# Matrices

## I. Matrix Arithmetic

a. Adding (2 matrix must have same size)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

b. Multiplying

i. By a scalar

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

ii. By a matrix (dot product of rows in left hand side matrix and the columns in right hand side matrix)

$$A_{m\times n} \times B_{n\times k} = C_{m\times k}$$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} g & h & i & j \\ k & l & m & n \end{bmatrix} = \begin{bmatrix} ag + bk & ah + bl & ai + bm & aj + bn \\ cg + dk & ch + dl & ci + dm & cj + dn \\ eg + fk & eh + fl & ei + fm & ej + fn \end{bmatrix}$$

## II. Matrix Invertibility

a. If $AB = I$ (the identity matrix), we say

i. $A$ is left inverse of $B$

ii. $B$ is right inverse of $A$

b. If $AB = I$ and $BA = I$, we say $A$ is invertible and write $B = A^{-1}$

c. If $A$ is invertible, it has a unique inverse

d. If $M$ & $N$ are invertible $m \times n$ matrices, then $MN$ is also invertible. That means: $(MN)^{-1} = N^{-1}M^{-1}$

e. Example:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 + R_1,\ \frac{-1}{2}R_2]{-3R_1 + R_2} \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{pmatrix}$$

## III. Matrix Elementary Row Operations:

a. Swapping 2 rows

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \to \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

b. Add multiple of 1 row to another

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 5 & 8 \end{pmatrix}$$

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c. Multiplying one row with a scalar

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 9 & 12 \end{pmatrix}$$

## IV. <u>Elementary Matrix</u>

a. An elementary matrix is obtained by performing a single elementary row operation to the identity matrix
b. Every elementary matrix is invertible, and its inverse is an elementary matrix
c. If E is an elementary matrix and EA is defined, then EA is the matrix defined by applying E's operation to A.
d. Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \xrightarrow{-9R_1 + R_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & -8 & -16 & -24 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -9 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

## V. <u>LU Factorizations</u>

a. When a square matrix A can be **<u>brought to echelon form without any row interchanges</u>**, then there are matrices L and U for which A=LU, L is a lower triangular square matrix with 1's on the diagonal, U is an echelon matrix.
b. Example:

$$A = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 3 & 2 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 \\ ? & ? & 1 & 0 \\ ? & ? & ? & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 \\ ? & ? & 1 & 0 \\ ? & ? & ? & 1 \end{pmatrix} = \begin{pmatrix} 4/4 & 0 & 0 & 0 \\ 4/4 & -1/-1 & 0 & 0 \\ 2/4 & -1/-1 & 2/2 & 0 \\ 1/4 & 1/-1 & -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{4} & -1 & \frac{-1}{2} & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{4} & -1 & \frac{-1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 3 & 2 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

## VI. <u>Matrix Determinants</u>

a. Determinant of a matrix A (Using cofactor expansion across the first row) is defined as following:

$$\det(A) = (a_{11}\det(c_{11}))(-1)^2 + (a_{12}\det(c_{12}))(-1)^3 + \cdots + (a_{1n}\det(c_{1n}))(-1)^{n+1}$$

$$a_{12}: \text{element at row 1, column 2}$$

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$$c_{12}\text{: matrix with } 1^{st}\text{row and } 2^{nd}\text{column removed}$$

b. Example:

$$\det([2]) = 2$$

$$\det\left(\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}\right) = (2*8) - (4*6) = -8$$

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right)$$

$$= 1\det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right)(-1)^2 + 2\det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}\right)(-1)^3 + 3\det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right)(-1)^4$$

$$= 1\det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right)(-1)^2 + 2\det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}\right)(-1)^3 + 3\det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right)(-1)^4$$

$$= (1)(-3)(-1)^2 + (2)(-6)(-1)^3 + (3)(-3)(-1)^4$$

$$= (1)(-3)(1) + (2)(-6)(-1) + 3(-3)(1)$$

$$= -3 + 12 - 9 = 0$$

c. Note:

i. Row interchange: change the determinant sign

ii. Add multiple of one row to another: determinant is not changed

iii. Multiply row by a constant: multiply determinant by that constant

iv. Some properties:

1. $\det(AB) = \det(A)\det(B)$ ; A, B are square matrices of same size
2. $\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1$
3. $\det(A^{-1}) = \frac{1}{\det(A)}$
4. $A = LU \rightarrow \det(A) = \det(L)\det(U) = \det(U)$
5. $\det(A^T) = \det(A)$
6. $\det(cA) = c^n \det(A)$ for an $n \times n$ matrix

v. When a matrix be reduced to echelon form (not reduced echelon form), the determinant of that matrix is the product of the diagonal of the echelon matrix.

$$\det\begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 3 & 2 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} = \det\begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 4*0*0*0 = 0$$

vi. The determinant of an upper or lower triangular matrix is the product of the diagonal of that matrix

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$$\det\left(\begin{bmatrix}1 & 8 & 9\\0 & 4 & 2\\0 & 0 & 6\end{bmatrix}\right) = 1*4*6 = 24$$

$$det\left(\begin{bmatrix}4 & 0 & 0\\3 & 8 & 0\\5 & 7 & 2\end{bmatrix}\right) = 4*8*2 = 64$$

vii. A matrix is invertible if and only if the determinant of that matrix $\neq 0$

## VII. Matrix Transposition

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{pmatrix}^T = \begin{pmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{pmatrix}$$

a. $det(A) = det(A^T)$; $A^T = the\ transpose\ matrix\ A$

b. Some properties:

 i. $(AB)^T = B^T A^T$

 ii. $(A)^{-1} = \frac{1}{det(A)} adj(A)$

 iii. $adj(A)$ is called adjugate (or adjoint, adjunct) matrix of A. It is the transpose of the cofactor matrix of A

$$\mathbf{A} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix}+\begin{vmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{vmatrix} & -\begin{vmatrix}a_{21} & a_{23}\\a_{31} & a_{33}\end{vmatrix} & +\begin{vmatrix}a_{21} & a_{22}\\a_{31} & a_{32}\end{vmatrix}\\ -\begin{vmatrix}a_{12} & a_{13}\\a_{32} & a_{33}\end{vmatrix} & +\begin{vmatrix}a_{11} & a_{13}\\a_{31} & a_{33}\end{vmatrix} & -\begin{vmatrix}a_{11} & a_{12}\\a_{31} & a_{32}\end{vmatrix}\\ +\begin{vmatrix}a_{12} & a_{13}\\a_{22} & a_{23}\end{vmatrix} & -\begin{vmatrix}a_{11} & a_{13}\\a_{21} & a_{23}\end{vmatrix} & +\begin{vmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{vmatrix}\end{pmatrix}$$

$$\mathrm{adj}(\mathbf{A}) = \mathbf{C}^\mathsf{T} = \begin{pmatrix}+\begin{vmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{vmatrix} & -\begin{vmatrix}a_{12} & a_{13}\\a_{32} & a_{33}\end{vmatrix} & +\begin{vmatrix}a_{12} & a_{13}\\a_{22} & a_{23}\end{vmatrix}\\ -\begin{vmatrix}a_{21} & a_{23}\\a_{31} & a_{33}\end{vmatrix} & +\begin{vmatrix}a_{11} & a_{13}\\a_{31} & a_{33}\end{vmatrix} & -\begin{vmatrix}a_{11} & a_{13}\\a_{21} & a_{23}\end{vmatrix}\\ +\begin{vmatrix}a_{21} & a_{22}\\a_{31} & a_{32}\end{vmatrix} & -\begin{vmatrix}a_{11} & a_{12}\\a_{31} & a_{32}\end{vmatrix} & +\begin{vmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{vmatrix}\end{pmatrix}.$$

## VIII. Eigenvalue & Eigen Vector

a. Given a $n \times n$ matrix $A$, an eigenvector for a matrix $A$ is a vector $v \in R^n$ with property $Av = \lambda v$ for some scalar $\lambda$ for which $Av = \lambda v$ for some $v \neq 0$

b. That means, if:

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$(A - \lambda I)v = 0$$

c. If $\lambda$ is an eigenvalue of A, then all eigenvectors $v$ associated with $\lambda$ are the vectors in null space of $A - \lambda I$

d. We define the characteristic polynomial as following:

$$p(\lambda) = (\lambda - c_1)^{a_1}(\lambda - c_2)^{a_2} \dots (\lambda - c_k)^{a_k}$$

   i. Eigenvalues: $c_1, c_2, c_3, \dots, c_k$
   ii. Algebraic multipliers: $a_1, a_2, a_3, \dots, a_k$
   iii. Geometric multipliers (Dimensions of eigenspaces): $b_1, b_2, b_3, \dots, b_k$
   iv. Note:
      1. $1 \le b_i \le a_i$

e. We also have:
   i. Product of eigenvalues equals determinant of matrix $A$

$$det(A) = (c_1)^{a_1} \times (c_2)^{a_2} \times \dots \times (c_k)^{a_k}$$

   ii. Sum of the eigenvalues = Trace of matrix $A$

$$c_1 + c_2 + \cdots + c_k = a_{11} + a_{22} + \cdots + a_{nn}$$

f. Example:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 & -1 \\ -1 & \lambda - 3 & -1 \\ -1 & -1 & \lambda - 3 \end{vmatrix} = \begin{vmatrix} \lambda - 2 & 0 & -\lambda + 2 \\ 0 & \lambda - 2 & -\lambda + 2 \\ -1 & -1 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 2)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = (\lambda - 2)^2(\lambda - 5)$$

$$With\ \lambda = 5, we\ have:$$

$$(\lambda I - A) = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{cases} x = z \\ y = z \\ z\ free \end{cases}$$

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$$So, the\ vector\ solution\ is: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$So\ the\ eigenspace\ at\ \ \lambda = 5\ is\ the\ basis \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$dim \left( \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \right) = 1 = algebraic\ multiplier\ at\ \lambda = 5$$

i. Finally, we have:

$$trace(A) = 3 + 3 + 3 = 9$$

$$det(A) = (2^2)(5^1) = 20$$

$$p(\lambda) = (\lambda - 2)^2 (\lambda - 5)$$

$$\begin{bmatrix} \begin{cases} \lambda_1 = 2 \\ a_1 = 2 \\ b_1 = 2 \end{cases} \\ \begin{cases} \lambda_2 = 5 \\ a_2 = 1 \\ b_2 = 1 \end{cases} \end{bmatrix}$$

IX. **Similarity**

a. Theorem: We say $A$ and $B$ are similar if there is an invertible matrix $P$ for which $B = P^{-1}AP$

b. Theorem: If $A, B$ are similar, then $\det(A) = \det(B)$ and $trace(A) = trace(B)$

X. **Diagonalizability**

a. We say $A$ is diagonalizable if $A$ is similar to a diagonal matrix

b. Theorem: $A$ is diagonalizable if and only if $\mathbb{R}^n$ has a basis consisting of eigenvectors of $A$. In fact, if $v_1, v_2, \ldots, v_n$ are linear independent eigenvectors, then $P = (v_1 | v_2 | \ldots | v_n)$ diagonalizes A.

c. Example:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}\ has\ eigenvalues\ of\ 1\ and\ 7$$

$$\lambda = 1, after\ row\ reduce\ the\ matrix\ A - \lambda I, we\ have\ basis \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

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$$\lambda = 7, after\ row\ reduce\ the\ matrix\ A - \lambda I, we\ have\ basis\ \left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$So, P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$The\ diagonal\ matrix\ D = P^{-1}AP = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Note: The\ diagonal\ contains\ eigenvalues\ of\ A$$