

GRADED BETTI NUMBERS OF THE JACOBIAN ALGEBRA OF SURFACES IN \mathbb{P}^3

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ABSTRACT. We compute an explicit closed formula for the Hilbert polynomial of the Jacobian algebra $M(f)$ of a reduced surface $X : f = 0$ in \mathbb{P}^3 in terms of the graded Betti numbers of the algebra $M(f)$. When X has only isolated singularities, a result by A. du Plessis and C. T. C. Wall yields new necessary condition for a set of positive integers to be the graded Betti numbers of the Jacobian algebra of such a surface. The comparison with the plane curve case is discussed in detail and additional information is given in the case of nodal surfaces.

1. INTRODUCTION

Let $S = \mathbb{C}[x, y, z, t]$ be the polynomial ring in four variables x, y, z, t with complex coefficients, and let $X : f = 0$ be a reduced surface of degree $d \geq 3$ in the complex projective space \mathbb{P}^3 . We denote by J_f the Jacobian ideal of f , i.e. the homogeneous ideal in S spanned by the partial derivatives f_x, f_y, f_z, f_t of f , and by $M(f) = S/J_f$ the corresponding graded quotient ring, called the Jacobian (or Milnor) algebra of f . Consider the general form of the minimal resolution of the Milnor algebra $M(f)$ of a reduced surface $X : f = 0$

$$(1.1) \quad 0 \longrightarrow \bigoplus_{k=1}^r S(1-d-b_k) \longrightarrow \bigoplus_{j=1}^q S(1-d-c_j) \longrightarrow \bigoplus_{i=1}^p S(1-d-d_i) \longrightarrow S^4(1-d) \longrightarrow S.$$

where $p \geq 3$, $q \geq 0$ and $r \geq 0$.

We call the ordered sequence of degrees

$$\mathbf{d} = (d_1, \dots, d_p), \quad \mathbf{c} = (c_1, \dots, c_q) \quad \text{and} \quad \mathbf{b} = (b_1, \dots, b_r)$$

the *graded Betti numbers of the Jacobian algebra $M(f)$* , since they determine and are determined by the usual graded Betti numbers of the Jacobian algebra $M(f)$ as defined for instance in [?].

It is known that there is a unique polynomial $P(M(f))(u) \in \mathbb{Q}[u]$, called the *Hilbert polynomial* of $M(f)$, and an integer $k_0 \in \mathbb{N}$ such that

$$(1.2) \quad H(M(f))(k) = P(M(f))(k)$$

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for all $k \geq k_0$. We denote by Σ the singular subscheme of X , which is defined by the Jacobian ideal $J(f)$. The general theory of Hilbert polynomials says that the degree of $P(M(f))$ is given by the dimension of the support of \mathcal{O}_Σ , the coherent sheaf associated to the graded S -module $M(f)$. Hence the assumption $\dim \Sigma = 0$ implies that the polynomial $P(M(f))$ is a constant, namely the total Tjurina number of X , given by

$$(1.3) \quad P(M(f)) = \tau(X) = \sum_{s \in \Sigma} \tau(X, s),$$

where $\tau(X, s)$ denotes the Tjurina number of the isolated singularity (X, s) , and $\dim \Sigma = 1$ implies that

$$(1.4) \quad P(M(f))(u) = au + b,$$

where $a = \deg(\Sigma)$, the degree of the subscheme Σ .

The first main result of this note is the following computation of the Hilbert polynomial $P(M(f))$ in terms of the graded Betti numbers of the Jacobian algebra $M(f)$ introduced in (??).

Theorem 1.1. *For the minimal resolution (??) of the Jacobian algebra $M(f)$ of a reduced surface $X : f = 0$ of degree d in \mathbb{P}^3 , one has the following.*

(1) *For any such surface, one has*

$$p + r = q + 3 \text{ and } \sum_{i=1}^p d_i - \sum_{j=1}^q c_j + \sum_{k=1}^r b_k = d - 1.$$

(2) *The surface X has at most isolated singularities if and only if*

$$(d - 1)^2 + \sum_{i=1}^p d_i^2 - \sum_{j=1}^q c_j^2 + \sum_{k=1}^r b_k^2 = 0.$$

If this is the case, then the total Tjurina number of X is given by the formula

$$6\tau(X) = (d - 1)^3 - \sum_{i=1}^p d_i^3 + \sum_{j=1}^q c_j^3 - \sum_{k=1}^r b_k^3.$$

(3) *The surface X is smooth if and only if $p = 6$, $q = 4$, $r = 1$ and*

$$\mathbf{d} = (d - 1)_6, \quad \mathbf{c} = (2d - 2)_4 \text{ and } \mathbf{b} = 3(d - 1).$$

(4) *If the surface $X : f = 0$ has a 1-dimensional singularity subscheme Σ , then the Hilbert polynomial $P(M(f))$ of the Jacobian algebra $M(f)$ is given by*

$$P(M(f))(u) = \frac{A}{2}u - B,$$

where

$$A = (d - 1)^2 + \sum_{i=1}^p d_i^2 - \sum_{j=1}^q c_j^2 + \sum_{k=1}^r b_k^2$$

and

$$B = \frac{d(d-3)^2 - 4}{3} + \frac{(d-3)}{2} \left(\sum_{i=1}^p d_i^2 - \sum_{j=1}^q c_j^2 + \sum_{k=1}^r b_k^2 \right) + \frac{1}{6} \left(\sum_{i=1}^p d_i^3 - \sum_{j=1}^q c_j^3 + \sum_{k=1}^r b_k^3 \right).$$

When the surface X has only isolated singularities, that is when $\dim \Sigma = 0$, then using a result by A. du Plessis and C.T.C. Wall quoted below in Theorem ??, we obtain the following new restrictions on the graded Betti numbers of the Jacobian algebra $M(f)$.

Corollary 1.2. *If the surface X has isolated singularities, then*

$$6d_1(d-d_1-1)(d-1) \leq \sum_{i=1}^p d_i^3 - \sum_{j=1}^q c_j^3 + \sum_{k=1}^r b_k^3 \leq 6d_1(d-1)^2.$$

Theorem ?? is proved in Section 2 and Corollary ?? is proved in Section 3.

In Section 4 a comparison with the plane curve case is discussed in detail. Surprisingly, many facts holding for the graded Betti numbers of curves fail in the case of surfaces, see for instance Proposition ?? and Remark ???. The similar behavior of the graded Betti numbers of maximal Tjurina curves and the graded Betti numbers of surfaces with large number of nodes, such as the Cayley surface from Example ?? and the Kummer surface from Example ?? is highlighted in Remark ?? and it may be an interesting direction of further research. However, Example ?? and Proposition ?? show that this analogy is rather subtle.

In Section 5 we collect a number of additional examples. All the minimal resolutions corresponding to (??) are computed in this note using the Computer Algebra softwares CoCoA [?] and SINGULAR [?].

2. PROOF OF THEOREM ??

We start with the following two Lemmas, whose proofs are elementary and straightforward, so we leave them to the reader.

Lemma 2.1. *For any integer a and $k > |a|$, one has*

$$\dim S_{k+a} = \binom{k+a+3}{3} = \frac{k^3 + 3(a+2)k^2 + (3a^2 + 12a + 11)k + a^3 + 6a^2 + 11a + 6}{6}.$$

Using the resolution (??), we get the following

$$(2.1) \quad \dim M(f)_{s+d-1} = \dim S_{s+d-1} - 4 \dim S_s + \sum_{i=1}^p \dim S_{s-d_i} - \sum_{j=1}^q \dim S_{s-c_j} + \sum_{k=1}^r \dim S_{s-b_k},$$

for any s large enough. Using now Lemma ??, we get the following.

Lemma 2.2. *With the above notation, for any large integer s , one has the equality*

$$6 \dim M(f)_{s+d-1} =$$

$$\begin{aligned}
&= (p-q+r-3)s^3 + 3(d-1-d_1-d_2-d_3 - \sum_{i=4}^p (d_i-2) + \sum_{j=1}^q (c_j-2) - \sum_{k=1}^r (b_k-2))s^2 + \\
&\quad + (3d^2+6d+2-44 + \sum_{i=1}^p (3d_i^2-12d_i+11) - \sum_{j=1}^q (3c_j^2-12c_j+11) + \sum_{k=1}^r (3b_k^2-12b_k+11))s + \\
&\quad + d^3+3d^2+2d-24 - \sum_{i=1}^p (d_i^3-6d_i^2+11d_i-6) + \sum_{j=1}^q (c_j^3-6c_j^2+11c_j-6) - \sum_{k=1}^r (b_k^3-6b_k^2+11b_k-6).
\end{aligned}$$

Using these computations, we can prove Theorem ?? as follows. By definition of the Hilbert polynomial $P(M(f))$ we have

$$6P(M(f))(s+d-1) = 6 \dim M(f)_{s+d-1}.$$

Since the surface X is reduced, we have

$$\deg P(M(f)) = \dim \Sigma \leq 1,$$

and hence the coefficients of s^3 and of s^2 in Lemma ?? must vanish. This proves the claim (1).

To prove the claim (2), we note that X has at most isolated singularities if and only if $\dim \Sigma < 1$. This last condition is equivalent to the vanishing of the coefficients of s in Lemma ??, in addition to the vanishings from the claim (1). When all these vanishing holds, then we conclude the proof of claim (2) by using (??) and the expression of the constant term in Lemma ??, simplified by using the equalities in (1) and the first equality in (2).

To prove the claim (3), assume first that $X : f = 0$ is smooth. Then the partial derivatives f_x, f_y, f_z and f_t form a regular sequence in S and the resolution of $M(f)$ is well known in this case, and has the form

$$0 \rightarrow S(4-4d) \rightarrow S(3-3d)^4 \rightarrow S(2-2d)^6 \rightarrow S(1-d)^4 \rightarrow S.$$

It follows that **d**, **c** and **b** are given by the equalities in claim (3) when X is smooth. Conversely, if **d**, **c** and **b** are given by the equalities in claim (3), then using the claim (2) we see that X has at most isolated singularities and that $\tau(X) = 0$. Therefore the surface X is smooth.

The proof of claim (4) follows directly from (??) and Lemma ??, where s has to be replaced by $u - (d-1)$.

3. PROOF OF COROLLARY ??

One has the following result, see [?, Theorem 5.3].

Theorem 3.1. *If the surface X has at most isolated singularities, then*

$$(d-1)^3 - d_1(d-1)^2 \leq \tau(X) \leq (d-1)^3 - d_1(d-d_1-1)(d-1).$$

In the case of plane curves, the corresponding result was obtained in [?], and played a key role in the understanding of free curves. Indeed, the reduced curve C is free if and only if

$$\tau(C) = (d-1)^2 - d_1(d-d_1-1),$$

i.e. the upper bound is attained, see [?, ?] for related results. A free surface X has necessarily non-isolated singularities, and so freeness must be related to other invariants, see for instance [?].

Remark 3.2. The lower bound in Theorem ?? is attained for any pair (d, d_1) . Indeed, it is enough to find a degree d , reduced curve $C : f'(x, y, z) = 0$ such that d_1 is the minimal exponent of C and

$$\tau(C) = (d - d_1 - 1)(d - 1),$$

and then take $X : f = 0$, with

$$f(x, y, z, t) = f'(x, y, z) + t^d.$$

The existence of curves C as above is shown in [?, Example 4.5] and a complete characterization of them is given in [?, Theorem 3.5 (1)].

Remark 3.3. The upper bound in Theorem ?? is attained for any pair (d, d_1) with $2d_1 < d$, since for such pairs (d, d_1) the existence of free plane curves $C : f' = 0$ of degree d and with exponents (d_1, d_2) is shown in [?] and then one constructs the surface X as in Remark ?? above. It is an interesting *open question* to improve the upper bound in Theorem ?? when $2d_1 \geq d$. The best upper bound for such pairs is (at least conjecturally) known in the case of plane curves, see [?, ?], and is given by the stronger inequality

$$\tau(C) \leq (d - 1)^2 - d_1(d - d_1 - 1) - \binom{2d_1 + 2 - d}{2}.$$

If we start with a degree d and a reduced curve $C : f'(x, y, z) = 0$ such that $d_1 \geq d/2$ and take $X : f = 0$, with

$$f(x, y, z, t) = f'(x, y, z) + t^d,$$

then f and f' have the same minimal exponent d_1 and

$$(3.1) \quad \tau(X) \leq (d - 1)^3 - d_1(d - d_1 - 1)(d - 1) - \binom{2d_1 + 2 - d}{2}(d - 1).$$

However, this stronger inequality fails for surfaces not constructed as suspensions of plane curves, as the following examples show.

Example 3.4. Consider the Cayley surface

$$X : f = xyz + xyt + xzt + yzt = 0$$

in \mathbb{P}^3 having four A_1 -singularities. Then $d = 3$, $d_1 = 2 > d/2$ and $\tau(X) = 4$. Indeed, the minimal resolution of the Jacobian algebra is given by

$$0 \rightarrow S[-6]^2 \rightarrow S[-5]^8 \rightarrow S[-4]^9 \rightarrow S[-2]^4 \rightarrow S$$

and hence

$$\mathbf{d} = (2_9), \mathbf{c} = (3_8) \text{ and } \mathbf{b} = (4_2).$$

The inequality in Theorem ?? is in this case $0 \leq \tau(X) \leq 8$, while the bound given by (??) is 2, which is clearly not good.

Example 3.5. Consider the Kummer surface

$$X : f = x^4 + y^4 + z^4 + t^4 - y^2z^2 - z^2x^2 - x^2y^2 - x^2t^2 - y^2t^2 - z^2t^2 = 0$$

in \mathbb{P}^3 having sixteen A_1 -singularities. Then $d = 4$, $d_1 = 3 > d/2$ and $\tau(X) = 16$. Indeed, the minimal resolution of the Jacobian algebra is given by

$$0 \rightarrow S(-8)^3 \rightarrow S(-7)^{12} \rightarrow S(-6)^{12} \rightarrow S(-3)^4 \rightarrow S \rightarrow 0$$

and hence

$$\mathbf{d} = (3_{12}), \mathbf{c} = (4_{12}) \text{ and } \mathbf{b} = (5_3).$$

The inequality in Theorem ?? is in this case $0 \leq \tau(X) \leq 27$, while the bound given by (??) is 15, which is clearly not good.

Example 3.6. Consider finally the octic surface with 144 nodes

$$\begin{aligned} X : f = 16(x^8 + y^8 + z^8 + t^8) + 224(x^4y^4 + x^4z^4 + x^4t^4 + y^4z^4 + y^4t^4 + z^4t^4) + \\ + 2688x^2y^2z^2t^2 - 9(x^2 + y^2 + z^2 + t^2)^4 = 0. \end{aligned}$$

This is a special case of Chebyshev hypersurfaces, which are classical examples of nodal hypersurfaces with many singularities. They were introduced by Chmutov to construct complex projective hypersurfaces with a large number of nodes, see [?] Vol. 2, p. 419, [?] as well as [?, Corollary 3.2, (iii)]. The minimal resolution of the Jacobian algebra is given by

$$0 \rightarrow S(-20)^4 \oplus S(-22) \rightarrow S(-17)^4 \oplus S(-18)^{13} \rightarrow S(-14)^6 \oplus S(-16)^9 \rightarrow S(-7)^4 \rightarrow S$$

and hence

$$\mathbf{d} = (7_6, 9_9), \mathbf{c} = (10_4, 11_{13}) \text{ and } \mathbf{b} = (13_4, 15).$$

The inequality in Theorem ?? is in this case $0 \leq \tau(X) = 144 \leq 343$, while the bound given by (??) is 147, which is also good.

4. COMPARISON TO THE CURVE CASE

Let $R = \mathbb{C}[x, y, z]$ and consider the general form of the minimal resolution of the Milnor algebra $M(f')$ of a reduced plane curve $C : f' = 0$, which is assumed not to be free

$$(4.1) \quad 0 \longrightarrow \bigoplus_{j=1}^{q'} R(1-d-c'_j) \longrightarrow \bigoplus_{i=1}^{p'} R(1-d-d'_i) \longrightarrow R^3(1-d) \longrightarrow R,$$

with $c'_1 \leq \dots \leq c'_{q'}$ and $d'_1 \leq \dots \leq d'_{p'}$, where $d = \deg f'$, see for instance [?]. One has

$$p' = q' + 2,$$

which corresponds to the first equality in Theorem ?? (1) above. It follows from [?, Lemma 1.1] that

$$(4.2) \quad \epsilon_j = d'_{j+2} - c'_j \geq 1 \quad j = 1, \dots, q'.$$

Using [?, Formula (13)], one obtains the relation

$$(4.3) \quad d'_1 + d'_2 = d - 1 + \sum_{j=1}^{q'} \epsilon_j$$

or, equivalently,

$$(4.4) \quad \sum_{i=1}^{p'} d_i - \sum_{j=1}^{q'} c'_j = d - 1,$$

which corresponds to the second equality in Theorem ?? (1) above.

A new invariant was recently introduced for a reduced plane curve C , namely the type of C defined by

$$(4.5) \quad t(C) = d'_1 + d'_2 - d + 1,$$

see [?]. When the curve C is not free, then one clearly has

$$(4.6) \quad t(C) = \sum_{j=1}^{q'} \epsilon_j.$$

This invariant has nice properties, for instance a reduced plane curve C is free (resp. plus-one generated) if and only if $t(C) = 0$ (resp. $t(C) = 1$). More, one has $t(C) \geq 0$ for any reduced curve, and the curves with $t(C) = 2$ and $t(C) = 3$ have been studied in detail in [?, ?]. One may try to extend this invariant to surfaces X in \mathbb{P}^3 by setting

$$(4.7) \quad t(X) = d_1 + d_2 + d_3 + 1 - d.$$

The next result shows that this case is much more complicated.

Proposition 4.1. *Let $X : f = 0$ be a reduced surface in \mathbb{P}^3 with the minimal resolution (??). Then*

$$t(X) = d_1 + d_2 + d_3 + 1 - d \geq 0$$

if there are first order Jacobian syzygies ρ_1, ρ_2 and ρ_3 which are linearly independent over the field of fractions K of the polynomial ring S and such that $\deg \rho_j = d_j$ for $j = 1, 2, 3$.

Note that if the surface X is tame with respect to the pair (ρ_1, ρ_2) as in [?, Definition 1.2], then the first order Jacobian syzygies ρ_1, ρ_2 and ρ_3 are linearly independent over the field K for any new additional syzygy ρ_3 . However, unlike the curve case, Example ?? shows that in general one may have $t(X) < 0$.

Proof. Let $M(E, \rho_1, \rho_2, \rho_3)$ be the 4×4 matrix with the first row (x, y, z, t) and the j -th row, for $j = 2, 3, 4$ being given by the components of the syzygy ρ_{j-1} . Then

$$g = \det M(E, \rho_1, \rho_2, \rho_3) \neq 0$$

if and only if the syzygies ρ_1, ρ_2 and ρ_3 are linearly independent over the field K . On the other hand, at any smooth point $s \in X$, the rows of $M(E, \rho_1, \rho_2, \rho_3)$ evaluated at s are tangent vectors to X at s . Since their number is larger than the dimension of this vector space, it follows that $g(s) = 0$ at any smooth point of X . Therefore g

vanishes on X , and hence g is divisible by f , the surface X being reduced. It follows that

$$1 + d_1 + d_2 + d_3 = \deg g \geq \deg f = d,$$

which proves our claim. \square

To see how the equality (??) extends to the surface case, one may proceed as follows. If we set $\alpha_j = c_j - d_{j+3}$ for $j = 1, \dots, p-3$ and $\beta_k = b_k - c_{p-3+k}$ for $k = 1, \dots, r$, it follows from Theorem ?? (1) that one has

$$t(X) = \sum_{j=1}^{p-3} \alpha_j - \sum_{k=1}^r \beta_k.$$

Remark 4.2. It is not true that for any reduced surface X one has $\alpha_j \geq 1$ for $j = 1, \dots, p-3$ and $\beta_k \geq 1$ for $k = 1, \dots, r$, see Examples ??, ?? and ?. This is surprising if compared with (??).

Remark 4.3. A reduced plane curve is said to be *Tjurina maximal* if $2d'_1 \geq d$ and

$$\tau(C) = (d-1)^2 - d'_1(d-d'_1-1) - \binom{2d'_1 + 2 - d}{2}.$$

It is known that for such a curve one has $\epsilon_j = 1$ for all $j = 1, \dots, q'$ and $d'_1 = \dots = d'_{p'}$, see [?, Theorem 3.1]. It is interesting to note that the Cayley and the Kummer surfaces considered in Examples ?? and ?? enjoy similar properties for their graded Betti numbers, that is the d_i 's are all equal, and $\alpha_j = \beta_k = 1$ for $j = 1, \dots, d-3$ and $k = 1, \dots, r$. It would be nice to have a theoretical explanation for this fact.

The nodal surface of degree $d = 8$ in Example ?? has two values for the degrees d_i 's, namely one has in this case $\mathbf{d} = (7_6, 9_9)$. A partial explanation of these two values is given by the following result. Before stating it, we need some notation. Consider the graded S -module of *all Jacobian relations* of f or, equivalently, the module of derivations killing f , namely

(4.8)

$$AR(f) = \{\theta \in Der(S) : \theta(f) = 0\} = \{\rho = (a_0, a_1, \dots, a_n) \in S^{n+1} : \sum_0^n a_j f_j = 0\}.$$

Inside this S -module, there is the graded S -submodule of *Koszul type relations* $KR(f)$ generated by the N Koszul type derivations $\theta_{i,j}$ of degree $d-1$, where

$$N = \binom{4}{2} = 6$$

and $\{i, j\}$ is a subset of $\{x, y, z, t\}$ having 2 elements and

$$\theta_{i,j}(f) = f_i f_j - f_j f_i = 0.$$

Let $ER(f) = AR(f)/KR(f)$ be the quotient module and let

$$mdr(f) = \min\{k : ER(f)_k \neq 0\}.$$

Proposition 4.4. *Let $X : f = 0$ be a nodal surface in \mathbb{P}^3 of degree $d \geq 5$. Then*

$$\text{mdr}(f) \geq 2d - \left\lfloor \frac{d}{2} \right\rfloor - 3 > d - 1.$$

In particular, the graded Betti numbers \mathbf{d} of the surface X satisfy

$$d_i = d - 1 \text{ for } i = 1, \dots, N$$

and

$$d_i \geq 2d - \left\lfloor \frac{d}{2} \right\rfloor - 3$$

for $i > N$.

Proof. The inequality

$$\text{mdr}(f) \geq 2d - \left\lfloor \frac{d}{2} \right\rfloor - 3$$

follows from [?, Equation (4.5) and Corollary 4.3]. The inequality

$$2d - \left\lfloor \frac{d}{2} \right\rfloor - 3 \geq d - 1$$

clearly holds for any $d \geq 5$. Since the N Koszul type derivations θ_{ij} are linearly independent in $AR(f)_{d-1}$, we need to have exactly N elements of degree $d - 1$ in any minimal set of generators for $AR(f)$. The other generators we add yield nonzero elements in $ER(f)$, and hence their degree is at least $\text{mdr}(f)$. \square

Remark 4.5. (i) Note that in Example ?? we have a nodal surface of degree $d = 8$ and the corresponding module $AR(f)$ has $N = 6$ generators of degree $d - 1 = 7$ and the remaining generators have degree $d_i = 9$ for $i > 6$. Since in this case

$$2d - \left\lfloor \frac{d}{2} \right\rfloor - 3 = 9,$$

we see that our Proposition ?? is sharp.

(ii) A similar result to Proposition ?? may be stated in the more general situation when the surface X has only isolated weighted homogeneous singularities, by using [?, Theorem 9], which extends the results in [?] to this setting.

5. EXAMPLES

Example 5.1. Consider the cubic surface

$$X : f = xyz - t^3 = 0.$$

A direct computation using CoCoA or SINGULAR shows that the corresponding minimal resolution for $M(f)$ is given by

$$0 \rightarrow S(-5)^2 \rightarrow S(-3)^2 \oplus S(-4)^3 \rightarrow S(-2)^4 \rightarrow S.$$

It follows that $p = 5$, $q = 2$ and $r = 0$, and the graded Betti numbers of this surface are

$$\mathbf{d} = (1_2, 2_3) \text{ and } \mathbf{c} = (3_2),$$

with the convention that u_v means u is repeated v times. The surface has 3 singularities A_3 , hence $\tau(X) = 6$. All the claims in Theorem ?? hold. With the notation from Remark ?? one has

$$\alpha_1 = c_1 - d_4 = 3 - 2 = 1 \text{ and } \alpha_2 = c_2 - d_5 = 3 - 2 = 1.$$

Example 5.2. Consider the cubic surface

$$X : f = txz + y^2z + x^3 - z^3 = 0.$$

A direct computation shows that the corresponding minimal resolution for $M(f)$ is given by

$$0 \rightarrow S(-6) \rightarrow S(-5)^5 \rightarrow S(-3) \oplus S(-4)^6 \rightarrow S(-2)^4 \rightarrow S.$$

It follows that $p = 7$, $q = 5$ and $r = 1$, and the graded Betti numbers of this surface are

$$\mathbf{d} = (1, 2_6), \mathbf{c} = (3_5) \text{ and } \mathbf{b} = (4).$$

The surface has $\tau(X) = 5$. All the claims in Theorem ?? hold and one has

$$\alpha_1 = c_1 - d_4 = 3 - 2 = 1, \alpha_2 = c_2 - d_5 = 3 - 2 = 1, \alpha_3 = c_3 - d_6 = 3 - 2 = 1,$$

$$\alpha_4 = c_4 - d_7 = 3 - 2 = 1, \beta_1 = b_1 - c_5 = 4 - 3 = 1.$$

Example 5.3. Consider the sextic surface

$$X : f = x^5z + y^6 + x^4yt + xy^5 = 0.$$

A direct computation shows that the corresponding minimal resolution for $M(f)$ is given by

$$0 \rightarrow S(-9) \rightarrow S(-6) \oplus S(-7) \oplus S(-8)^2 \rightarrow S(-5)^4 \rightarrow S.$$

It follows that $p = 4$, $q = 1$ and $r = 0$, and the graded Betti numbers of this surface are

$$\mathbf{d} = (1, 2, 3_2) \text{ and } \mathbf{c} = (4).$$

The surface X is nearly-free and has a 1-dimensional singular locus, with the Hilbert polynomial

$$P(M(f))(u) = 16u - 27.$$

All the claims in Theorem ?? hold and one has $\alpha_1 = c_1 - d_4 = 4 - 3 = 1$.

Example 5.4. Consider the surface of degree $d = 9$ given by

$$X : f = (x^3 - yzt)^3 + (t^3 - xyz)^3 = 0.$$

Then X is a union of 3 cubic surfaces in \mathbb{P}^3 and a direct computation shows that the corresponding minimal resolution for $M(f)$ is given by

$$\begin{aligned} 0 &\rightarrow S(-18) \oplus S(-19) \rightarrow S(-13) \oplus S(-16) \oplus S(-17)^3 \oplus S(-18)^2 \rightarrow \\ &\rightarrow S(-9) \oplus S(-12)^2 \oplus S(-15)^2 \oplus S(-16)^3 \rightarrow S(-8)^4 \rightarrow S. \end{aligned}$$

It follows that $p = 8$, $q = 7$ and $r = 2$, and the graded Betti numbers of this surface are

$$\mathbf{d} = (1, 4_2, 7_2, 8_3), \mathbf{c} = (5, 8, 9_3, 10_2) \text{ and } \mathbf{b} = (10, 11).$$

All the claims in Theorem ?? hold, in particular one has

$$P(M(f))(u) = 38u - 119.$$

With the notation from Remark ?? one has

$$\beta_1 = b_1 - c_6 = 10 - 10 = 0.$$

Example 5.5. Consider the surface of degree $d = 12$ given by

$$X : f = (x^3 - yzt)^4 + (t^3 - xyz)^4 = 0.$$

Then X is a union of 4 cubic surfaces in \mathbb{P}^3 and a direct computation shows that the corresponding minimal resolution for $M(f)$ is given by

$$\begin{aligned} 0 \rightarrow S(-25)^2 &\rightarrow S(-16) \oplus S(-23)^2 \oplus S(-24)^4 \rightarrow \\ &\rightarrow S(-12) \oplus S(-15)^2 \oplus S(-22)^5 \rightarrow S(-11)^4 \rightarrow S. \end{aligned}$$

It follows that $p = 8$, $q = 7$ and $r = 2$, and the graded Betti numbers of this surface are

$$\mathbf{d} = (1, 4_2, 11_5), \mathbf{c} = (5, 12_2, 13_4) \text{ and } \mathbf{b} = (14_2).$$

All the claims in Theorem ?? hold, in particular one has

$$P(M(f))(u) = 81u - 491.$$

With the notation from Remark ?? one has

$$\alpha_1 = c_1 - d_4 = 5 - 11 = -6, \alpha_2 = \alpha_3 = 1, \alpha_4 = \alpha_5 = 2$$

and

$$\beta_1 = b_1 - c_6 = 14 - 13 = 1 = b_2 - c_7 = \beta_2.$$

It follows that $t(S) = -2 < 0$ in this case.

Example 5.6. Consider the surface of degree $d = 16$ given by

$$X : f = (x^4 - yzt^2)^4 + (t^4 - xyz^2)^4 = 0.$$

Then X is a union of 4 quartic surfaces in \mathbb{P}^3 and a direct computation shows that the corresponding minimal resolution for $M(f)$ is given by

$$\begin{aligned} 0 \rightarrow S(-29) \oplus S(-30) \oplus S(-32) &\rightarrow S(-22)^2 \oplus S(-23) \oplus S(-28)^4 \oplus S(-29)^3 \oplus S(-31)^2 \rightarrow \\ &\rightarrow S(-20)^2 \oplus S(-21)^3 \oplus S(-27)^4 \oplus S(-28)^2 \oplus S(-30) \rightarrow S(-15)^4 \rightarrow S. \end{aligned}$$

It follows that $p = 12$, $q = 12$ and $r = 3$, and the graded Betti numbers of this surface are

$$\mathbf{d} = (5_2, 6_3, 12_4, 13_2, 15), \mathbf{c} = (7_2, 8, 13_4, 14_3, 16_2) \text{ and } \mathbf{b} = (14, 15, 17).$$

All the claims in Theorem ?? hold, in particular one has

$$P(M(f))(u) = 147u - 1382.$$

With the notation from Remark ?? one has

$$\alpha_9 = c_9 - d_{12} = 14 - 15 = -1,$$

$$\beta_1 = b_1 - c_{10} = 14 - 14 = 0,$$

and

$$\beta_2 = b_2 - c_{11} = 15 - 16 = -1.$$

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