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# The robust estimation of the density of a forest stand using a new conditioned distance method

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## SUMMARY

Estimators for the density of points in a plane may be unbiased for the spatial pattern for which they were designed but are usually biased for most others. Two new estimators are proposed which are approximately unbiased for a wide range of patterns, and are used on data in the form of the positions of the centres of trees in a forest stand.

*Some key words:* Density estimation; Distance method; Forest stand; Nearest neighbour; Robustness; Spatial pattern; Two-dimensional point process.

## 1. INTRODUCTION

Holgate (1972) has reviewed distance methods for estimating the density of events considered as a stationary two-dimensional point process, the main use being in forestry where an estimate of the number of trees or plants per unit area is required. It has been pointed out by many authors that the estimators considered are generally biased, Persson (1971) quantifying the matter. Diggle (1975) has an improved estimator which uses two distance measurements, biased in opposite directions, combined to reduce the bias. Lewis (1975) also combines two distance measurements to form an estimator which is unbiased for two different types of pattern and reduces bias for others. We propose two new estimators which are approximately unbiased for a wide range of spatial patterns.

Cox & Lewis (1976) propose a conditioned distance method for testing spatial patterns varying from regular to random to aggregated. In the present paper we use the same conditioned distance measurements for estimating density.

Within the forest stand or plant community,  $N$  random points are selected, which will be referred to as the sampling origins. From each of these we measure the distance,  $X$ , to the nearest tree, and from this tree we measure the distance,  $Y$ , to the nearest neighbour. The  $N$  pairs of measurements  $(X_1, Y_1), \dots, (X_N, Y_N)$  are then split into two sets  $A$  and  $B$ , where

$$A = \{(X_i, Y_i): Y_i \leq 2X_i \ (i = 1, \dots, n)\},$$

$$B = \{(X_j, Y_j): Y_j > 2X_j \ (j = 1, \dots, m)\} \quad (m + n = N).$$

For convenience we relabel the members of  $A$  and  $B$  as  $(X_{1i}, Y_{1i})$  and  $(X_{2j}, Y_{2j})$  respectively. We define the random variables,

$$Z_{1i}^2 = X_{1i}^2 \{2\pi + \sin B_{1i} - (\pi + B_{1i}) \cos B_{1i}\} / \pi \quad (i = 1, \dots, n),$$

where  $\sin(\frac{1}{2}B_{1i}) = \frac{1}{2}Y_{1i}/X_{1i}$ . The pairs of random variables  $(X_1, Y_1)$  and  $(X_2, Y_2)$ , together with the geometrical interpretation of  $Z_1^2$ , are illustrated by Cox & Lewis (1976). Our aim is to find estimators of the density,  $\lambda$  trees per unit area, or equivalently the mean area per tree or inverse density,  $\theta = \lambda^{-1}$ , based on the random variables just defined.

## 2. THE POISSON FOREST

A random stand or Poisson forest arises when the trees, considered as points, form a two-dimensional Poisson process. As shown by Cox & Lewis (1976), the joint probability density function, p.d.f., of  $X_1$  and  $B_1$  is

$$\frac{8}{3}\pi(\pi + \beta_1)\theta^{-2}x_1^3\sin\beta_1\exp[-x_1^2\theta^{-1}\{2\pi + \sin\beta_1 - (\pi + \beta_1)\cos\beta_1\}], \quad (1)$$

and the probability,  $p$ , that  $Y > 2X$  is  $\frac{1}{4}$ .

Also the joint p.d.f. of  $X_2$  and  $Y_2$  is

$$16\pi^2\theta^{-2}x_2y_2\exp(-\pi\theta^{-1}y_2^2) \quad (0 < x_2 < \infty, 2x_2 < y_2 < \infty). \quad (2)$$

The likelihood of the observations  $(x_{1i}, y_{1i})$  and  $(x_{2j}, y_{2j})$  is thus

$$\binom{N}{m} \prod_{i=1}^n \frac{8}{3}\pi(\pi + \beta_{1i})\theta^{-2}x_{1i}^3\sin\beta_{1i}\exp[-x_{1i}^2\theta^{-1}\{2\pi + \sin\beta_{1i} - (\pi + \beta_{1i})\cos\beta_{1i}\}] \\ \times \prod_{j=1}^m 16\pi^2\theta^{-2}x_{2j}y_{2j}\exp(-\pi y_{2j}^2\theta^{-1}),$$

and hence the maximum likelihood estimator of  $\theta$  is

$$\hat{\theta}_L = \frac{1}{2}\pi \left( \sum_{i=1}^n z_{1i}^2 + \sum_{j=1}^m y_{2j}^2 \right) / N.$$

Because the first partial derivative of the log likelihood with respect to  $\theta$  is  $2N(\hat{\theta}_L - \theta)/\theta^2$ ,  $\hat{\theta}_L$  is unbiased, and fully efficient with variance  $\frac{1}{2}\theta^2/N$ .

From (1) and (2) the marginal p.d.f.'s of  $Z_1$  and  $Y_2$  are given, respectively, by

$$2\pi^3\theta^{-2}z_1^3\exp(-\pi\theta^{-1}z_1^2) \quad (0 \leq z_1 < \infty), \\ 2\pi^2\theta^{-2}y_2^3\exp(-\pi\theta^{-1}y_2^2) \quad (0 \leq y_2 < \infty).$$

Thus  $Z_1^2$  and  $Y_2^2$  are identically distributed with mean  $2\theta/\pi$  and variance  $2\theta^2/\pi^2$ .

We now modify  $\hat{\theta}_L$  so that it is approximately unbiased for a wide range of spatial patterns.

3. THE ESTIMATOR  $\hat{\theta}$ 

If  $\hat{\theta}_L$  from §2 were used for patterns other than the Poisson forest, it would generally be biased. In order to overcome this we propose varying the coefficient of the estimator according to some quantitative feature of the spatial pattern, whose value varies with the type of pattern. The feature we choose is  $p$ , the probability that  $Y > 2X$ , which varies from 0.91 for points at the vertices of a triangular lattice, i.e. extreme regularity, through 0.25 for a Poisson forest, to zero for the case of extreme aggregation in which the trees form clumps of coincident points. In the sample situation we can estimate  $p$  from our  $N$  pairs of measurements  $(X, Y)$  by  $m/N$ , the relative frequency of the event  $Y > 2X$ . Let us first consider the estimator,

$$\hat{\theta}' = (a + bm/N)\hat{\theta}_L = \frac{1}{2}\pi(a + bm/N) \left( \sum_{i=1}^n z_{1i}^2 + \sum_{j=1}^m y_{2j}^2 \right) / N,$$

where  $a$  and  $b$  are constants chosen to make  $\hat{\theta}'$  effectively unbiased for a wide range of spatial patterns.

The expectation of  $\hat{\theta}'$ , conditional on  $m$ , is  $\frac{1}{2}\pi(a + bm/N)(n\mu_z + m\mu_y)/N$ , where  $\mu_z = E(Z_1^2)$ ,  $\mu_y = E(Y_2^2)$ , and hence

$$E(\hat{\theta}') = \frac{1}{2}\pi\{(a + bp)(q\mu_z + p\mu_y) + bpq(\mu_y - \mu_z)/N\},$$

where  $q = 1 - p$ .

Thus  $\hat{\theta}'$  is asymptotically unbiased for any particular spatial pattern if,

$$a + bp = 2\theta/\{\pi(q\mu_z + p\mu_y)\} = V, \quad (3)$$

say, for that spatial pattern. Here  $V$  is a parameter with a value for each spatial pattern. In particular, for a Poisson forest,  $V = 1$  and  $E(\hat{\theta}') = \theta$  exactly if  $a + \frac{1}{4}b = 1$ .

For a forest formed by the vertices of a triangular lattice of side  $d$ , the p.d.f. of  $X$  is

$$\begin{aligned} 4\pi x/(\sqrt{3}d^2) & \quad (0 \leq x \leq \tfrac{1}{2}d), \\ 4x\{\pi - 6\cos^{-1}(\tfrac{1}{2}d/x)\}/(\sqrt{3}d^2) & \quad (\tfrac{1}{2}d \leq x \leq d/\sqrt{3}). \end{aligned}$$

Also  $\text{pr}(Y = d) = 1$ ,  $\theta = \frac{1}{2}\sqrt{3}d^2$ , and  $p = \frac{1}{2}\pi/\sqrt{3}$ . Similar results are easily obtained for the square and hexagonal lattices. The value of  $p$ , and the means and variances of  $Z_1^2$  and  $Y_2^2$  are given in Table 1 for the triangular, square and hexagonal lattices, and also for the Poisson forest.

Table 1. *The values of  $p$ ,  $\mu_z$ ,  $\sigma_z^2$ ,  $\mu_y$ ,  $\sigma_y^2$  and  $V$  for various patterns, where  $\mu_z = E(Z_1^2)$ ,  $\sigma_z^2 = \text{var}(Z_1^2)$ ,  $\mu_y = E(Y_2^2)$ ,  $\sigma_y^2 = \text{var}(Y_2^2)$  and  $V = 2\theta/\{\pi(q\mu_z + p\mu_y)\}$*

Pattern	$p$	$\mu_z/\theta$	$\sigma_z^2/\theta^2$	$\mu_y/\theta$	$\sigma_y^2/\theta^2$	$V$
Triangular lattice	0.91	1.16	$8.0 \times 10^{-5}$	1.15	0.0	0.55
Square lattice	0.79	1.03	$1.1 \times 10^{-3}$	1.00	0.0	0.63
Hexagonal lattice	0.60	0.88	$9.7 \times 10^{-3}$	0.77	0.0	0.78
Poisson forest	0.25	0.64	0.20	0.64	0.20	1.00
Thomas process, $\alpha = 0.1$	0.23	0.70	0.25	0.70	0.25	0.91
Thomas process, $\alpha = 0.5$	0.15	0.95	0.46	0.95	0.46	0.67
Thomas process, $\alpha = 1.0$	0.09	1.27	0.81	1.27	0.81	0.50
Thomas process, $\alpha = 2.0$	0.03	1.91	1.82	1.91	1.82	0.33
Thomas process, $\alpha = 3.0$	0.01	2.55	3.24	2.55	3.24	0.25
Thomas process, $\alpha = 5.0$	0.002	3.82	7.30	3.82	7.30	0.17

For an aggregation pattern we consider a Thomas process (Thomas, 1949). Here clump centres, considered as trees, occur as a Poisson forest and each clump has a Poisson number, with mean  $\alpha$ , of 'offspring' trees which are coincident with the clump centre. This model is not completely realistic, but it does provide a good aggregation model when the pattern is almost a Poisson forest, i.e. when  $\alpha$  is small. For the Thomas process  $p = \frac{1}{4}e^{-\alpha}$  and  $E(\hat{\theta}') = (a + \frac{1}{4}be^{-\alpha})(1 + \alpha)\theta$ .

Figure 1 shows values of  $V$  plotted against the values of  $p$  for various spatial patterns. The points fall effectively on two straight lines, and thus the constants  $a$  and  $b$  from (3) can be calculated for each of these lines. Hence we can consider the following two estimators based on  $\hat{\theta}'$ :

$$\begin{aligned} \hat{\theta}_1 &= \tfrac{1}{2}\pi(1.17 - 0.68m/N) \left( \sum_{i=1}^n z_{1i}^2 + \sum_{j=1}^m y_{2j}^2 \right) / N, \\ \hat{\theta}_2 &= \tfrac{1}{2}\pi(0.20 + 3.20m/N) \left( \sum_{i=1}^n z_{1i}^2 + \sum_{j=1}^m y_{2j}^2 \right) / N. \end{aligned}$$

Both  $\hat{\theta}_1$  and  $\hat{\theta}_2$  are approximately unbiased, and have variance given by

$$\tfrac{1}{4}\pi^2[(a_i + b_i p)(p\sigma_y^2 + q\sigma_z^2) + pq\{(a_i + b_i - 2b_i q)\mu_z - (a_i + 2b_i p)\mu_y\}^2]/N + O(1/N^2) \quad (i = 1, 2), \quad (4)$$

where  $a_1 = 1.17$ ,  $b_1 = -0.68$ ,  $a_2 = 0.20$  and  $b_2 = 3.20$ .

If we knew beforehand whether the spatial pattern underlying our data was on the regular or aggregated side of a random process, we could use  $\hat{\theta}_1$  or  $\hat{\theta}_2$  respectively to estimate  $\theta$ . In fact we can base this decision on the value of  $m/N$ , and therefore we define a new estimator by  $\hat{\theta} = \hat{\theta}_1$  ( $m \geq \frac{1}{4}N$ ),  $\hat{\theta} = \hat{\theta}_2$  ( $m < \frac{1}{4}N$ ).

The expectation and variance of  $\hat{\theta}$  are not the same as those for  $\hat{\theta}_1$  or  $\hat{\theta}_2$  because of the risk of misclassification when  $p > \frac{1}{2}$  and  $m/N < \frac{1}{2}$  and vice versa. After much algebra we have

$$E(\hat{\theta}) \simeq \frac{1}{2}\pi\left\{\left(1.17 - 0.68p\right) - 0.97(1 - 4p)\Phi(L) - 3.88\left(\frac{1}{2}pq/\pi\right)^{\frac{1}{2}}e^{-\frac{1}{2}L^2}/N^{\frac{1}{2}}\right\}(q\mu_z + p\mu_y) \\ + pq(\mu_y - \mu_z)\{-0.68 + 3.88\Phi(L)\}/N],$$

where  $L = N(\frac{1}{4} - p)/(Npq)^{\frac{1}{2}}$ , and  $\Phi(\cdot)$  is the standard normal distribution function. As  $N \rightarrow \infty$ ,

$$\lim E(\hat{\theta}) = \begin{cases} \frac{1}{2}\pi(1.17 - 0.68p)(q\mu_z + p\mu_y) & (p > \frac{1}{2}), \\ \frac{1}{2}\pi(0.75\mu_z + 0.25\mu_y) & (p = \frac{1}{2}), \\ \frac{1}{2}\pi(0.20 + 3.2p)(q\mu_z + p\mu_y) & (p < \frac{1}{2}). \end{cases}$$

Thus for patterns for which  $2\theta/\{\pi(q\mu_z + p\mu_y)\}$  lies on one of the straight lines in Fig. 1, we have that  $\hat{\theta}$  is approximately unbiased, except when  $p$  is near  $\frac{1}{2}$  and for small  $N$ . However Table 2 shows that for the Poisson forest with  $N = 500$ , the bias is only  $0.03\theta$ .

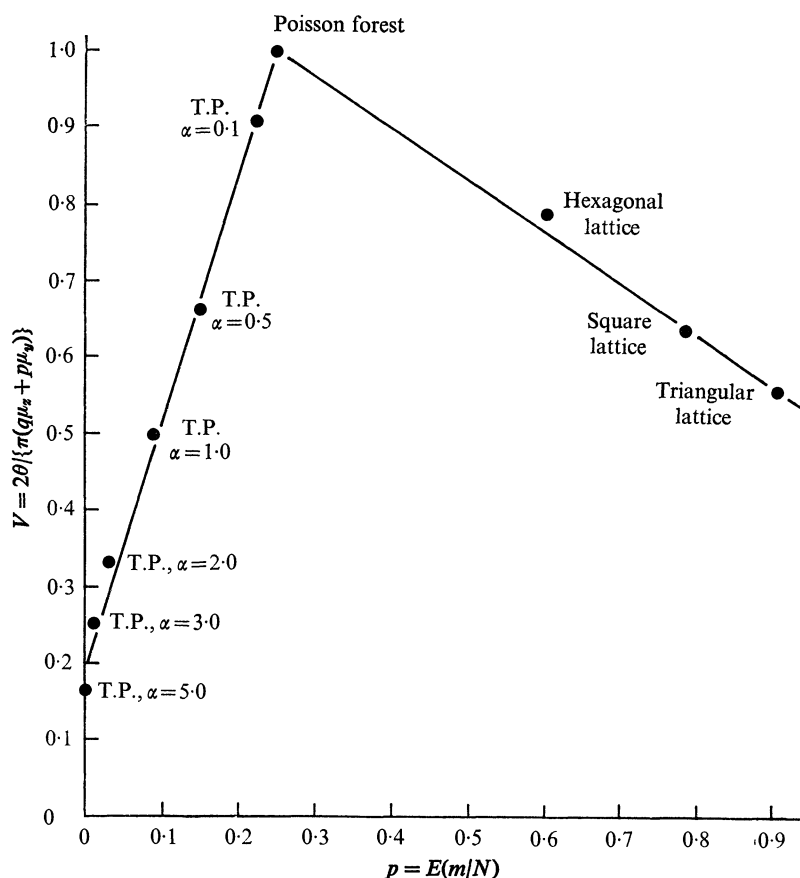


Fig. 1. The values of  $V = 2\theta/\{\pi(q\mu_z + p\mu_y)\}$  plotted against  $p = \text{pr}\{Y > 2X\}$  for the triangular, square and hexagonal lattices, the Poisson forest, and some Thomas processes, T.P., with mean number of offspring  $\alpha$ .

The expression for  $\text{var}(\hat{\theta})$  proves to be extremely long and awkward. Details can be obtained from the author.

Table 2 shows  $\text{var}(\hat{\theta})$  evaluated for the patterns considered in Table 1, for comparison with  $\text{var}(\hat{\theta}_i)$ , for the appropriate  $i$ . Also given in Table 2 are  $E(\hat{\theta})$  and  $E(\hat{\theta}_i)$ , for the appropriate  $i$ .

It can be shown that  $\text{var}(\hat{\theta}) \simeq 1.055\theta^2/N$  for the Poisson forest, and so  $\hat{\theta}$  is 47 % efficient compared to  $\hat{\theta}_L$  which is fully efficient. This is approximately midway between the efficiencies of  $\hat{\theta}_1$  and  $\hat{\theta}_2$  which are 85 % and 21 %. However, given that we know the spatial pattern is a Poisson forest,  $\hat{\theta}_L$  should be used as an estimator of  $\theta$ . From Table 2 it can be seen that  $\text{var}(\hat{\theta}) \simeq \text{var}(\hat{\theta}_i)$  for  $p$  not near  $\frac{1}{4}$  and for large  $N$ . Thus for this case  $\text{var}(\hat{\theta})$  can be estimated using an estimate of  $\text{var}(\hat{\theta}_i)$  based on (3), namely

$$\frac{1}{4}\pi^2[(a_i + b_i\tilde{p})^2(\tilde{p}\tilde{\sigma}_y^2 + \tilde{q}\tilde{\sigma}_z^2) + \tilde{p}\tilde{q}\{(a_i + b_i - 2b_i\tilde{q})\tilde{\mu}_z - (a_i + 2b_i\tilde{p})\tilde{\mu}_y\}^2]/N,$$

where

$$\tilde{p} = 1 - \tilde{q} = m/N, \quad \tilde{\mu}_z = n^{-1}\sum z_{1i}^2, \quad \tilde{\mu}_y = m^{-1}\sum y_{2j}^2, \quad \tilde{\sigma}^2 = n^{-1}\sum z_{1i}^4 - \tilde{\mu}_z^2, \\ \tilde{\sigma}_y^2 = m^{-1}\sum y_{2j}^4 - \tilde{\mu}_y^2.$$

In §5 the estimator  $\hat{\theta}$ , together with another to be described next, will be used on data in the form of the coordinates of trees. Estimated standard errors will also be given.

Table 2. *The values of  $E(\hat{\theta})$  and  $\text{var}(\hat{\theta})$  for comparison with  $E(\hat{\theta}_i)$ ,  $\text{var}(\hat{\theta}_i)$  ( $i = 1$  or  $2$ ), for the lattice pattern, Poisson forest and Thomas processes, each with  $\theta = 1$*

Pattern	$i$	$N = 50$				$N = 500$			
		$E(\hat{\theta}_i)$	$E(\hat{\theta})$	$\text{var}(\hat{\theta}_i)$	$\text{var}(\hat{\theta})$	$E(\hat{\theta}_i)$	$E(\hat{\theta})$	$\text{var}(\hat{\theta}_i)$	$\text{var}(\hat{\theta})$
Triangular lattice	1	1.00	1.00	$3.0 \times 10^{-3}$	$3.0 \times 10^{-3}$	1.00	1.00	$3.0 \times 10^{-4}$	$3.0 \times 10^{-4}$
Square lattice	1	1.01	1.01	$4.0 \times 10^{-3}$	$4.0 \times 10^{-3}$	1.01	1.01	$4.0 \times 10^{-4}$	$4.0 \times 10^{-4}$
Hexagonal lattice	1	0.97	0.97	$5.0 \times 10^{-3}$	$5.0 \times 10^{-3}$	0.97	0.97	$5.0 \times 10^{-4}$	$5.0 \times 10^{-4}$
Poisson forest	1	1.00	0.91	0.01	0.02	1.00	0.97	$1.0 \times 10^{-3}$	$2.0 \times 10^{-3}$
Poisson forest	2	1.00	0.91	0.05	0.02	1.00	0.97	$5.0 \times 10^{-3}$	$2.0 \times 10^{-3}$
Thomas process, $\alpha = 0.1$	2	1.02	0.96	0.05	0.03	1.02	1.01	$5.0 \times 10^{-3}$	$5.0 \times 10^{-3}$
Thomas process, $\alpha = 0.5$	2	1.03	1.02	0.07	0.07	1.03	1.03	$7.0 \times 10^{-3}$	$7.0 \times 10^{-3}$
Thomas process, $\alpha = 1.0$	2	0.99	0.99	0.08	0.08	0.99	0.99	$8.0 \times 10^{-3}$	$8.0 \times 10^{-3}$
Thomas process, $\alpha = 2.0$	2	0.92	0.92	0.07	0.07	0.92	0.92	$7.0 \times 10^{-3}$	$7.0 \times 10^{-3}$
Thomas process, $\alpha = 3.0$	2	0.96	0.96	0.05	0.05	0.96	0.96	$5.0 \times 10^{-3}$	$5.0 \times 10^{-3}$
Thomas process, $\alpha = 5.0$	2	1.23	1.23	0.03	0.03	1.23	1.23	$3.0 \times 10^{-3}$	$3.0 \times 10^{-3}$

#### 4. THE SECOND ESTIMATOR

We now propose an approximately unbiased estimator,  $\tilde{\lambda}$ , of  $\lambda = \theta^{-1}$ , the density of the trees, namely

$$\tilde{\lambda} = 4(\pi N)^{-1} \sum_{j=1}^m (y'_{2j})^{-2},$$

where  $Y'_{2j} = Y_{2j}$ , given that  $Y_{2j}$  is greater than some small fixed value  $\epsilon$ . Thus for the Poisson forest,  $Y'_{2j}$  has the p.d.f.  $2\pi^2\lambda^2y^3\exp(-\lambda\pi y^2)/\{(1+\lambda\pi\epsilon^2)\exp(-\lambda\pi\epsilon^2)\}$ , and hence

$$E\{(Y'_{2j})^{-2}\} = \lambda\pi/(1+\lambda\pi\epsilon^2),$$

$$\text{var}\{(Y'_{2j})^{-2}\} = \lambda^2\pi^2\{(1+\lambda\pi\epsilon^2)\text{Ei}(\lambda\pi\epsilon^2) - 1\}/(1+\lambda\pi\epsilon^2)^2,$$

where  $\text{Ei}(\cdot)$  is the exponential integral.

Now  $E(m/N)$  is no longer  $\frac{1}{2}N$  for the Poisson forest, but is reduced by the factor  $\text{pr}(Y_2 > \epsilon)$ , which is  $(1+\lambda\pi\epsilon^2)\exp(-\lambda\pi\epsilon^2)$ . Hence for the Poisson forest,  $E(\tilde{\lambda}) = \lambda\exp(-\lambda\pi\epsilon^2)$ , and thus  $E(\tilde{\lambda}) = \lambda + O(\epsilon^2)$ . Hence  $\tilde{\lambda}$  is unbiased for the Poisson forest as  $\epsilon \rightarrow 0$ . The reason for using  $Y'_2$  instead of  $Y_2$  is to prevent very small values of  $Y_2$  from inflating the estimator. Note that the variance of  $Y_2^{-2}$  is infinite, while, although that of  $(Y'_{2j})^{-2}$  is not infinite, it is very large for small  $\epsilon$ .

Another criticism of this estimator is that  $m$  will be small for aggregated patterns, and thus much of the available information will be lost; even for the Poisson forest, 75% of the information is lost. It is a high price to pay, but the estimator will be shown to be approximately unbiased for a wide range of spatial patterns.

For the triangular, square and hexagonal lattices we have the following:

$$E(m/N) = \frac{1}{2}\pi/\sqrt{3}, \quad E\{(Y'_2)^{-2}\} = \frac{1}{2}\sqrt{3}\lambda \quad (\text{triangular lattice});$$

$$E(m/N) = \frac{1}{4}\pi, \quad E\{(Y'_2)^{-2}\} = \lambda \quad (\text{square lattice});$$

$$E(m/N) = \frac{1}{3}\pi/\sqrt{3}, \quad E\{(Y'_2)^{-2}\} = \frac{2}{3}\sqrt{3}\lambda \quad (\text{hexagonal lattice}).$$

Thus  $E(\tilde{\lambda}) = \lambda$  for these three lattices and the variance of  $\tilde{\lambda}$  is zero for each. We now consider an aggregation model.

Suppose the trees are aggregated in very small areas of average size  $\delta$ , the areas being distributed with density  $\gamma$ . The sampling origin,  $O$ , will fall outside a clump with probability  $1 - \gamma\delta$ , and  $Y$  will almost certainly be much less than  $2X$ . With probability  $\gamma\delta$ ,  $X$  and  $Y$  will both be measured within a clump, and then  $\tilde{\lambda}$  will estimate the density,  $\sigma$  say, of the trees within a clump. Hence overall,  $E(\tilde{\lambda}) = \gamma\delta\sigma$ . However in a large area  $K$ , there are  $\gamma K$  clumps on average, and hence  $\gamma K\sigma\delta$  trees, and so the overall density,  $\lambda$ , equals  $\gamma\sigma\delta$ . Thus  $E(\tilde{\lambda}) = \lambda$ .

We have seen that  $\tilde{\lambda}$  is unbiased for models from extreme regularity to extreme aggregation. Hence we suggest that  $\tilde{\lambda}$  is unbiased for a wide range of spatial patterns.

The variance of  $\tilde{\lambda}$  is  $16(\pi^2N)^{-1}p[\text{var}\{(Y'_2)^{-2}\} + q\{E\{(Y'_2)^{-2}\}\}^2]$ , and can thus be estimated in a similar manner to  $\text{var}(\hat{\theta})$ , provided  $m$  is not zero.

## 5. RESULTS

The same data were used as by Cox & Lewis (1976). These were coordinates of trees from: (i) rectangular lattices with death of some of the individuals; (ii) Lansing Woods, Michigan, U.S.A.; (iii) simulated modified Thomas processes, with mean number of offspring  $\alpha$  and 'radial dispersion'  $s$ . Table 3 summarizes the results obtained for  $\tilde{\lambda}$  and  $\hat{\theta}$  with  $N = 100$ ,  $\epsilon = 0.01$ . The estimated standard errors are also given.

In spite of the large loss of information when using  $\tilde{\lambda}$ , this estimator seems to work remarkably well for patterns of a more regular type. The estimator  $\hat{\theta}$  did reasonably well although there were a few poor estimates, the worst being for the black oaks and the last of the modified Thomas processes. Better results may be obtainable for a larger sample size, although one must bear in mind the practical aspects of collecting actual data.



Table 3. True values of  $\lambda$ ,  $\theta$  and their estimates  $\tilde{\lambda}$ ,  $\hat{\theta}$  for various sets of data, with sample size  $N = 100$ 

Type of forest	$m$	$\lambda$	$\tilde{\lambda}$	Standard error of $\tilde{\lambda}$	$\theta$	$\hat{\theta}$	Standard error of $\hat{\theta}$
Lattice 1	25	1.54	1.08	0.21	0.65	0.58	0.05
Lattice 2	61	1.34	0.99	0.09	0.75	1.02	0.06
Lattice 3	50	1.34	1.35	0.14	0.75	0.70	0.04
Red oaks	27	1.61	1.81	0.47	0.62	0.75	0.06
Black oaks	24	0.60	0.47	0.12	1.65	2.98	0.40
White oaks	26	2.04	2.53	1.13	0.49	0.51	0.05
Hickories	21	3.18	2.66	1.15	0.31	0.46	0.07
Maples	20	2.36	3.61	1.19	0.42	0.56	0.08
Miscellaneous	11	0.48	0.45	0.22	2.10	3.00	0.53
M.T.P.† ( $\alpha = 5$ , $s = 1.0$ )	23	2.20	1.68	0.50	0.46	0.51	0.06
M.T.P. ( $\alpha = 2$ , $s = 1.0$ )	16	1.18	0.65	0.20	0.85	0.63	0.11
M.T.P. ( $\alpha = 2$ , $s = 0.2$ )	7	1.25	0.44	0.24	0.80	0.42	0.10

† Modified Thomas process

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