Entropy formula for surface diffeomorphisms

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Abstract

Let f be a C^r $(r>1)$ diffeomorphism on a compact surface M with $h_{\text{top}}(f) \geq \frac{\lambda^+(f)}{r}$ where $\lambda^+(f) := \lim_{n\to+\infty} \frac{1}{n} \max_{x\in M} \log \left\| Df_x^n \right\|$. We establish an equivalent formula for the topological entropy:

$$h_{\text{top}}(f) = \lim_{n\to+\infty} \frac{1}{n} \log \int_M \left\| Df_x^n \right\| dx.$$

Our approach builds on the key ideas developed in the works of Buzzi-Crovisier-Sarig (*Invent. Math.*, 2022) and Burguet (*Ann. Henri Poincaré*, 2024) concerning the continuity of the Lyapunov exponents.

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1 Introduction

Let f be a diffeomorphism on a compact surface M. Write

$$\lambda^+(f) := \lim_{n\to+\infty} \frac{1}{n} \max_{x\in M} \log \left\| Df_x^n \right\|.$$

The sequence $\left(\max_{x\in M} \log \left\| Df_x^n \right\|\right)_{n\geq 1}$ is sub-additive, and hence the above limit exists. For surface diffeomorphisms with large entropy, we establish an equivalent formula for the topological entropy in terms of the *volume growth of the tangent cocycle* :

Theorem A. *For any C^r $(r>1)$ diffeomorphism f on a compact surface M with $h_{\text{top}}(f) \geq \frac{\lambda^+(f)}{r}$, we have*

$$h_{\text{top}}(f) = \lim_{n\to+\infty} \frac{1}{n} \log \int_M \left\| Df_x^n \right\| dx.$$

Let us first make some remarks about Theorem **??**.

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- Note that the integral is taken with respect to the Lebesgue measure. The Lyapunov regular set (that is, the set of points with well-defined Lyapunov exponents) usually has zero Lebesgue measure and therefore does not contribute to the integral above. Even for ergodic volume-preserving Anosov systems-where Lebesgue almost every point has a positive Lyapunov exponent equal to the entropy of the volume measure (Lebesgue measure)-the integral appearing in Theorem **??** does not converge to the entropy of the volume measure. Instead, it converges to the entropy of the measure of maximal entropy. In this sense, for sufficiently large time n, it is the *exceptional* points that dominate the integral. The order of the operations log and $\int$ is thus crucial (see Corollary **??**).

- In the special case $r = \infty$, no additional term is added to the topological entropy, and our result thus implies the main results of Kozlovski [**?**] for surface diffeomorphisms. Also note that Theorem **??** is formulated directly in terms of the derivative Df_x^n, rather than the induced map $(Df_x^n)^\wedge$ between exterior algebras of the tangent spaces which are considered by Kozlovski [**?**]. Although the equivalence of these two quantities for evaluating the topological entropy is not apparent from their respective forms, it is nonetheless natural to work directly with Df_x^n in the two-dimensional setting, after a more careful analysis. The use of the derivative Df_x^n in Theorem **??** might provide a more effective approach for numerically estimating the topological entropy.

- The threshold $\frac{\lambda^+(f)}{r}$ for the topological entropy is typically not expected to be attained in Yomdin-type results. For instance:

  - For C^r $(r > 1)$ surface diffeomorphisms with $h_{\text{top}}(f) > \frac{\lambda^+(f)}{r}$, Buzzi-Crovisier-Sarig [**?**] (see also subsequent extensions by Buzzi-Luo-Yang [**?**], Burguet-Luo-Yang [**?**] and Zang [**?**]) showed that there are at most finitely many ergodic measures of maximal entropy.
  - For C^r $(r > 1)$ surface diffeomorphisms with $h_{\text{top}}(f) > \frac{\lambda^+(f)}{r}$, Burguet [**?**] showed if μ_n is a sequence of ergodic measures with $h(f, \mu_n) \to h_{\text{top}}(f)$ and $\mu_n \to \mu$ for some invariant measure μ, then μ is a measure of maximal entropy and the positive Lyapunov exponent of μ_n converges to that of μ.
  - For C^r $(r > 1)$ surface diffeomorphisms, Burguet [**?**] also showed that if Lebesgue almost every point has upper Lyapunov exponent strictly larger than $\frac{\lambda^+(f)}{r}$, then the system admits an SRB measure.

  All these results rely on the assumption that certain invariants-such as the topological entropy or Lyapunov exponents-are *strictly* larger than the threshold $\frac{\lambda^+(f)}{r}$. The critical value $\frac{\lambda^+(f)}{r}$ itself is delicate and remains poorly understood. In contrast, Theorem **??** does not suffer from this limitation, as it concerns a formula for the topological entropy itself.

- - The case of systems with low regularity $(r = 1)$ is currently not understood by us.
  - The condition $h_{\text{top}}(f) \geq \frac{\lambda^+(f)}{r}$ appears to be critical. There are examples for interval maps (see Kozlovski [**?**]) with $h_{\text{top}}(f) < \frac{\lambda^+(f)}{r}$, in which the topological entropy vanishes while the volume growth of the tangent cocycle appearing in Theorem **??**

is positive. Although this does not immediately yield counterexamples in our setting of surface diffeomorphisms, we do not expect a fundamental gap between the situations of interval maps and surface diffeomorphisms. We also note that Kozlovski [?, Page 2] mentioned, without providing a precise reference, that Misiurewicz constructed counterexamples for surface diffeomorphisms where the topological entropy is strictly smaller than the volume growth.

– Recently, S. Ben Ovadia and Burguet [?] studied the Viana conjecture in arbitrary dimensions using a refined high-dimensional version of Yomdin theory. This work may shed light on possible extensions of Theorem ?? to higher dimensions.

The study of the volume growth of the tangent cocycle:

$$\frac{1}{n}\log\int \|(Df_x^n)^\wedge\|\,dx$$

appears to go back to Sacksteder-Shub[?]. Here $(Df_x^n)^\wedge$ denotes the linear map induced by Df_x^n on the exterior algebras of the tangent spaces T_xM and $T_{f^nx}M$, and $\|\cdot\|$ is the operator norm induced by the Riemannian metric.

Based on Pesin theory, an inequality was obtained by Przytycki [?] for C^r $(r>1)$ diffeomorphisms and by Newhouse [?] for C^r $(r>1)$ maps on a compact manifold M of arbitrary dimension:

$$\liminf_{n\to+\infty}{}^{1}\frac{1}{n}\log\int \|(Df_x^n)^\wedge\|\,dx \geq h_{\mathrm{top}}(f). \tag{1}$$

Subsequently, Kozlovski [?](see also Klapper-L. S. Young [?]) showed that, for C^∞ diffeomorphisms, by Yomdin theory [?], a lower bound for the topological entropy is provided, i.e.,

$$\limsup_{n\to+\infty}\frac{1}{n}\log\int \|(Df_x^n)^\wedge\|\,dx \leq h_{\mathrm{top}}(f). \tag{2}$$

Combining (??) and (??), we obtain an equality for the topological entropy. One might wonder what the C^r version of the above formula is for finite r. Indeed, in the C^r setting, one may expect an inequality that combines the topological entropy with the Yomdin term:

$$\limsup_{n\to+\infty}\frac{1}{n}\log\int \|(Df_x^n)^\wedge\|\,dx \leq^{2} h_{\mathrm{top}}(f)+\frac{\lambda^+(f)}{r}. \tag{3}$$

In the C^∞ case, the Yomdin term $\frac{\lambda^+(f)}{r}$ vanishes, which yields the inequality (??). In contrast, for finite r, the inequality (??) cannot be reduced to inequality (??), as the Yomdin term does not vanish in general. Rather than the *additive* bound of the topological entropy and the Yomdin term appearing in (??), we obtain a *maximum* bound that yields a substantial improvement (see Theorem ??):

$$\limsup_{n\to+\infty}\frac{1}{n}\log\int_M \|Df_x^n\|\,dx \leq \max\left\{h_{\mathrm{top}}(f),\ \frac{\lambda^+(f)}{r}\right\}. \tag{4}$$

[1] Przytycki established the inequality in terms of a lim sup and remarked at the end of the paper that it remains valid with lim inf.

[2] Although this formula is not stated explicitly in Kozlovski [?] or elsewhere (as far as we know), it can be deduced from the arguments in [?] as an application of Yomdin theory in the C^r case.

If $h_{\text{top}}(f) \geq \frac{\lambda^+(f)}{r}$, inequality (**??**) immediately implies inequality (**??**). Together with inequality (**??**) which holds for any C^r $(r>1)$ diffeomorphism, this yields Theorem **??**.

There are also related results in the partially hyperbolic setting (see the works of Saghin [**?**], Yang-Zang [**?**] and Guo-Liao-Sun-Yang [**?**]), as well as in the setting of random dynamical systems (see Ma [**?**]).

The order of the operations log and $\int$ is crucial in the following sense.

Corollary B. *For any C^r $(r>1)$ diffeomorphism f on a compact surface M with $h_{\text{top}}(f) > \frac{\lambda^+(f)}{r}$* [3]*, if*

$$\limsup_{n\to+\infty} \frac{1}{n}\int_M \log\|Df_x^n\|\, dx = \limsup_{n\to+\infty} \frac{1}{n}\log\int_M \|Df_x^n\|\, dx,$$

then there exists finitely many ergodic SRB measures. Moreover, each of them is a measure of maximal entropy.

Proof. By Reverse Fatou Lemma (see Lemma **??**) and Theorem **??**,

$$\begin{aligned}\int_M \limsup_{n\to+\infty}\frac{1}{n}\log\|Df_x^n\|\, dx &\geq \limsup_{n\to+\infty}\frac{1}{n}\int_M \log\|Df_x^n\|\, dx\\ &= \limsup_{n\to+\infty}\frac{1}{n}\log\int_M \|Df_x^n\|\, dx\\ &= h_{\text{top}}(f)\\ &> \frac{\lambda^+(f)}{r}.\end{aligned}$$

We claim that for Lebesgue almost every x,

$$\limsup_{n\to+\infty}\frac{1}{n}\log\|Df_x^n\| = h_{\text{top}}(f). \tag{5}$$

Because otherwise, since the integral of the left side is no less than $h_{\text{top}}(f)$, there must exist some set A with positive Lebesgue measure such that

$$\limsup_{n\to+\infty}\frac{1}{n}\log\|Df_x^n\| > h_{\text{top}}(f) > \frac{\lambda^+(f)}{r}, \quad \forall x\in A.$$

By Burguet's result [**?**], there is an ergodic SRB measure μ whose positive Lyapunov exponent is strictly larger than $h_{\text{top}}(f)$. This is a contradiction since the positive exponent of an ergodic SRB measure coincides with its entropy.

Again, by Burguet's result [**?**], under condition (**??**), there are countably many ergodic SRB measures $\{\mu_i\}_{i\in\mathbb{N}}$ such that Lebesgue almost every x lies in the basin of some μ_i and each μ_i carries maximal entropy. Indeed, in our setting there are only finitely many such ergodic SRB measures, as a direct consequence of the main results of Buzzi-Crovisier-Sarig [**?**], which assert that there are at most finitely many ergodic measures of maximal entropy. □

[3] Note that here we have to assume a strict inequality. Otherwise, Burguet's result [**?**] does not guarantee the existence of an SRB measure.

We also provide a formula expressing the topological entropy in terms of the *volume growth of sub-manifolds* which extends the classical results in Yomdin theory [**?**] in C^∞ setting.

The *volume* (i.e., arc length) of a curve $\sigma : [0,1] \to M$ is denoted by $\mathrm{Vol}(\sigma)$.

Theorem C. *For any C^r $(r > 1)$ diffeomorphism f on a compact surface M with $h_{\mathrm{top}}(f) \geq \frac{\lambda^+(f)}{r}$, we have*

$$\begin{aligned} h_{\mathrm{top}}(f) &= \lim_{n\to+\infty} \frac{1}{n} \log \sup_{\sigma} \mathrm{Vol}\left(f^n(\sigma)\right) \\ &= \sup_{\sigma} \limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Vol}\left(f^n(\sigma)\right) \\ &= \sup_{\sigma} \liminf_{n\to+\infty} \frac{1}{n} \log \mathrm{Vol}\left(f^n(\sigma)\right) \end{aligned}$$

where the supremum is taken over all C^r embedded curves $\sigma : [0,1] \to M$ with $\|\sigma\|_{C^r} \leq 1$ and $\|d_t\sigma\| \geq 1/2$, $t \in [0,1]$ [4].

We make some comments.

- In general, beyond our C^r setting together with the condition $h_{\mathrm{top}}(f) \geq \frac{\lambda^+(f)}{r}$, it is quite subtle to establish an equivalence between the two kinds of volume growth of sub-manifolds in Theorem **??** :

$$\limsup_{n\to+\infty} \frac{1}{n} \log \sup_{\sigma} \mathrm{Vol}\left(f^n(\sigma)\right) \stackrel{?}{=} \sup_{\sigma} \limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Vol}\left(f^n(\sigma)\right).$$

  This question was implicitly raised in an unpublished manuscript of Buzzi [**?**, Remark 1.2].

- If the topological entropy is strictly larger than $\frac{\lambda^+(f)}{r}$, then there exists an invariant measure μ of maximal entropy ([**?**, **?**]). Moreover, by a result of Cogswell [**?**] (see also Zang [**?**]), the supremum in Theorem **??** is attained by the local unstable manifold $W^u_{\mathrm{loc}}(x)$ for μ-almost every point x.

- It is worth noting that Theorem **??** actually implies Theorem **??** via a Fubini-type argument (see the discussions in Section **??**). However, we choose to prove Theorem **??** using a more fundamental convex bound, namely Theorem **??**. This approach offers a deeper insight into the mechanism governing the volume growth of tangent cocycle. There is no essential difference between these two approaches, since both ultimately follow from Theorem **??**.

We believe that the ideas of Theorem **??** and Theorem **??** might extend to other topological invariants (see Llibre–Saghin [**?**] for a survey of various invariants measuring dynamical complexity), such as the growth rate of hyperbolic periodic points. This has been established by Burguet [**?**] in the C^∞ setting and it was conjectured by Burguet in the C^r case:

[4] These conditions ensure that the curves are neither degenerate nor highly oscillatory and are imposed only for technical convenience. One may allow more general families.

Conjecture ([**?**]). *For any C^r $(r>1)$ diffeomorphism f on a compact surface M with $h_{\mathrm{top}}(f) > \frac{\lambda^+(f)}{r}$. For any $0<\delta<h_{\mathrm{top}}(f)$,*

$$h_{\mathrm{top}}(f) = \limsup_{n\to+\infty} \frac{1}{n} \log \#\mathscr{P}_\delta^n$$

where $\mathscr{P}_\delta^n$ is the set of n-periodic points with Lyapunov exponents δ-away from zero.

2 Convex bounds for volume growth

In this section, we present the main results used to prove Theorem **??** and Theorem **??**.

2.1 Mechanism behind convex bounds for volume growth

Theorem 2.1. *Let f be a C^r $(r>1)$ diffeomorphism on a compact surface M. There is some $\alpha\in[0,1]$ such that*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \sup_{\sigma} \mathrm{Vol}(f^n\sigma) \le \alpha\cdot h_{\mathrm{top}}(f) + (1-\alpha)\cdot \frac{\lambda^+(f)}{r}.$$

Equivalently,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \sup_{\sigma} \mathrm{Vol}(f^n\sigma) \le \max\left\{h_{\mathrm{top}}(f),\ \frac{\lambda^+(f)}{r}\right\}.$$

Here the supremum is taken over all C^r embedded curves $\sigma:[0,1]\to M$ with $\|\sigma\|_{C^r}\le 1$ and $\|d_t\sigma\|\ge 1/2$, $t\in[0,1]$.

Theorem 2.2. *Let f be a C^r $(r>1)$ diffeomorphism on a compact surface M. There is some $\alpha\in[0,1]$ such that*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \int_M \|Df_x^n\|\, dx \le \alpha\cdot h_{\mathrm{top}}(f) + (1-\alpha)\cdot \frac{\lambda^+(f)}{r}.$$

Equivalently,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \int_M \|Df_x^n\|\, dx \le \max\left\{h_{\mathrm{top}}(f),\ \frac{\lambda^+(f)}{r}\right\}.$$

The reason for presenting Theorems **??** and **??** in a convex-combination form is that our proof actually yields a more refined upper bound, in which the coefficient $\alpha\in[0,1]$ reflects how the volume growth is distributed between the entropic contribution and the Lyapunov-type contribution (the Yomdin term). In particular, the argument implicitly determines an admissible choice of α and thus contains strictly more information than the crude bound given by the maximum. However, making this choice explicit would require introducing a substantial amount of additional technical machinery. For this reason, we do not pursue an explicit description of α here in the statements. As a consequence, the statements of the theorems may be equivalently weakened to the simpler form involving $\max\{h_{\mathrm{top}}(f), \lambda^+(f)/r\}$.

These convex-type upper bounds originate from the idea that dynamical invariants such as volume growth measure complexity from two different perspectives: a global one and a local one. To illustrate this idea, let us consider the volume growth of a curve σ and write

$$\mathrm{Vol}(f,\sigma) := \limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Vol}(f^n\sigma).$$