proj1_b

September 15, 2016

Note: this is a continuation of Part A of this project's documentation, which should be included alongside this document. Please refer to that document for project boilerplate, execution instructions and requirements, and other projectwide information.

0.1 Part B – Errors in a Forward Finite Difference Approximation

Part B of this project concerns itself with forward difference estimates, used to approximate the derivative for a function at a point, given the function's zeroth derivative (base function) and an "offset," which is effectively used as a Δx value, such that as it becomes smaller, it approaches dx, and the value of $\delta^+ f(x)$ approaches $f'(x)$.

0.1.1 Goals

In this instance, we are given the function $f(x) = x^{-3}$ where $x = 3$, and we are to approximate $f'(3)$ without being able to directly use the derivative function. The increment $h = 2^{-n}$ and we iterate $n = 1 \ldots 52$ such that n shrinks, thus allowing $\delta^+ f(3)$ to approximate $f'(3)$.

0.1.2 Math Background

Forward Finite Difference Estimate The forward finite difference estimate as a means of approximating a derivative is defined as follows:

$$\delta^+ f(a) = \frac{f(a+h) - f(a)}{h}$$

Where a is the derivative value to calculate, and h is the increment (the smaller, the better).

0.1.3 Implementation Requirements

The requirements for this program, once parsed from the source documentation, are extremely straightforward:

Input Functions The function $f(x)$, as well as $f'(x)$ and $f''(x)$, ought to be implemented somewhere in the codebase for FDE and error calculation.

$$f(x) = x^{-3},\ f'(x) = -3x^{-4},\ f''(x) = 12x^{-5}$$

Forward Difference Estimate The FDE's arithmetic should be implemented within the codebase to replicate the math as described above.

Error Calculation The relative error function should be implemented as described:

$$r = \left| \frac{f'(a) - \delta_{DP}^{+} f(a)}{f'(a)} \right|$$

The upper bound of relative error in the FDE approximating $f'(a)$ was given as follows:

$$R = c_1 h + c_2 \frac{1}{h},\ c_1 = \left| \frac{f''(a)}{2f'(a)} \right|,\ c_2 = \left| \frac{f(a)\epsilon_{DP}}{f'(a)} \right|$$

0.1.4 Implementation

Input Functions These are implemented as `f`, `f1`, and `f2` in the C++ source.

Forward Difference Estimate FDE is implemented as an extremely simple arithmetic function within the project part B file. Refer to `forwardDifferenceEstimate` in the C++ source below.

Error Calculation Relative error calculation is implemented as a simple arithmetic function in `relativeError` in the source.

The upper bound function is similarly available at `relativeErrorUpperBound`.

```
In [1]: %cat ../src/proj1_b.cpp
```

```
//
//  proj1_b.cpp
//  HPSCProject1
//
//  Created by Paul Herz on 9/3/16.
//  Copyright © 2016 Paul Herz. All rights reserved.
//

#include <iostream>
#include <cmath>

#include "Vector.h"

using namespace PH;


// System double-precision epsilon.
// determined to be 2^-52 during testing.
double EDP = __DBL_EPSILON__;


double f(double x) {
        return std::pow(x, -3.0);
}
```

```
double f1(double x) {
        // first derivative of f(x)
        return -3.0 * std::pow(x, -4.0);
}


double f2(double x) {
        // second derivative of f(x)
        return 12 * std::pow(x, -5.0);
}


double forwardDifferenceEstimate(double a, double h) {
        // for a sufficiently smooth function f(x), the F.D.E. of f'(a) is def.
        // δ+f(a) = [f(a+h)-f(a)] / h

        return (f(a+h) - f(a)) / h;
}


double relativeError(double a, double h) {
        // relative DP approximation error δ+DP f ~ f'(a)
        // r = |[f'(a)-δ+f(a)]/f'(a)|
        auto f1a = f1(a);
        auto fde = forwardDifferenceEstimate(a, h);

        return std::abs((f1a-fde)/f1a);
}


double relativeErrorUpperBound(double a, double h) {
        // the upper bound on the relative error r is
        // R = c_1 * h + c_2 * (1/h)
        //
        // where c_1 = |[f"(a)]/[2*f'(a)]|,
        //   and c_2 = |[f(a)*ϵ_DP]/[f'(a)]|

        auto fa = f(a);
        auto f1a = f1(a);
        auto f2a = f2(a);

        auto c_1 = std::abs((f2a)/(2*f1a));
        auto c_2 = std::abs((fa*EDP)/(f1a));

        return c_1 * h + c_2 * (1.0/h);
}
```

```
int main(int argc, const char * argv[]) {

        // for each of these increment values, compute:
        // • the relative error r in your approximation
        // • the upper bound on the relative error, R,
        //   from the derivation in (7)


        // GOAL: compute the forward difference estimates δ+f(3) (see 1) of
        // f'(3), with the sequence of increments h.
        //
        // f(x) = x^-3
        // h = 2^-n (n = 1, 2, 3, ..., 52)

        auto n = Vector::linSpace(1, 52, 52);
        Vector h = 2 ^ -n;

        auto a = 3.0;
        Vector estimates(52), r(52), R(52);

        // for each increment value `h_i`...
        for(Index i = 0; i < h.size(); ++i) {
                // ...calculate relative error `r` in your approximation...
                r[i] = relativeError(a, h[i]);

                // ...and the upper bound on `r`: `R`.
                R[i] = relativeErrorUpperBound(a, h[i]);

        }

        // save n, h, r, and R as
        // n.txt, h.txt, r.txt, and R.txt.
        std::string prefix = "../data/b/";
        n.saveTo(prefix + "n.txt");
        h.saveTo(prefix + "h.txt");
        r.saveTo(prefix + "r.txt");
        R.saveTo(prefix + "bigr.txt");

        return 0;
}
```

```
In [2]: %pylab inline
        pylab.rcParams['figure.figsize'] = (10, 6)
        matplotlib.rcParams.update({'font.size': 16})
        matplotlib.rcParams.update({'axes.labelsize': 20})
        matplotlib.rcParams.update({'xtick.labelsize': 12})
        matplotlib.rcParams.update({'ytick.labelsize': 12})
```

```
        matplotlib.rcParams.update({
                'font.family': 'Helvetica, Arial, sans-serif'
        })

        %config InlineBackend.figure_format = 'retina'
```

```
Populating the interactive namespace from numpy and matplotlib
```

In [3]:
```
        names = ['n', 'h', 'r', 'bigr']
        v = {}
        for name in names:
            v[name] = loadtxt('../data/b/' + name + '.txt')
```

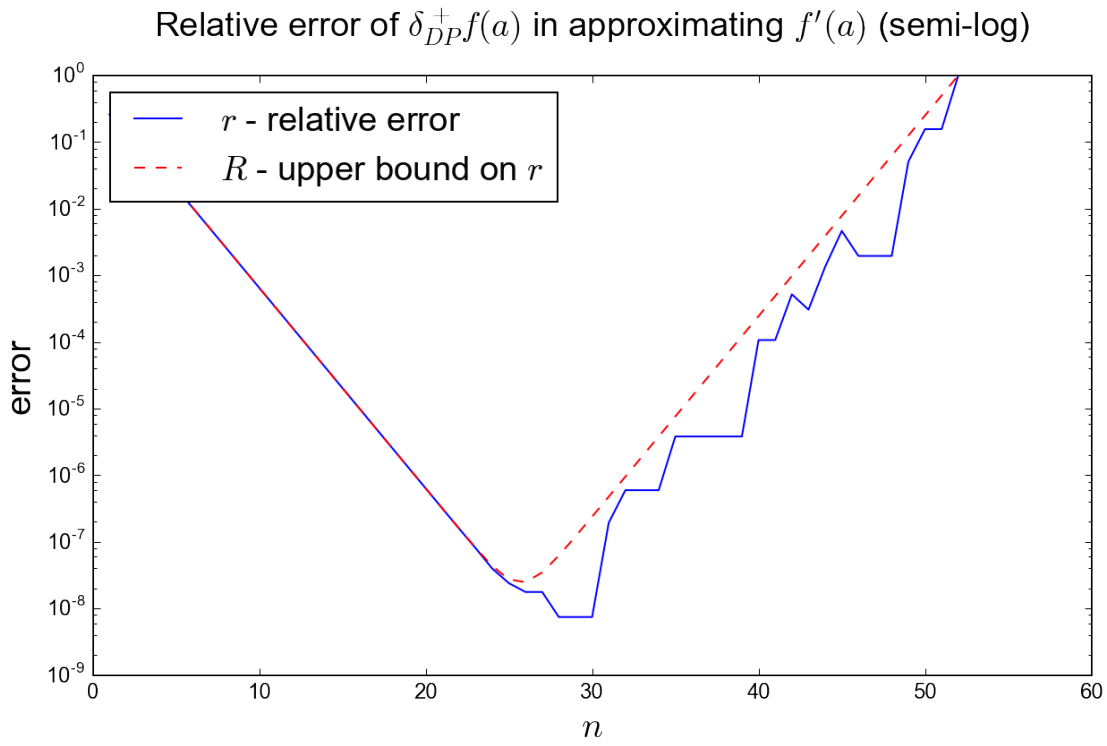
In [4]:
```
        # create a semilogy plot that overlays r versus n with
        # a solid blue line, and R versus n with a red dashed
        # line...

        pylab.semilogy(v['n'], v['r'], '-b')
        pylab.semilogy(v['n'], v['bigr'], '--r')

        pylab.legend(('$r$ - relative error',
                      '$R$ - upper bound on $r$'), loc=2)

        pylab.xlabel('$n$')
        pylab.ylabel('error')
        pylab.title('Relative error of $\delta^+_{DP}f(a)$ in approximating $f\'(a)
```

Out[4]:
```
<matplotlib.text.Text at 0x10f3a9208>
```



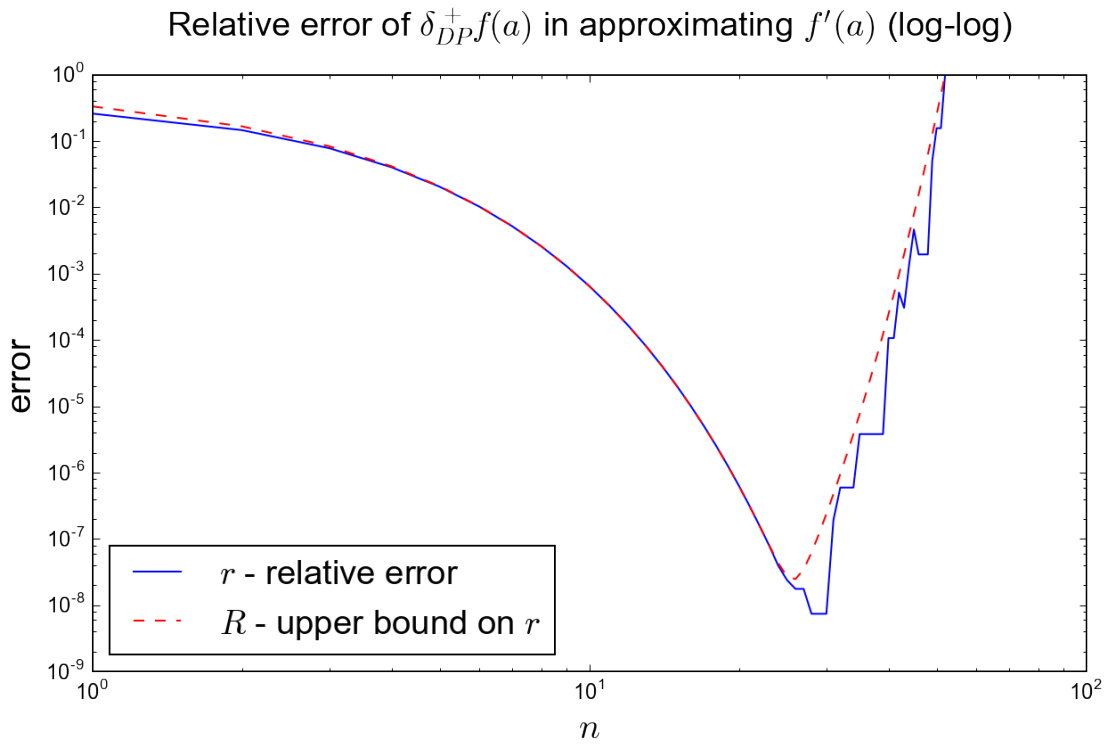

```
In [5]: # create a different loglog plot that overlays
        # r versus h with a solid blue line, and
        # R versus h with a red dashed line...

        pylab.loglog(v['n'], v['r'], '-b')
        pylab.loglog(v['n'], v['bigr'], '--r')

        pylab.legend(('$r$ - relative error',
                      '$R$ - upper bound on $r$'), loc=3)

        pylab.xlabel('$n$')
        pylab.ylabel('error')
        pylab.title('Relative error of $\delta^+_{DP}f(a)$ in approximating $f\'(a)
```

```
Out[5]: <matplotlib.text.Text at 0x10fb73550>
```



0.1.5 Analysis

As the integer n increments, and h as such decreases, the accuracy of the FDE improves until n reaches about the $3/4$ point of the interval, and h is still decreasing but not at its smallest value yet. At sufficiently small values of h, there may be a loss-of-significance error at any number of steps including: (1) the evaluation of h itself, (2) the evaluation of $a + h$ relative to a, or *especially* (3) division by h, which causes to grow exponentially which may outpace the numerator of the FDE function when loss-of-significance occurs.

0.1.6 Conclusion

The arithmetic for this section was simple once past understanding the theory of FDEs and the relatively advanced error calculation when considering double point error, etc., but was a strong demonstration of the importance of loss-of-significance errors and similar computing issues that arise in such calculations, and how such factors influence mathematical decisions (such as selection of n) compared to such decisions made in a pen-and-paper math environment.