Inequalities

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Theorem 1. *(**Markov's inequalities**) Let X be a non-negative random variable and suppose $\mathbb{E}(X)$ exists. For any, $t > 0$,*

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}$$

Proof. fk8y We have $X > 0$ so:

$$\begin{aligned}
\mathbb{E}(x) &= \int_0^{+\infty} x f(x)\, dx \\
&= \int_0^{t} x f(x)\, dx + \int_t^{+\infty} x f(x)\, dx \\
&\geq \int_t^{+\infty} x f(x)\, dx \\
&\geq t \int_t^{+\infty} f(x) dx \\
&= t\, \mathbb{P}(X > t)
\end{aligned}$$

$\square$

Theorem 2. *(**Chebysev's inequality**) Let $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathbb{V}(X)$. Then,*

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2} \quad \textit{and} \quad \mathbb{P}(|Z| \geq k) \leq \frac{1}{k^2}$$

where $Z = (X - \mu)/\sigma$.

Proof.

$$\mathbb{P}(|X - \mu| \geq t) = \mathbb{P}(|X - \mu|^2 \geq t^2) \leq \frac{\mathbb{E}((X - \mu)^2)}{t^2} = \frac{\sigma^2}{t^2}$$

$$\mathbb{P}(|Z| \geq k) = \mathbb{P}(|X - \mu| \geq \sigma k) \leq \frac{\sigma^2}{\sigma^2 k^2} = \frac{1}{k^2}$$

$\square$

Theorem 3. *(**Hoeffding's inequality**) Let $Y_1, Y_2, \ldots, Y_n$ be independent observations such that $\mathbb{E}(X_i) = 0$ and $a_i \leq Y_i \leq b_i$. Let $\epsilon > 0$. Then, for any $\epsilon > 0$,*

$$\mathbb{P}(\sum_{i=1}^{n} Y_i \geq \epsilon) \leq e^{-t\epsilon} \prod_{i=1}^{n} e^{t^2 (b_i - a_i)^2 / 8}$$

Let $X_1, X_2, \ldots, X_n \sim$ Bernulli(p). Then, for any $\epsilon > 0$,

$$\mathbb{P}(|\overline{X_n} - p| > \epsilon) \leq 2e^{-2n\epsilon^2}$$

where $\overline{X_n} = \frac{X_1+X_2+\ldots+X_n}{n}$

Theorem 4. *(**Cauchy-Schwarz inequality**) If X and Y have finite variances then*

$$\mathbb{E}(|XY|) \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$$

Theorem 5. *(**Jensen's Inequality**) If g is **convex** then*

$$\mathbb{E}g(X) \geq g(\mathbb{E}(X))$$

*if g is **concave** then*

$$\mathbb{E}g(X) \leq g(\mathbb{E}(X))$$