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Author(s): W. G. S. Hines and R. J. O'Hara Hines
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The Eberhardt statistic and the detection of nonrandomness of spatial point distributions

BY W. G. S. HINES AND R. J. O'HARA HINES
Department of Mathematics and Statistics, University of Guelph, Ontario

SUMMARY

Critical values are tabulated for Eberhardt's statistic. A variation of this statistic, adapted to T-square sampling, is found to perform well as a test of the complete randomness of spatial distributions, in power comparisons with other statistics recommended in the literature.

Some key words: Distance method; Eberhardt statistic; Point process; Spatial pattern; T-square sampling.

1. INTRODUCTION

A region D contains a large number of small items at unknown locations. In order to infer the approximate nature of the mechanism which has determined these locations, certain distance measurements are to be made at and near sampling locations completely randomly selected within a sampling frame S. This sampling frame is sufficiently remote from all boundaries of D to have rendered edge effects negligible. As item locations are unknown, procedures such as one proposed by Hopkins (1954) which require complete enumeration of the items are precluded.

The simplest possible hypothesis about the mechanism determining item locations is H_0; that the items have been distributed completely randomly within D. Collected data which are insufficiently informative to cause rejection of H_0 are unlikely to provide any clear indication of appropriate alternatives. In this paper we demonstrate that a variation of the Eberhardt statistic, adapted to T-square sampling, compares well with other statistics at detecting some forms of nonrandomness.

2. THE EBERHARDT STATISTIC AND ITS CRITICAL VALUES

For m sampling locations $O_1, \ldots, O_m$ completely randomly selected within S, Euclidean distances X_i from each O_i to the nearest item, at P_i, are measured. For low sampling intensities, under H_0 the X_i can be assumed independent and identically distributed with

$$\mathrm{pr}\,(X_i \leqslant x) = 1 - e^{-\lambda x^2}$$

(Holgate, 1972), where λ/π is the unknown mean number of items per unit area. Eberhardt's statistic A, defined by

$$A = m \sum_{j=1}^{m} X_i^2 \bigg/ \left(\sum_{j=1}^{m} X_j\right)^2,$$

has sampling distribution independent of λ under H_0. For single values of m, critical values of A can be estimated by Monte Carlo simulations such as carried out for a version of A using item-to-item distances by P. D. M. MacDonald (Bartlett, 1975, p. 61). For the routine use of A, a table of critical values such as Table 1 is needed. These critical values, while approximate, have been tested in two sets of simulations; 5000 replications for $m = 10$, 20, 30, 50 and 100, and 100,000 replications for $m = 10$, 20 and 40. There was significant evidence that, for

$m = 10$, low nominal levels of significance, i.e. 2·5%, 1·0%, 0·5%, for critical values for regular alternatives were conservative, being up to twice as large as actual levels. All other nominal levels of significance were found to have small relative errors, usually of a few percent or less. The tabulated critical values, superior in performance to critical values estimated using either a direct normal or a Pearson curve approximation, were obtained by fitting a normal distribution to a power law transformation of A, with the aid of moments obtained by relating A to a Dirichlet distribution (Wilks, 1962, § 7·7).

Table 1. *Critical values for* $A = m\Sigma X_i^2/(\Sigma X_i)^2$, *where m is the sample size*

	Regular alternative				Aggregate alternative			
m \ α	0·005	0·01	0·025	0·05	0·05	0·025	0·01	0·005
10	1·0340	1·0488	1·0719	1·0932	1·4593	1·5211	1·6054	1·6727
12	1·0501	1·0644	1·0865	1·1069	1·4472	1·5025	1·5769	1·6354
14	1·0632	1·0769	1·0983	1·1178	1·4368	1·4872	1·5540	1·6060
16	1·0740	1·0873	1·1080	1·1268	1·4280	1·4743	1·5352	1·5821
18	1·0832	1·0962	1·1162	1·1344	1·4203	1·4633	1·5195	1·5623
20	1·0912	1·1038	1·1232	1·1409	1·4136	1·4539	1·5061	1·5456
22	1·0982	1·1105	1·1293	1·1465	1·4078	1·4456	1·4945	1·5313
24	1·1044	1·1164	1·1348	1·1515	1·4025	1·4384	1·4844	1·5189
26	1·1099	1·1216	1·1396	1·1559	1·3978	1·4319	1·4755	1·5080
28	1·1149	1·1264	1·1439	1·1598	1·3936	1·4261	1·4675	1·4983
30	1·1195	1·1307	1·1479	1·1634	1·3898	1·4209	1·4604	1·4897
35	1·1292	1·1399	1·1563	1·1710	1·3815	1·4098	1·4454	1·4715
40	1·1372	1·1475	1·1631	1·1772	1·3748	1·4008	1·4333	1·4571
50	1·1498	1·1593	1·1738	1·1868	1·3644	1·3870	1·4151	1·4354
60	1·1593	1·1682	1·1818	1·1940	1·3565	1·3768	1·4017	1·4197
70	1·1668	1·1753	1·1882	1·1996	1·3504	1·3689	1·3915	1·4077
80	1·1730	1·1811	1·1933	1·2042	1·3455	1·3625	1·3833	1·3981
90	1·1782	1·1859	1·1976	1·2080	1·3414	1·3572	1·3765	1·3903
100	1·1826	1·1900	1·2013	1·2112	1·3379	1·3528	1·3709	1·3837
150	1·1979	1·2043	1·2139	1·2223	1·3260	1·3377	1·3519	1·3619
200	1·2073	1·2130	1·2215	1·2290	1·3189	1·3289	1·3408	1·3492
300	1·2187	1·2235	1·2307	1·2369	1·3105	1·3184	1·3279	1·3344
400	1·2257	1·2299	1·2362	1·2417	1·3055	1·3122	1·3203	1·3258
600	1·2341	1·2376	1·2429	1·2474	1·2995	1·3049	1·3113	1·3158
800	1·2391	1·2422	1·2468	1·2509	1·2960	1·3006	1·3061	1·3099
1000	1·2426	1·2454	1·2496	1·2532	1·2936	1·2977	1·3025	1·3059

3. SIMILAR DISTANCE BASED STATISTICS

A number of statistics, summarized in Table 2, were used in a simulation study of the relative power of the Eberhardt statistic, and of a variation, for testing H_0. Statistics II, IV, VI and X were not originally intended to provide tests of regularity, but have been adapted for such tests in this study.

Two versions of the univariate statistics, I and III, and II and IV, the latter with twice the sample size of the former, have been included in recognition of the lesser effort required in taking univariate rather than bivariate measurements at each sampling location.

In addition to the measurements X_i of § 2, the unrestricted bivariate sampling statistics require corresponding distances, Y_i, from each P_i to its nearest neighbouring item. For reasons discussed by Cox & Lewis (1976) and by Cormack (1977), of the m pairs of distance measurements, only those m' pairs for which $Y_i \leqslant 2X_i$, say for $i \in I$, are retained, and the statistics are then defined in terms of R_i, $i \in I$, obtained via

$$\theta_i = 2\sin^{-1}(\tfrac{1}{2}Y_i/X_i), \quad W_i = \{2\pi + \sin\theta_i - (\pi+\theta_i)\cos\theta_i\}^{-1}, \quad R_i = \tfrac{4}{3}(1-\pi W_i).$$

In this study, samples in which $m' = 0$ are taken as not rejecting H_0, although a composite test is possible in principle.

Table 2. *Statistics considered, with the notation of the cited references, and approximate or exact distributions under* H_0

Statistic	Distribution under H_0
(*a*) *Univariate sampling*	
I. $m\Sigma X_i^2/(\Sigma X_j)^2 = A$ (Eberhardt, 1967)	See critical values in Table 1
II. $12m(7m+1)^{-1}\{m \log \Sigma(X_i^2/m) - \Sigma \log X_i^2\} = M/C$ (Pollard, 1971)	χ^2_{m-1}
III. Statistic I, but with twice sample size	See critical values in Table 1
IV. Statistic II, but with twice sample size	χ^2_{2m-1}
(*b*) *Unrestricted bivariate sampling*	
V. Minimum of R_i $(i \in I)$ (Cox & Lewis, 1976)	Beta $(1, m')$
VI. $\Sigma_{i\in I} R_i/m' = \bar{R}$ (Cox & Lewis, 1976)	Normal, mean $\frac{1}{2}$, variance $(12m')^{-1}$
(*c*) T-*square sampling*	
VII. $m^{-1}\Sigma 2X_i^2/(2X_i^2+Z_i^2) = t_N$ (Diggle, Besag & Gleaves, 1976)	Normal, mean $\frac{1}{2}$, variance $(12m)^{-1}$
VIII. $2m^{-1}\Sigma\{\min(2X_i^2, Z_i^2)/(2X_i^2+Z_i^2)\} = t_N^*$ (Diggle, Besag & Gleaves, 1976)	Normal, mean $\frac{1}{2}$, variance $(12m)^{-1}$
IX. $\Sigma 2X_i^2/(\Sigma 2X_j^2+Z_j^2) = t_B$ (Diggle, Besag & Gleaves, 1976)	Beta (m, m)
X. $48m(13m+1)^{-1}[m \log \Sigma\{(2X_i^2+Z_i^2)/m\} - \Sigma \log(2X_i^2+Z_i^2)] = M/C$ (Diggle, 1977)	χ^2_{m-1}
XI. $2m(\Sigma 2X_i^2+Z_i^2)/(\Sigma X_j\sqrt{2}+Z_j)^2$, original with this paper	See critical values in Table 1, sample size $2m$

Notation: m, sample size; X_i, distance from sampling location to nearest item, at P_i; m', effective sample size for unrestricted bivariate sampling; R_i, transformation of ratio of distance from the item at P_i to its nearest item, to X_i; Z_i, distance from nearest item, subject to T-square sampling. Unless otherwise indicated, sums are over i or j from 1 to m.

Under regular alternatives statistics V, VI and VIII tend to be large, while all other statistics tend to be small. Under aggregate alternatives, statistics V and VI tend to be small, and all others large. Statistic V was not recommended by Cox & Lewis (1976) for detecting aggregation, and is not so used in the present paper.

The final five statistics are based on T-square sampling (Besag & Gleaves, 1973), in which a nearest neighbour distance Z_i is measured from P_i to the nearest item, say at Q_i, such that the angle $O_i P_i Q_i$ is at least a right angle. All m pairs of measurements (X_i, Z_i) are used. The final statistic tabulated, XI, is a variation of the Eberhardt statistic adapted to T-square sampling.

Two of the statistics, VI and VII, or $\bar{R}$ and t_N in the references cited, are very similar in form. The quantities R_i and $Z_i^2/(2X_i^2+Z_i^2)$ are the ratios of area searched for nearest neighbour to total area searched for unrestricted bivariate and for T-square sampling respectively. For $2X_i = Y_i = Z_i$ and trivially for $Y_i = Z_i = 0$, the area searched with the unrestricted procedure is 3/2 times that for T-square sampling. This suggests the approximation

$$R_i \simeq \tfrac{3}{2} Y_i^2/(2X_i^2 + Y_i^2),$$

which is found to have an absolute error never more than 0·01. The similarity in form between $\bar{R}$ and t_N is reflected by similarity of performance in the following, in spite of differences in measurement procedure and in effective sample size. A possible exception to this is a difference in power for the medium packing intensity regular alternative, where a conservative comparison of power finds a difference significant at a level of 1·4%, as in Table 3.

4. A comparison of statistical power

Four types of populations were examined: random, aggregate, regular and lattice cluster. Observed powers for the various statistics over ranges of alternative, nonrandom population types are summarized in Table 3 and in Figs 1 and 2. All hypothesis tests used were at 5% levels, and were one sided when testing for aggregate or for regular alternatives, or when statistics V and VIII were used. Other tests were two sided. The use of a one sided test for statistic V was in keeping with a recommendation of its proposers, Cox & Lewis (1976).

Aside from the completely regular and lattice cluster populations, which could be considered continued indefinitely, all populations were generated on a $1·4 \times 1·4$ square containing a centred $1·0 \times 1·0$ square sampling frame. Each generated population was used for a single sampling replication. Population sizes were roughly 1000 or greater, and sample sizes were $m = 20$, or 40 for statistics III and IV.

As a check both on the roughly 40 critical values to be used and on the statistical assumptions on which these values rested, 1000 replications of completely random populations were first generated and sampled. With the exception of 39 rejections of H_0 for the lower 2·5% critical value for statistic XI instead of the expected 25, no anomalies were found, and this one exception, significant at the 0·3% level, was considered unsurprising in view of the large number of tests being conducted. Hence, both critical values and underlying statistical assumptions were considered acceptable under H_0.

Aggregate populations were generated by the doubly stochastic Matérn cluster process (Matérn, 1971) with population sizes of just under 1200, or 2400 with a compensating scale change when sampling for statistics III and IV. Items were distributed completely randomly on discs of common diameter d, the disc centres having been completely randomly distributed on the $1·4 \times 1·4$ square. Powers observed in 200 replications for a range of disc diameters, d, and of the mean μ of the Poisson distributed cluster sizes are summarized in Fig. 1. Consequences of oversampling were investigated for $\mu = 8·0$, for which oversampling was probably most severe, by generating and sampling 200 replications of populations of 6000 items for d corresponding to 0·10, 0·15 and 0·20, but adjusted to compensate for increased population size. For $d = 0·10$ and 0·20, the statistics VIII and IX respectively had reductions in estimated powers, going from 73·5% to 59·5% and 35% to 21%. Aside from these differences, significant at 0·4% and 0·3% levels respectively, no significant or systematic effects were observed. The effect through resulting oversampling of the aggregation departure from H_0 therefore appears slight.

Populations with varying degrees of regularity were generated by a process termed the simple sequential inhibition process by Diggle, Besag & Gleaves (1976). For this process, which Ripley (1977) describes as a variant of one defined by Matérn (1960), item locations were considered distributed as centres of a sequence of nonoverlapping but otherwise completely randomly located discs of diameter d. Population generations continued until population sizes were 1000 or until 100 successive attempts to locate a next disc had failed. Because of the computing effort involved, only three values of d could be used, resulting in the discs covering approximately $\pi/20$, $3\pi/20$ and $5\pi/20$ of the total area available, with 100 replicates per value of d. Fortunately, these values of d, corresponding to low, medium and high packing intensity, provided examples of relative performances in situations where low,

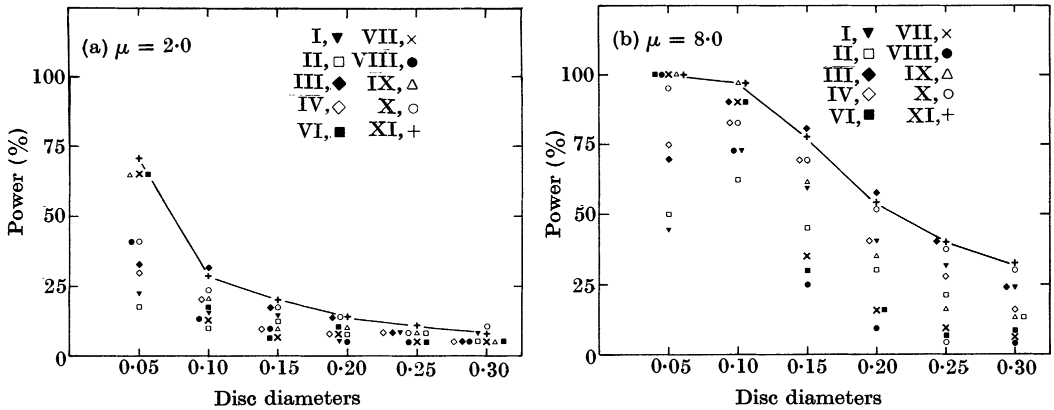

Fig. 1. Estimated powers, based on 200 replications, for the statistics of Table 2 suitable for aggregate alternatives, for increasing cluster dispersion, or disc diameter, and cluster size, μ, with side of square 1·4 units. All statistics use one sided tests with a 5% level of significance, and sample sizes of 20, or 40 for statistics III and IV. Statistic XI is emphasized.

Table 3. *Estimated powers for the statistics of Table 2, for four examples of populations listed in order of increasing regularity*

Statistic	Simple sequential inhibition process Packing density Low	Medium	High	Square lattice population
I	8	9	34	58·6
II	9	12	28	50·3
III	10	28	60	88·3
IV	7	25	47	74·5
V	38	97	99	97·0
VI	33	91	100	96·3
VII	23	76	98	100·0*
VIII	0	0	0	9·8
IX	16	45	81	100·0*
X	13	91	100	100·0*
XI	29	89	98	96·6

For simple sequential inhibition process, 100 replications were used; square lattice, 1000 replications.
* Power values are significantly greater than the corresponding value for statistic XI at a 1% level, as tested using McNemar's procedure for comparing frequencies in matched samples (Everitt, 1977, § 2·5).

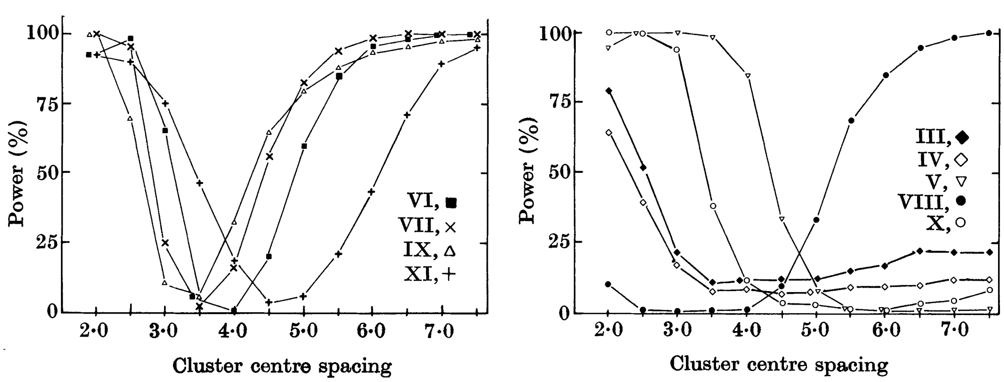

Fig. 2. Estimated powers, based on 1000 replications, for all statistics of Table 2 except I and II, which had similar but inferior powers to III and IV, for increasing cluster separation. All statistics use a 5% test, two sided except for statistics V and VIII; and use sample size of 20, or 40 for statistics III and IV.

moderate and high power were possible. The results are summarized in Table 3, along with results of 1000 replications for completely regular square lattice populations.

Possessing both regularity and aggregation, the square lattice cluster populations provided a severe test of all of the statistics considered, and one in which none of the statistics demonstrated uniformly good power. In Fig. 2 the powers of all statistics except I and II are displayed; these were similar but inferior to those of III and of IV respectively, and have been deleted for generally greater clarity. The population parameter varied in Fig. 2 is the ratio of the distance between cluster centres to the edge length of the square.

5. DISCUSSION

Foremost among the implications of the simulation study is perhaps the emergence of the Eberhardt statistic, XI, adapted to T-square sampling, as a useful statistic for the analysis of spatial data. The impression from Fig. 1 and from Table 3 that this statistic performs very well in detecting both aggregation and regularity is supported by statistical analyses using McNemar's test for comparing frequencies in related samples (Everitt, 1977, § 2·5). With the minor exception noted in Table 3, no statistic was ever found to possess significantly higher power, at a 1% level, at detecting either aggregate or regular alternatives, and only one, V, had significantly higher power at a 5% level for detecting regularity in the medium packing intensity. In contrast, in tests of individual levels of significance of 1%, statistic XI proved to have better power at detecting aggregation, for each μ, for at least some disc diameter, than any of the other statistics considered. This considerable success is the more surprising as the Eberhardt statistic is only one of a great many possible ways for testing whether a sampled distribution is exponential and the statistic does not appear to be distinguished by any theoretical motivation such as the likelihood ratio test of homogeneity of variances (Bartlett, 1937) on which statistics II, IV and X are based. While further improvement in power by the use of some other test of exponentiality is in principle possible, the results just described do suggest that such increases are likely to be slight, at least for tests of H_0 against aggregate or regular alternatives using bivariate distance measurements.

The example of the lattice cluster demonstrates that procedures based on single statistics for detecting nonrandomness are unlikely to prove universally suitable, and that, as Diggle (1977) has discussed, the use of compound procedures incorporating a number of statistics sensitive to different alternatives merits consideration. In the case of the lattice cluster, of course, examination of nearest neighbour distances alone, perhaps using the Eberhardt statistic or statistics described by Brown & Rothery (1978) would easily detect the nonrandomness present.

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