structure of U is straightforward, and is left to the reader to verify. We will focus our attention on verifying quasitriangularity and the ribbon structure.

Let us verify equation (2.26) first. The left-hand side is:

$$\mathcal{R}_{12} /\!\!/ \Delta^2_{23} = \exp(\mathbf{b}_1(\mathbf{a}_2 + \mathbf{a}_3)) \exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{y}_1(\mathbf{x}_2 + \mathbf{x}_3)\right) \tag{2.46}$$

Equality with the right-hand side follows by commutativity of $\mathbf{b}_1$ and $\mathbf{y}_1$:

$$\begin{aligned}\mathcal{R}_{13}\mathcal{R}_{42} /\!\!/ m^{14}_1 &= \exp(\mathbf{b}_1\mathbf{a}_3) \exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{y}_1\mathbf{x}_3\right) \exp(\mathbf{b}_1\mathbf{a}_2) \exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{y}_1\mathbf{x}_2\right)\\ &= \exp(\mathbf{b}_1(\mathbf{a}_2 + \mathbf{a}_3)) \exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{y}_1(\mathbf{x}_2 + \mathbf{x}_3)\right)\end{aligned} \tag{2.47}$$

Next we verify equation (2.25), whose left-hand side is:

$$\mathcal{R}_{13} /\!\!/ \Delta^1_{12} = \exp((\mathbf{b}_1 + \mathbf{b}_2)\mathbf{a}_3) \exp\left(\left(\mathbf{B}_2\frac{1-\mathbf{B}_1}{\mathbf{b}_1}y_1 + \frac{1-\mathbf{B}_2}{\mathbf{b}_2}y_2\right)\mathbf{x}_3\right) \tag{2.48}$$

On the right-hand side, we have

$$\begin{aligned}\mathcal{R}_{13}\mathcal{R}_{24} /\!\!/ m^{34}_3 &= \exp(\mathbf{b}_1\mathbf{a}_3) \exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{y}_1\mathbf{x}_3\right) \exp(\mathbf{b}_2\mathbf{a}_3) \exp\left(\frac{1-\mathbf{B}_2}{\mathbf{b}_2}\mathbf{y}_2\mathbf{x}_3\right)\\ &= \exp((\mathbf{b}_1 + \mathbf{b}_2)\mathbf{a}_3) \exp\left(\frac{1-\mathbf{B}_1}{\mathbf{b}_1}\mathbf{B}_2\mathbf{y}_1\mathbf{x}_3\right) \exp\left(\frac{1-\mathbf{B}_2}{\mathbf{b}_2}\mathbf{y}_2\mathbf{x}_3\right)\end{aligned} \tag{2.49}$$

We use equation (2.45) to write the expression in a canonical order. Finally, the right two exponentials may be combined since each variable commutes with the others, either by belonging to separate tensor factors, or in the case of $\mathbf{b}$, being central. The verifications of equation (2.27) and equations (2.33) to (2.37) follow with similar computations. □

2.6 MORPHISMS BETWEEN META-OBJECTS

When equipped with meta-structures on both tangles and an algebraic object, we can define a tangle invariant by considering a morphism between the meta-objects.

Definition 2.35 (morphism of meta-objects)**.** Let $\{A_X\}_X$ and $\{B_X\}_X$ be compatible meta-objects (i.e. ones with the same operations and relations between the operations). A <u>morphism</u> ϕ between these meta-objects is map $\phi\colon \{A_X\}_X \to \{B_X\}_X$ sending $A_X \mapsto B_X$ such that for each map f^X_Y in A, $\phi(f^X_Y) = f^X_Y$ in B.

Upright tangle invariants from a ribbon meta-Hopf algebra

We define a U-valued tangle invariant in the following way:

1. Given a open tangle, disconnect each crossing from its neighbours, as well as each arc with a nonzero rotation number.
2. Replace each crossing with an $\mathcal{R}$-matrix $\mathcal{R}_{ij} \in U_{\{i,j\}}$, and each rotation of an arc with a spinner $C_i \in U_{\{i\}}$.
3. For each disconnection, there is a corresponding stitching operation required to bring the tangle back to its original state. Replace each stitching operation with a multiplication operation in U.

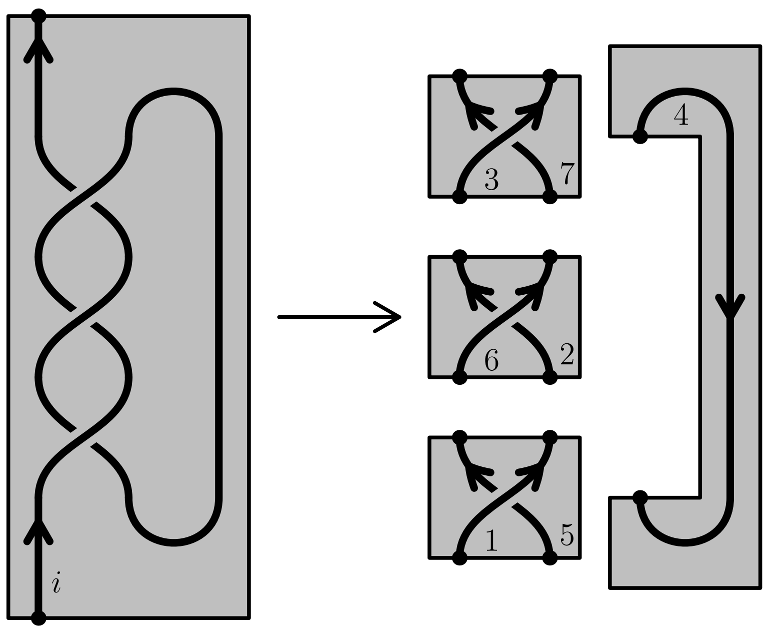

Figure 2.22: Breaking up a tangle into its constituent components. The left-hand tangle is obtainable by the right-hand one by applying the map $m_i^{1,2,\ldots,7}$.

3

PERTURBED GAUSSIANS

We now summarize the work of Bar-Natan and van der Veen in [BNvdVb], which develops a universal knot invariant using perturbed Gaußians.

3.1 EXPRESSING MORPHISMS AS GENERATING FUNCTIONS

The invariant we have defined takes the form of building a knot out of several crossings and spinners, then replacing each crossing with an $\mathcal{R}$-matrix $\mathcal{R}_{ij}$ and spinners C_k. Finally, we apply a series of multiplication operations to join the various tensor factors into the final value. In order to be able to apply the operations efficiently, we need a compact way of encoding morphisms. In [BNvdVb], Bar-Natan and van der Veen achieve this through the use of generating functions, whose definition we reproduce below:

For A and B finite sets, consider the set $\hom(\mathbb{Q}[z_A], \mathbb{Q}[z_B])$ of linear maps between multivariate polynomial rings. Such a map is determined by its values on the monomials $z_A^{\mathbf{n}}$ for each multi-index $\mathbf{n} \in \mathbb{N}^A$.

Definition 3.1 (Exponential generating function)**.** The exponential generating function of a linear map $\Phi\colon \mathbb{Q}[z_A] \to \mathbb{Q}[z_B]$ between polynomial spaces is

$$\mathcal{G}(\Phi) := \sum_{\mathbf{n}\in\mathbb{N}^A} \frac{\Phi(z_A^{\mathbf{n}})}{\mathbf{n}!}\zeta_A^{\mathbf{n}} \in \mathbb{Q}[z_B][\![\zeta_A]\!] \tag{3.1}$$

Remark 3.2*.* We may extend the domain of Φ from $\mathbb{Q}[z_B]$ to $\mathbb{Q}[z_B][\![\zeta_A]\!]$ via extension of scalars $\mathbb{Q} \to \mathbb{Q}[\![\zeta_A]\!]$. This extension allows us to write the exponential generating function in a new way:

$$\mathcal{G}(\Phi) = \Phi\left(\sum_{\mathbf{n}\in\mathbb{N}^A} \frac{(z_A\zeta_A)^{\mathbf{n}}}{\mathbf{n}!}\right) = \Phi\Big(\mathcal{G}(\mathrm{id}_{\mathbb{Q}[z_A]})\Big) \tag{3.2}$$

By the PBW theorem, we know that U is isomorphic as a vector space to the polynomial ring $\mathbb{Q}[y,b,a,x]$ by choosing an ordering of the generators (following [BNvdVb], we use $(\mathbf{y},\mathbf{b},\mathbf{a},\mathbf{x})$):

$$\begin{aligned} \mathbb{O}\colon \mathbb{Q}[y,b,a,x] &\xrightarrow{\sim} U \\ y^{n_1}b^{n_2}a^{n_3}x^{n_4} &\mapsto \mathbf{y}^{n_1}\mathbf{b}^{n_2}\mathbf{a}^{n_3}\mathbf{x}^{n_4} \end{aligned} \tag{3.3}$$

To efficiently describe $\Bbbk$-linear maps between tensor powers of the algebra U, we define categories $\mathcal{U}$, $\mathcal{H}$, and $\mathcal{C}$ with objects finite sets and morphisms:

$$\mathrm{Hom}_{\mathcal{U}}(J,K) \coloneqq \mathrm{Hom}_{\Bbbk}(U^{\otimes J}, U^{\otimes K}) \tag{3.4}$$

$$\mathrm{Hom}_{\mathcal{H}}(J,K) \coloneqq \mathrm{Hom}_{\Bbbk}(\mathbb{Q}[z_J], \mathbb{Q}[z_K]) \tag{3.5}$$

$$\mathrm{Hom}_{\mathcal{C}}(J,K) \coloneqq \mathbb{Q}[z_K][\![\zeta_J]\!] \tag{3.6}$$

Where equations (3.4) and (3.5) explicitly denote vector space maps, not just algebra or ring homomorphisms. There exist monoidal isomorphisms between these categories, namely $\mathbb{O}\colon \mathcal{H} \xrightarrow{\sim} \mathcal{U}$ and $\mathcal{G}\colon \mathcal{H} \xrightarrow{\sim} \mathcal{C}$ as introduced in equations (3.1) and (3.3).

We use this formulation because of the existence of a computationally amenable subcategory of $\mathcal{C}$ which contains the image of this invariant.

Formulating composition in other categories

Composing operations in $\mathcal{U}$ or $\mathcal{H}$ is straightforward to define, but lacks a closed form. However, on $\mathcal{C}$, the corresponding definition of composition takes the following form (quoted from [BNvdVb, Lemma 3]):

Lemma 3.3 (Composition of generating functions)**.** *Suppose J, K, L are finite sets and $\phi \in \mathrm{Hom}(\mathbb{Q}[z_J], \mathbb{Q}[z_K])$ and $\psi \in \mathrm{Hom}(\mathbb{Q}[z_K], \mathbb{Q}[z_L])$. We have*

$$\mathcal{G}(\phi \mathbin{/\!\!/} \psi) = \left(\mathcal{G}(\phi)\big|_{z_K \mapsto \partial_{\zeta_K}} \mathcal{G}(\psi) \right)\Big|_{\zeta_K = 0} \tag{3.7}$$

Since the above notation will occur several times, we will use the notion of <u>contraction</u> used by Bar-Natan and van der Veen (taken from [BNvdVb, Definition 4]):

Definition 3.4 (Contraction)**.** Let $f \in \Bbbk[\![r,s]\!]$ be a powerseries. The contraction of $f = \sum_{k,l} c_{k,l} r^k s^l$ along the pair (r,s) is:

$$\langle f \rangle_{(r,s)} := \sum_k c_{k,k} k! = \sum_{k,l} c_{k,l} \partial_s^k s^l \Big|_{s=0} \tag{3.8}$$

Further, this notation is to be extended to allow for multiple consecutive contractions for $f \in \Bbbk[\![r_i, s_i]\!]_{i \le n}$:

$$\langle f \rangle_{((r_i)_i,(s_i)_i)} := \left\langle \left\langle \langle f \rangle_{(r_1,s_1)} \right\rangle_{(r_2,s_2)} \cdots \right\rangle_{(r_n,s_n)} \tag{3.9}$$

It is important to note that contraction does not always define a convergent expression. We will focus our attention on cases when convergence is well-defined, and especially those where the computation is accessible.

The theorem we will rely heavily on in this thesis is the following, taken from [BNvdVb, Theorem 6]:

Theorem 3.5 (Contraction theorem)**.** *For any $n \in \mathbb{N}$, consider the ring $R_n = \mathbb{Q}[r_j, g_j][\![s_j, W_{ij}, f_j \mid 1 \le i,j \le n]\!]$. Then*

$$\langle \mathbf{e}^{gs+rf+rWs} \rangle_{r,s} = \det(\tilde{W}) \mathbf{e}^{g\tilde{W}f} \tag{3.10}$$

where $\tilde{W} = (1-W)^{-1}$.

The main takeaway of this theorem is this: morphisms whose generating functions are Gaußians have a clean formula for composition. Furthermore, this formula is computationally reasonable, growing only polynomially in complexity with n. This is contrasted with the conventional approach of choosing a representation V of U. When one considers morphisms between large tensor powers $V^{\otimes n}$, the computational complexity is exponential in n.

Expressing Hopf algebra operations as perturbed Gaußians

Using this vector space isomorphism, [BNvdVb] expresses all Hopf algebra operations as power series in a closed form, namely as perturbed Gaußians.

Theorem 3.6 (The meta-Hopf structure of U is Gaußian)**.** *Each of the meta-Hopf algebra operations for U as defined in section 2.5 all have the form of a perturbed Gaußian. That is, when the generators (y,b,a,x) are assigned* *weights of* $(1,0,2,1)$ *respectively, and their dual variables are assigned complementary weights so that* $\operatorname{wt} z + \operatorname{wt} z^* = 2$*, we have the following expressions which are Gaußian or central:*

$$\begin{aligned}
\mathcal{G}\left(m_k^{ij}\right) = \; &\exp\Bigg((\alpha_i+\alpha_j)a_k + (\beta_i+\beta_j+\xi_i\eta_j)b_k \\
&\quad + \left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right)x_k + \left(\frac{\eta_j}{\mathcal{A}_i}+\eta_i\right)y_k\Bigg) && (3.11)\\
\mathcal{G}(\eta_i) &= 1 && (3.12)\\
\mathcal{G}\left(\Delta^i_{jk}\right) = \; &\exp\Bigg(\eta_i\frac{b_j+b_k}{1-B_jB_k}\left(B_k\frac{1-B_j}{b_j}y_j + \frac{1-B_k}{b_k}y_k\right) && (3.13)\\
&\quad + \beta_i(b_j+b_k) + \alpha_i(a_j+a_k) + \xi_i(x_j+x_k)\Bigg)\\
\mathcal{G}(\epsilon^i) &= 1 && (3.14)\\
\mathcal{G}(S^i_i) &= \exp(-a_i\alpha_i - b_i\beta_i - \eta_i\mathcal{A}_iy_i - \mathcal{A}_i\xi_ix_i + \eta_i\mathcal{A}_i\xi_ib_i) && (3.15)\\
\mathcal{G}(\mathcal{R}_{ij}) &= \exp\left(a_jb_i + \frac{1-B_i}{b_i}y_ix_j\right) && (3.16)\\
\mathcal{G}(C_i) &= \sqrt{B_i} && (3.17)\\
\mathcal{G}(\nu_i) &= \sqrt{B_i}\exp\left(a_ib_i + \frac{1-B_i}{b_i}x_iy_i\right) && (3.18)
\end{aligned}$$

Proof. To prove equation (3.11), we use equations (2.44) and (2.45), which allows us to commute exponentials past each other to bring expressions into $ybax$-order. Below we omit the index k for readability:

$$\begin{aligned}
\mathcal{G}(m^{ij}) &= \left(\mathbb{O}^{-1}\circ m^{ij}\circ\mathbb{O}\right)\left(\mathbf{e}^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i}\mathbf{e}^{\eta_jy_j+\beta_jb_j+\alpha_ja_j+\xi_jx_j}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{\eta_i\mathbf{y}}\mathbf{e}^{\beta_i\mathbf{b}}\mathbf{e}^{\alpha_i\mathbf{a}}\mathbf{e}^{\xi_i\mathbf{x}}\mathbf{e}^{\eta_j\mathbf{y}}\mathbf{e}^{\beta_j\mathbf{b}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\xi_j\mathbf{x}}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{\eta_i\mathbf{y}}\mathbf{e}^{\beta_i\mathbf{b}}\mathbf{e}^{\alpha_i\mathbf{a}}\left(\mathbf{e}^{-\xi_i\eta_j\mathbf{b}}\mathbf{e}^{\eta_j\mathbf{y}}\mathbf{e}^{\xi_i\mathbf{x}}\right)\mathbf{e}^{\beta_j\mathbf{b}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\xi_j\mathbf{x}}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)\mathbf{b}}\mathbf{e}^{\eta_i\mathbf{y}}\mathbf{e}^{\frac{\eta_j}{\mathcal{A}_i}\mathbf{y}}\mathbf{e}^{\alpha_i\mathbf{a}}\mathbf{e}^{\xi_i\mathbf{x}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\xi_j\mathbf{x}}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)\mathbf{b}}\mathbf{e}^{(\eta_i+\frac{\eta_j}{\mathcal{A}_i})\mathbf{y}}\mathbf{e}^{\alpha_i\mathbf{a}}\mathbf{e}^{\alpha_j\mathbf{a}}\mathbf{e}^{\frac{\xi_i}{\mathcal{A}_j}\mathbf{x}}\mathbf{e}^{\xi_j\mathbf{x}}\right) \qquad (3.19)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{\left(\eta_i+\frac{\eta_j}{\mathcal{A}_i}\right)\mathbf{y}}\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)\mathbf{b}}\mathbf{e}^{(\alpha_i+\alpha_j)\mathbf{a}}\mathbf{e}^{\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right)\mathbf{x}}\right)\\
&= \mathbf{e}^{\left(\eta_i+\frac{\eta_j}{\mathcal{A}_i}\right)y}\mathbf{e}^{(\beta_j+\beta_i-\xi_i\eta_j)b}\mathbf{e}^{(\alpha_i+\alpha_j)a}\mathbf{e}^{\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right)x}
\end{aligned}$$

Since this expression is now written in the **ybax**-order, we conclude that the corresponding generating function is this same expression, but written with commuting variables.

The other computation we must verify is the antipode, which follows similarly:

$$\begin{aligned}
\mathcal{G}(S) &= \left(\mathbb{O}^{-1}\circ S\circ \mathbb{O}\right)\left(\mathrm{e}^{\eta y+\beta b+\alpha a+\xi x}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\alpha\mathbf{a}}\mathbf{e}^{-\beta\mathbf{b}}\mathbf{e}^{-\eta\mathbf{y}}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{-\alpha\mathbf{a}}\mathbf{e}^{-\beta\mathbf{b}}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{-\xi\mathcal{A}\eta\mathbf{b}}\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\alpha\mathbf{a}}\mathbf{e}^{-\beta\mathbf{b}}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{-\xi\mathcal{A}\eta\mathbf{b}}\mathbf{e}^{-\beta\mathbf{b}}\mathbf{e}^{-\xi\mathbf{x}}\mathbf{e}^{-\alpha\mathbf{a}}\right)\\
&= \mathbb{O}^{-1}\left(\mathbf{e}^{-\mathcal{A}\eta\mathbf{y}}\mathbf{e}^{-(\xi\mathcal{A}\eta+\beta)\mathbf{b}}\mathbf{e}^{-\alpha\mathbf{a}}\mathbf{e}^{-\xi\mathcal{A}\mathbf{x}}\right)
\end{aligned} \tag{3.20}$$

Finally, equations (3.12), (3.14) and (3.16) to (3.18) follow immediately. □

Notational conventions

The generating function of a tangle is not the entirety of this definition, for the additional data is the domain and codomain of the corresponding morphism.

We will thereby write a morphism with domain D, codomain C, and generating function $f(\zeta_D, z_C)$ as $f(\zeta_D, z_C)^D_C$.

When applied to knots, this invariant computes the Alexander polynomial Δ:

$$Z(K_{3,1}) = \left(\frac{1}{B_1^{-1}+1+B_1^1}\right)^{\emptyset}_{\{1\}} = \Delta(K_{3,1})^{-1} \tag{3.21}$$

Here is a second example, where we define $D_i^n := B_i^n + B_i^{-n}$ to emphasize the palindromic nature of the Alexander polynomial:

$$Z(K_{11a10}) = \left(\frac{1}{2D_1^3-11D_1^2+25D_1^1-31}\right)^{\emptyset}_{\{1\}} = \Delta(K_{11a10})^{-1} \tag{3.22}$$

Since each tangle is expressed as an object, the domains in these examples are empty.

4

CONSTRUCTING THE TRACE

4.1 EXTENDING AN OPEN TANGLE INVARIANT TO LINKS AND GENERAL TANGLES

Thus far, the algebraic setting we have defined allows us to describe invariants of tangles with no closed components. We now extend the notion of a meta-Hopf algebra to include closed components.

Definition 4.1 (traced meta-algebra)**.** A traced meta-algebra is a family of meta-algebras: for each finite set L, we assign one meta-algebra $\{A_{L,S}\}_S$.[†1] The multiplication maps $m_k^{i,j}[L]$ then take the form:

$$m_k^{i,j}[L][S]\colon A_{\{i,j\}\sqcup S,L} \to A_{\{k\}\sqcup S,L} \tag{4.1}$$

for i, j, k disjoint from both S and L.

There is an additional structure, called a trace. The compatibility of the families of meta-algebras is governed this trace in the following way: $\mathrm{tr}^i\colon A_{\{i\}\sqcup S,L} \to A_{S,\{i\}\sqcup L}$ which is universal with respect to the cyclic property:

$$m_k^{i,j} \mathbin{/\!\!/} \mathrm{tr}^k = m_k^{j,i} \mathbin{/\!\!/} \mathrm{tr}^k \tag{4.2}$$

Furthermore, tr^i is a morphism of meta-coalgebras. That is:

$$\Delta^i_{jk} \mathbin{/\!\!/} \mathrm{tr}^j \mathbin{/\!\!/} \mathrm{tr}^k = \mathrm{tr}^i \mathbin{/\!\!/} \Delta^i_{jk} \tag{4.3}$$

$$\mathrm{tr}^i \mathbin{/\!\!/} \epsilon^i = \epsilon^i \tag{4.4}$$

The first example we give is that of mixed tangles.

Definition 4.2 (mixed upright tangles)**.** Let $\overline{\mathcal{T}}^{\mathrm{up}}_{L,S}$ be the set of upright tangles with open strands indexed by S and closed strands indexed by L.

†1 These sets index the “strands” S and the “loops” L.

The operations $\phi[L][S]$ are defined analogously to the $\phi[S]$ given in **??**. (Here ϕ varies over m, η, Δ, ϵ, S, $\mathcal{R}$, and C.)

Lemma 4.3 (tangles as a traced algebra). *The collection of all $\overline{\mathcal{T}}^{up}_{L,S}$ is a traced ribbon meta-Hopf algebra, with trace map given by closing a strand into a loop.*

Proof. When $L = \emptyset$, the situation is exactly the case of theorem 2.32, so $\overline{\mathcal{T}}^{\mathrm{up}}_{\emptyset,S} = \mathcal{T}^{\mathrm{up}}_{S}$ is a meta-Hopf algebra. Furthermore, since the Reidemeister moves are local operations, the presence of closed components does not affect our ability to verify the identities on the Hopf-algebra operations.

The last point to verify is that closing a strand into a loop is a cyclic operation. Given two strands, we must verify that stitching one end together, then tracing the other yields the same diagram as stitching the other ends together, then taking the trace. However, by definition of trace, these two actions yield identical diagrams, the two strands replaced by the same closed loop. □

Lemma 4.4 (coinvariants as a trace map). *Let A be an algebra, and denote by $A_A := A/[A, A]$ its set of coinvariants. Then define $A_{S,L} := A^{\otimes S} \otimes A_A^{\otimes L}$. Then A defines a traced meta-algebra with trace map given by $\mathrm{tr}^i_j \colon A_i \to (A_A)_j$.*

Proof. Observe that for any choice of L, extending morphisms by the identity yield an isomorphism of traced meta-Hopf algebras:

$$\begin{aligned} \phi_L \colon \{A^{\otimes S}\}_S &\xrightarrow{\sim} \{A^{\otimes S} \otimes A_A^{\otimes L}\}_S \\ A^{\otimes S} &\mapsto A^{\otimes S} \otimes A_A^{\otimes L} \\ f &\mapsto f \otimes \mathrm{id}_{A_A}^{\otimes L} \end{aligned} \tag{4.5}$$

Next, we must show that $m^{ij}_k \mathbin{/\!\!/} \mathrm{tr}^k = m^{ji}_k \mathbin{/\!\!/} \mathrm{tr}^k$. This amounts to showing that, given $u, v \in A$, that $\overline{uv} = \overline{vu} \in A_A$. However, by the construction of the coinvariants, $\overline{uv} - \overline{vu} = \overline{uv - vu} = \overline{0} \in A$, and we are done. □

4.2 THE COINVARIANTS OF U

We start with a result which simplifies working with coinvariants:

Lemma 4.5 (Coinvariant simplification). *Let $\mathfrak{h}$ be a Lie algebra. Then* $\mathfrak{U}(\mathfrak{h})_{\mathfrak{U}(\mathfrak{h})} = \mathfrak{U}(\mathfrak{h})_{\mathfrak{h}}$.

Proof. First, observe that for any u, v, $f \in \mathfrak{U}(\mathfrak{h})$, $\mathrm{ad}_{uv}(f) = \mathrm{ad}_u(vf) + \mathrm{ad}_v(fu)$. Proceeding inductively, for any monomial $\mu \in \mathfrak{U}(\mathfrak{h})$, $\mathrm{ad}_\mu(u)$ is a linear combination of elements of $[\mathfrak{h}, \mathfrak{U}(\mathfrak{h})]$. By linearity of ad, we conclude $[\mathfrak{U}(\mathfrak{h}), \mathfrak{U}(\mathfrak{h})] = [\mathfrak{h}, \mathfrak{U}(\mathfrak{h})]$. □

Theorem 4.6. *The coinvariants of U, U_U, has basis $\{y^n a^k x^n\}_{n,k\geq 0}$.*

Proof. Using lemma 4.5, we need only compute $[\mathfrak{g}, U]$ to determine U_U. Given a polynomial f, we have the following relations in U:

$$f(a)y^r = y^r f(a-r) \qquad\qquad x^r f(a) = f(a-r)x^r \tag{4.6}$$

Next we compute the adjoint actions of y, a, and x. (Recall b is central.)

$$\mathrm{ad}_a\, f(x) = xf'(x) \qquad\qquad \mathrm{ad}_a\, f(y) = -yf'(y) \tag{4.7}$$
$$\mathrm{ad}_x\, f(y) = bf'(y) \qquad\qquad \mathrm{ad}_x\, f(a) = -\nabla[f](a)x \tag{4.8}$$
$$\mathrm{ad}_y\, f(x) = -bf'(x) \qquad\qquad \mathrm{ad}_y\, f(a) = y\nabla[f](a) \tag{4.9}$$

(Here ∇ is the backwards finite difference operator $\nabla[f](x) := f(x) - f(x-1)$.) Observe for any n, m, k, and polynomials f and g:

$$\mathrm{ad}_a(y^m g(b,a)x^n) = (n-m)y^m g(b,a)x^n \tag{4.10}$$
$$\mathrm{ad}_x(y^{n+1}b^{m-1}f(a)x^k) = (n+1)y^n b^m f(a)x^k - y^{n+1}b^{m-1}\nabla[f](a)x^{k+1} \tag{4.11}$$
$$\mathrm{ad}_y(y^n b^{m-1}f(a)x^{k+1}) = -(k+1)y^n b^m f(a)x^k + y^{n+1}b^{m-1}\nabla[f](a)x^{k+1} \tag{4.12}$$

By equation (4.10), any monomial whose powers of y and x differ vanish in $U_{\mathfrak{g}}$. As a consequence, in equations (4.11) and (4.12), the only nontrivial case is when $n = k$, resulting in the same relation. By induction on n, we conclude that:

$$y^n b^m f(a)x^k \sim \delta_{nk}\frac{n!}{(n+m)!}y^{n+m}\nabla^m[f](a)x^{n+m} \tag{4.13}$$

where $\sim$ refers to equivalence in the set of coinvariants. Observing when f is a monomial in equation (4.13), we see $U_{\mathfrak{g}}$ is spanned by $\{y^n a^k x^n\}_{n,k\geq 0}$.

Finally, all that remains to show is this set is linearly independent. This is equivalent to no two $y^n a^k x^n$'s with distinct choices of exponents being related under a sequence of relations. Since lemma 4.5 allows us to only consider sequences of relations of the form ad_z for z a one-letter word in $\{y, b, a, x\}$, inspection of the above comprehensive summary of all one-letter relations (in particular, equation (4.13) with $m = 0$) allows us to conclude that this set is indeed linearly independent. □

A generating function for the coinvariants

In order to define a generating function, we need to choose an appropriate basis for the space of coinvariants. We define an isomorphism from the space of coinvariants to a polynomial space, tweaking the earlier-defined basis by scalar multiples. Since it plays the role of the ordering map, we also name it $\mathbb{O}$.

$$\begin{aligned} \mathbb{O} \colon \mathbb{Q}[a, z] &\xrightarrow{\sim} U_U \\ a^n z^k &\mapsto \frac{1}{k!} y^k a^n x^k \\ k! \nabla^m [f](a) z^{k+m} &\mapsfrom y^k b^m f(a) x^k \end{aligned} \tag{4.14}$$

This defines a commutative square upon whose bottom edge $\tau = \mathbb{O} \mathbin{/\!\!/} \mathrm{tr} \mathbin{/\!\!/} \mathbb{O}^{-1}$ we compute the generating function:

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{tr}} & U_U \\ {\scriptstyle \mathbb{O}}\uparrow & & {\scriptstyle \mathbb{O}}\uparrow \\ \mathbb{Q}[y, b, a, x] & \xrightarrow{\tau} & \mathbb{Q}[a, z] \end{array} \tag{4.15}$$

We begin with a result on finite differences:

Lemma 4.7 (finite differences of exponentials)**.** *The finite difference operator acts in the following way on exponentials:*

$$\nabla^n [\mathbf{e}^{\alpha a}](a) = (1 - \mathbf{e}^{-\alpha})^n \mathbf{e}^{\alpha a} \tag{4.16}$$

Proof. Using the fact that $\nabla^n [f](x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x - k)$, we see that $\nabla^n [\mathbf{e}^{\alpha a}](a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbf{e}^{\alpha a - \alpha k} = (1 - \mathbf{e}^{-\alpha})^n \mathbf{e}^{\alpha a}$. □

We now are ready to compute the generating function for the trace:

Theorem 4.8 (Generating function for the trace of U)**.**

$$\mathcal{G}(\mathrm{tr}) = \exp\Big(\alpha a + \big(\eta\xi + \beta(1-\mathbf{e}^{-\alpha})\big)z\Big) \tag{4.17}$$

Proof. Using lemma 4.7 and the extension of scalars of tr to $\mathbb{Q}[\![\eta,\beta,\alpha,\xi]\!]$, we see

$$\begin{aligned}
&\mathcal{G}(\mathbb{O} \mathbin{/\!\!/} \mathrm{tr} \mathbin{/\!\!/} \mathbb{O}^{-1}) = (\mathbf{e}^{\eta y}\mathbf{e}^{\beta b}\mathbf{e}^{\alpha a}\mathbf{e}^{\xi x}) \mathbin{/\!\!/} \mathrm{tr} \mathbin{/\!\!/} \mathbb{O}^{-1} \\
&= \mathbb{O}^{-1}\sum_{i,j,k}\mathrm{tr}\left(\frac{(\eta y)^i}{i!}\frac{(\beta b)^j}{j!}\mathbf{e}^{\alpha a}\frac{(\xi x)^k}{k!}\right) \\
&= \sum_{i,j}\frac{\eta^i\beta^j\xi^i}{i!j!}(1-\mathbf{e}^{-\alpha})^j\mathbf{e}^{\alpha a}z^{i+j} = \mathbf{e}^{\alpha a+(\eta\xi+\beta(1-\mathbf{e}^{-\alpha}))z} \qquad \square
\end{aligned} \tag{4.18}$$

Evaluation of the trace on a generic element

Here we will outline a computation involving the trace by using Bar-Natan and van der Veen's Contraction Theorem.

A typical value for a tangle invariant that arises is of the form:

$$P\mathbf{e}^{c+\alpha a_i+\beta b_i+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i} \tag{4.19}$$

Here, c, α, and β denote constants with respect to the variables y_i, b_i, a_i, and x_i (collectively referred to as "v_i"s), while ξ, η, and λ are potentially b_i-dependent, and P is a (rational) function in (the square root of) B_i.

Theorem 4.9 (The trace of a Gaußian)**.** *With symbols as defined above, let* $f(y_i,b_i,a_i,x_i) = P(B_i)\mathbf{e}^{c+\alpha a_i+\beta b_i+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i}$*. Then*

$$\left\langle f(y_i,b_i,a_i,x_i)\,\mathrm{tr}^i\right\rangle_{v_i} = \frac{P(\mathbf{e}^{-\mu})}{1-\lambda(\mu)\bar{z}_i}\mathbf{e}^{c+\alpha\bar{a}_i+\beta\mu+\frac{\eta(\mu)\xi(\mu)\bar{z}_i}{1-\lambda(\mu)\bar{z}_i}} \tag{4.20}$$

where $\mu := (1-\mathbf{e}^{-\alpha})\bar{z}_i$.

Proof. Let us compute the trace of equation (4.19). For clarity, we will put bars over the coinvariants variables a_i and z_i, as they do not play a role in the contraction.

$$\langle P(B_i)\mathbf{e}^{c+\alpha a_i+\beta b_i+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i}\,\mathrm{tr}^i\rangle_{v_i}$$
$$= \langle P(B_i)\mathbf{e}^{c+\beta b_i+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i+\eta_i\xi_i\bar{z}_i+\beta_i(1-\mathbf{e}^{-\alpha_i})\bar{z}_i}\mathbf{e}^{\alpha a_i+\alpha_i\bar{a}_i}\rangle_{v_i}$$
$$= \mathbf{e}^{\alpha\bar{a}_i}\langle P(B_i)\mathbf{e}^{c+\xi(b_i)x_i+\eta(b_i)y_i+\lambda(b_i)x_iy_i+\eta_i\xi_i\bar{z}_i}\mathbf{e}^{\beta b_i+\beta_i(1-\mathbf{e}^{-\alpha})\bar{z}_i}\rangle_{b_i,x_i,y_i}$$

In what follows, we let $\mu := (1-\mathbf{e}^{-\alpha})\bar{z}_i$:

$$= \mathbf{e}^{c+\alpha\bar{a}_i+\beta\mu}P(\mathbf{e}^{-\mu})\langle \mathbf{e}^{\eta(\mu)y_i}\mathbf{e}^{(\xi(\mu)+\lambda(\mu)y_i)x_i+\xi_i\eta_i\bar{z}_i}\rangle_{x_i,y_i}$$
$$= \mathbf{e}^{c+\alpha\bar{a}_i+\beta\mu}P(\mathbf{e}^{-\mu})\langle \mathbf{e}^{\eta(\mu)y_i+\xi(\mu)\bar{z}_i\eta_i+\lambda(\mu)\bar{z}_i\eta_iy_i}\rangle_{y_i}$$
$$= \frac{P(\mathbf{e}^{-\mu})}{1-\lambda(\mu)\bar{z}_i}\mathbf{e}^{c+\alpha\bar{a}_i+\beta\mu+\frac{\eta(\mu)\xi(\mu)\bar{z}_i}{1-\lambda(\mu)\bar{z}_i}} \tag{4.21}$$

□

We point out that the outcome of this computation is not guaranteed to be a Gaußian. This puts a limitation on the applicability of this formula to links with more than two components, explored in chapter 5.

Computational examples

Using the formula given in equation (4.20), let us do some preliminary examples:

$$\mathrm{tr}^i(R_{ij}) = 1 \tag{4.22}$$
$$\mathrm{tr}^j(R_{ij}) = \mathbf{e}^{b_i\bar{a}_j} \tag{4.23}$$
$$\mathrm{tr}^2\left(\sqrt{B_2}\mathbf{e}^{-a_2b_1-a_1b_2+\frac{(B_1-1)x_2y_1}{b_1B_1}+\frac{(B_2-1)x_1y_2}{b_2B_2}}\right) = \mathbf{e}^{\frac{a_1(\bar{z}_2-B_1\bar{z}_2)}{B_1}-b_1\bar{a}_2+\frac{e^{B_1\bar{z}_2}(x_1y_1e^{B_1^{-1}\bar{z}_2}-x_1y_1e^{\bar{z}_2})}{b_1}+\frac{1}{2}B_1^{-1}\bar{z}_2-\frac{\bar{z}_2}{2}} \tag{4.24}$$

Equations (4.22) and (4.23) are the values one obtains for the two (virtual) one-crossing, two-component link, while equation (4.24) is the value of the invariant on the Hopf link.

When computing this on a link, however, it is important to keep track of which strands are open, and which are closed. We will extend the notation from the previous section to differentiate between open and closed indices. We write a morphism with domain $D = D_\mathrm{o} \sqcup D_\mathrm{c}$, codomain $C = C_\mathrm{o} \sqcup C_\mathrm{c}$ (here D_o denotes domain indices which are open, while D_c those which are closed, with the same convention for C) and generating function $f(\zeta_D, z_C)$ as $f(\zeta_D, z_C)^{(D_\mathrm{o}, D_\mathrm{c})}_{(C_\mathrm{o}, C_\mathrm{c})}$.

5

CONCLUSIONS

Limitations of this definition

For some inputs to the trace, expressions involving the Lambert W-function appear, which complicates attempts to keep the invariant valued in the space of perturbed Gaußians.

5.1 COMPARISON WITH THE MULTIVARIABLE ALEXANDER POLYNOMIAL

Given that the long knot (i.e. one-component) case of this invariant encodes the Alexander Polynomial, it was suspected that the invariant on long links (i.e. multiple components, one of which is long) formed by adding the trace would encode the MVA. However, there are links which the MVA separates which this invariant does not.

On all two-component links with at most 11 crossings (a collection of size 914), the trace map attains 878 distinct values, while the MVA attains only 778. However, the two invariants are incomparable in terms of their strength.

The links L_{5a1} and L_{10a43} are not distinguished by their partial traces, with both returning a value of:

$$\left(\left(\frac{B_1}{B_1^2 t_2 - 2B_1 t_2 + B_1 + t_2}\right)_{(\{1\},\{2\})}, \left(\frac{B_2^{3/2}}{B_2^2 t_1 - 2B_2 t_1 + B_2 + t_1}\right)_{(\{2\},\{1\})}\right) \tag{5.1}$$

The values of these links under the MVA are, however

$$\frac{(B_1-1)(B_2-1)}{\sqrt{B_1}\sqrt{B_2}} \text{ and } -\frac{(B_1-1)(B_2-1)(B_1+B_2-1)(B_2B_1-B_1-B_2)}{B_1^{3/2}B_2^{3/2}} \tag{5.2}$$

respectively.

In the other direction, there are also pairs of links in the same fibre of the MVA which this traced invariant can distinguish. In particular L_{5a1} and L_{7n2} both have the same value under the MVA:

$$\frac{(B_1-1)(B_2-1)}{\sqrt{B_1}\sqrt{B_2}} \tag{5.3}$$

The trace yields the following values (respectively):

$$\left(\left(\frac{B_1}{B_1^2t_2-2B_1t_2+B_1+t_2}\right)_{(\{1\},\{2\})}, \left(\frac{B_2^{3/2}}{B_2^2t_1-2B_2t_1+B_2+t_1}\right)_{(\{2\},\{1\})}\right) \tag{5.4}$$

$$\left(\left(\frac{B_1}{B_1^2t_2-2B_1t_2+B_1+t_2}\right)_{(\{1\},\{2\})}, \left(\frac{B_2^{5/2}}{B_2^2t_1-2B_2t_1+B_2^2-B_2+t_1+1}\right)_{(\{2\},\{1\})}\right) \tag{5.5}$$

This example also serves to highlight that the information provided by leaving one strand open is not enough to recover the value of a different strand being left open.

5.2 FURTHER WORK

While all other Hopf algebra operations in U are expressed by [BNvdVb] as perturbed Gaußians, the form in equation (4.17) does not to conform to the same structure. Further work is needed to either implement this operation into the established framework, or to suitably extend the framework (perhaps with the use of Lambert W-functions).

A

CODE

A.1 IMPLEMENTATION OF THE INVARIANT Z

This is a Mathematica™ implementation by Bar-Natan and van der Veen in [BNvdVb], modified by the author. The full source code is available at https://github.com/phro/GDO. We begin by setting some variables, as well as a method for modifying associations.

```
γ = 1; ℏ = 1; $k = 0;
setValue[value_,obj_,coord_]:=Module[
        {b=Association@@obj},
        b[coord] = value; Head[obj]@@Normal@b
]
```

We introduce notation `PG[L, Q, P]` to be interpreted as the Perturbed Gaußian $P\mathrm{e}^{L+Q}$. The function `fromE` serves as a compatibility layer between a former version of the code.

```
toPG[L_, Q_, P_] := PG["L"->L, "Q"->Q, "P"->P]
fromE[e_\[DoubleStruckCapitalE]] := toPG@@e/.
        Subscript[(v:y|b|t|a|x|B|T|η|β|τ|α|ξ|A), i_] -> v[i]
```

We define the Kronecker-δ function next.

```
δ[i_,j_] := If[SameQ[i,j],1,0]
```

Next we introduce helper functions for the reading and manipulating of `PG`-objects:

```
getL[pg_PG] := Lookup[Association@@pg,"L",0]
getQ[pg_PG] := Lookup[Association@@pg,"Q",0]
getP[pg_PG] := Lookup[Association@@pg,"P",1]

setL[L_][pg_PG] := setValue[L, pg, "L"];
```

```
setQ[Q_][pg_PG] := setValue[Q, pg, "Q"];
setP[P_][pg_PG] := setValue[P, pg, "P"];

applyToL[f_][pg_PG] := pg//setL[pg//getL//f]
applyToQ[f_][pg_PG] := pg//setQ[pg//getQ//f]
applyToP[f_][pg_PG] := pg//setP[pg//getP//f]
```

Next is a function CF, which bring objects into canonical form allows us to compare for equality effectively. This is defined by Bar-Natan and van der Veen.

```
CCF[e_] := ExpandDenominator@ExpandNumerator@Together[
        Expand[e] //. E^x_ E^y_ :> E^(x + y) /. E^x_ :>
         ↪  E^CCF[x]
];
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[e_] := Module[
        {vs = Union[
                Cases[e, (y|b|t|a|x|η|β|τ|α|ξ)[_], ∞],
                {y, b, t, a, x, η, β, τ, α, ξ}
        ]},
        Total[CoefficientRules[Expand[e], vs] /.
                (ps_ -> c_) :> CCF[c] (Times @@ (vs^ps))
        ]
];
CF[e_PG] := e//applyToL[CF]//applyToQ[CF]//applyToP[CF]
```

We must also define the notion of equality for PG-objects, as well as what it means to multiply them.

```
Congruent[x_, y_, z__] := And[Congruent[x, y], Congruent[y, z]]
PG /: Congruent[pg1_PG, pg2_PG] := And[
        CF[getL@pg1 == getL@pg2],
        CF[getQ@pg1 == getQ@pg2],
        CF[Normal[getP@pg1-getP@pg2] == 0]
]

PG /: pg1_PG * pg2_PG := toPG[
        getL@pg1 + getL@pg2,
```

```
        getQ@pg1 + getQ@pg2,
        getP@pg1 * getP@pg2
]

setEpsilonDegree[k_Integer][pg_PG] :=
    setP[Series[Normal@getP@pg,{ϵ, 0, k}]][pg]
```

The variables y, b, t, a, and x are paired with their dual variables η, β, τ, α, and ξ. This applies as well when they have subscripts.

```
ddsl2vars = {y, b, t, a, x, z};
ddsl2varsDual = {η, β, τ, α, ξ, ζ};

Evaluate[Dual/@ddsl2vars] = ddsl2varsDual;
Evaluate[Dual/@ddsl2varsDual] = ddsl2vars;
Dual@z = ζ;
Dual@ζ = z;
Dual[u_[i_]]:=Dual[u][i]
```

Since various exponentials of the lowercase variables frequently appear, we introduce capital variable names to handle various exponentiated forms.

```
U2l = {
        B[i_]^p_. :> E^(-p ℏ γ b[i]), B^p_. :> E^(-p ℏ γ b),
        W[i_]^p_. :> E^(w[i])       , W^p_. :> E^(p w),
        T[i_]^p_. :> E^(-p ℏ t[i])  , T^p_. :> E^(-p ℏ t),
        A[i_]^p_. :> E^(p γ α[i])   , A^p_. :> E^(-p γ α)
};
l2U = {
        E^(c_. b[i_] + d_.) :> B[i]^(-c/(ℏ γ))E^d,
        E^(c_. b + d_.)     :> B^(-c/(ℏ γ))E^d,
        E^(c_. t[i_] + d_.) :> T[i]^(-c/ℏ)E^d,
        E^(c_. t + d_.)     :> T^(-c/ℏ)E^d,
        E^(c_. α[i_] + d_.) :> A[i]^(c/γ)E^d,
        E^(c_. α + d_.)     :> A^(c/γ)E^d,
        E^(c_. w[i_] + d_.) :> W[i]^(c)E^d,
        E^(c_. w + d_.)     :> W^(c)E^d,
        E^expr_             :> E^Expand@expr
};
```

Below the notion of differentiation is defined for expressions which involve both upper- and lower-case variables.

```
DD[f_, b]     := D[f, b   ] - ℏ γ B    D[f, B   ];
DD[f_, b[i_]] := D[f, b[i]] - ℏ γ B[i] D[f, B[i]];

DD[f_, t    ] := D[f, t   ] - ℏ T    D[f, T   ];
DD[f_, t[i_]] := D[f, t[i]] - ℏ T[i] D[f, T[i]];

DD[f_, α    ] := D[f, α   ] + γ A    D[f, A   ];
DD[f_, α[i_]] := D[f, α[i]] + γ A[i] D[f, A[i]];

DD[f_, v_] := D[f, v];
DD[f_, {v_,0}] := f;
DD[f_, {}] := f;
DD[f_, {v_,n_Integer}] := DD[DD[f,v],{v,n-1}];
DD[f_, {l_List, ls___}] := DD[DD[f, l], {ls}];
```

What follows now is the implementation of contraction as introduced in definition 3.4. We begin with the introduction of contractions of (finite) polynomials.

```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[expr_, ζ_] := Collect[expr, ζ];

Zip[{}][P_] := P;
Zip[ζs_List][Ps_List] := Zip[ζs]/@Ps;
Zip[{ζ_,ζs___}][P_] := (collect[P // Zip[{ζs}],ζ] /.
        f_. ζ^d_. :> DD[f,{Dual[ζ], d}]) /.
        Dual[ζ] -> 0 /.
        ((Dual[ζ] /. {b->B, t->T, α -> A}) -> 1)
```

We define contraction along the variables x and y (here packaged into the matrix Q).

```
QZip[ζs_List][pg_PG] := Module[{Q, P, ζ, z, zs, c, ys, ηs, qt,
 ↪  zrule, ζrule},
        zs = Dual/@ζs;
        Q  = pg//getQ;
```

```
        P  = pg//getP;
        c  = CF[Q/.Alternatives@@Union[ζs, zs]->0];
        ys = CF/@Table[D[Q,ζ]/.Alternatives@@zs->0,{ζ,ζs}];
        ηs = CF/@Table[D[Q,z]/.Alternatives@@ζs->0,{z,zs}];
        qt = CF/@#&/@(Inverse@Table[
                δ[z, Dual[ζ]] - D[Q,z,ζ],
                {ζ,ζs},{z,zs}
        ]);
        zrule = Thread[zs -> CF/@(qt . (zs + ys))];
        ζrule = Thread[ζs -> ζs + ηs . qt];
        CF@setQ[c + ηs.qt.ys]@setP[Det[qt] Zip[ζs][P /.
         ↪  Union[zrule, ζrule]]]@pg
]
```

We define contraction along the variables a and b (here packaged into the matrix L).

```
LZip[ζs_List][pg_PG] := Module[
        {
                L, Q, P, ζ, z, zs, Zs, c, ys, ηs, lt,
                zrule, Zrule, ζrule, Q1, EEQ, EQ, U
        },
        zs = Dual/@ζs;
        {L, Q, P} = Through[{getL, getQ, getP}@pg];
        Zs = zs /. {b -> B, t -> T, α -> A};
        c  = CF[L/.Alternatives@@Union[ζs,
         ↪  zs]->0/.Alternatives@@Zs -> 1];
        ys = CF/@Table[D[L,ζ]/.Alternatives@@zs->0,{ζ,ζs}];
        ηs = CF/@Table[D[L,z]/.Alternatives@@ζs->0,{z,zs}];
        lt = CF/@#&/@Inverse@Table[
                δ[z, Dual[ζ]] - D[L,z,ζ],
                {ζ,ζs},{z,zs}
        ];
        zrule = Thread[zs -> CF/@(lt . (zs + ys))];
        Zrule = Join[zrule, zrule /.
                r_Rule :> ( (U = r[[1]] /. {b -> B, t -> T, α
                 ↪  -> A}) ->
                (U /. U2l /. r //. l2U))
```

```
        ];
        \[Zeta]rule = Thread[\[Zeta]s -> \[Zeta]s + \[Eta]s . lt];
        Q1 = Q /. Union[Zrule, ζrule];
        EEQ[ps___] :=
                EEQ[ps] = (
                        CF[E^-Q1 DD[E^Q1,Thread[{zs,{ps}}]] /.
                                {Alternatives@@zs -> 0, Alternatives @@Zs -> 1}]
                );
        CF@toPG[
                c + ηs.lt.ys,
                Q1 /. {Alternatives@@zs -> 0, Alternatives@@Zs -> 1},
                Det[lt] (Zip[ζs][(EQ@@zs) (P /. Union[Zrule,ζrule])] /.
                        Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ ->1
                )
        ]

]
```

The function `Pair` combines the above zipping functions into the final contraction map.

```
Pair[{}][L_PG,R_PG] := L R;
Pair[is_List][L_PG,R_PG] := Module[{n},
        Times[
                L /. ((v: b|B|t|T|a|x|y)[#] -> v[n@#]&/@is),
                R /. ((v: β|τ|α|A|ξ|η)[#] -> v[n@#]&/@is)
        ] // LZip[Join@@Table[Through[{β, τ, a}[n@i]],{i, is}]] //
        QZip[Join@@Table[Through[{ξ, y}[n@i]],{i, is}]]
]
```

Our next task is to provide domain and codomain information for the `PG`-objects. These will be packaged inside a `GDO`, (Gaußian Differential Operator).

The four lists' names refer to whether it is a domain or a codomain, and whether the index corresponds to an open strand or a closed one.

```
toGDO[do_List,dc_List,co_List,cc_List,L_,Q_,P_] := GDO[
        "do" -> do,
        "dc" -> dc,
        "co" -> co,
        "cc" -> cc,
        "PG" -> toPG[L, Q, P]
]

toGDO[do_List,dc_List,co_List,cc_List,pg_PG] := GDO[
        "do" -> do,
        "dc" -> dc,
        "co" -> co,
        "cc" -> cc,
        "PG" -> pg
]
```

Next are defined functions for accessing and modifying sub-parts of `GDO`-objects. The last argument of `Lookup` is the default value if nothing is specified. This means that a morphism with empty domain or codomain may be specified as such by omitting that portion of the definition.

```
getDO[gdo_GDO] := Lookup[Association@@gdo, "do", {}]
getDC[gdo_GDO] := Lookup[Association@@gdo, "dc", {}]
getCO[gdo_GDO] := Lookup[Association@@gdo, "co", {}]
getCC[gdo_GDO] := Lookup[Association@@gdo, "cc", {}]

getPG[gdo_GDO] := Lookup[Association@@gdo, "PG", PG[]]

getL[gdo_GDO] := gdo//getPG//getL
getQ[gdo_GDO] := gdo//getPG//getQ
getP[gdo_GDO] := gdo//getPG//getP

setPG[pg_PG][gdo_GDO] := setValue[pg, gdo, "PG"]

setL[L_][gdo_GDO] := setValue[setL[L][gdo//getPG], gdo, "PG"]
```

```
setQ[Q_][gdo_GDO] := setValue[setQ[Q][gdo//getPG], gdo, "PG"]
setP[P_][gdo_GDO] := setValue[setP[P][gdo//getPG], gdo, "PG"]

setDO[do_][gdo_GDO] := setValue[do, gdo, "do"]
setDC[dc_][gdo_GDO] := setValue[dc, gdo, "dc"]
setCO[co_][gdo_GDO] := setValue[co, gdo, "co"]
setCC[cc_][gdo_GDO] := setValue[cc, gdo, "cc"]

applyToDO[f_][gdo_GDO] := gdo//setDO[gdo//getDO//f]
applyToDC[f_][gdo_GDO] := gdo//setDC[gdo//getDC//f]
applyToCO[f_][gdo_GDO] := gdo//setCO[gdo//getCO//f]
applyToCC[f_][gdo_GDO] := gdo//setCC[gdo//getCC//f]

applyToPG[f_][gdo_GDO] := gdo//setPG[gdo//getPG//f]

applyToL[f_][gdo_GDO] := gdo//setL[gdo//getL//f]
applyToQ[f_][gdo_GDO] := gdo//setQ[gdo//getQ//f]
applyToP[f_][gdo_GDO] := gdo//setP[gdo//getP//f]
```

The canonical form function (`CF`) and the contraction mapping (`Pair`) we extend to include `GDO`-objects. Furthermore, on the level of `GDO`-objects we can compose morphisms and keep track of the corresponding domains and codomains, using the left-to-right composition operator "$/\!\!/$".

```
CF[e_GDO] := e//
        applyToDO[Union]//
        applyToDC[Union]//
        applyToCO[Union]//
        applyToCC[Union]//
        applyToPG[CF]

Pair[is_List][gdo1_GDO, gdo2_GDO] := GDO[
        "do" -> Union[gdo1//getDO, Complement[gdo2//getDO, is]],
        "dc" -> Union[gdo1//getDC, gdo2//getDC],
        "co" -> Union[gdo2//getCO, Complement[gdo1//getCO, is]],
        "cc" -> Union[gdo1//getCC, gdo2//getCC],
```

```
        "PG" -> Pair[is][gdo1//getPG, gdo2//getPG]
]

gdo1_GDO // gdo2_GDO := Pair[Intersection[gdo1//getCO,gdo2//getDO]][gdo1,gdo2];
```

We also define notions of equality and multiplication (by concatenation) for GDO’s.

```
GDO /: Congruent[gdo1_GDO, gdo2_GDO] := And[
        Sort@*getDO/@Equal[gdo1, gdo2],
        Sort@*getDC/@Equal[gdo1, gdo2],
        Sort@*getCO/@Equal[gdo1, gdo2],
        Sort@*getCC/@Equal[gdo1, gdo2],
        Congruent[gdo1//getPG, gdo2//getPG]
]

GDO /: gdo1_GDO gdo2_GDO := GDO[
        "do" -> Union[gdo1//getDO, gdo2//getDO],
        "dc" -> Union[gdo1//getDC, gdo2//getDC],
        "co" -> Union[gdo1//getCO, gdo2//getCO],
        "cc" -> Union[gdo1//getCC, gdo2//getCC],
        "PG" -> (gdo1//getPG)*(gdo2//getPG)
]
```

For the sake of compatibility with Bar-Natan and van der Veen’s program, we introduce several conversion functions between the two notations.

```
setEpsilonDegree[k_Integer][gdo_GDO] :=
        setP[Series[Normal@getP@gdo,{ϵ,0,k}]][gdo]

fromE[Subscript[\[DoubleStruckCapitalE],{do_List, dc_List}->{co_List, cc_List}][
        L_, Q_, P_
]] := toGDO[do, dc, co, cc, fromE[\[DoubleStruckCapitalE][L, Q, P]]]

fromE[Subscript[\[DoubleStruckCapitalE], dom_List->cod_List][
        L_, Q_, P_
```

```
]] := GDO["do" -> dom, "co" -> cod,
        "PG" -> fromE[\[DoubleStruckCapitalE][L, Q, P]]
]
```

It is at this point that we implement the morphisms of the algebra U. Each operation is prepended with a “c” to emphasize that this is a classical algebra, not a quantum deformation.

```
fromLog[l_] := CF@Module[
        {L, l0 = Limit[l, ϵ->0]},
        L = l0 /. (η|y|ξ|x)[_]->0;
        PG[
                "L" -> L,
                "Q" -> l0 - L
        ]/.l2U
]

cΛ = (η[i] + E^(-γ α[i] - ϵ β[i]) η[j]/(1+γ ϵ η[j]ξ[i])) y[k] +
     (β[i] + β[j] + Log[1 + γ ϵ η[j]ξ[i]]/ϵ            ) b[k] +
     (α[i] + α[j] + Log[1 + γ ϵ η[j]ξ[i]]/γ            ) a[k] +
     (ξ[j] + E^(-γ α[j] - ϵ β[j]) ξ[i]/(1+γ ϵ η[j]ξ[i])) x[k];

cm[i_, j_, k_] = GDO["do" -> {i,j}, "co" -> {k}, "PG" ->
 ↪ fromLog[cΛ]];

cη[i_] = GDO["co" -> {i}];
cσ[i_,j_] = GDO["do"->{i},"co"->{j},
        "PG"->fromLog[β[i] b[j] + α[i] a[j] + η[i] y[j] + ξ[i]
         ↪ x[j]]
];
cϵ[i_] = GDO["do" -> {i}];
cΔ[i_, j_, k_] = GDO["do"->{i}, "co"->{j, k},
        "PG" -> fromLog[
                β[i](b[j] + b[k]) +
                α[i](a[j] + a[k]) +
                η[i]
                        ((b[j]+b[k])/(1-B[j]B[k]))
                        (
```

```
                              B[k]((1-B[j])/b[j])y[j]+
                                  ((1-B[k])/b[k])y[k]
                      ) +
              ξ[i](x[j] + x[k])
        ]
];

sY[i_, j_, k_, l_, m_] = GDO["do"->{i}, "co"->{j, k, l, m},
        "PG" -> fromLog[β[i]b[k] + α[i]a[l] + η[i]y[j] + ξ[i]x[m]]
];

sS[i_] = GDO["do"->{i},"co"->{i},
        "PG"->fromLog[-(β[i] b[i] + α[i] a[i] + η[i] y[i] + ξ[i] x[i])]
];

cS[i_] = sS[i] // sY[i, 1, 2, 3, 4] // cm[4,3, i] // cm[i, 2, i] // cm[i, 1, i];
cS[i_] = GDO["do"->{i},"co"->{i},
        "PG"->fromLog[-(β[i] b[i] + α[i] a[i] + η[i] B[i] y[i] + ξ[i] x[i])]
];

cR[i_, j_] = GDO[
        "co" -> {i,j},
        "PG" -> toPG[ℏ a[j] b[i], (B[i]-1)/(-b[i]) x[j] y[i], 1]
]

cRi[i_, j_] = GDO[
        "co" -> {i,j},
        "PG" -> toPG[-ℏ a[j] b[i], (B[i]-1)/(B[i] b[i]) x[j] y[i], 1]
]
```

```
CC[i_] :=  GDO["co"->{i},"PG"->PG["P"->B[i]^( 1/2)]]
CCi[i_] := GDO["co"->{i},"PG"->PG["P"->B[i]^(-1/2)]]

cKink[i_]  = Module[{k}, cR[i,k] CCi[k] // cm[i, k, i]]
cKinki[i_] = Module[{k}, cRi[i,k] CC[k] // cm[i, k, i]]

cKinkn[0][i_]  = cη[i]
cKinkn[1][i_]  = cKink[i]
cKinkn[-1][i_] = cKinki[i]
cKinkn[n_Integer][i_] :=
 ↪ Module[{j},cKinkn[n-1][i]cKink[j]//cm[i,j,i]]/; n > 1
cKinkn[n_Integer][i_] :=
 ↪ Module[{j},cKinkn[n+1][i]cKinki[j]//cm[i,j,i]]/; n < -1

uR[i_, j_]  = Module[{k}, cR[i,j] cKinki[k] // cm[i, k, i]]
uRi[i_, j_] = Module[{k}, cRi[i,j] cKink[k] // cm[i, k, i]]
```

A.2 IMPLEMENTATION OF THE TRACE

Now we implement the trace. We introduce several functions which extract the various coefficients of a `GDO`, so that we may apply equation (4.20). Coefficients are extracted based on whether they belong to the matrix `L` or the matrix `Q`.

```
getConstLCoef::usage = "getConstLCoef[i][gdo] returns the terms
 ↪ in the L-portion of a GDO expression which are not a
 ↪ function of y[i], b[i], a[i], nor x[i]."
getConstLCoef[i_][gdo_] :=
        (SeriesCoefficient[#, {b[i],0,0}]&) @*
        (Coefficient[#, y[i], 0]&) @*
        (Coefficient[#, a[i], 0]&) @*
        (Coefficient[#, x[i], 0]&) @*
        ReplaceAll[U2l] @*
        getL@
        gdo

```

```
getConstQCoef::usage = "getConstQCoef[i][gdo] returns the terms
↪ in the Q-portion of a GDO expression which are not a
↪ function of y[i], b[i], a[i], nor x[i]."
getConstQCoef[i_][gdo_][bb_] :=
        ReplaceAll[{b[i]->bb}] @*
        (Coefficient[#, y[i], 0]&) @*
        (Coefficient[#, a[i], 0]&) @*
        (Coefficient[#, x[i], 0]&) @*
        ReplaceAll[U2l] @*
        getQ@
        gdo

getyCoef::usage = "getyCoef[i][gdo][b[i]] returns the linear
↪ coefficient of y[i] as a function of b[i]."
getyCoef[i_][gdo_][bb_] :=
        ReplaceAll[{b[i]->bb}] @*
        ReplaceAll[U2l] @*
        (Coefficient[#, x[i],0]&) @*
        (Coefficient[#, y[i],1]&) @*
        getQ@
        gdo

getbCoef::usage = "getbCoef[i][gdo] returns the linear
↪ coefficient of b[i]."
getbCoef[i_][gdo_] :=
        (SeriesCoefficient[#, {b[i],0,1}]&) @*
        (Coefficient[#, a[i],0]&) @*
        (Coefficient[#, x[i],0]&) @*
        (Coefficient[#, y[i],0]&) @*
        ReplaceAll[U2l] @*
        getL@
        gdo

getPCoef::usage = "getPCoef[i][gdo] returns the perturbation P
↪ of a GDO as a function of b[i]."
getPCoef[i_][gdo_][bb_] :=
```

```
        ReplaceAll[{b[i]->bb}] @*
        (Coefficient[#, a[i],0]&) @*
        (Coefficient[#, x[i],0]&) @*
        (Coefficient[#, y[i],0]&) @*
        ReplaceAll[U2l] @*
        getP@
        gdo

getaCoef::usage = "getaCoef[i][gdo] returns the linear coefficient of a[i]."
getaCoef[i_][gdo_] :=
        (SeriesCoefficient[#, {b[i],0,0}]&) @*
        (Coefficient[#, a[i],1]&) @*
        ReplaceAll[U2l] @*
        getL@
        gdo

getxCoef::usage = "getxCoef[i][gdo][b[i]] returns the linear coefficient of x[i] as a function of b[i]."
getxCoef[i_][gdo_][bb_] :=
        ReplaceAll[{b[i]->bb}] @*
        ReplaceAll[U2l] @*
        (Coefficient[#, y[i],0]&) @*
        (Coefficient[#, x[i],1]&) @*
        getQ@
        gdo

getabCoef::usage = "getabCoef[i][gdo] returns the linear coefficient of a[i]b[i]."
getabCoef[i_][gdo_] :=
        (SeriesCoefficient[#,{b[i],0,1}]&) @*
        (Coefficient[#,a[i],1]&) @*
        ReplaceAll[U2l] @*
        getL@
        gdo
```

```
getxyCoef::usage = "getxyCoef[i][gdo][b[i]] returns the linear coefficient of x[i]y[i] as a function of b[i]."
getxyCoef[i_][gdo_][bb_] :=
        ReplaceAll[{b[i]->bb}] @*
        ReplaceAll[U2l] @*
        (Coefficient[#,x[i],1]&) @*
        (Coefficient[#,y[i],1]&) @*
        getQ@
        gdo
```

In order to run more efficiently, limits are first computed by direct evaluation, unless such an operation is ill-defined. In such a case, the corresponding series is computed and evaluated at the limit point.

```
safeEval[f_][x_] := Module[{fx, x0},
        If[(fx=Quiet[f[x]]) === Indeterminate,
                Series[f[x0],{x0,x,0}]//Normal,
                fx
        ]
]

(* safeEvalVars[f_][vars_List][vals_List] := Module[{fx, x0}, *)
        (* If[(fx=Quiet[f/.Thread[vars->vals]]) === Indeterminate, *)
                (* Series[f, Sequence@@MapThread[{#1,#2,0}&,vars,vals]]//Normal, *)
                (* fx *)
        (* ] *)
(* ] *)

(* ηs = CF/@Table[Limit[D[Q,z], (#1->0&)/@ζs],{z,zs}]; *)
safeEvalVars[f_][vars_List][val_] := Module[{fx},
        If[(fx=Quiet[f/.Thread[vars->ConstantArray[val, Length[vars]]]]) ===
                Indeterminate,
```

```
                Series[f,
                 ↪  Sequence@@MapThread[{#1,#2,0}&,vars,vals]]//Normal,
                fx
        ]
]

closeComponent[i_][gdo_GDO]:=gdo//
        setCO[Complement[gdo//getCO,{i}]]//
        setCC[Union[gdo//getCC,{i}]]
```

Now we come to the implementation of the trace map. The current implementation requires that the coefficient of $a_i b_i$ be zero. (See chapter 5 for how this restriction limits computability.)

```
tr::usage = "tr[i] computes the trace of a GDO element on
 ↪  component i. Current implementation assumes the Subscript[a,
 ↪  i] Subscript[b, i] term vanishes and $k=0."
tr::nonzeroSigma = "tr[`1`]: Component `1` has writhe: `2`,
 ↪  expected: 0."
tr[i_][gdo_GDO] := Module[
        {
                cL = getConstLCoef[i][gdo],
                cQ = getConstQCoef[i][gdo],
                βP = getPCoef[i][gdo],
                ηη = getyCoef[i][gdo],
                ββ = getbCoef[i][gdo],
                αα = getaCoef[i][gdo],
                ξξ = getxCoef[i][gdo],
                λ = getxyCoef[i][gdo],
                ta
        },
        ta = (1-Exp[-αα]) z[i];
        expL = cL + αα w[i] + ββ ta;
        expQ = safeEval[cQ[#] + z[i]ηη[#]ξξ[#]/(1-z[i]
         ↪  λ[#])&][ta];
        expP = safeEval[βP[#]/(1-z[i] λ[#])&][ta];
        CF[(gdo//closeComponent[i]//
         ↪  setL[expL]//setQ[expQ]//setP[expP])//.l2U]
```

```
] /; Module[
        {σ = getabCoef[i][gdo]},
        If[σ == 0,
                True,
                Message[tr::nonzeroSigma, i, ToString[σ]];
                 ↪  False
        ]
]
```

Here we introduce some formatting to display the output more aesthetically.

```
Format[gdo_GDO] := Subsuperscript[\[DoubleStruckCapitalE],
        Row[{gdo//getCO, ",", gdo//getCC}],
        Row[{gdo//getDO, ",", gdo//getDC}]
][gdo//getL, gdo//getQ, gdo//getP];
Format[pg_PG] := \[DoubleStruckCapitalE][pg//getL, pg//getQ,
 ↪  pg//getP];

SubscriptFormat[v_] := (Format[v[i_]] := Subscript[v, i]);

SubscriptFormat/@{y,b,t,a,x,z,w,η,β,α,ξ,A,B,T,W};
```

Implementing the full invariant

Now we are in a position to implement the Z invariant to tangles with a closed component. We begin by defining an object representing an isolated strand with arbitrary integer rotation number, `CCn`:

```
CCn[i_][n_Integer]:=Module[{j},
  If[n==0,
    GDO["co"->{i}],
    If[n>0,
      If[n==1,
        CC[i],
        CC[j]//CCn[i][n-1]//cm[i,j,i]
      ],
      If[n==-1,
        CCi[i],
```

```
        CCi[j]//CCn[i][n+1]//cm[i,j,i]
      ]
    ]
  ]
]
```

Since multiplication is associative, we may implement a generalized multiplication which can take any number of arguments. It is also named `cm`, with a first argument given as an ordered list of indices to be concatenated.

```
cm[{}, j_] := cη[j]
cm[{i_}, j_] := cσ[i,j]
cm[{i_, j_}, k_] := cm[i,j,k]
cm[ii_List, k_] := Module[
        {
                i  = First[ii],
                is = Rest[ii],
                j  ,
                js ,
                l
        },
        j  = First[is];
        js = Rest[is];
        cm[i,j,l] // cm[Prepend[js, l], k]
]
```

The function `toGDO` serves as the invariant for the generators of the tangles. We define its value on crossings and on concatenations of elements.

```
toGDO[Xp[i_,j_]] := cR[i,j]
toGDO[Xm[i_,j_]] := cRi[i,j]
toGDO[xs_Strand] := cm[List@@xs, First[xs]]
toGDO[xs_Loop]   := Module[{x = First[xs]}, cm[List@@xs,
 ↪  x]//tr[x]]

getIndices[RVT[cs_List, _List, _List]] :=
 ↪  Sort@Catenate@(List@@@cs)

TerminalQ[cs_List][i_] := MemberQ[Last/@cs,i];
```

```
next[cs_List][i_]:=If[TerminalQ[cs][i],
        Nothing,
        Extract[cs,
         ↪  ((#/.{c_,j_}->{c,j+1}&)@FirstPosition[i]@cs)]
]

InitialQ[cs_List][i_] := MemberQ[First/@cs,i];
prev[cs_List][i_]:=If[InitialQ[cs][i],
        Nothing,
        Extract[cs,
         ↪  ((#/.{c_,j_}->{c,j-1}&)@FirstPosition[i]@cs)]
]
```

To minimize the size of computations, whenever adjacent indices are present in the partial computation, they are to be concatenated before more crossings are introduced.

```
MultiplyAdjacentIndices[{cs_List,calc_GDO}]:=Module[
        { is=getCO[calc]
        , i
        , i2
        },
        i = SelectFirst[is,MemberQ[is,next[cs][#]]&];
        If[Head[i]===Missing,
                {cs,calc},
                i2 = next[cs][i];
                {DeleteCases[cs,i2,2], calc//cm[i,i2,i]}
        ]
]

MultiplyAllAdjacentIndices[{cs_List, calc_GDO}] :=
        FixedPoint[MultiplyAdjacentIndices, {cs, calc}]

generateGDOFromXing[x:_Xp|_Xm,rs_Association]:=Module[
        {p, i,j, in, jn},
        {i,j} = List@@x;
        {in,jn} = Lookup[rs,{i,j},0];
```

```
        toGDO[x]*CCn[p[i]][in]*CCn[p[j]][jn]
         //cm[p[i],i,i]//cm[p[j],j,j]
]

addRotsToXingFreeStrands[rvt_RVT] := GDO[] * Times @@ (
        CCn[#][Lookup[rvt[[3]], #, 0]] & /@
        First /@ Select[rvt[[1]], Length@# == 1 &]
)
```

Next we implement the framed link invariant `ZFramed`.

```
ZFramedStep[{_List,{},_Association,calc_GDO}]:={{},{},<||>,calc};
ZFramedStep[{cs_List,xs_List,rs_Association,calc_GDO}]:=Module[
        { x=First[xs], xss=Rest[xs]
        , csOut, calcOut
        , new
        },
        new=calc*generateGDOFromXing[x,rs];
        {csOut,calcOut} = MultiplyAllAdjacentIndices[{cs,new}];
        {csOut,xss,rs,calcOut}
]

ZFramed[rvt_RVT] := Last@FixedPoint[ZFramedStep, {Sequence @@
    rvt,
        addRotsToXingFreeStrands[rvt]}]
ZFramed[L_] := ZFramed[toRVT@L]
```

Finally, when we wish to consider the unframed invariant, we apply the function `Unwrithe`, defined below.

```
Z[rvt_RVT] := Unwrithe@Last@FixedPoint[ZFramedStep, {Sequence
    @@ rvt, GDO[]}]
Z[L_] := Z[toRVT@L]

combineBySecond[l_List] := mergeWith[Total,#]& /@ GatherBy[l,
    First];
combineBySecond[lis___] := combineBySecond[Join[lis]]

mergeWith[f_, l_] := {l[[1, 1]], f@(#[[2]] & /@ l)}
```

```
Reindex[RVT[cs_, xs_, rs_]] := Module[
  {
    sf,
    cs2, xs2, rs2,
    repl, repl2
  },
   sf = Flatten[List@@#&/@cs];
   repl = (Thread[sf -> Range[Length[sf]]]);
   repl2 = repl /. {(a_ -> b_) -> ({a, i_} -> {b, i})};
   cs2 = cs /. repl;
   xs2 = xs /. repl;
   rs2 = rs /. repl2;
   RVT[cs2, xs2, rs2]
]

UnwritheComp[i_][gdo_GDO] := Module[
        {n = gdo//getL//SeriesCoefficient[#,{a[i]b[i],0,1}]&, j},
        gdo//(cKinkn[-n][j])//cm[i,j,i]
]

Unwrithe[gdo_GDO]:=(Composition@@(UnwritheComp/@(gdo//getCO)))@gdo

toRVT[L_RVT] := L
```

The partial trace is what we use to close a subset of the strands in a tangle. It takes the trace of all but one component, then returns the collection of all such ways of leaving one component open. (As described in ??).

```
ptr[L_RVT] := Module[
        {
                ZL = Z[L],
                cod
        },
        cod = getCO@ZL;
        Table[(Composition@@Table[tr[j], {j,Complement[cod,{i}]}])[ZL], {i,cod}]
```

```
]
ptr[L_] := ptr[toRVT[L]]
```

In order to be able to compare `GDO`'s properly, we require a way to canonically represent them. This is achieved by reindexing the strands of the link and selecting one who's resulting invariant comes first in an (arbitrarily-selected) order, in this case the built-in ordering of expressions as defined by Mathematica™.

```
getGDOIndices[gdo_GDO]:=Sort@Catenate@Through[{getDO, getDC, getCO, getCC}@gdo]

isolateVarIndices[i_ -> j_] := (v:y|b|t|a|x|η|β|α|ξ|A|B|T|w|z|W)[i]->v[j];

ReindexBy[f_][gdo_GDO] := Module[
        {
        replacementRules,
        varIndexFunc,
        repFunc,
        indices = getGDOIndices[gdo]
        },
        replacementRules = Thread[indices->(f/@indices)];
        repFunc = ReplaceAll[replacementRules];
        varIndexFunc = ReplaceAll[Thread[isolateVarIndices[replacementRules]]];
        gdo//applyToPG[varIndexFunc]//
                applyToCO[repFunc]//
                applyToDO[repFunc]//
                applyToDC[repFunc]//
                applyToCC[repFunc]
]

fromAssoc[ass_] := Association[ass][#] &

ReindexToInteger[gdos_List] := Module[
        {is = getGDOIndices@gdos[[1]], f},
        f = fromAssoc@Thread[is -> Range[Length[is]]];
```

```
        ReindexBy[f]/@gdos
]

getReindications[gdos_List] := Module[
        {
                gdosInt = ReindexToInteger[gdos],
                is,
                fs,
                ls
        },
        is = getGDOIndices[gdosInt[[1]]];
        fs = (fromAssoc@*Association@*Thread)/@(is -> # & /@ Permutations[is]);
        ls = CF@ReindexBy[#]/@gdosInt&/@fs;
        Sort[Sort/@ls]
]

getCanonicalIndex[gdo_] := First@getReindications@gdo

deleteIndex[i_][expr_] := SeriesCoefficient[expr/.U2l, Sequence @@ ({#[i], 0, 0} & /@ {
        y, b, t, a, x, z, w
})]/.l2U
```

Here we introduce functions to further verify the co-algebra structure of a traced ribbon meta-Hopf algebra. In particular, the counit is responsible for deleting a strand. This has further applications in determining whether the invariants of individual components are contained in those of more complex links.

```
deleteIndexPG[i_][pg_PG] := pg//
        applyToL[deleteIndex[i]]//
        applyToQ[deleteIndex[i]]//
        applyToP[deleteIndex[i]]

deleteLoop[i_][gdo_] := gdo//
        applyToCC[Complement[#,{i}]&]//
        applyToPG[deleteIndexPG[i]]
```

A.3 IMPLEMENTATION OF ROTATION NUMBER ALGORITHM

Description of algorithm for knots

Bar-Natan and van der Veen develop an algorithm to convert a classical long knot into an upright tangle. It involves passing a line segment, called the front, over the knot, requiring that everything behind the front is in upright form. For example, consider the link: By pulling the crossings along

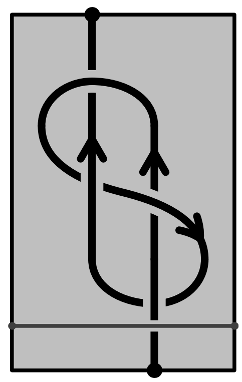

Figure A.1: A knot which is not in upright form. The front is written in grey.

the arc which touches the front, we can bring the knot to upright form.

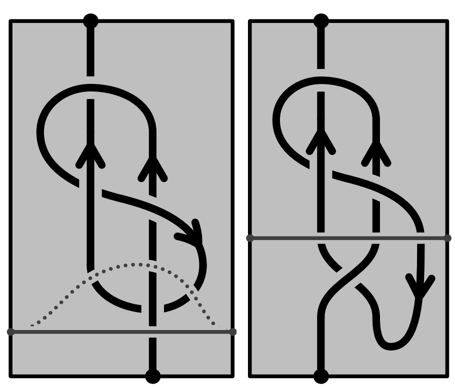

Figure A.2: By advancing the front over a crossing, we bring a crossing into upright form. A dashed front indicates where the front is advancing to.

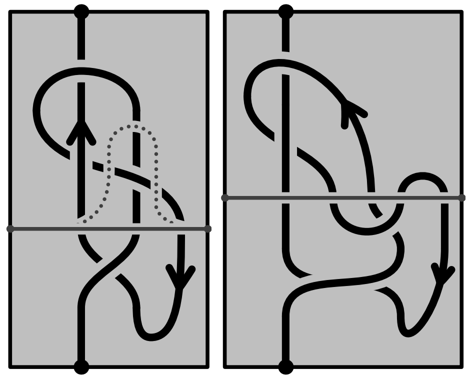

Figure A.3: By advancing the front over a crossing, we bring a crossing into upright form. A dashed front indicates where the front is advancing to.

Extending the algorithm to multiple components

An algorithm to convert a classical knot diagram into an upright knot diagram was implemented by Bar-Natan and van der Veen. Here we generalize the algorithm to convert a classical tangle with one open component to an upright tangle diagram. This generalization allows us to consider tangle diagrams with multiple components.

Lemma A.1 ([BNvdVb], Lemma 43)**.** *For each classical tangle with one open component, there exists a unique upright tangle whose unbounded arcs have rotation numbers* 0*.*

This is a Haskell implementation[†1] of the algorithm `toRVT`,[†2] which takes a classical tangle and produces an upright tangle by computing a compatible choice of rotation numbers for each arc. This follows largely the same logic as above, except the leftmost strand is always prioritized for absorption, regardless of which component it belongs to.

Use case

For example, to query the `SX`-form of a link (i.e. its skeleton-crossing form), one writes:

†1 The full source code is available at `https://github.com/phro/KnotTheory`.

†2 Here, the acronym RVT stands for "Rotational Virtual Tangle", which is another term for "Upright Tangle".

```
>>> link 4 True 1
SX [Loop [1,2,3,4],Loop [5,6,7,8]]
   [Xm 1 6,Xm 3 8,Xm 5 2,Xm 7 4]
```

To convert from the `SX` form to an upright tangle form (here written `RVT`), we must first replace one of the closed `Loop`s with an open `Strand` (accomplished by `openFirstStrand`):

```
>>> toRVT . openFirstStrand $ link 4 True 1
RVT [Strand [1,2,3,4],Loop [5,6,7,8]]
    [Xm 1 6,Xm 3 8,Xm 5 2,Xm 7 4]
    [(5,-1),(6,1),(8,1)]
```

Reading off the final line, we see that arc 5 has rotation number -1, arcs 6 and 8 have rotation number 1, and the rest of the arcs have rotation number 0.

Implementation

We begin with a series of imports of common functions, relating to list manipulations and type-wrangling. The exact details are not too important.

```
{-# LANGUAGE DeriveFunctor #-}
module KnotTheory.PD where
import Data.Maybe (listToMaybe, catMaybes, mapMaybe, fromMaybe,
 ↪ fromJust)
import Data.List (find, groupBy, sortOn, partition, intersect,
 ↪ union, (\\))
import Data.Tuple (swap)
import Data.Function (on)
import Control.Monad ((>=>))
import Control.Arrow ((>>>))
```

Next, we introduce the crossing type, which can be either positive `Xp` or negative `Xm` (using the mnemonic "plus" and "minus"):

```
type Index = Int
data Xing i = Xp i i | Xm i i -- | Xv i i
  deriving (Eq, Show, Functor)
```

We define several functions which extract basic data from a crossing.

```
sign :: (Integral b) => Xing Index -> b
sign (Xp _ _) = 1
sign (Xm _ _) = -1

isPositive :: Xing i -> Bool
isPositive (Xp _ _) = True
isPositive (Xm _ _) = False

isNegative :: Xing i -> Bool
isNegative (Xp _ _) = False
isNegative (Xm _ _) = True

overStrand :: Xing i -> i
overStrand (Xp i _) = i
overStrand (Xm i _) = i

underStrand :: Xing i -> i
underStrand (Xp _ i) = i
underStrand (Xm _ i) = i
```

Next, we introduce the notion of a planar diagram, whose data is comprised of a collection of `Strand`s and `Loop`s (indexed by some type `i`, typically an integer). The `Skeleton` of a planar diagram is defined to be the collection of `Component`s, each of which is either an open `Strand` or a closed `Loop`.

```
type Strand i = [i]
type Loop i = [i]
data Component i = Strand (Strand i) | Loop (Loop i)
  deriving (Eq, Show, Functor)

type Skeleton i = [Component i]
```

Next, we introduce the notion of a `KnotObject`, which has its components labelled by the same type `i`. We further define a function `toRVT` which converts a generic `KnotObject` into an upright tangle (in this codebase, the term Rotational Virtual Tangle (RVT) is frequently used for the notion of an upright tangle). We call an object a <u>planar diagram</u> (or **PD**) if it has a notion of **Skeleton** and a collection of crossings.

```
class KnotObject k where
  toSX  :: (Ord i) => k i -> SX i
  toRVT :: (Ord i) => k i -> RVT i
  toRVT = toRVT . toSX

class PD k where
  skeleton :: k i -> Skeleton i
  xings :: k i -> [Xing i]

```

The **SX** form of a diagram just contains the **Skeleton** and the **Xing**s (crossings), while the **RVT** form also assigns each arc an integral rotation number.

```
data SX  i = SX  (Skeleton i) [Xing i] deriving (Show, Eq, Functor)
data RVT i = RVT (Skeleton i) [Xing i] [(i,Int)] deriving (Show, Eq, Functor)
```

Given any labelling of the arcs in a diagram, we can re-label the arcs using consecutive whole numbers. This is accomplished with `reindex`:

```
reindex :: (PD k, Functor k, Eq i) => k i -> k Int
reindex k = fmap (fromJust . flip lookup table) k
  where
    table = zip (skeletonIndices s) [1..]
    s = skeleton k
```

Most importantly, we now declare that a diagram expressed in **SX** form (that is, without any rotation data) may be assigned rotation numbers to each of its arcs in a meaningful way. The bulk of the work is done by `getRotNums`, which is defined farther below. We handle the case where the entire tangle is a single crossingless strand separately.

```
instance KnotObject SX where
  toSX = id
  toRVT k@(SX cs xs) = RVT cs xs rs where
    rs = filter ((/=0) . snd) . mergeBy sum $ getRotNums k f1
```

```
    i1 = head . toList $ s
    Just s = find isStrand cs
    f1 = case next i1 (toList s) of
             Just _  -> [(Out,i1)]
             Nothing -> []

instance KnotObject RVT where
  toRVT = id
  toSX (RVT s xs _) = SX s xs

instance PD SX where
  skeleton (SX s _) = s
  xings (SX _ xs)   = xs

instance PD RVT where
  skeleton (RVT s _ _) = s
  xings (RVT _ xs _)   = xs
```

Next, we include a series of functions which answer basic questions about planar diagrams. Note in `rotnum`, if a rotation number is not present in the table of values, it is assumed to be 0.

```
rotnums :: RVT i -> [(i,Int)]
rotnums (RVT _ _ rs) = rs

rotnum :: (Eq i) => RVT i -> i -> Int
rotnum k i = fromMaybe 0 . lookup i . rotnums $ k

isStrand :: Component i -> Bool
isStrand (Strand _) = True
isStrand _          = False

isLoop :: Component i -> Bool
isLoop (Loop _) = True
isLoop _        = False

toList :: Component i -> [i]
toList (Strand is) = is
```

```
toList (Loop is)   = is

skeletonIndices :: Skeleton i -> [i]
skeletonIndices = concatMap toList

involves :: (Eq i) => Xing i -> i -> Bool
x `involves` k = k `elem` [underStrand x, overStrand x]

otherArc :: (Eq i) => Xing i -> i -> Maybe i
otherArc x i
  | i == o      = Just u
  | i == u      = Just o
  | otherwise = Nothing
  where o = overStrand x
        u = underStrand x

next :: (Eq i) => i -> Strand i -> Maybe i
next e = listToMaybe . drop 1 . dropWhile (/= e)

prev :: (Eq i) => i -> Strand i -> Maybe i
prev e = next e . reverse

nextCyc :: (Eq i) => i -> Loop i -> Maybe i
nextCyc e xs = next e . take (length xs + 1). cycle $ xs

prevCyc :: (Eq i) => i -> Loop i -> Maybe i
prevCyc e xs = prev e . take (length xs + 1). cycle $ xs

isHeadOf :: (Eq i) => i -> [i] -> Bool
x `isHeadOf` ys = x == head ys

isLastOf:: (Eq i) => i -> [i] -> Bool
x `isLastOf` ys = x == last ys

nextComponentIndex :: (Eq i) => i -> Component i -> Maybe i
nextComponentIndex i (Strand is) = next i is
```

```haskell
nextComponentIndex i (Loop is) = nextCyc i is

prevComponentIndex :: (Eq i) => i -> Component i -> Maybe i
prevComponentIndex i (Strand is) = prev i is
prevComponentIndex i (Loop is) = prevCyc i is

isHeadOfComponent :: (Eq i) => i -> Component i -> Bool
isHeadOfComponent _ (Loop   _ ) = False
isHeadOfComponent i (Strand is) = i `isHeadOf` is

isLastOfComponent :: (Eq i) => i -> Component i -> Bool
isLastOfComponent _ (Loop   _ ) = False
isLastOfComponent i (Strand is) = i `isLastOf` is

isTerminalOfComponent :: (Eq i) => Component i -> i ->  Bool
isTerminalOfComponent c i = i `isHeadOfComponent` c || i `isLastOfComponent` c

isTerminalIndex :: (Eq i) => Skeleton i -> i ->  Bool
isTerminalIndex cs i = any (`isTerminalOfComponent` i) cs

nextSkeletonIndex :: (Eq i) => Skeleton i -> i -> Maybe i
nextSkeletonIndex s i = listToMaybe . mapMaybe (nextComponentIndex i) $ s

prevSkeletonIndex :: (Eq i) => Skeleton i -> i -> Maybe i
prevSkeletonIndex s i = listToMaybe . mapMaybe (prevComponentIndex i) $ s
```

In order to obtain all the crossing indices, we must take every combination of the under- and over-strands and their following indices:

```haskell
getXingIndices :: (Eq i) => Skeleton i -> Xing i -> [i]
getXingIndices s x = catMaybes
        [ f a | f <- [id, (>>= nextSkeletonIndex s)], a <- [o, u] ]
        where o = return (overStrand x)
              u = return (underStrand x)
```

```haskell

δ :: (Eq a) => a -> a -> Int
δ x y
  | x == y    = 1
  | otherwise = 0

mergeBy :: (Ord i) => ([a] -> b) -> [(i,a)] -> [(i,b)]
mergeBy f = map (wrapIndex f) . groupBy ((==) `on` fst) . sortOn fst
  where
    wrapIndex :: ([a] -> b) -> [(i,a)] -> (i,b)
    wrapIndex g xs@(x:_) = (fst x, g . map snd $ xs)
```

Here we come to the main function, `getRotNums`, for which we have the following requirements (not expressed in the code):

1. The diagram `k` is a $(1, n)$-tangle (a tangle with only one open component)
2. The underlying graph of `k` is a planar.
3. The diagram `k` is a connected.

Only in this case will the function `toRVT` will then output a planar $(1, n)$-upright tangle which corresponds to a classical (i.e. planar) diagram.

This function involves taking a simple open curve (a Jordan curve passing through infinity) called the `Front`, and passing it over arcs in the diagram. This curve is characterized by the arcs it passes through, together with their orientations. Each intersection of the `Front` with the diagram provides a different `View`, either `In` or `Out` of the `Front` when following the orientation of the intersecting arc.

```haskell
type Front i = [View i]
type View  i = (Dir, i)
```

We obtain the rotation numbers by successively passing the front across new crossings (achieved by `advanceFront`), keeping track of the rotation numbers of arcs which have already passed by the front. Once the front has passed across every crossing, all the rotation numbers have been computed.

Next, we define `converge`, which iterates a function until a fixed point is achieved.

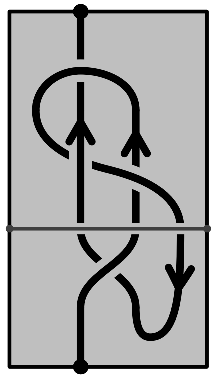

Figure A.4: A tangle with a front passing over it. The portion of the tangle below the front is in upright form.

```
converge :: (Eq a) => (a -> a) -> a -> a
converge f x
        | x == x'   = x
        | otherwise = converge f x'
        where x' = f x
```

The function `convergeT` wraps `converge` in monadic transformations. In our context, the monad will be used to keep track of rotation numbers of the arcs.

```
convergeT :: (Monad m, Eq (m a)) => (a -> m a) -> a -> m a
convergeT f = return >>> converge (>>= f)
```

The implementation of `getRotNums` takes a front and advances it along a diagram until no more changes occur.

```
getRotNums :: (Eq i) => SX i -> Front i -> [(i,Int)]
getRotNums k = convergeT (advanceFront k) >>> fst
```

When advancing the **`Front`**, we start by absorbing arcs that intersect with the front twice until the leftmost **`View`** no longer connects directly back to the **`Front`**. At this point, we can absorb a crossing into the front.

```
advanceFront :: (Eq i) => SX i -> Front i -> ([(i,Int)], Front
 ↪  i)
advanceFront k = convergeT (absorbArc k) >=> absorbXing k
```

We next check for the case where the leftmost arc connects back to the **`Front`**. If it is pointing **`Out`** (and therefore connects back **`In`** further to the right), we adjust the rotation number of the arc by -1. Otherwise, we leave both the `Front` and the rotation numbers unchanged.

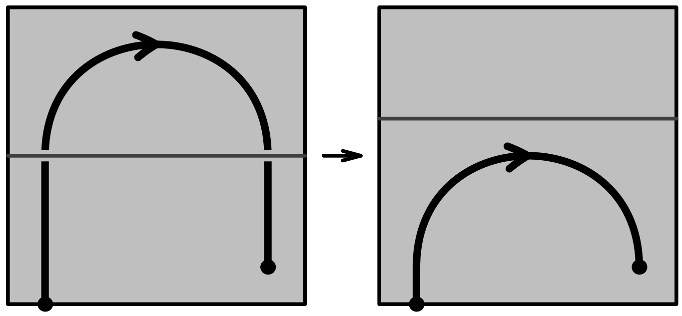

Figure A.5: Example of absorbing an arc which intersects the front multiple times. If the horizontal tangent vector points to to right, as in this picture, then the rotation number of the arc is decreased by 1. Otherwise, no change in the rotation number is recorded.

```
absorbArc :: (Eq i) => SX i -> Front i -> ([(i,Int)],Front i)
absorbArc k []      = return []
absorbArc k f@(f1:fs) = case fs1 of
        ( In,i):_ -> (return (i,-1), fss)
        (Out,i):_ -> return fss           -- No new rotation numbers
        []        -> return f
        where (fs1,fss) = partition (((==) `on` snd) f1) fs
```

Our goal is to repeat this operation until we get a fixed point, which is encoded in `absorbArcs`:

```
absorbArcs :: (Eq i) => SX i -> Front i -> ([(i,Int)],Front i)
absorbArcs k = convergeT (absorbArc k)
```

Absorb a crossing involves expanding one's view at an arc from looking at a crossing to all the views one gets when looking in every direction at the crossing (namely, to the left, along the arc, and to the right). The function `absorbXing` performs this task on the leftmost `View` on the `Front`. The transverse strand receives a positive rotation number if it moves from left to right. The arc receiving the rotation depends on how the crossing is oriented.

```
absorbXing :: (Eq i) => SX i -> Front i -> ([(i,Int)],Front i)
absorbXing _ [] = return []
```

```
absorbXing k (f:fs) = (rs,newFront++fs) where
        newFront = catMaybes [l, a, r]
        l = lookLeft k f
        a = lookAlong k f
        r = lookRight k f
        rs = case (l,f,r) of
                (Just (In,i), (Out,_),_           ) -> [(i,1)]
                (_          , (In ,_),Just (Out, j)) -> [(j,1)]
                _                                   -> [     ]

data Dir = In | Out
  deriving (Eq, Show)
```

The following functions take a **View**, returning the **View** one has when looking in the corresponding direction. Since it is possible for the resulting gaze to be merely the boundary, it is possible for these functions to return **Nothing**.

```
lookAlong :: (Eq i, PD k) => k i -> View i -> Maybe (View i)
lookAlong k (d, i) = case d of
        Out -> sequence (Out, nextSkeletonIndex s i)
        In  -> sequence (In , prevSkeletonIndex s i)
        where s = skeleton k

lookSide :: (Eq i, PD k) => Bool -> k i -> View i -> Maybe
 ↪  (View i)
lookSide isLeft k di@(Out,i) = do
        x <- findNextXing k di
        j <- otherArc x i
        if isLeft == ((underStrand x == i) == isPositive x)
        then return (In, j)
        else sequence (Out, nextSkeletonIndex (skeleton k) j)
lookSide isLeft k (In,i) =
        sequence (Out, prevSkeletonIndex (skeleton k) i) >>=
        lookSide (not isLeft) k

lookLeft  :: (Eq i, PD k) => k i -> View i -> Maybe (View i)
lookLeft = lookSide True
```

```
lookRight :: (Eq i, PD k) => k i -> View i -> Maybe (View i)
lookRight = lookSide False

findNextXing :: (Eq i, PD k) => k i -> View i -> Maybe (Xing i)
findNextXing k (Out,i) = find (`involves` i) $ xings k
findNextXing k (In ,i) = do
  i' <- prevSkeletonIndex (skeleton k) i
  find (`involves` i') $ xings k
```

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COLOPHON

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