CS 5751: Assignment #1

Due on January 23, 2018

Eleazar 11:00am

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Exercise #1

Consider the function $z = f(x, y) = x^2+y^2$. Now, write a pdf file named yourlastname_hw1_q2.pdf containing the answers to the following tasks:

(1)

Compute the gradient $\nabla f(x, y)$. For this task you need to explain every step of your computation of the gradient. You cannot simply write the vector.

We have $f(x, y) = x^2 + y^2$. Since $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$, we will have the following:

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, 2y)$$

Therefore, the gradient of $f(x, y) = (2x, 2y)$.

(2)

Compute the Hessian of $\nabla f(x, y)$. For this task you need to explain every step of your computation of the Hessian. You cannot simply write it the matrix.

We have the function $f(x, y) = x^2 + y^2$. In part 1, we know that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$. Since $f : \mathbb{R}^2 \to \mathbb{R}$, the Hessian matrix of f is a square matrix with a dimension 2×2, that is:

$$\begin{aligned}
\nabla^2 f &= \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial x}2x & \frac{\partial}{\partial y}2x \\ \frac{\partial}{\partial x}2y & \frac{\partial}{\partial y}2y \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
\end{aligned}$$

(3)

Write the second order Taylor series expansion for $f(x, y)$ around $x_0 = 0$. Explain your answer.

Exercise #1 [(3)] continued on next page. . .

We have the Taylor series expansion around $x_0 = 0$ for $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x_0 + h) = f(x_0) + \Big(\nabla f(x_0)\Big)h + \frac{1}{2}\Big(h^T\Big)\Big[\Big(\nabla^2 f(x_0)\Big)h\Big]$$

We know that x_0 and h are vectors with the following values: $x_0 = \begin{bmatrix}0\\0\end{bmatrix}$ and $h = \begin{bmatrix}h_0\\h_1\end{bmatrix}$. From section 1 and 2 above, we also know that the gradient $\nabla f = (2x, 2y)$ and the Hessian of f is $\nabla^2 f = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$. Therefore, the Taylor series expansion can be written as follows:

$$\begin{aligned}
f(x_0 + h) &= f(x_0) + \Big(\nabla f(x_0)\Big)h + \Big(\frac{1}{2}\Big)\big(h^T\big)\left[\Big(\nabla^2 f(x_0)\Big)\big(h\big)\right] \\
&= f(0,0) + \Big(\nabla f(0,0)\Big)h + \Big(\frac{1}{2}\Big)\big(h^T\big)\left[\Big(\nabla^2 f(0,0)\Big)\big(h\big)\right] \\
&= 0 + \begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}h_0\\h_1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}h_0 & h_1\end{bmatrix}\left(\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}h_0\\h_1\end{bmatrix}\right) \\
&= \begin{bmatrix}\frac{h_0}{2} & \frac{h_1}{2}\end{bmatrix}\begin{bmatrix}2h_0\\2h_1\end{bmatrix} \\
&= h_0^2 + h_1^2
\end{aligned}$$

(4)

Consider Figure 2, which is a plot of $f(x, y)$. What do you observe in this plot? In which direction does the gradient point? Why does this happen?

The plot display a surface of a convex function $f(x, y) = x^2 + y^2$. The color of the surface indicates the value z ranging from blue (small value) to red (big value). The arrows on the z-surface is vector field indicating the gradient of $f(x, y)$ at different coordinates (x, y). At the origin $(x, y) = (0, 0)$ is the global minimum of this convex function. The gradient of this function is diverging from the origin $(0, 0)$. This is because this surface has only 1 minimum at $(x, y) = (0, 0)$, as we move further away from this point, the slope of the surface becomes higher, which means the gradient will increase.