BLACK-SCHOLES MODEL

HOW WE INTERPRET AND DEPLOY BLACK-SCHOLES MODEL FOR CW VALUATION AND RISK HEDGING

DERIVATION OF BLACK-SCHOLES

DERIVATION

Black-Scholes model assumes that log-return of underlying stock follows normal distribution → stock price follows log-normal distribution. From that:

$$\frac{dS}{S} = \mu dt + \sigma dW \quad [1]$$

where W is a Wiener process, which is characterized by the following notable properties:

- W has Gaussian increments: $W_{t+\Delta t} - W_t \sim \mathcal{N}(0, \Delta t) \quad [2]$
- W has independent increments: $W_{t+\Delta t} - W_t \amalg W_{t'}$ for any $t' < t \quad [3]$

Continue with (1):

$$E\left(\frac{dS}{S}\right) = E(\mu dt + \sigma dW) = E(\mu dt) + E(\sigma dW) = \mu dt \ \ (from\ [2])$$

$$Var\left(\frac{dS}{S}\right) = Var(\mu dt + \sigma dW) = Var(\mu dt) + Var(\sigma dW) = \sigma^2 dt \ \ (from\ [2])$$

DERIVATION

Since V(S,t) can be extended by Itô drift-diffusion process, it follows Itô lemma on 2-variable functions:

$$dV(S,t) = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2\right) dt + \sigma S \frac{\partial V}{\partial S} dW \quad [4]$$

Consider the delta-hedge portfolio in which we short one option and long $\frac{\partial V}{\partial S}$ shares. The value of these holdings is:

$$\Pi = -V + \frac{\partial V}{\partial S} S \quad [5]$$

Over an infinitesimal period of time Δt, the profit and loss is:

$$\Delta\Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S \quad [6]$$

DERIVATION

Discretize [1] and [4]:

$\Delta S = \mu S \Delta t + \sigma S \Delta W$ [7]

$$\Delta V(S,t) = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2\right)\Delta t + \sigma S \frac{\partial V}{\partial S}\Delta W \quad [8]$$

Substitute [7] and [8] into [6]:

$$\Delta \Pi = -\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2\right)\Delta t - \sigma S \frac{\partial V}{\partial S}\Delta W + \frac{\partial V}{\partial S}(\mu S \Delta t + \sigma S \Delta W)$$

$$\Leftrightarrow \Delta \Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2\right)\Delta t \quad [9]$$

Uncertainties surrounding W has vanished, these holdings have determined profit and hence is actually risk-free

DERIVATION

The risk-free profit [9] must equal to any other risk-free holdings. Denoting risk-free return by r, the following equation must hold:

$$r\Pi\Delta t = \Delta\Pi \quad [10]$$

Substitute [5] and [9] to [10]:

$$r\left(-V + \frac{\partial V}{\partial S}S\right)\Delta t = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\Delta t$$

$$\Leftrightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\frac{\partial V}{\partial S} - rV = 0 \quad [11]$$

The partial differential equation [11] is called "Black-Scholes Equation". Our initial problem is reduced to solving for V in [11] as a function of S and t.

DERIVATION

The boundary conditions of [11] are:

$$\begin{cases} V(0,t) = 0 \; for \; all \; t \\ V(S,t) \longrightarrow S \; as \; S \longrightarrow \infty \\ V(S,T) = \max\{S-K, 0\} \end{cases}$$

Equation [11] can be transformed to standard diffusion equation by change-of-variable technique:

$\tau = T - t$

$u = Ve^{r\tau}$

$x = ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)r$

Then [11] becomes:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} \qquad [12]$$

DERIVATION

Solution of diffusion equation [12] has been proved for its existence and uniqueness, which has the form of:

$$u(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u_0(y)\, exp\left[-\frac{(x-y)^2}{2\sigma^2\tau}\right] dy \quad [13]$$

in which:

$u_0(y) \coloneqq K(e^{\max\{x,0\}-1})$

Solution [13] $\Rightarrow u(x,\tau) = Ke^{x+\frac{1}{2}\sigma^2\tau} N(d_1) - KN(d_2) \quad [14]$

where $N(\cdot)$is the standard normal cumulative distribution function and

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\left(x+\frac{1}{2}\sigma^2\tau\right)+\frac{1}{2}\sigma^2\tau\right] ; d_2 = \frac{1}{\sigma\sqrt{\tau}}\left[\left(x+\frac{1}{2}\sigma^2\tau\right)-\frac{1}{2}\sigma^2\tau\right]$$

DERIVATION

Revert back to the original variables and realize the fact that V is actually the price of a call option (denoted by C), we derive to Black-Scholes formula:

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)} \quad [15]$$

where:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right];$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

DELTA-GAMMA HEDGING

DELTA-GAMMA HEDGING

In order to delta-gamma hedge a position, we firstly need to create a Π neutral portfolio.
It must be clear that this objective cannot be accomplished by simply trading underlying stock since $\Pi_{stock} = \frac{\partial^2 S}{\partial S^2} = 0$. As a result, another option with non-zero Π is required.

Let the price of such option be denoted by $\tilde{V}(S,t)$. Then:

$$\tilde{\Delta}(S,t) = \frac{\partial}{\partial S}\tilde{V}(S,t);\ \tilde{\Pi}(S,t) = \frac{\partial^2}{\partial S^2}\tilde{V}(S,t)$$

Let $\tilde{n}(S,t)$ be the number of these options to hold at time t with underlying stock price S.
Let $n(S,t)$ be the number of underlying shares to hold

Suppose we need to hedge n_0 number of options. To have Π neutrality, it must satisfy:

$$n_0\Pi(S,t) + \tilde{n}(S,t)\tilde{\Pi}(S,t) = 0$$

with $\Pi(S,t) = \frac{\partial^2}{\partial S^2}V(S,t)$;

$$\Rightarrow \tilde{n}(S,t) = -n_0\frac{\Pi(S,t)}{\tilde{\Pi}(S,t)}$$

DELTA-GAMMA HEDGING

To establish Δ neutrality, we need to trade underlying stock. Let $\mathrm{n}(S, t)$ be the required number of shares. It must satisfy:

$$n_0(\mathrm{S}, t)\Delta(S, t) + \tilde{n}(S, t)\tilde{\Delta}(S, t) + n(S, t) = 0$$

$$\Rightarrow n(S, t) = -\tilde{n}(S, t)\tilde{\Delta}(S, t) - n_0(\mathrm{S}, t)\Delta(S, t)$$

In conclusion, to delta-gamma hedge a position of n_0 option, we need to maintain:

$$\begin{cases} Number\ of\ equivalent\ options\colon\ \tilde{n}(S, t) = -n_0 \dfrac{\Pi(S, t)}{\tilde{\Pi}(S, t)} \\ Number\ of\ underlying\ stocks\colon\ \ n(S, t) = -\tilde{n}(S, t)\tilde{\Delta}(S, t) - n_0(S, t)\Delta(S, t) \end{cases}$$

EXAMPLE

For two Covered Warrants (or European call options), CW-I and CW-II, on FPT. Our calculation provides:

	CW-I	CW-II
Delta	0.5825	0.7773
Gamma	0.0651	0.0746

Suppose Phu Hung Securities sold 100 units of CW-I
Determine the numbers of CW-II and FPT Phu Hung must buy or sell in order to both delta-hedge and gamma-hedge the position in CW-I.

EXAMPLE

<u>Solution:</u>

Phu Hung <u>sold</u> 100 CW-I $\Rightarrow n_0 = -100$

Required number of CW-II:

$$\tilde{n}(S,t) = -n_0 \frac{\Pi(S,t)}{\tilde{\Pi}(S,t)} = -(-100)\frac{0.0651}{0.0746} = 87.2654$$

Required number of FPT shares:

$$n(S,t) = -\tilde{n}(S,t)\tilde{\Delta}(S,t) - n_0(S,t)\Delta(S,t) = -87.2654 \times 0.7773 - (-100) \times 0.5825 = -9.5814$$

Hence, Phu Hung must buy 87.2654 units of CW-II and sell 9.5814 units of FPT shares

EXAMPLE

Check:

Delta of total portfolio:
$\Delta_{total} = n_0(S,t)\Delta(S,t) + \tilde{n}(S,t)\tilde{\Delta}(S,t) + n(S,t) = -100 \times 0.5825 + 87.2654 \times 0.7773 - 9.5814 = 0$

Gamma of total portfolio:
$\Pi_{total} \ = n_0\Pi(S,t) + \tilde{n}(S,t)\tilde{\Pi}(S,t) = -100 \times 0.0651 + 87.2654 \times 0.0746 = 0$

Hence, after delta-gamma hedging the position, Phu Hung exposes to zero risk of <u>change in stock price</u> and <u>change in rate of change of stock price</u>

DELTA-GAMMA HEDGING

As can be seen in the above example, there is a single solution to completely hedge both Δ and Π risk. In practice, particularly in such emerging market as Vietnam, the derivative market is not always liquid enough for issuers to trade on derivative assets; furthermore, going short on underlying stocks is yet allowed in Vietnam. Hence, a single hard solution produced by delta-gamma hedging strategy might turn impossible at some point of time during the holding period. As the matter of this fact, we are encouraged to relax our model for the sake of feasibility with the trade-off of giving up some certain absolute neutrality of our holding. In order words, we allow ourselves to expose to some degrees of Π risk.

DELTA-GAMMA HEDGING

The Π–neutrality equation could then be adjusted as:

$$n_0\Pi(S,t) + \tilde{n}(S,t)\tilde{\Pi}(S,t) \le \alpha n_0 \quad \text{(where } 0 \le \alpha \le 1) \quad [*]$$

in which: α is some small number that represents the maximal Π risk we must take.

$$[*] \;\Rightarrow \tilde{n}(S,t) \le n_0 \frac{\alpha - \Pi(S,t)}{\tilde{\Pi}(S,t)} \quad (\text{since } \tilde{\Pi}(\mathrm{S},\mathrm{t})>0)$$

To avoid having to short the covered warrants, it requires that $\tilde{n}(S,t) \ge 0 \Rightarrow \alpha \le \Pi(S,t)$ (since $n_0 < 0$ for issuers)

criteration1

DELTA-GAMMA HEDGING

From the Δ–neutrality equation, we arrive at:

$$n(S,t) = -n_0\Delta(S,t) - \tilde{n}(S,t)\widetilde{\Delta}(S,t) \geq -n_0\Delta(S,t) - n_0\frac{[\alpha - \Pi(S,t)]}{\widetilde{\Pi}(S,t)}\,\widetilde{\Delta}(S,t)$$

It also requires that $n(S,t) \geq 0$, to ensure this condition is satisfied, let:

$$-n_0\Delta(S,t) - n_0\frac{[\alpha - \Pi(S,t)]}{\widetilde{\Pi}(S,t)}\,\widetilde{\Delta}(S,t) \geq 0 \;\Leftrightarrow\; -\Delta(S,t) - \frac{\alpha - \Pi(S,t)}{\widetilde{\Pi}(S,t)} \leq 0 \quad (\text{since } \mathrm{n}_0 < 0 \text{ for issuers})$$

$\Rightarrow \alpha \geq \Pi(S,t) - \Delta(S,t)\tilde{n}(S,t)$ criterion 2

DELTA-GAMMA HEDGING

In order to simultaneously satisfy criterion 1 and criterion 2, it requires:

$$\Pi(S,t) - \Delta(S,t)\widetilde{\Pi}(S,t) \leq \alpha \leq \Pi(S,t)$$

If we chose $\alpha = \alpha_0$, then:

$$\tilde{n}(S,t) = n_0 \frac{\alpha_0 - \Pi(S,t)}{\widetilde{\Pi}(S,t)}; \;\; n(S,t) = -\, n_0\, \Delta(S,t) - \tilde{n}(S,t)\widetilde{\Delta}(S,t)$$

DELTA-GAMMA HEDGING

Get back to the earlier example on FPT, the table below presents feasible solutions with varying α.

	CW-I	CW-II
Delta	0.5825	0.7773
Gamma	0.0651	0.0746

minimum alpha	0.0216
maximum alpha	0.0651

n0	-100

alpha	n_FPT	n_CWII
0.0216	13.0	58.3
0.0250	16.5	53.8
0.0300	21.7	47.1
0.0350	26.9	40.3
0.0400	32.1	33.6
0.0450	37.3	26.9
0.0500	42.5	20.2
0.0550	47.7	13.5
0.0600	52.9	6.8
0.0650	58.1	0.1
0.0651	58.3	0.0

THANK YOU

SOMEONE@EXAMPLE.COM