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A conditioned distance ratio method for analyzing spatial patterns

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SUMMARY

A new distance-based method is proposed for investigating the pattern in the plane formed by points, which may be assumed to be the positions of centres of trees in a forest stand. For each of a number of randomly selected origins the distance from the origin to the nearest tree, and the distance of this tree to its nearest neighbour are measured, and their ratio calculated. The values of this ratio which exceed one-half are shown to have the property of distinguishing between regular, random and clustered patterns; the values which are less than one-half are shown to have the same distribution for such patterns. A number of sets of data on tree positions are analyzed.

Some key words: Aggregation; Distance method; Forest stand; Lattice pattern; Nearest neighbour; Poisson forest; Two-dimensional point process.

1. INTRODUCTION

The estimation of the density of a forest stand, or plant community, using distance methods is affected by the spatial pattern of the trees, considered as points in the plane. Estimators when used for patterns different from those for which they were designed are biased, as discussed by Persson (1971). For use with such estimators one can envisage a 'scale' of pattern, ranging from: extreme regularity where trees are situated at the vertices of a triangular lattice, which is in a sense the most regular of the three types of lattice, triangular, square and hexagonal; through randomness, or a Poisson forest, where the positions of the trees form a two-dimensional Poisson process; to extreme aggregation, where the trees form tight clumps. Usually the Poisson forest is used as a starting point and departures from randomness are then considered.

The regular-random-aggregation 'scale' has obvious inadequacies as a descriptive framework, e.g. it does not accommodate a pattern composed of pairs of coincident points situated at the vertices of a triangular lattice. However, various indices relating to this scale have been introduced, the commonest of which is the variance to mean ratio for quadrat counts, which is summarized with other indices, such as Lloyd's index of mean crowding, Morisita's index of dispersion, Hopkins's and Skellam's index, etc., by Pielou (1969).

We describe in §§2, 3 and 4 a new distance method for looking at a spatial pattern with a view to density estimation. A diagnostic test of aggregation is proposed using this method and is compared with a test introduced by Holgate (1965) which has been considered with the other tests of randomness by Diggle, Besag & Gleaves (1976). In §5 we use our method of analysis on two sets of real data and one simulated set.

2. THE METHOD

2·1. *The distance measurements*

Our method of investigating the spatial pattern of the trees involves the selection of N random points in the forest, which we will call the sampling origins. From each of these origins we measure the distance, X, to the nearest tree, and then from this tree we measure the distance, Y, to the nearest neighbour. The N pairs of measurements $(X_1, Y_1), \ldots, (X_N, Y_N)$ are then split into two sets, A and B, where

$$A = \{(X_i, Y_i): Y_i \leqslant 2X_i \ (i = 1, \ldots, n)\}$$
$$B = \{(X_j, Y_j): Y_j > 2X_j \ (j = 1, \ldots, m)\} \quad (m+n = N).$$

For convenience we relabel the members of A and B as (X_{1i}, Y_{1i}) and (X_{2j}, Y_{2j}) respectively. We define the random variables W_{1i} and W_{2j} as

$$W_{1i} = 1/\{2\pi + \sin \mathrm{B}_i - (\pi + \mathrm{B}_i) \cos \mathrm{B}_i\} \quad (i = 1, \ldots, n),$$

where $\sin(\frac{1}{2}\mathrm{B}_i) = \frac{1}{2}Y_{1i}/X_{1i}$, and

$$W_{2j} = 4X_{2j}^2/Y_{2j}^2 \quad (j = 1, \ldots, m).$$

The W_{1i}'s will be used for the analysis of the spatial pattern, while the W_{2j}'s will be shown to have the same distribution for a wide range of patterns. That is, the information about spatial pattern resides essentially in the W_{1i}'s.

2·2. *The Poisson forest*

Figure 1 (*a*) illustrates the geometrical configuration for X_1 and Y_1. The random origin is O, P is the nearest tree to O, and Q is the nearest neighbour to P. Let angle POS be B.

The unconditional probability density function, p.d.f., of X is $2\lambda\pi x \exp(-\lambda\pi x^2)$ $(x \geqslant 0)$, where λ is the forest density, i.e. the rate or intensity of the Poisson process.

For $Y \leqslant 2X$ we work in terms of angle B and X_1, rather than Y_1 and X_1. For $0 \leqslant \mathrm{B} \leqslant \pi$, write

$$\mathrm{pr}\{x < X < x+dx,\ Y > 2x \sin(\tfrac{1}{2}\beta)\} = H_1(x, \beta)\, dx;$$

for $Y \geqslant 2X$, write

$$\mathrm{pr}\{x < X < x+dx,\ Y > y\} = H_2(x, y)\, dx.$$

Then

$$H_1(x, 0) = 2\lambda\pi x \exp(-\lambda\pi x^2), \quad H_1(x, \pi) = H_2(x, 2x), \quad \mathrm{pr}(Y > 2X) = \int_0^\infty H_1(x, \pi)\, dx.$$

From Figs. 1 (*a*) and (*b*),

$$H_1(x, \beta + d\beta) = H_1(x, \beta)\{1 - \lambda(\pi+\beta)\, 2x \sin(\tfrac{1}{2}\beta)\, x \cos(\tfrac{1}{2}\beta)\, d\beta\},$$
$$H_2(x, y+dy) = H_2(x, y)\,(1 - 2\lambda\pi y\, dy),$$

whence

$$H_1(x, \beta) = C_1(x) \exp[-\lambda x^2\{\sin\beta - (\pi+\beta)\cos\beta\}], \quad H_2(x, y) = C_2(x) \exp(-\lambda\pi y^2).$$

The boundary conditions give $C_1(x) = 2\lambda\pi x \exp(-2\lambda\pi x^2)$ and $C_2(x) = 2\lambda\pi x$; therefore $H_1(x, \pi) = 2\lambda\pi x \exp(-4\lambda\pi x^2)$ and $\mathrm{pr}(Y > 2X) = \frac{1}{4}$.

The joint p.d.f. of X_1 and B is thus

$$-\tfrac{4}{3}\,\partial H_1/\partial\beta = \tfrac{8}{3}\lambda^2\pi(\pi+\beta)\, x^3 \sin\beta \exp[-\lambda x^2\{2\pi + \sin\beta - (\pi+\beta)\cos\beta\}]$$
$$(0 \leqslant x \leqslant \infty,\ 0 \leqslant \beta \leqslant \pi), \qquad (1)$$

and the joint p.d.f. of X_2 and Y_2 is

$$-4\,\partial H_2/\partial y = 16\lambda^2\pi^2 xy \exp(-\lambda\pi y^2) \quad (0 \leqslant x \leqslant \infty,\ 2x < y \leqslant \infty). \qquad (2)$$

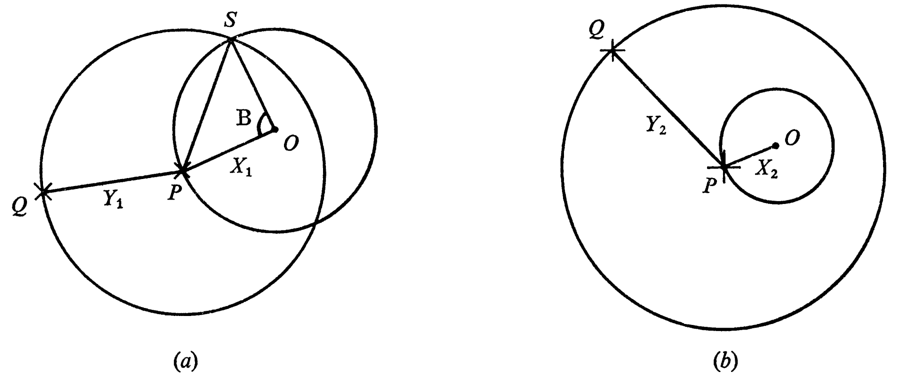

Fig. 1. The geometrical configurations (*a*) for the random variables (X_1, Y_1), and (*b*) for (X_2, Y_2). The random origin is O, the nearest tree to O is P, Q is P's nearest neighbour, and by definition $2X_1 \geqslant Y_1$ and $2X_2 < Y_2$.

If $W_1 = 1/\{2\pi + \sin \mathrm{B} - (\pi + \mathrm{B}) \cos \mathrm{B}\}$, the marginal p.d.f. of W_1 is, from (1),

$$\tfrac{8}{3}\pi\lambda^2 \int_0^\infty x^3 w^{-2} \exp(-\lambda x^2/w)\, dx \quad (\tfrac{1}{4}\pi^{-1} \leqslant w \leqslant \pi^{-1}),$$

which reduces to $\frac{4}{3}\pi$. Thus if we define $R_1 = \frac{4}{3}(1-\pi W_1)$, we have that R_1 is distributed uniformly over $[0, 1]$.

On letting $W_2 = 4X_2^2/Y_2^2$, we see from (2) that the marginal p.d.f. of W_2 is

$$32\pi^2\lambda^2 \int_0^\infty x^3 w^{-2} \exp(-4\lambda\pi x^2/w^2)\, dx \quad (0 \leqslant w \leqslant 1),$$

which reduces to unity. Thus W_2 is distributed uniformly over $[0, 1]$.

2·3. *Lattice patterns*

Consider trees occupying the vertices of a triangular lattice of side a. Obviously $Y = a$ with probability one, and the p.d.f. of X is

$$\begin{array}{ll} 4\pi x/(\surd 3a^2) & (0 \leqslant x \leqslant \tfrac{1}{2}a) \\ 4x\{\pi - 6\cos^{-1}(\tfrac{1}{2}a/x)\}/(\surd 3a^2) & (\tfrac{1}{2}a \leqslant x \leqslant a/\surd 3). \end{array}$$

Hence $\mathrm{pr}(Y > 2X) = \frac{1}{2}\pi/\surd 3 \simeq 0{\cdot}907$, giving the p.d.f. of X_2 as $8x/a^2$ $(0 \leqslant x \leqslant \frac{1}{2}a)$.

Thus W_2 is again distributed uniformly over $[0, 1]$ for the triangular lattice. Similar arguments show that $\mathrm{pr}(Y > 2X)$ is $\frac{1}{4}\pi \simeq 0{\cdot}785$ and $\frac{1}{3}\pi/\surd 3 \simeq 0{\cdot}605$, for the square and hexagonal lattices respectively. Also the p.d.f. of X_2 is likewise $8x/a^2$ for both. Thus W_2 is distributed uniformly over $[0, 1]$ for all three lattices.

The p.d.f. of $R_1 = \frac{4}{3}(1-\pi W_1)$ for the triangular lattice is

$$\tfrac{4}{3} - \tfrac{1}{3}\pi(3\beta - 2\pi)\{2\pi + \sin\beta - (\pi+\beta)\cos\beta\}^2/\{(1-\tfrac{1}{2}\pi/\surd 3)(\pi+\beta)\sin^4(\tfrac{1}{2}\beta)\},$$

where

$$r = \tfrac{4}{3}[1 - \pi/\{2\pi + \sin\beta - (\pi+\beta)\cos\beta\}],$$

for

$$\tfrac{4}{3}(11\pi + 3\surd 3)(17\pi + 3\surd 3)^{-1} \leqslant r \leqslant 1,$$

that is $0{\cdot}904 \leqslant r \leqslant 1$.

Similar results can be obtained for the square and hexagonal lattices, the ranges of r being $[\frac{4}{3}(\pi+1)(2\pi+1)^{-1}, 1]$ and $[\frac{4}{3}(2\pi+3\sqrt{3})(8\pi+3\sqrt{3})^{-1}, 1]$, that is $[0{\cdot}785, 1]$ and $[0{\cdot}505, 1]$, respectively. Table 1 gives the p.d.f.'s for the three lattices and for the Poisson forest.

Table 1. *The p.d.f.'s of the triangular, square and hexagonal lattices, and also the Poisson forest*

r	0	0·5	0·60	0·65	0·70	0·75	0·80	0·85	0·90	0·95	1·0
Triangular lattice	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	8·9	29·22
Square lattice	0·0	0·0	0·0	0·0	0·0	0·0	1·36	2·92	4·57	6·53	10·98
Hexagonal lattice	0·0	0·59	1·03	1·42	1·72	2·01	2·27	2·58	2·92	3·39	4·59
Poisson forest	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0

Now consider a pattern where the trees are in clumps, each clump consisting of at least two individuals occupying the same point. This can be considered as an idealized extreme aggregation pattern. In this case R_1 is zero with probability one.

The mass of the R_1 distribution is therefore uniformly spread between zero and one in the random case, concentrated entirely at zero for the extreme aggregation case, and concentrated more and more towards one for the cases of increasing regularity. This suggests that aggregation patterns can be regarded as characterized by R_1, with distributions concentrated toward zero.

2·4. *An aggregation case*

Suppose we have 'clump centres' forming a Poisson forest in the plane with density λ_1, and suppose a proportion of these clumps are single trees, while the remainder are small local Poisson forests with local density μ, where $\mu \gg \lambda_1$. There are three situations arising in the sampling procedure.

(i) If the random sampling origin, O, falls within one of the local Poisson forest clumps, then so do P and Q, where P is the nearest tree to O and Q is P's nearest neighbour. Thus in effect we are taking observations in a Poisson forest of density μ, and the R_1 value will belong to the uniform distribution on $[0, 1]$.

(ii) If O falls outside one of the clumps, and P is a single tree clump, we are in effect taking observations in a Poisson forest of density λ_1, and the R_1 value will again belong to the uniform distribution on $[0, 1]$.

(iii) If O falls outside one of the clumps and P is a tree on the periphery of a clump with several trees, then the same argument which led to the uniform distribution for R_1 in §2·2 applies, except that the density for Y is μ instead of λ_1.

Hence the random variable

$$Z = 1/\{(1+\lambda_1\mu^{-1})\pi + \sin \mathrm{B} - (\pi+\mathrm{B})\cos \mathrm{B}\}$$

is uniformly distributed over the range $[\mu(3\mu\pi+\lambda_1\pi)^{-1}, \mu(\lambda_1\pi)^{-1}]$, and we note that $\mathrm{pr}(Y \leqslant 2X) = 3\mu(3\mu+\lambda_1)^{-1}$, which, of course, tends to 1 as $\mu/\lambda_1 \to \infty$, and equals $\frac{3}{4}$ when $\mu/\lambda_1 = 1$.

The distribution of W_1 conditioned on case (iii) is therefore given by

$$\begin{aligned}\mathrm{pr}(W_1 \leqslant w) &= \mathrm{pr}\{2\pi + \sin \mathrm{B} - (\pi+\mathrm{B})\cos \mathrm{B} \geqslant w^{-1}\} \\ &= \mathrm{pr}\,[Z \leqslant \{w^{-1} - (1-\lambda_1\mu^{-1})\pi\}^{-1}].\end{aligned}$$

Hence the p.d.f. of R_1 in case (iii) is

$$f(r) = (1+k)(1+kr)^{-2} \quad (0 \leqslant r \leqslant 1), \quad k = \tfrac{3}{4}(\mu-\lambda_1)/\lambda_1,$$

i.e. a censored Pareto distribution.

We assume that case (i) or case (ii) happens with probability p; the unconditioned p.d.f. of R_1 is then

$$g(r; p, k) = p + (1-p)(1+k)(1+kr)^{-2} \quad (0 \leqslant r \leqslant 1, k > 0). \tag{3}$$

We can consider p and k as parameters of aggregation. Thus p indicates to what extent the population tends towards clumping and k measures the actual clumping, increasing as clumping increases.

We now consider the distribution of W_2 in the three cases above. In case (i), W_2 will be defined in a local Poisson forest and is thus distributed uniformly over $[0, 1]$. In case (ii), W_2 is defined in a Poisson forest of density λ_1 and is again distributed uniformly over $[0, 1]$. No values of W_2 arise in case (iii), since $\text{pr}(Y > 2X)$ may be assumed negligible. Overall therefore W_2 is distributed uniformly in this aggregation case.

3. Estimation of p and k

Given observations $r_1, \ldots, r_n$ in the aggregation case, rough estimates $\tilde{p}$ and $\tilde{k}$ of p and k can be obtained by equating the sample mean, $\bar{r}$, and the proportion, $\bar{s}$, of r's greater than $\frac{1}{2}$ to the theoretical values:

$$\bar{r} = \tfrac{1}{2}\tilde{p} + (1-\tilde{p})(1+\tilde{k})\{\log(\tilde{k}+1) - \tilde{k}(\tilde{k}+1)^{-1}\}/\tilde{k}^2,$$
$$\bar{s} = \tfrac{1}{2}\tilde{p} + (1-\tilde{p})(2+\tilde{k})^{-1}.$$

Hence $\tilde{k}$ is the solution of the equation

$$\bar{r} = (2+\tilde{k})[\bar{s}\tilde{k}^{-1} + (\tilde{k} - 2\bar{s})\{(\tilde{k}+1)\log(\tilde{k}+1) - \tilde{k}\}\tilde{k}^{-4}],$$

and then $\tilde{p} = 2\{(2+\tilde{k})\bar{s} - 1\}/\tilde{k}$.

The log likelihood of the observations is given by

$$L(p, k) = \sum_{i=1}^{n} \log\{p + (1-p)(k+1)(1+kr_i)^{-2}\},$$

and hence the maximum likelihood estimates $\hat{p}$ and $\hat{k}$ of p and k can be found by calculating values of $L(p, k)$ over a grid of values for p and k surrounding the initial estimates $\tilde{p}$ and $\tilde{k}$, and fitting a quadratic surface over the grid points in the neighbourhood of the observed maximum. The maximum point of this fitted surface gives the values of $\hat{p}$ and $\hat{k}$, and the coefficients of the second-degree terms give the estimated terms of the information matrix, and hence by inversion the estimated variance-covariance matrix of $\hat{p}$ and $\hat{k}$.

4. Testing hypotheses about the patterns

First consider the two simple hypotheses,

H_0: the r_i's are from a Poisson forest, that is R_1 is uniform on $[0, 1]$,

H_1: the r_i's are from a square lattice population.

As a test of H_0 against H_1, we reject H_0 if

$$M = \min(r_1, \ldots, r_n) \geqslant \tfrac{4}{3}(\pi+1)(2\pi+1)^{-1} = 0{\cdot}758 = m.$$

Under H_0, $\text{pr}(M \geqslant m) = (1-m)^n = (0{\cdot}242)^n$ and under H_1, $\text{pr}(M \geqslant m) = 1{\cdot}0$. Thus the power of this test is unity and the type I error is $(0{\cdot}242)^n$ which converges to zero rapidly as n increases.

Similar considerations apply if the pattern under the alternative hypothesis is a triangular lattice or a hexagonal lattice; the critical values for M will be 0·904 and 0·505 respectively.

Consider now the problem of testing for the presence of aggregation. Take as null hypothesis H_0 that R_1 is uniformly distributed on $[0, 1]$, corresponding to the assumption of a Poisson forest, and as alternative H_1 that R_1 has the density $g(r; p, k)$ of (3), corresponding to the above described aggregation model with parameters p and k. A conventional large-sample test of $\hat{p}$ and $\hat{k}$, against their hypothetical values under H_0, using their estimated variance-covariance matrix is not possible, since $p = 1$ and $k = 0$ separately imply H_0 and the likelihood surface contains horizontal generators lying in the planes $p = 1$ and $k = 0$. However, all that is required is to test the sample $(r_1, \ldots, r_n)$ for uniformity over $[0, 1]$ against an alternative implying concentration of the R distribution toward zero. A suitable procedure is a one-tailed test of $(\frac{1}{2} - \bar{r})\sqrt{(12n)}$ as $N(0, 1)$ (Cox, 1955). If the test leads to the rejection of H_0, the estimates $\hat{p}, \hat{k}$ and their estimated variance-covariance matrix can be used for estimation of nonzero p and k, the log likelihood surface having the required paraboloidal form in the neighbourhood of the true parameter values.

Note that by basing our inference on the observed values of R_1, we do in fact have a loss of information inasmuch as we are only using n out of the N observations of X and Y. However we have seen that the expectation of n ranges from $0{\cdot}093N$ for the triangular lattice, to $\frac{3}{4}N$ for the Poisson forest, and then increases to N as the aggregation increases. In the light of the fact that lattice distributions are self-evident and the test of randomness is very powerful, the information loss is greatest where it can most be spared, and the smallest where it is most needed.

Holgate's statistic for testing for randomness is $T = \Sigma x_i^2/v_i^2$, where x_i is the distance of the nearest tree to the sampling origin, and v_i is the distance of the second nearest tree to the sampling origin. For a large sample, T is normally distributed with mean $\frac{1}{2}$ and variance $(12N)^{-1}$. Results using this statistic will be compared below to the testing of R_1 values described above.

5. THE DATA

We have used data of three types. The first type, kindly made available by Dr E. D. Ford of the Institute of Terrestrial Ecology, Penicuik, and described by Ford (1975), is in the form of the coordinates of trees planted on rectangular lattices with death of some of the individuals. The second type consists of the coordinates of the six types of trees in Lansing Woods, Michigan, kindly supplied by Professor D. J. Gerrard. Data of the third type were simulated by a modified Thomas process where clump centres, considered as trees, are distributed randomly, and a Poisson number of 'offspring', with mean μ, are allocated to the clumps, the radial dispersion being σ. Figure 2 shows sections of the data from the last two types. For each set of data, 100 sample origins were selected at random, giving 100 pairs (X, Y) for analysis. Tables 2 and 3 summarize the results obtained. Typical histograms of the r_i's are shown in Fig. 3.

The values of Cox's statistic for testing the r_i's in each data set for uniformity, and hence the spatial pattern for randomness, are shown in Table 2, together with the results of Holgate's randomness test for comparison. Table 2 also gives the results of a Kolmogorov–Smirnov test applied to each data set to test the hypothesis that the W_{2j}'s are from a uniform distribution.

Table 3 shows the estimated values of $\hat{p}, \hat{k}$ and their variance-covariance matrix, for the sets of data for which Cox's test rejected randomness in favour of clumping. Also given in Table 3 are the results of a χ^2 test, to test the fit of the aggregation model.

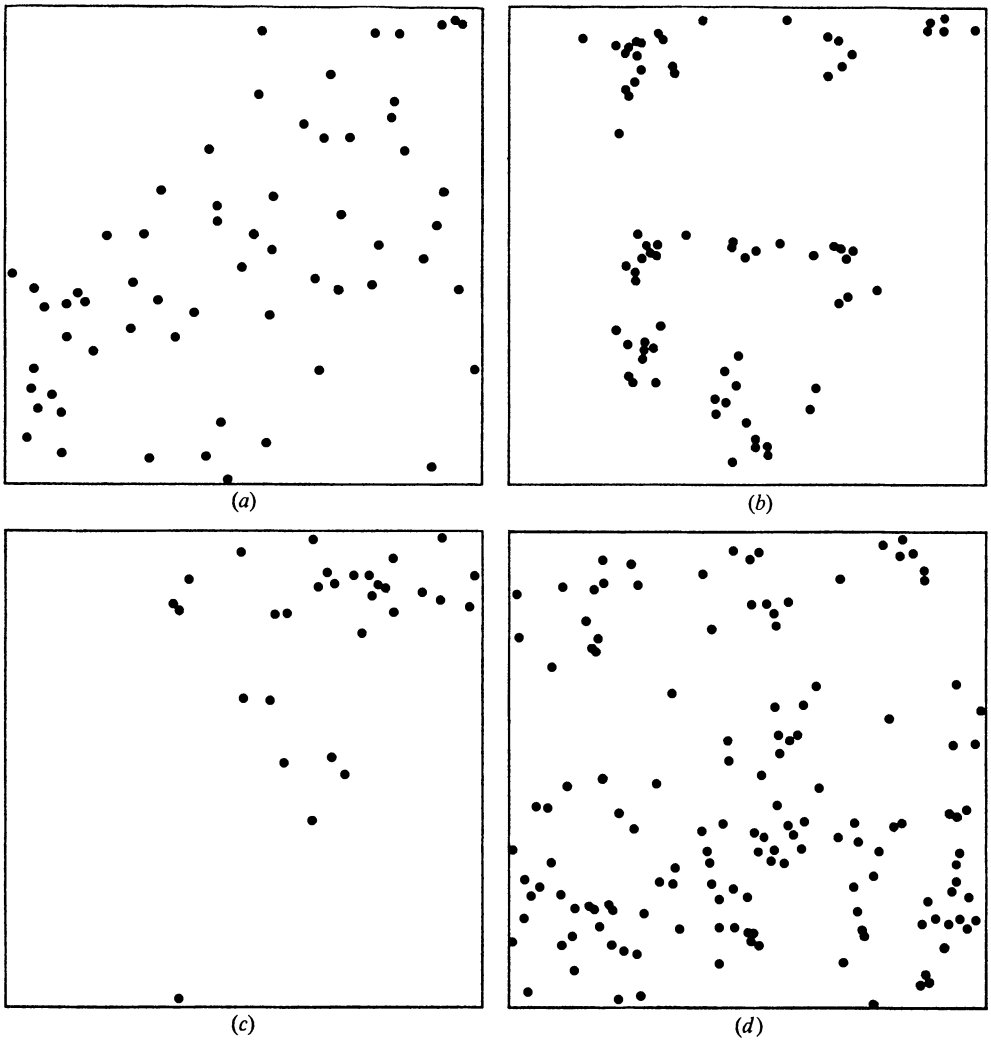

Fig. 2. Sections of four of the twelve data sets; (*a*), modified Thomas process, with mean number of offspring 5 and radial dispersion 1·0; (*b*) modified Thomas process, with mean number of offspring 5 and radial dispersion 0·2; (*c*) black oaks in Lansing Woods; (*d*) hickories in Lansing Woods.

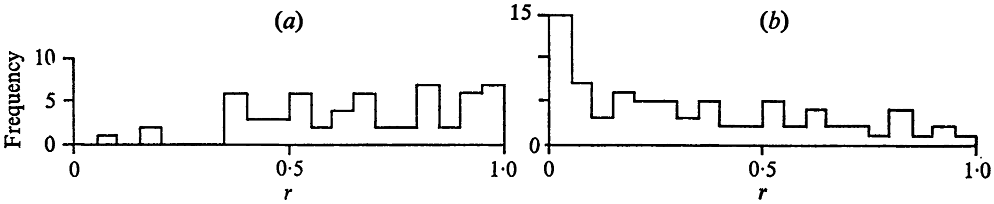

Fig. 3. Sample r_i's for two of the data sets, illustrating the method of distinguishing between regular and aggregated populations. (*a*) rectangular Lattice No. 1, (*b*) black oaks in Lansing Woods.

Inspection of the histograms of the r_i's for the rectangular lattices with death indicated that the pattern was obviously far from random. The effect of the missing individuals was shown up by values of the r_i's near to zero.

In the case of the simulated data, when the radial dispersion of the modified Thomas process becomes relatively large, the pattern becomes effectively indistinguishable from a

Poisson forest. This is illustrated by the sample with $\mu = 5\cdot0$ and $\sigma = 1\cdot0$, the density of the clump centres being 0·44. The two other modified Thomas processes fitted well to our aggregation model, as indeed they should do, the single trees occurring when there are no offspring in a clump, and the local Poisson forests occurring otherwise.

Table 2. *Cox's test and Holgate's test for randomness, together with a Kolmogorov–Smirnov* (KS) *test for uniformity of the* W_{2j}*'s*

Type of forest	n	Cox's test	Holgate's test	KS test for W_2's
Rectangular lattice No. 1	52	—	—	0·10
Rectangular lattice No. 2	59	—	—	0·07
Rectangular lattice No. 3	55	—	—	0·18
M.T.P.† ($\mu = 5\cdot0$, $\sigma = 0\cdot2$)	98	10·14**	0·71**	0·40
M.T.P. ($\mu = 2\cdot0$, $\sigma = 0\cdot2$)	89	9·26**	0·64**	0·10
M.T.P. ($\mu = 5\cdot0$, $\sigma = 1\cdot0$)	71	0·57	0·47	0·11
Hickories in Lansing Woods	74	0·06	0·45	0·11
White oaks in Lansing Woods	81	0·22	0·51	0·19
Red oaks in Lansing Woods	79	3·52**	0·89**	0·23
Black oaks in Lansing Woods	77	4·82**	0·57*	0·12
Maples in Lansing Woods	78	2·42**	0·56*	0·21
Miscellaneous in Lansing Woods	93	8·00**	0·69**	0·19

† Modified Thomas process   * Significant at 5 %   ** Significant at 1 %

Table 3. *The values of* $\hat{p}$, $\hat{k}$ *and their estimated covariance matrix, together with a* χ^2 *test for the fit of the aggregation model*

Type of forest	$\hat{p}$	$\hat{k}$	Covariance matrix	χ^2_7
M.T.P.,† $\mu = 5\cdot0$, $\sigma = 0\cdot2$	0·17	12·58	$\begin{bmatrix} 0\cdot02 & 0\cdot21 \\ 0\cdot21 & 5\cdot13 \end{bmatrix}$	2·98
M.T.P., $\mu = 2\cdot0$, $\sigma = 0\cdot2$	0·16	10·15	$\begin{bmatrix} 0\cdot001 & 0\cdot26 \\ 0\cdot26 & 29\cdot92 \end{bmatrix}$	4·63
Red oaks	0·92	16490	$\begin{bmatrix} 2\cdot0\times10^{-3} & 62\cdot6 \\ 62\cdot6 & 64\times10^{-7} \end{bmatrix}$	10·88
Black oaks	0·51	6·36	$\begin{bmatrix} 3\cdot0\times10^{-3} & 0\cdot93 \\ 0\cdot93 & 428 \end{bmatrix}$	3·00
Maples	0·92	9706	$\begin{bmatrix} 1\cdot0\times10^{-3} & 28\cdot2 \\ 28\cdot2 & 4\cdot0\times10^{7} \end{bmatrix}$	4·69
Miscellaneous trees	0·34	14·31	$\begin{bmatrix} 0\cdot04 & 0\cdot32 \\ 0\cdot32 & 6\cdot25 \end{bmatrix}$	7·21

† Modified Thomas process.

The data from Lansing Woods prove to be the most interesting. It appears that the hickories and white oaks each form a Poisson forest, while the red oaks, black oaks, maples and miscellaneous trees tend towards clumping. In each case $\hat{p}$ measures the tendency towards clumping and $\hat{k}$ is a measure of the clumps. Note that the red oaks and maples tend only to very slight clumping, that is $\hat{p}$ near 1, but with clumps which are very dense, that is $\hat{k}$ large.

It is pleasing to note that our test of randomness based on the r_i's gives the same results as Holgate's test of randomness. However we suggest that when aggregation does occur, our method may well give more insight into the pattern through $\hat{p}$ and $\hat{k}$.

In each of the twelve sets of data, which ranged from regular, to random, to aggregation, the Kolmogorov–Smirnov test gave no evidence for rejecting the hypothesis that the W_{2j}'s were from a uniform distribution. It would be interesting to know what is the class of stationary spatial point processes, where multiple events do not dominate, for which W_2 is not distributed uniformly on [0, 1]. This class would seem to be quite restricted.

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