Cost Functions in Fuzzy Analysis Clustering

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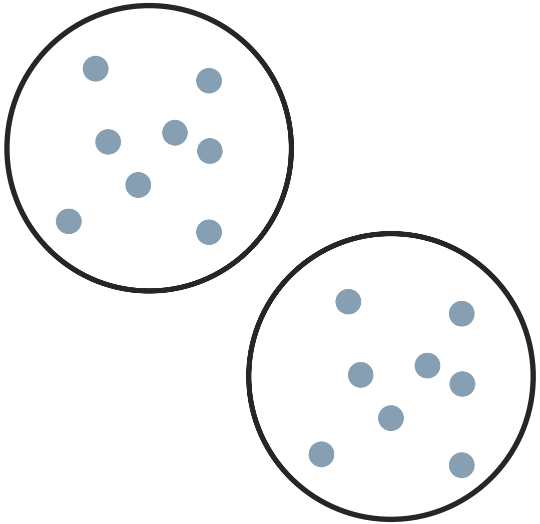
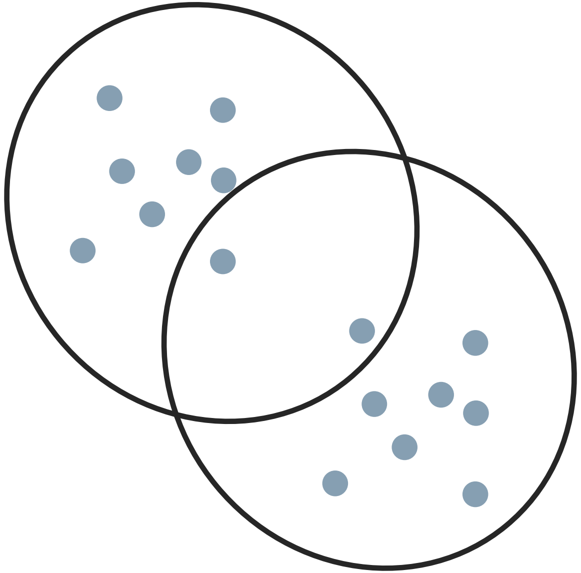
Basic Idea

- Fuzzy C-means clustering is a soft clustering algorithm where a data point can be assigned to more than one cluster using membership values.
- In simple terms, the objective (cost) function is a correctional function that measures the quality of the clustering by analyzing how wrong the cluster designations are.
- The process of clustering is to optimize this objective function.
- It is a weighted average of cluster variances, with weights that are proportional to cluster size in terms of the number of points.

Visualize



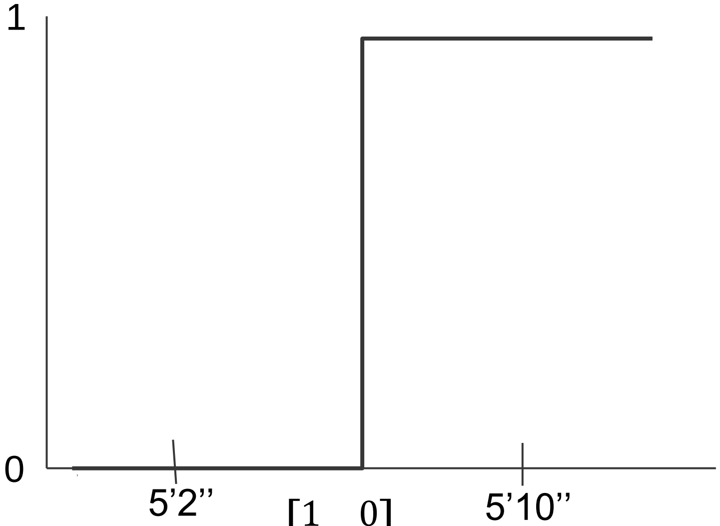
VISUALIZE



Membership function

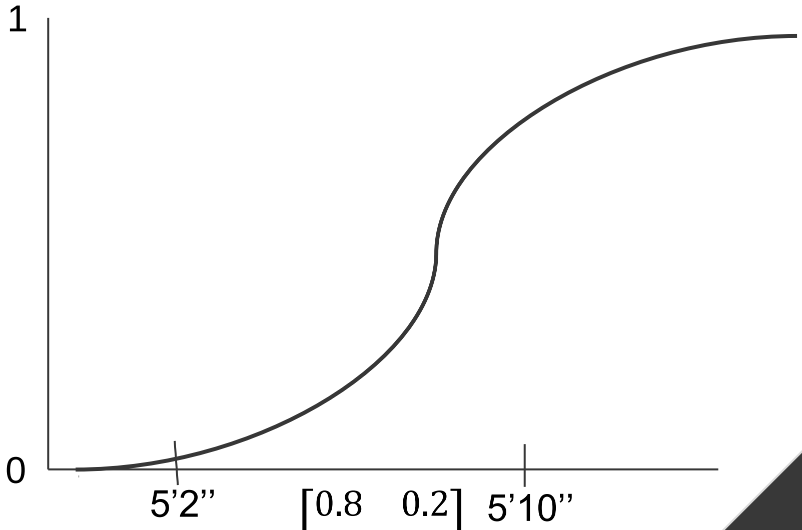
- What is the membership value for each data point to each cluster?
    - Falls between 0 to 1.

K-Means



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

C-Means



$$\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.7 & 0.3 \\ 0.9 & 0.1 \end{bmatrix}$$

Fuzzy logic

- In standard binary problems, a statement is either True or False… there is no in-between.
- Using fuzzy logic.. A statement can be True with an associated membership value and False with an associated membership value.

Basic Concepts

$$0 \le u_{ij} \le 1 \quad \forall \quad u_{ij}$$

Membership Value for each point

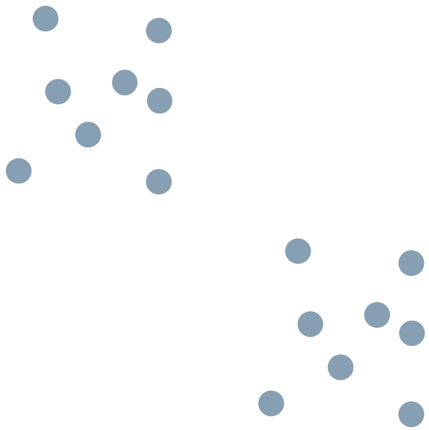
$$\sum_{j=1}^{c} u_{ij} = 1 \quad \forall \quad c = 1,2,\dots,n$$

Cluster level summation

$$0 < \sum_{i=1}^{n} u_{ij} < n \quad \forall \quad j = 1,2,\dots,c$$

Point level summation

VISUALIZE



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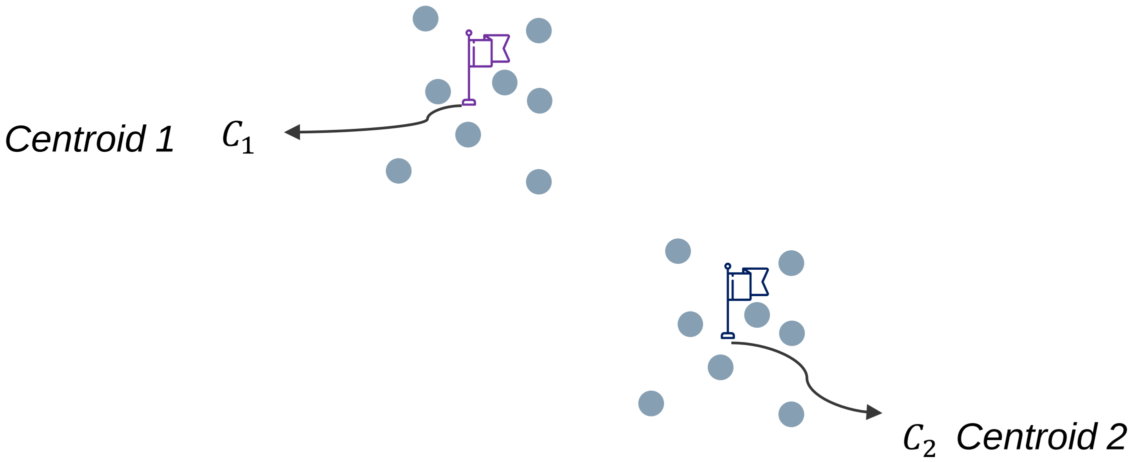

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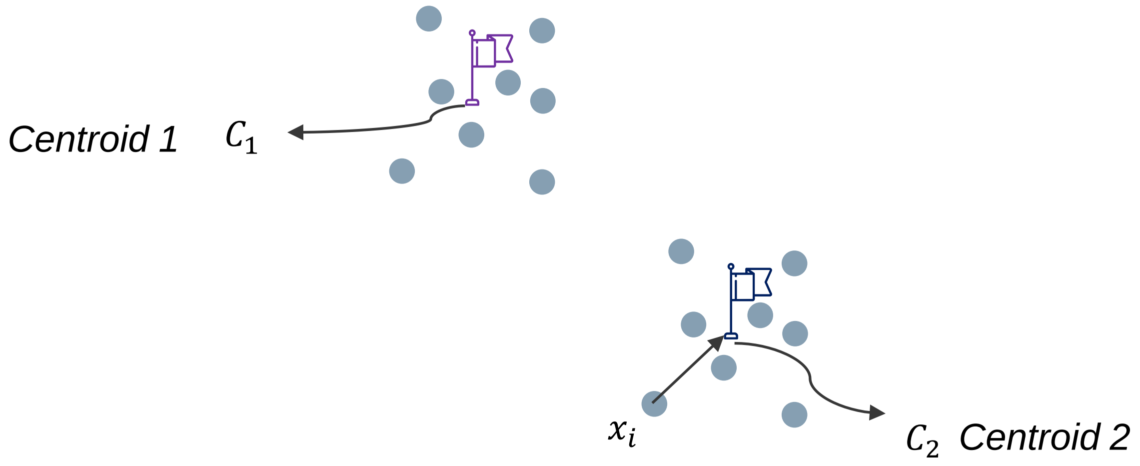

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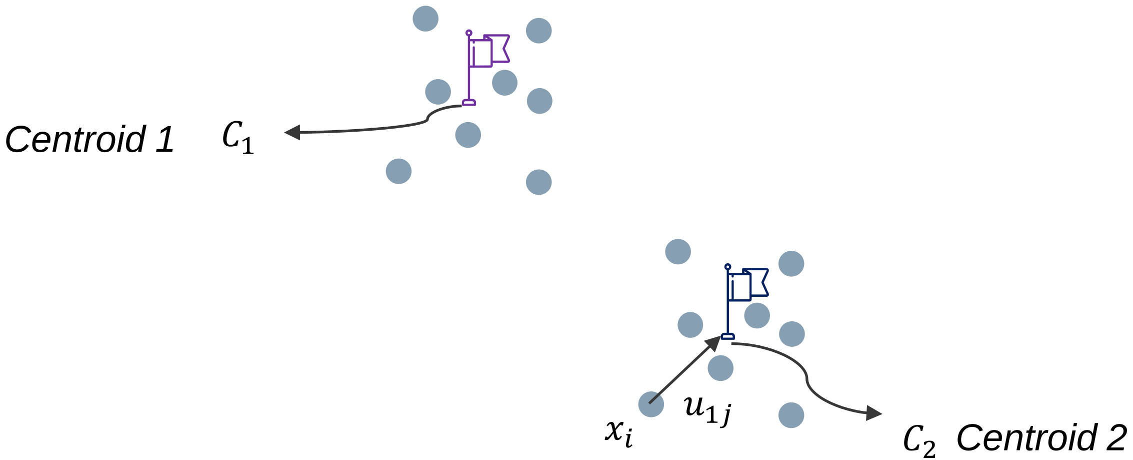

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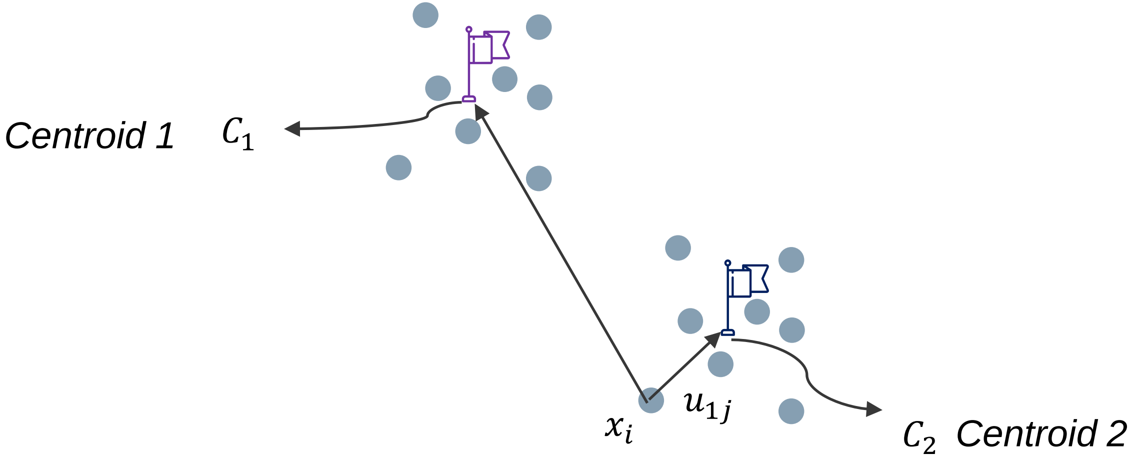

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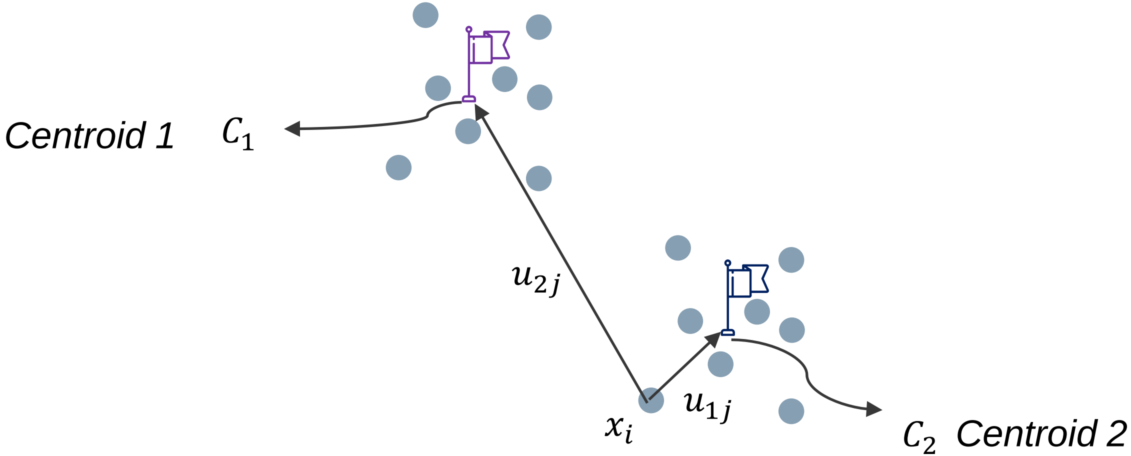
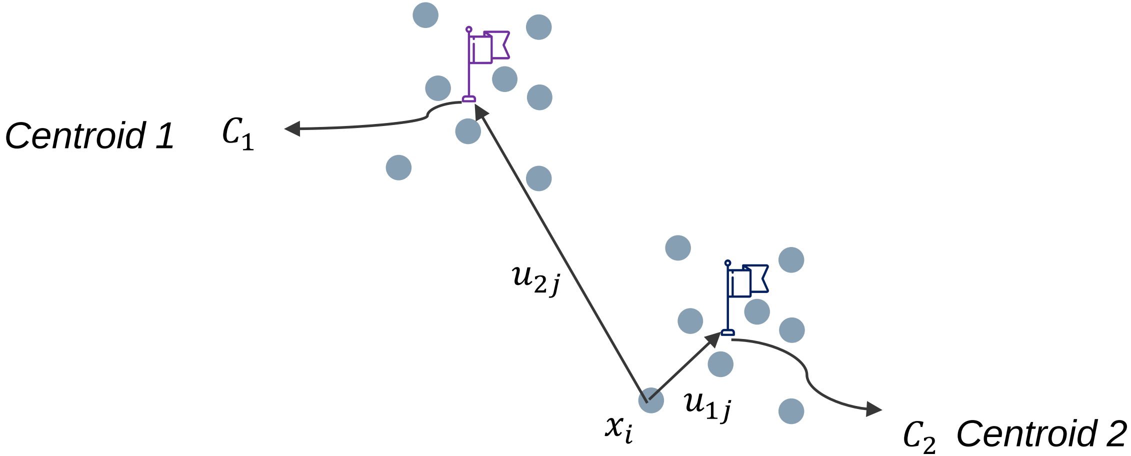


VISUALIZE

$$u_{1j} + u_{2j} = 1$$



VISUALIZE

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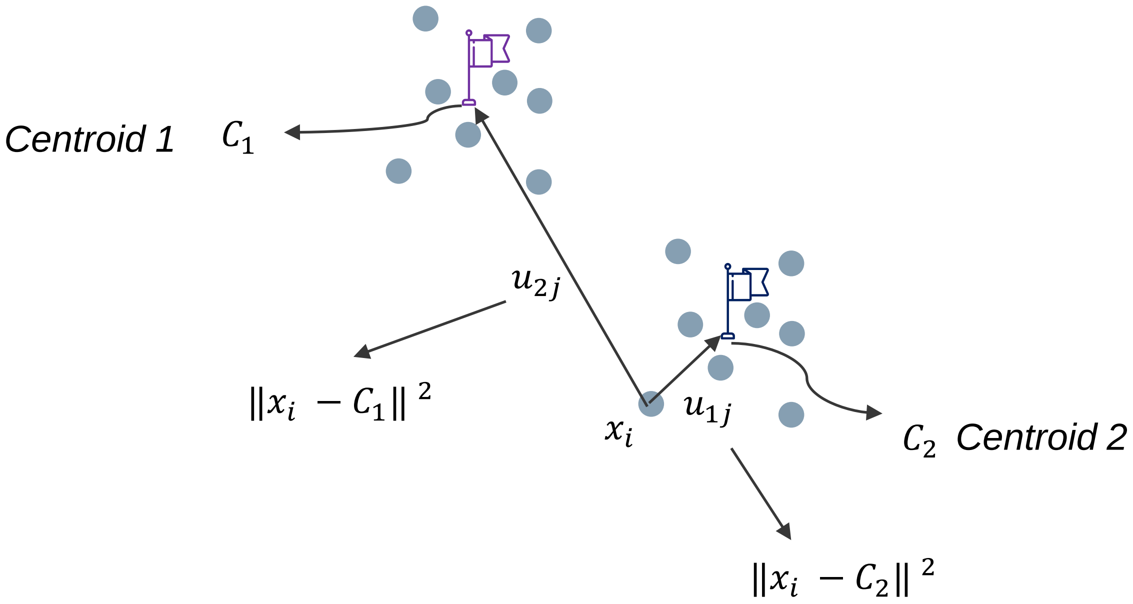


Triangle inequality

$$
\begin{aligned}
\| x_i - C_1 \|^2 &= \langle x_i - C_1, x_i - C_1 \rangle \\
&= \| x_i \|^2 - \langle x_i , C_1 \rangle - \langle C_1, x_i \rangle - \| C_1 \|^2 \quad where\ \langle C_1, x_i \rangle = \overline{\langle x_i, C_1 \rangle} \\
&= \| x_i \|^2 - 2Re\langle x_i , C_1 \rangle - \| C_1 \|^2 \\
&\leq \| x_i \|^2 - 2\, |\langle x_i, C_1 \rangle| - \| C_1 \|^2 \\
&\leq \| x_i \|^2 - 2 \| x_i \| \| C_1 \| - \| C_1 \|^2 \\
&= (\| x_i \| - \| C_1 \|)^2
\end{aligned}
$$

$Taking\ the\ square\ roots\ gives\ the\ triangle\ inequality$

$$\| x_i - C_1 \| \leq \| x_i \| - \| C_1 \|$$

https://wiki2.org/en/Cauchy%E2%80%93Schwarz_inequality

Objective (Cost) Function

$$\mathrm{J_m}(\mathrm{u_{ij}}, \mathrm{c_i}) = \sum_{i=1}^{N} \sum_{j=1}^{C} u_{ij}^{m} \, \|x_i - C_j\|^{2}$$

$$Let\ m > 1$$

$$\sum_{i=1}^{1} u_{i1} = u_{1,1} + u_{2,1} = 1$$

$$\vdots$$

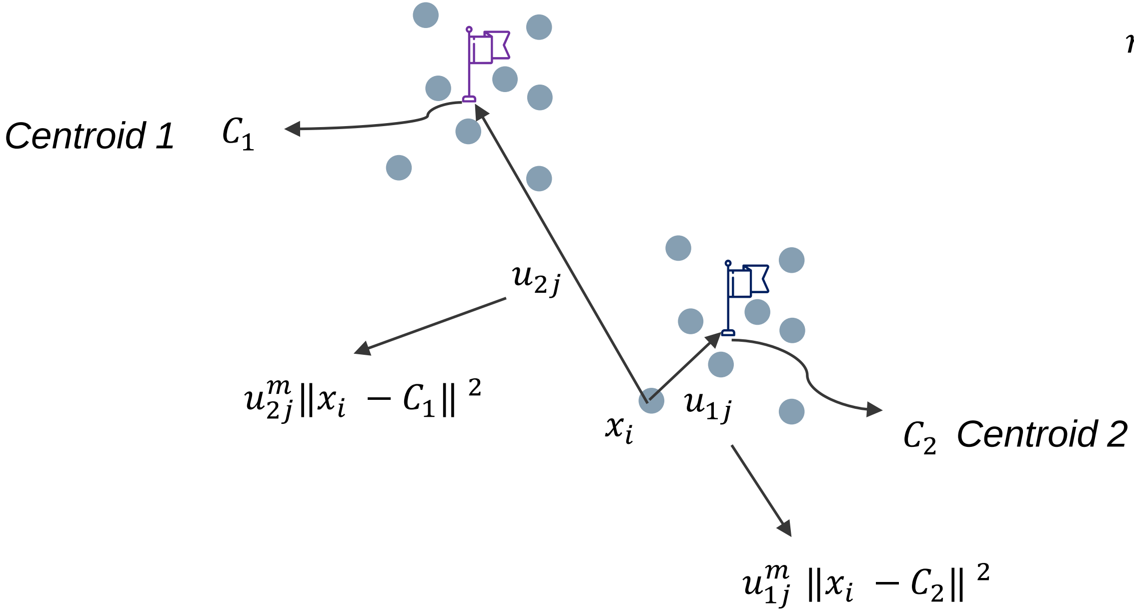
$$\sum_{i=1}^{N} u_{iN} = u_{1,N} + u_{2,N} = 1$$

$$V_i = \frac{\left(\sum_{u=1}^{n} x_i * u_{ij}^{r}\right)}{\left(\sum_{i=1}^{n} u_{ij}^{r}\right)} \qquad j = 1,2, \dots, c$$

VISUALIZE

$$u_{1j} + u_{2j} = 1$$

$$m > 1$$



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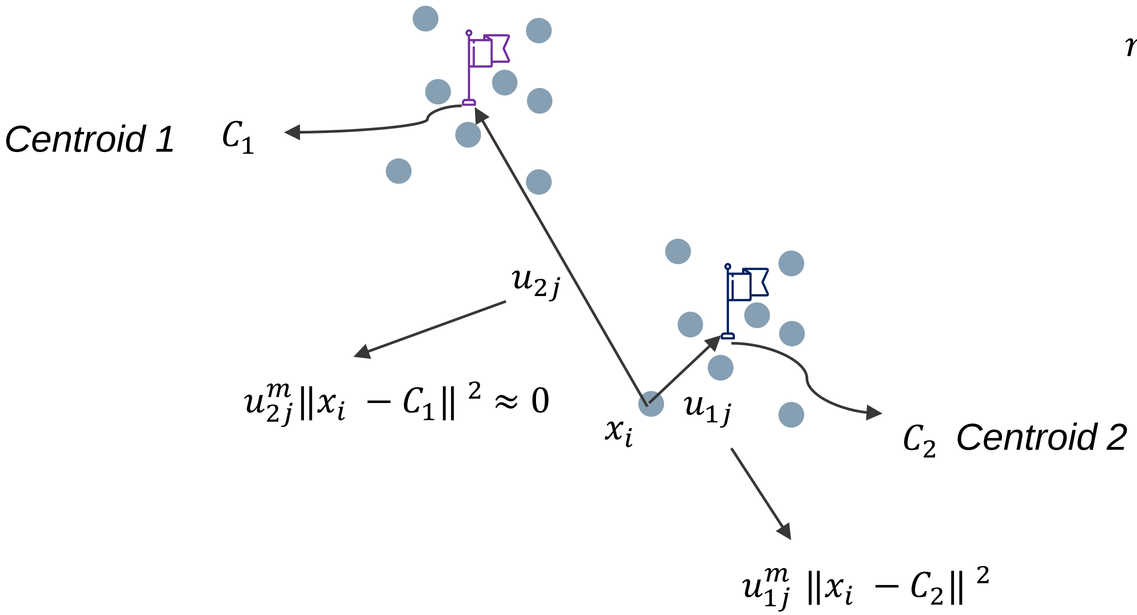
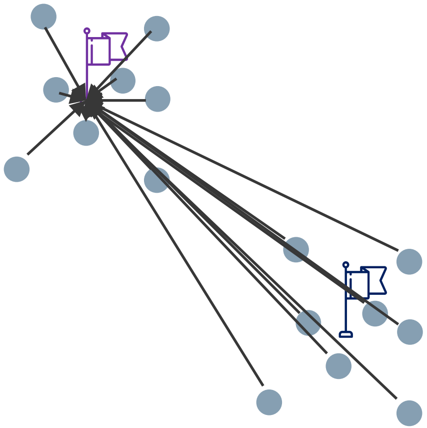


$$u_{1j} + u_{2j} = 1$$

$$m > 1$$

VISUALIZE

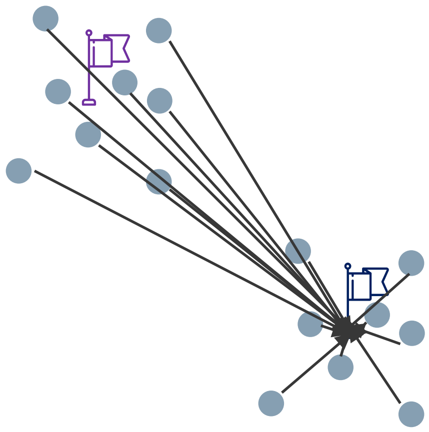


$u_{1j} + u_{2j} = 1$

$m > 1$

$$\sum_{i=1}^{N} u_{1j}^{m} \|x_i - C_1\|^2 = u_{1,1}^{m} \|x_1 - C_1\|^2 + u_{1,2}^{m} \|x_2 - C_1\|^2 + \ldots + u_{1,N}^{m} \|x_N - C_1\|^2$$

VISUALIZE

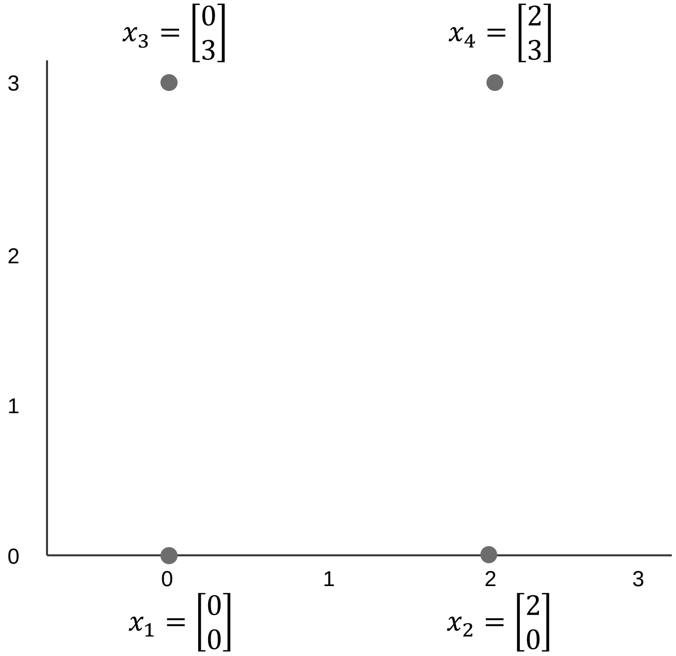


$u_{1j} + u_{2j} = 1$

$m > 1$

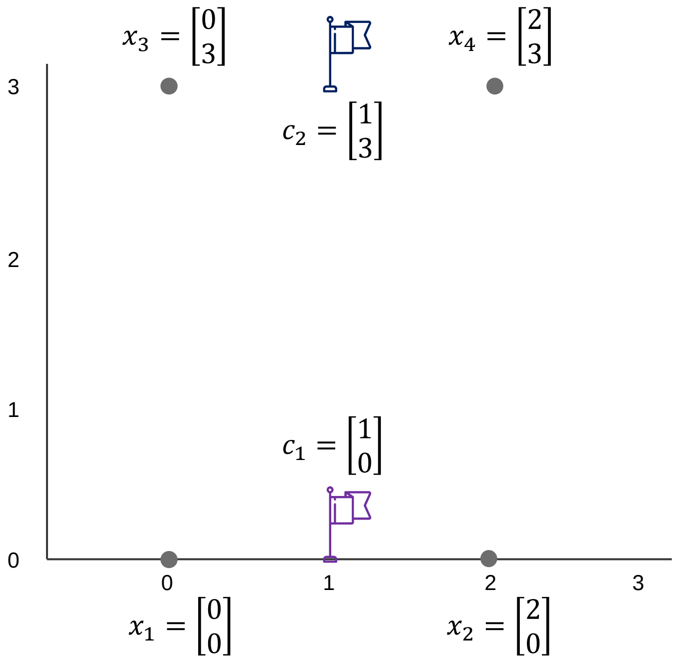
$$\sum_{i=1}^{N} u_{1j}^{m} \|x_i - C_2\|^2 = u_{1,1}^{m} \|x_1 - C_2\|^2 + u_{1,2}^{m} \|x_2 - C_2\|^2 + \ldots + u_{1,N}^{m} \|x_N - C_2\|^2$$

EXAMPLE


$x_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$x_4 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
3
2
1
0
0
1
2
3
$x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$x_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

EXAMPLE

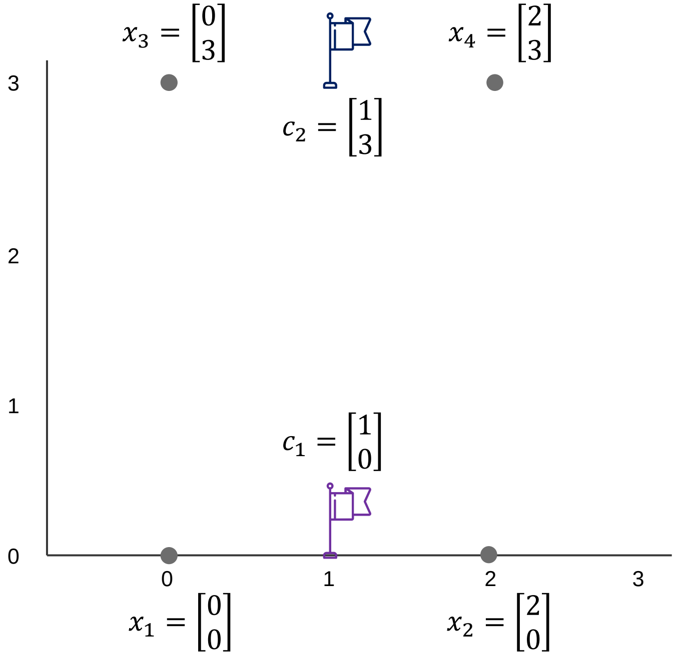
$$J(u_{ij}, c_i) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \|x_i - C_j\|^2$$



EXAMPLE

$$J(\mathrm{u_{ij}}, \mathrm{c_i}) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \, \|x_i - C_j\|^2$$

$$= u_{1,1}^{m}\|x_1 - C_1\|^2 + u_{1,2}\|x_2 - C_1\|^2$$

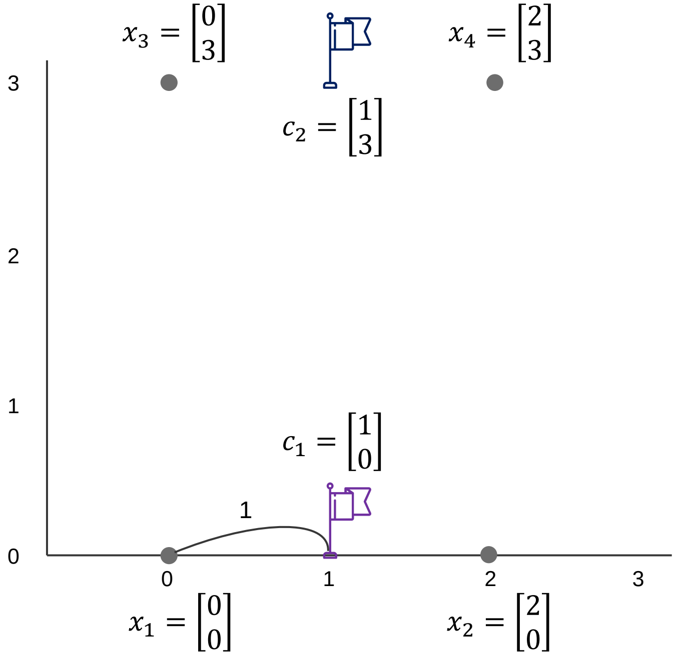


EXAMPLE

$$J(\mathrm{u_{ij}}, \mathrm{c_i}) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \, \|x_i - C_j\|^{2}$$

1

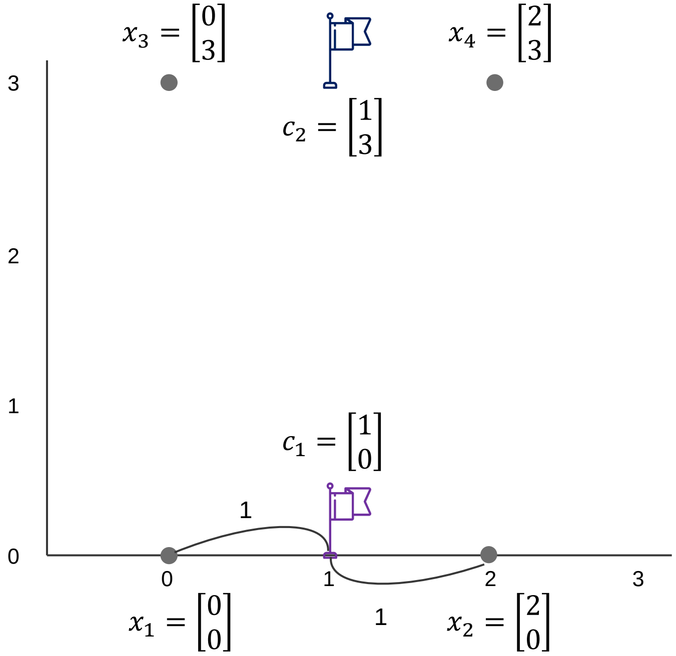
$$= u_{1,1}^{m}\|x_1 - C_1\|^2 + u_{1,2}\|x_2 - C_1\|^2$$



EXAMPLE

$$J(u_{ij}, c_i) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \|x_i - C_j\|^2$$

$$= u_{1,1}^{m}\|x_1 \overset{1}{-} C_1\|^2 + u_{1,2}\|x_2 \overset{1}{-} C_1\|^2$$



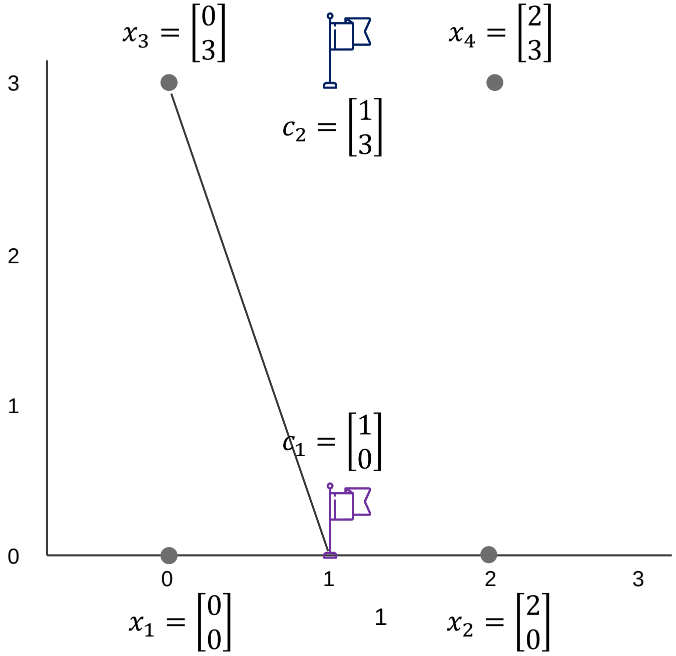
EXAMPLE

$$J(u_{ij}, c_i) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \|x_i - C_j\|^2$$

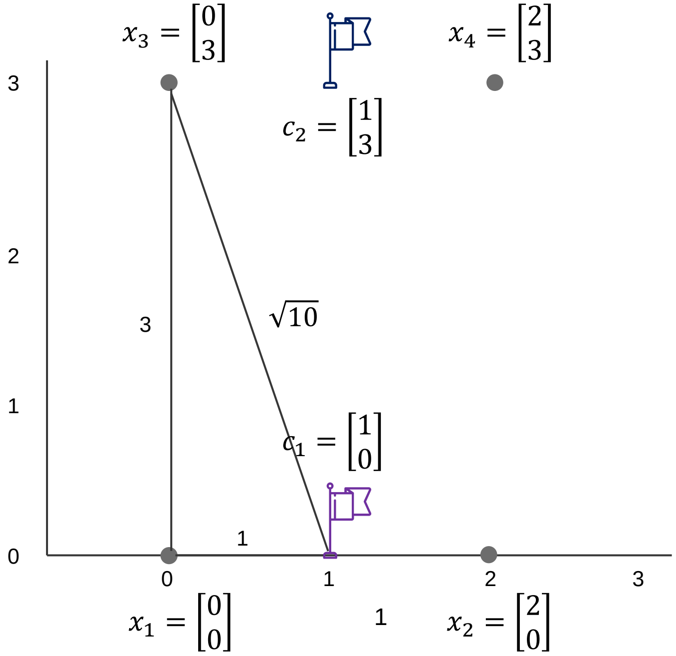
1 1

$$= u_{1,1}^{m} \|x_1 - C_1\|^2 + u_{1,2} \|x_2 - C_1\|^2$$

$$+ u_{1,3}^{m} \|x_3 - C_1\|^2 + u_{1,4} \|x_4 - C_1\|^2$$



EXAMPLE



$$J(u_{ij}, c_i) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \|x_i - C_j\|^{2}$$

$$= u_{1,1}^{m} \overset{1}{\|x_1 - C_1\|^2} + u_{1,2} \overset{1}{\|x_2 - C_1\|^2}$$

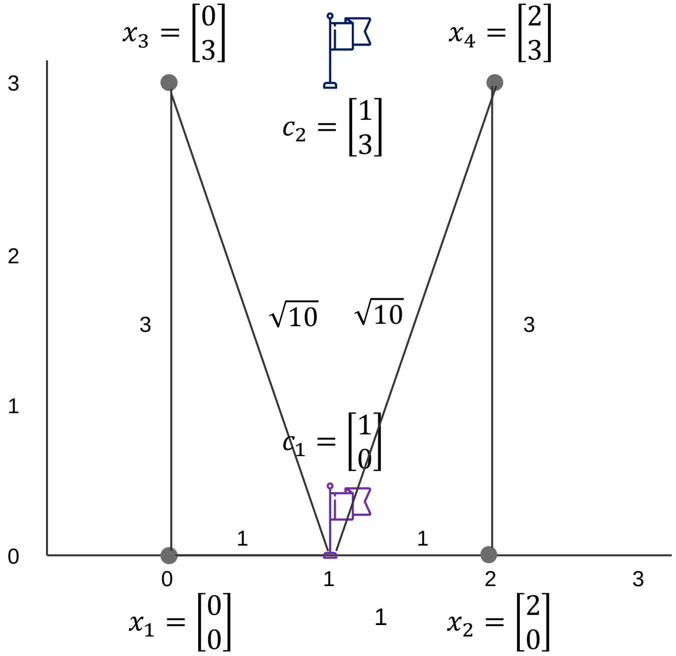
$$+ u_{1,3}^{m} \overset{10}{\|x_3 - C_1\|^2} + u_{1,4} \|x_4 - C_1\|^2$$

EXAMPLE

$$J(u_{ij}, c_i) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \|x_i - C_j\|^2$$

$$= u_{1,1}^{m} \overset{1}{\|x_1 - C_1\|^2} + u_{1,2} \overset{1}{\|x_2 - C_1\|^2}$$

$$+ u_{1,3}^{m} \overset{10}{\|x_3 - C_1\|^2} + u_{1,4} \overset{10}{\|x_4 - C_1\|^2}$$



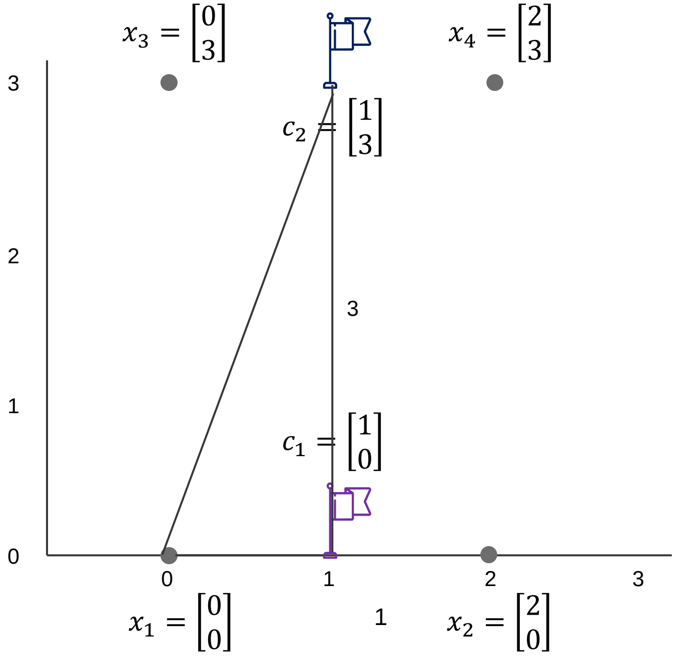
EXAMPLE

$$J(u_{ij}, c_i) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \|x_i - C_j\|^2$$

$$= u_{1,1}^{m}\|x_1 \overset{1}{-} C_1\|^2 + u_{1,2}\|x_2 \overset{1}{-} C_1\|^2$$

$$+ u_{1,3}^{m}\|x_3 \overset{10}{-} C_1\|^2 + u_{1,4}\|x_4 \overset{10}{-} C_1\|^2$$

$$+ u_{2,1}^{m}\|x_1 \overset{10}{-} C_2\|^2 + u_{2,2}\|x_2 - C_2\|^2$$



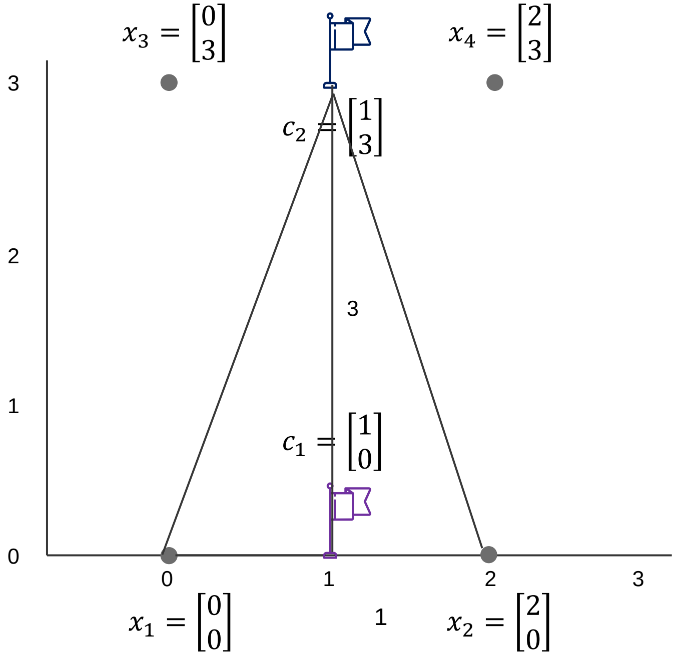
EXAMPLE

$$J(u_{ij}, c_i) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \|x_i - C_j\|^2$$

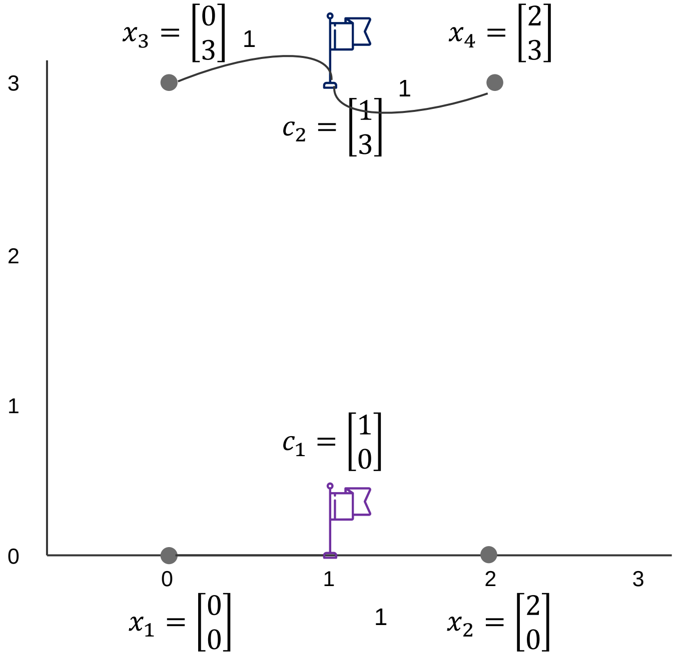
$$= u_{1,1}^{m}\|\overset{1}{x_1 - C_1}\|^2 + u_{1,2}\|\overset{1}{x_2 - C_1}\|^2$$

$$+ u_{1,3}^{m}\|\overset{10}{x_3 - C_1}\|^2 + u_{1,4}\|\overset{10}{x_4 - C_1}\|^2$$

$$+ u_{2,1}^{m}\|\overset{10}{x_1 - C_2}\|^2 + u_{2,2}\|\overset{10}{x_2 - C_2}\|^2$$



EXAMPLE



$$\mathrm{J}(\mathrm{u_{ij}}, \mathrm{c_i}) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \left\| x_i - C_j \right\|^{2}$$

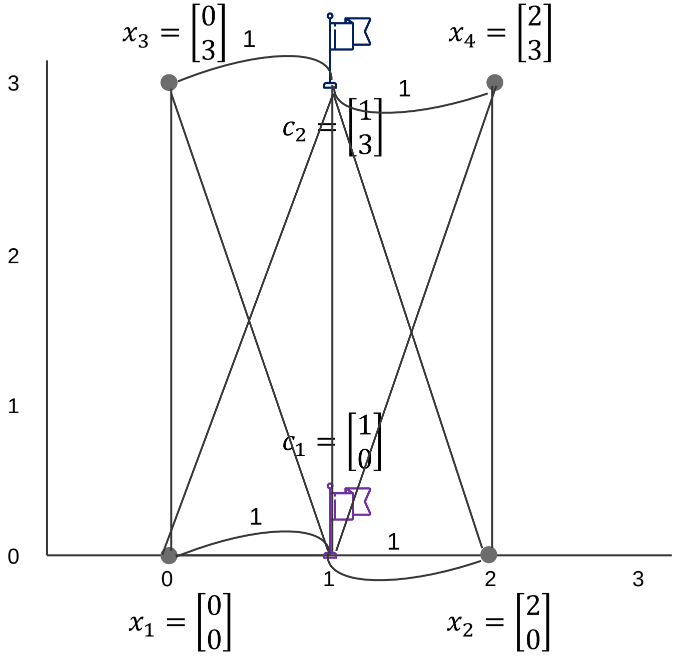
$$= u_{1,1}^{m} \overset{1}{\| x_1 - C_1 \|^2} + u_{1,2} \overset{1}{\| x_2 - C_1 \|^2}$$

$$+ u_{1,3}^{m} \overset{10}{\| x_3 - C_1 \|^2} + u_{1,4} \overset{10}{\| x_4 - C_1 \|^2}$$

$$+ u_{2,1}^{m} \overset{10}{\| x_1 - C_2 \|^2} + u_{2,2} \overset{10}{\| x_2 - C_2 \|^2}$$

$$+ u_{2,3}^{m} \overset{1}{\| x_3 - C_2 \|^2} + u_{2,4} \overset{1}{\| x_4 - C_2 \|^2}$$

Example



$$\mathrm{J}(\mathrm{u_{ij}}, \mathrm{c_i}) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \, \|x_i - C_j\|^{2}$$

$$= u_{1,1}^{m} \overset{1}{\|x_1 - C_1\|^2} + u_{1,2} \overset{1}{\|x_2 - C_1\|^2}$$

$$+ u_{1,3}^{m} \overset{10}{\|x_3 - C_1\|^2} + u_{1,4} \overset{10}{\|x_4 - C_1\|^2}$$

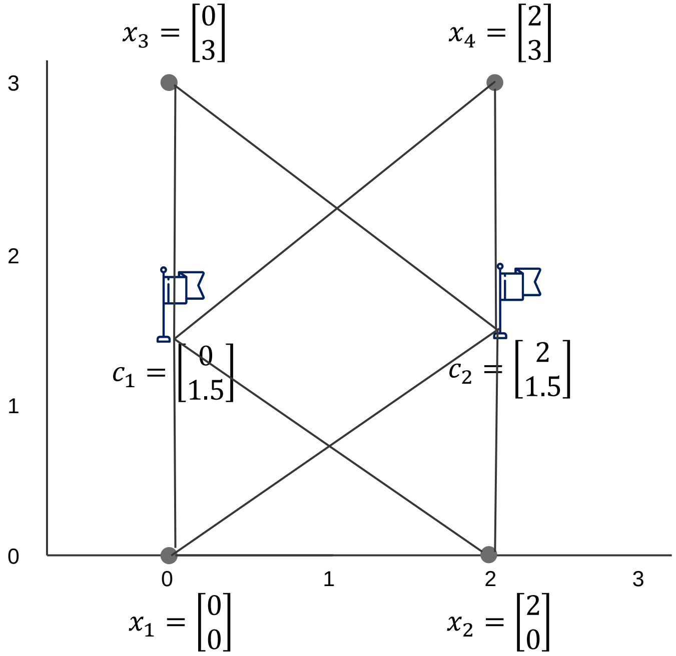
$$+ u_{2,1}^{m} \overset{10}{\|x_1 - C_2\|^2} + u_{2,2} \overset{10}{\|x_2 - C_2\|^2}$$

$$+ u_{2,3}^{m} \overset{1}{\|x_3 - C_2\|^2} + u_{2,4} \overset{1}{\|x_4 - C_2\|^2}$$

$$= 1u_{1,1}^{m} + 1u_{1,2}^{m} + 10u_{1,3}^{m} + 10u_{1,4}^{m}$$
$$+10u_{2,1}^{m} + 10u_{2,2}^{m} + 1u_{2,3}^{m} + 1u_{2,4}^{m}$$

$$= 44$$

EXAMPLE



$$\mathrm{J}(\mathrm{u_{ij}}, \mathrm{c_i}) = \sum_{j=1}^{K} \sum_{i=1}^{N} u_{ji}^{m} \, \|x_i - C_j\|^{2}$$

$$= u_{1,1}^{m} \overset{3}{\|x_1 - C_1\|^2} + u_{1,2} \overset{6.25}{\|x_2 - C_1\|^2}$$

$$+ u_{1,3}^{m} \overset{3}{\|x_3 - C_1\|^2} + u_{1,4} \overset{6.25}{\|x_4 - C_1\|^2}$$

$$+ u_{2,1}^{m} \overset{6.25}{\|x_1 - C_2\|^2} + u_{2,2} \overset{3}{\|x_2 - C_2\|^2}$$

$$+ u_{2,3}^{m} \overset{6.25}{\|x_3 - C_2\|^2} + u_{2,4} \overset{3}{\|x_4 - C_2\|^2}$$

$$= 3u_{1,1}^{m} + 6.25u_{1,2}^{m} + 3u_{1,3}^{m} + 6.25u_{1,4}^{m}$$
$$+6.25u_{2,1}^{m} + 3u_{2,2}^{m} + 6.25u_{2,3}^{m} + 3u_{2,4}^{m}$$

$$= 37$$

Now What

- Two paths to take once the cost function has been minimized.
  - Set the maximum membership value to 1 for each point. This turns the fuzzy clustering into a hard clustering were values are 0 and 1.
  - Allows the use of overlapping clusters since the same point has membership values for each cluster.

    - We don’t want to always say that a point is assigned to a particular cluster 100% of the time.

    - We might not want to force a point to a singular cluster… knowing that a point can be associated with multiple clusters can be helpful.