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ON DISTANCE ESTIMATORS OF DENSITY IN RANDOMLY DISTRIBUTED FORESTS

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SUMMARY

In various journals a number of methods are described which utilize spacing distances instead of fixed-area plots for estimating plant densities. Some of these methods are studied in this paper. Maximum likelihood (ML) estimators for the forest density are given and the distributions and moments of these estimators are derived. Other distance estimators of density have been used in the past, but they do not have the advantages of ML estimators, and it even seems difficult to determine their moments.

A distance method has the advantage over a fixed plot method that the sample size does not depend upon the density being measured. There are many practical difficulties in using them, however, and some of these are discussed.

The trees in a natural forest will not be distributed uniformly at random, and some care must be exercised in using a distance method. The difficulty can sometimes be overcome by using a stratified sampling method.

The model is described in terms of trees and forests, but it is clear that it can be applied to many different types of problem.

1. INTRODUCTION

Consider an infinite forest in which the trees are distributed at random with a uniform average density of λ per unit area. A forestry official wishes to estimate λ by a distance method. He selects a random point in the forest and notes the distance to the nearest tree. If the forest is dense, this distance will tend to be rather small, and if the forest is sparse, the converse is likely to be true.

Let us denote the distance random variable by R, and consider the situation depicted in Figure 1. The circle of radius r (and area πr^2) will contain zero trees with the Poisson probability $\exp(-\pi r^2\lambda)$, and the ring of width dr will contain one tree with the Poisson probability $2\pi r\lambda\, dr$. These two events are independent, and it follows that the probability density function of the distance R from a random point to the nearest tree is

$$p(r) = 2\pi r\lambda \exp(-\pi r^2\lambda). \tag{1}$$

This is a well-known result (Skellam [1952]). It is easy to prove that the expected value of R is given by

$$E(R) = \tfrac{1}{2}\lambda^{-\frac{1}{2}}. \tag{2}$$

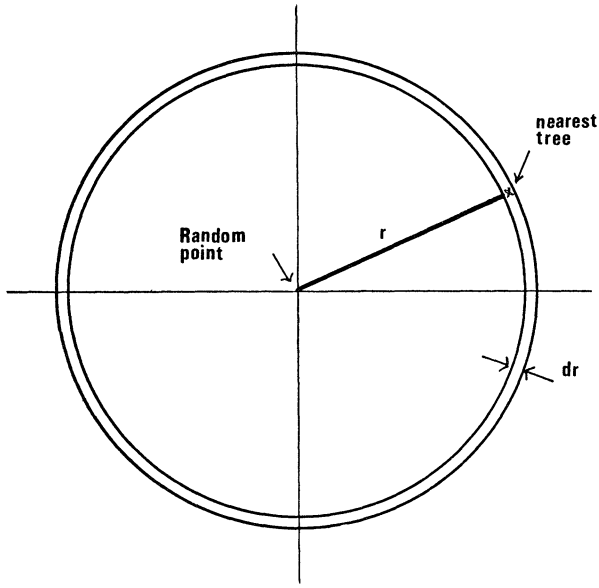

FIGURE 1
THE NEAREST TREE TO A RANDOM POINT

Equation (2) suggests that we should seek an estimator of λ proportional to the square of the reciprocal of R. However, $E(R^{-2})$ is infinite. Cottam and Curtis [1956] sought to avoid this problem by defining the mean area per tree M as the reciprocal of λ. There is no difficulty in estimating M; however, Cottam and Curtis chose to estimate $\sqrt{M}$ using n random points and the estimator

$$\widehat{\sqrt{M}} = \frac{2}{n}\sum_{i=1}^{n} r_i \ . \tag{3}$$

The appropriate theoretical result was proved by Morisita [1954]. An estimate of M obtained by squaring the estimate of $\sqrt{M}$ will be biased, and unless the size n of the sample is rather large, such an estimate of M is unlikely to be very reliable.

In practice, it is usually the parameter λ which is of interest. An estimate of this parameter obtained by taking the reciprocal of the square of the estimate of $\sqrt{M}$ will again be biased, and it is unlikely to lead to reliable results except with rather large samples.

A simple maximum likelihood (ML) distance estimator for λ exists, and this estimator is described in section 2.

2. A MAXIMUM LIKELIHOOD DISTANCE ESTIMATOR FOR λ

It will be assumed in this section that n random points have been chosen in the forest, and that the distance random variable associated with the

jth point is R_j ($j = 1, 2, \cdots, n$). The likelihood of sample distances r_1, r_2, $\cdots$, r_n is given by the formula

$$L = (2\pi\lambda)^n \exp\left(-\pi\lambda \sum_{j=1}^{n} r_j^2\right)(r_1 r_2 \cdots r_n). \tag{4}$$

This likelihood is maximized by differentiating L with respect to λ and equating to zero. The following ML estimator emerges:

$$\hat{\lambda} = n \Big/ \left(\pi \sum_{j=1}^{n} r_j^2\right). \tag{5}$$

Moore [1954] also suggests this estimator which is proportional to the reciprocal of the mean squared distance. The estimator of λ implied by Cottam and Curtis [1956] is proportional to the squared mean distance, and it does not seem possible to give theoretical formulae for its moments.

The statistic (5) does not have a well-known distribution, although it will be proved in section 6 that its reciprocal is proportional to a χ^2_{2n} variable.

It is of interest to determine the mean and variance of $\hat{\lambda}$. Clearly,

$$E(\hat{\lambda}) = \int_0^\infty \cdots \int_0^\infty \frac{n}{\pi \sum_{j=1}^{n} r_j^2} \exp\left(-\pi\lambda \sum_{j=1}^{n} r_j^2\right)(2\pi\lambda)^n r_1 \cdots r_n \, dr_1 \cdots dr_n .$$

This multiple integral may be evaluated by making the transformation

$$u_i = \sum_{j=1}^{i} r_j^2 \qquad i = 1, 2, \cdots, n, \tag{6}$$

and the Jacobian of the transformation is readily seen to be

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(r_1, r_2, \cdots, r_n)} = 2^n(r_1 r_2 \cdots r_n). \tag{7}$$

The limits of integration cause no difficulty: the variable u_n is integrated between 0 and ∞, and each u_i ($1 \leq i < n$) is integrated between 0 and u_{i+1}. Eventually a simple formula emerges:

$$E(\hat{\lambda}) = (n\lambda)/(n - 1). \tag{8}$$

The ML estimator is therefore biased, but it is of course asymptotically unbiased. For the expectation to be finite, n must be greater than 1.

The second moment of $\hat{\lambda}$ is determined in a similar manner, and it leads to the following formula for the variance:

$$\text{var}\,(\hat{\lambda}) = n^2\lambda^2/[(n - 1)^2(n - 2)]. \tag{9}$$

The sample size n must be greater than 2.

3. AN UNBIASED ESTIMATOR OF λ

The bias in the ML estimator for λ may be removed by multiplying $\hat{\lambda}$ by $(n - 1)/n$ and defining the following estimator T:

$$T = (n - 1) \Big/ \left(\pi \sum_{i=1}^{n} r_i^2\right). \tag{10}$$

This statistic has expectation λ and its variance is given by

$$\text{var } (T) = \lambda^2/(n - 2). \tag{11}$$

Again, n must be greater than 2.

4. THE NUMBER OF RANDOM POINTS REQUIRED

The sampler needs to decide upon a suitable value of n in order that the estimate of λ will have a prescribed accuracy. Let us imagine that we require the estimate to be within 10 per cent of the true value with probability greater than or equal to 0.95. That is,

$$\Pr\left(-0.1 < \frac{T - \lambda}{\lambda} < 0.1\right) \geq 0.95. \tag{12}$$

The statistic T is proportional to the ML estimator $\hat{\lambda}$, and is therefore asymptotically normally distributed. The value of n required in the present problem is fairly large and hence it is convenient and accurate to assume that T is a normal random variable with mean λ and variance $\lambda^2/(n - 2)$. It follows that $0.1\sqrt{(n - 2)}$ must be greater than or equal to 1.96 and therefore n must be greater than or equal to 386. This sample size does not depend upon the density λ.

5. THE POINT-CENTRED QUARTER METHOD

With this method, the plane around each random point is divided into 4 quadrants and the distance to the nearest tree in each quadrant is noted. If n random points are used, and $4n$ distances are measured, the ML estimator of λ may be shown to be

$$\hat{\lambda}^{(4)} = 16n \Big/ \left(\pi \sum_{i=1}^{n} \sum_{j=1}^{4} r_{ij}^2\right), \tag{13}$$

where r_{ij} represents the distance from the ith random point to the nearest tree in the jth quadrant of the random point.

The ML estimator is again biased, and it is convenient to use the following unbiased estimator:

$$T^{(4)} = 4(4n - 1) \Big/ \left(\pi \sum_{i=1}^{n} \sum_{j=1}^{4} r_{ij}^2\right), \tag{14}$$

with variance

$$\text{var } (T^{(4)}) = \lambda^2/(4n - 2). \tag{15}$$

This result may be generalised by dividing the angle 2π into m angles each of size $2\pi/m$. The unbiased estimator $T^{(m)}$ will be

$$T^{(m)} = m(mn - 1) \Big/ \left(\pi \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij}^2\right) \tag{16}$$

with mean λ and variance

$$\text{var } (T^{(m)}) = \lambda^2/(mn - 2). \tag{17}$$

The quarter method appears to have no advantage over the simple distance method described in section 3. It is true that to obtain the same accuracy, the quarter method requires fewer random points (one quarter as many) but this advantage is far out-weighed by the great practical difficulty of having to divide the plane into 4 quadrants at each random point. It will often be difficult to decide in which quadrant a certain tree belongs.

6. THE kTH NEAREST TREE

In this section a single random point will be considered and the distance from the random point to the kth nearest tree will be denoted by the random variable $R_{(k)}$. Figure 1 must now be modified somewhat. The circle of radius $r_{(k)}$ will contain $k - 1$ trees with probability

$$\frac{(\pi r_{(k)}^2 \lambda)^{k-1}}{(k - 1)!} \exp\,(-\pi r_{(k)}^2 \lambda),$$

and the ring of width $dr_{(k)}$ will contain one tree with probability $2\pi\lambda r_{(k)}\, dr_{(k)}$. These probabilities are independent and it follows that the density function associated with $R_{(k)}$ is

$$p_{(k)}(r_{(k)}) = \frac{2(\pi\lambda)^k r_{(k)}^{2k-1}}{(k - 1)!} \exp\,(-\pi r_{(k)}^2 \lambda). \tag{18}$$

It is soon apparent from this equation that $2\pi\lambda R_{(k)}^2$ is a χ_{2k}^2 random variable (Skellam [1952]; Thompson [1956]).

The joint distributions of these distances are also of interest. Let us consider therefore the joint distribution of $R_{(1)}$ and $R_{(2)}$ and examine the situation depicted in Figure 2. The joint density of $R_{(1)}$ and $R_{(2)}$ is seen to be

$$\exp\,(-\pi r_{(2)}^2 \lambda) \cdot (2\pi\lambda)^2 r_{(1)} r_{(2)} \ , \tag{19}$$

and if the following transformation is made:

$$Z_{(1)} = 2\pi\lambda R_{(1)}^2 \ , \qquad Z_{(2)} = 2\pi\lambda(R_{(2)}^2 - R_{(1)}^2), \tag{20}$$

the joint density of $Z_{(1)}$ and $Z_{(2)}$ is found to be

$$\tfrac{1}{2} \exp\,(-\tfrac{1}{2} z_{(1)}) \tfrac{1}{2} \exp\,(-\tfrac{1}{2} z_{(2)}), \tag{21}$$

where $0 \leq z_{(i)} < \infty$ $(i = 1, 2)$. This joint density is the product of the density functions of two mutually independent χ_2^2 random variables. It follows that $Z_{(1)}$ and $Z_{(2)}$ are two mutually independent χ_2^2 random variables.

More generally, $Z_{(1)}$, $Z_{(2)}$, $\cdots$, $Z_{(k)}$ are k mutually independent χ_2^2

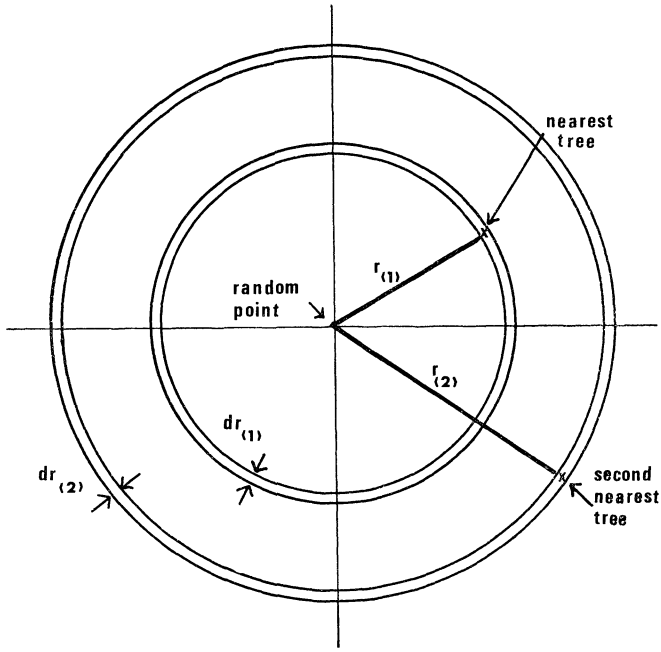

FIGURE 2

THE TWO NEAREST TREES TO A RANDOM POINT

random variables where

$$Z_{(1)} = 2\pi\lambda R_{(1)}^2$$

and (22)

$$Z_{(j)} = 2\pi\lambda(R_{(j)}^2 - R_{(j-1)}^2) \quad j > 1.$$

Using equation (18) and the definition of T in equation (10), it is clear that $2\lambda(n - 1)T^{-1}$ is a χ^2 with $2n$ D.F. This particular result is given by Pielou ([1969] p. 113). (It could have been used in section 4; however the number of degrees of freedom is large and it is still necessary to use a normal approximation.)

7. DENSITY ESTIMATORS USING THE kTH NEAREST TREE

Consider n random points in the uniform randomly-distributed forest and let us assume that the investigator records exactly one distance from each random point. With the jth point ($j = 1, 2, \cdots, n$) he records the distance from the point to the k_jth nearest tree where k_j takes some positive integral value which is predetermined. The distance random variable associated with the jth random point will be denoted by $R_{j(k_j)}$ and a particular observed value of this random variable will be denoted by $r_{j(k_j)}$.

It is simple, although slightly tedious, to write down the likelihood of a given sample in a formula like equation (4). This likelihood may be differentiated with respect to λ to yield the following ML estimator for λ:

$$\hat{\lambda} = K \Big/ \left(\pi \sum_{i=1}^{n} r_{i(k_i)}^2 \right), \tag{23}$$

where

$$K = \sum_{i=1}^{n} k_i \, . \tag{24}$$

The methods of section 2 may be used to determine the moments of $\hat{\lambda}$. The multiple integral is rather more tedious because although the same transformation is made, the first $n - 1$ integrals are beta integrals. It may be shown that

$$E(\hat{\lambda}) = K\lambda/(K - 1) \tag{25}$$

and

$$E(\hat{\lambda}^2) = K^2\lambda^2/[(K - 1)(K - 2)]. \tag{26}$$

The ML estimator is again biased, and so we define the unbiased estimator T as follows:

$$T = (K - 1) \Big/ \left(\pi \sum_{i=1}^{n} r_{i(k_i)}^2 \right), \tag{27}$$

where K is defined in equation (24). The estimator has variance

$$\text{var } (T) = \lambda^2/(K - 2). \tag{28}$$

This result is rather interesting. It shows that the use of the k_ith nearest tree ($k_i > 1$) instead of the nearest tree results in an estimator of λ with a smaller sampling variance. If, for example, the distance to the second nearest tree is measured for each random point (i.e. $k_i = 2$ for all j) and n is large, the variance of the estimator is approximately halved. The standard deviation is reduced by about 30%. The use of the third nearest tree to each random point will result in a reduction of about 42% in the standard deviation of the estimate. This is probably about as far as this process can be carried without leading to serious practical difficulties in the field.

The variance (28) is also approximately the same as that obtained using n fixed plots with areas k_i/λ ($j = 1, 2, \cdots, n$), namely λ^2/K. This seems reasonable because the expected number of trees on the jth plot is k_i .

It is soon apparent, using the results of section 6, that $2\lambda(K - 1)T^{-1}$ is a χ^2 with $2K$ D.F.

8. THE RELATIVE DENSITIES OF TWO FORESTS

The chi square results of section 6 are useful when it comes to the problem of testing the relative densities of two forests. Let us denote the density

of the first forest by λ and the density of the second by λ', and assume that it is necessary to test the null hypothesis $\lambda' = 2\lambda$ against the alternative hypothesis $\lambda' \neq 2\lambda$.

A number of random points n will be selected in the first forest and the distance to the nearest tree from the jth point will be denoted by R_j ($j = 1, 2, \cdots, n$). Similarly, n' random points will be selected in the second forest and the appropriate distances denoted by R'_j ($j = 1, 2, \cdots, n'$).

It is simple to deduce from the results of section 6 that $2\pi\lambda \sum_{j=1}^{n} R_j^2$ is a χ^2_{2n} random variable and that $2\pi\lambda' \sum_{j=1}^{n'} R_j'^2$ is a $\chi^2_{2n'}$. These two random variables are independent, and it follows therefore that the ratio

$$F_{2n,2n'} = \left(n'\lambda \sum_{j=1}^{n} R_j^2\right) \Big/ \left(n\lambda' \sum_{j=1}^{n'} R_j'^2\right) \tag{29}$$

is an F variate with $2n$ and $2n'$ D.F. A simple F test may be applied to test the null hypothesis (Moore [1954]).

9. TESTS OF HOMOGENEITY

The test of homogeneity of variance due to Bartlett [1937] is usually encountered in connection with the analysis of variance, but it may be of use in the present context. The test statistic may be written in the form

$$M/C \tag{30}$$

where

$$M = N \ln\left(N^{-1} \sum_{j=1}^{n} \nu_j Y_j\right) - \sum_{j=1}^{n} \nu_j \ln Y_j , \tag{31}$$

$$N = \sum_{j=1}^{n} \nu_j , \tag{32}$$

$$C = 1 + \frac{1}{3(n-1)}\left(\sum_{j=1}^{n} \frac{1}{\nu_j} - \frac{1}{N}\right) , \tag{33}$$

and Y_j is a χ^2 with ν_j D.F. ($j = 1, 2, \cdots, n$). The Y_j are mutually independent.

Bartlett proved that the statistic M/C is approximately a χ^2 with $n - 1$ D.F. The approximation is not always satisfactory if some of the ν_j are 1, 2, or 3. (See Nair [1938]; Bishop and Nair [1939]; Pearson and Hartley ([1966]) p. 63).

Consider a sample of n random points in a uniform randomly-distributed forest. Let us denote the distance from the jth such point to the k_jth nearest tree by $R_{j(k_j)}$. According to the theory of section 6, $2\pi\lambda R^2_{j(k_j)}$ is a χ^2 with $2k_j$ D.F. Bartlett's test statistic may be used to test the homogeneity of the forest if $R^2_{j(k_j)}$ is substituted for Y_j in formula (31) and ν_j is replaced by $2k_j$ in (31), (32), and (33). If k_j is greater than 1 for all j, the approximation difficulties mentioned above are avoided.

10. A STRATIFIED-SAMPLING EXAMPLE

The following data have been obtained about a certain forest by aerial photography:

1. the forest should be stratified into 3 distinct large areas for sampling purposes;
2. the second area A_2 is twice the first area A_1 and the third area A_3 is three times A_1 ; and
3. the density λ_2 of A_2 is twice the density λ_1 of A_1 , and $\lambda_3 = 2\lambda_2$.

The aim is to estimate the total number of trees in the forest. The statistic T of section 3 is to be used in each area to estimate the density of trees in that area. Each density will be multiplied by the area involved to yield an estimate of the total number of trees in the forest. A total of 1026 random points are to be selected in the forest. How many points should be chosen at random in each area?

Let n_j denote the number of random points and T_j denote the estimate of λ_j in area j ($j = 1, 2, 3$). An estimate of the total number of trees in the forest is given by

$$N = \sum_{j=1}^{3} A_j T_j , \tag{34}$$

with variance

$$\text{var } (N) = \sum_{j=1}^{3} A_j^2 \text{ var } (T_j). \tag{35}$$

According to equation (11) and the aerial photography data

$$\text{var } (N) = A_1\lambda_1^2\left(\frac{1}{n_1 - 2} + \frac{16}{n_2 - 2} + \frac{144}{n_3 - 2}\right). \tag{36}$$

The variance in formula (36) is to be minimized by suitably selecting n_1 , n_2 , and n_3 . It may be shown that the minimal variance is achieved when

$$n_1 - 2 : n_2 - 2 : n_3 - 2 = 1 : 2 \times 2 : 4 \times 3. \tag{37}$$

It is given that the sum of n_1 , n_2 , and n_3 must be 1026, and we deduce from equation (37) that $n_1 = 62$, $n_2 = 242$, and $n_3 = 722$.

11. SOME PRACTICAL DIFFICULTIES

It was noted in section 4 that the number of sample points required to estimate the density of a forest with a prescribed accuracy using a distance method does not depend upon the density being measured. This is a considerable advantage. Some authors would claim other advantages for the distance methods (e.g. greater speed and lower cost). However, many practical difficulties arise, and some of these seem to outweigh the advantages.

Bias in the selection of a random point is very difficult to eliminate. This problem was encountered by ecology students at Macquarie University.

While the students obtained reliable estimates for the density of large trees in a relatively sparse forest, their estimates of plant density were completely unreliable. This was because they tried to avoid walking on and through the plants, and in so doing they over-estimated the mean squared distance and under-estimated the plant density.

The problem of non-uniform density is discussed in section 12.

12. NON-UNIFORM FOREST DENSITIES

Consider an infinite forest divided into two areas. (The generalization to m areas is straightforward). Trees are distributed uniformly at random in the first area with density λ_1 , and uniformly at random in the second area with density λ_2 . Both these densities and the relative sizes of the two areas are unknown.

An ecologist decides to estimate the average density of the forest using the estimator (10). Our aim is to examine the validity of the use of the estimator T in this situation. We shall assume that the first area is a proportion p_1 of the total forest, and that the second area is a proportion $p_2 = 1 - p_1$.

The ecologist selects a point in the forest at random and measures the distance to the nearest tree. The probability density function for this distance is

$$p(r) = 2\pi r p_1 \lambda_1 \exp(-\pi r^2 \lambda_1) + 2\pi r p_2 \lambda_2 \exp(-\pi r^2 \lambda_2), \tag{38}$$

and it is easy to deduce that the moment-generating function for the random variable representing π times the square of the distance is

$$m(\theta) = \frac{\lambda_1 p_1}{\lambda_1 - \theta} + \frac{\lambda_2 p_2}{\lambda_2 - \theta} = \frac{1 - \theta(p_1/\lambda_2 + p_2/\lambda_1)}{(1 - \theta/\lambda_1)(1 - \theta/\lambda_2)}. \tag{39}$$

If the ecologist selects n random points, and computes $X = \pi(r_1^2 + r_2^2 + \cdots + r_n^2)$, this random variable will have the moment-generating function

$$M(\theta) = [m(\theta)]^n. \tag{40}$$

The moments of X are readily determined from the cumulant-generating function of X

$$K(\theta) = \ln [M(\theta)] = n \ln [m(\theta)], \tag{41}$$

and we find that

$$E(X) = n\left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \left(\frac{p_1}{\lambda_2} + \frac{p_2}{\lambda_1}\right)\right]; \tag{42}$$

$$\text{var}(X) = n\left\{\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \left(\frac{p_1}{\lambda_2} + \frac{p_2}{\lambda_1}\right)^2\right\}. \tag{43}$$

The argument will now be restricted to the case in which n is large. The standard deviation of X will then be small relative to the mean, and it is possible to use the accurate approximation (Lindley [1965] p. 135)

TABLE 1

THE BIAS CAUSED BY NON-UNIFORM DENSITY

Sample size n	Densities λ_1	λ_2	Proportions p_1	p_2	Expectation of T $E(T)$	True mean density $p_1\lambda_1 + p_2\lambda_2$
1000	.10	.01	.5	.5	.018	.055
1000	.10	.20	.5	.5	.133	.150
1000	.10	.10	.9	.1	.100	.100
1000	.10	.01	.9	.1	. 053	.091

$$E\left(\frac{1}{X}\right) \doteqdot \frac{1}{E(X)}\left\{1 + \frac{\text{var }(X)}{[E(X)]^2}\right\}. \tag{44}$$

The expected value of the estimator T is

$$E(T) \doteqdot \frac{n-1}{E(X)}\left\{1 + \frac{\text{var }(X)}{[E(X)]^2}\right\}. \tag{45}$$

We need to compare this expectation with the mean density of the forest $\lambda = p_1\lambda_1 + p_2\lambda_2$. Let us consider the case in which $\lambda_1 = .1$, $\lambda_2 = .01$, $p_1 = p_2 = .5$, and $n = 1000$. We find that $E(T)$ is equal to .0182 which is much smaller than the mean density $\lambda = .055$. The estimator T is clearly biased. Further comparisons are given in Table 1, and the case in which $\lambda_1 = \lambda_2$ in particular should be noted.

It is apparent from these calculations that distance methods will provide biased estimates of density when a forest is not uniformly random. Considerable care is necessary if these methods are to be used at all.

SUR DES ESTIMATEURS DE LA DENSITE BASEE SUR LA DISTANCE DANS DES FORETS DISTRIBUEES AU HASARD

RESUME

Dans différents journaux, on trouve la description de méthodes qui utilisent les distances d'espacement au lieu de placettes d'aire fixe pour estimer les densités de plants. Certaines de ces méthodes sont étudiées dans ce papier. Les estimateurs du maximum de vraisemblance (ML) pour la densité de la forêt sont donnés et on calcule les distributions et les moments de ces estimateurs. D'autres estimateurs basés sur la distance ont été utilisés dans le passé mais ils n'ont pas les avantages des estimateurs du maximum de vraisemblance et il semble même difficile de déterminer leurs moments.

Une méthode de distance a l'avantage sur une méthode de placette que la taille d'échantillonnage ne dépend pas de la densité mesurée. Il y a néanmoins beaucoup de difficultés pratiques à l'utiliser et on discute plusieurs d'entre elles.

Les arbres dans une forêt naturelle ne seront pas distribués uniformément au hasard et on doit y faire attention en utilisant une méthode de distance. La difficulté peut quelquefois être surmontée en utilisant une méthode d'échantillonnage stratifiée.

Le modèle est décrit en termes d'arbres et forêts, mais il est clair qu'il peut être appliqué à des types différents et nombreux de problèmes.

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