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ORIGINAL PAPER

Inertial Splitting Methods Without Prior Constants for Solving Variational Inclusions of Two Operators

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Abstract

In this paper, we introduce two new inertial algorithms for solving a variational inclusion of the sum of two operators in a Hilbert space. The algorithms are constructed around the resolvent of a maximally monotone operator and the inertial technique. The algorithms work with or without a linesearch procedure. Using some stepsize rules in the algorithms allows them to be implemented easily without knowing previously the Lipschitz constant of operator. Theorems of weak convergence are established. We also present the applications of the obtained results to convex optimization problems, split feasibility problems and composite monotone inclusions. Some numerical results are reported to illustrate the numerical behavior of the new algorithms and compare them with others

Keywords Variational inclusion · Maximal monotonicity · Lipschitz continuity

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1 Introduction

The Tseng's extragradient method [38] or the Forward-Backward-Forward Splitting method (FBFS) was proposed for finding a zero point of the sum of two operator acting on a Hilbert space $\mathcal{H}$, namely

$$\text{Find } x^* \in \mathcal{H} \text{ such that } 0 \in \mathcal{A}x^* + \mathcal{B}x^*, \qquad \text{(VI)}$$

where $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximally monotone operator and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is a L - Lipschitz continuous operator. The FBFS method generates a sequence $\{x_n\}$ in $\mathcal{H}$, from a starting point $x_0 \in \mathcal{H}$, by the following manner,

$$\begin{cases} y_n = J_{\lambda_n \mathcal{A}}(x_n - \lambda_n \mathcal{B}x_n), \\ x_{n+1} = y_n - \lambda_n(\mathcal{B}y_n - \mathcal{B}x_n), \end{cases} \qquad \text{(FBFS)}$$

where $J_{\mathcal{A}}$ denotes the resolvent of $\mathcal{A}$, defined by $J_{\mathcal{A}} = (I + \mathcal{A})^{-1}$. The weak convergence of $\{x_n\}$ to a solution of problem (VI) under the condition $0 < \lambda_* \leq \lambda_n \leq \lambda^* < \frac{1}{L}$. Due to the importance of the FBFS method in applied sciences, in recent year the FBFS method has received a lot of attention by several authors who modified and improved it by different ways and under various types of conditions for numerous types of problems [2,3,10,11,14,16,20,23,33–35]. Some more general classes of the problem of finding zero points of the sum of three operators can be found, for example, in [13,18,21,31,39] and others in [4,22,25–28,32,36]. A interesting class of algorithms which has been widely and intensively investigated recently is inertial methods. The idea of inertial schemes originates from the implicit discretization of a differential system of second-order in time [1,24]. The authors in [1] introduced the following inertial proximal-point method for finding a zero point of a maximally monotone operator $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$,

$$x_{n+1} = J_{\lambda_n \mathcal{A}}(x_n + \alpha_n(x_n - x_{n-1})),$$

where $x_{-1},\ x_0 \in \mathcal{H}$ and $\lambda_n \geq \lambda > 0$, $0 \leq \alpha_n \leq \alpha < 1$ satisfy some suitable conditions. A characterization of inertial schemes is that the next iterate is constructed from the previous two or more ones. It also turns out that in the case $\alpha_n = 0$ the inertial-point proximal method reduces to the proximal-point method in [30]. Under the condition $\sum_{n=1}^{\infty} \alpha_n ||x_n - x_{n-1}||^2 < +\infty$, the sequence $\{x_n\}$ generated by the inertial proximal-point method converges weakly to some point in $\mathcal{A}^{-1}(0)$.

In this paper, we introduce two algorithms of inertial form for solving problem (VI) in a Hilbert space. The algorithms are developed from method FBFS in [38] and the inertial technique in [1]. Some simple stepsize rules are involved which can be implemented more easily in practise. Firstly, we propose an algorithm, namely the inertial FBFS method (IFBFSM), with a linesearch procedure and analyze the convergence of the obtained method under some mild conditions. Secondly, we present a simpler stepsize rule without any linesearch. The stepsizes in the second one are updated over each iteration by a cheap computation, and are based on the previous iterates. Our algorithms do not require any information on the Lipschitz constant of operator. We also present several applications of the obtained results to other known mathematical models as convex optimization problems, split feasibility problems and composite monotone inclusions. Finally, some numerical results are reported to show the effectiveness of the new algorithms over others.

The paper is organized as follows: Sect. 2 recalls some definitions and preliminary results used further in the paper. Sects. 3 and 4 deal with the description of new algorithms and their convergence analyses. Finally, some applications and fundamental numerical experiments are presented in Sects. 5 and 6.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued operator. The graph of $\mathcal{A}$ is the set in $\mathcal{H} \times \mathcal{H}$ defined by

$$\mathrm{Graph}(\mathcal{A}) := \{(x, u) : x \in \mathcal{H},\ u \in \mathcal{A}(x)\}.$$

An operator $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is called: (i) *monotone* if $\langle u - v, x - y\rangle \geq 0,\ \forall x,\ y \in \mathcal{H},\ \forall u \in \mathcal{A}(x),\ \forall v \in \mathcal{A}(y)$; (ii) *maximally monotone* if $\mathcal{A}$ is monotone and its graph is not a proper subset of the graph of any monotone operator; (iii) γ*- strongly monotone* if there exists a number $\gamma > 0$ such that $\langle u - v, x - y\rangle \geq \gamma||x - y||^2,\ \forall x,\ y \in \mathcal{H},\ \forall u \in \mathcal{A}(x),\ \forall v \in \mathcal{A}(y)$;

An operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is called L*- Lipschitz continuous* if there exists a number $L > 0$ such that $||\mathcal{B}(x) - \mathcal{B}(y)|| \leq L||x - y||,\ \forall x,\ y \in \mathcal{H}$.

Remark that a monotone operator $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone if and only if for each pair $(x, u) \in \mathcal{H}\times\mathcal{H}$ and $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in \mathrm{Graph}(\mathcal{A})$, then $u \in \mathcal{A}(x)$. We denote by I the identity mapping in $\mathcal{H}$. The resolvent of $\mathcal{A}$ is given by

$$J_{\mathcal{A}}(x) = (I + \mathcal{A})^{-1}(x),\ x \in \mathcal{H}.$$

We need the following results to prove the convergence of the new algorithms.

Lemma 2.1 [2, Corollary 2.14] *For all $x, y \in \mathcal{H}$ and $\alpha \in \mathcal{R}$, the following equality holds,*

$$||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2.$$

Lemma 2.2 [1] *Let $\{\Phi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, +\infty)$ such that*

$$\Phi_{n+1} \leq \Phi_n + \alpha_n(\Phi_n - \Phi_{n-1}) + \delta_n, \quad \forall n \geq 1,\ \sum_{n=1}^{+\infty}\delta_n < +\infty,$$

and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \geq 0$. Then the followings hold:

(1) $\sum_{n=1}^{+\infty}[\Phi_n - \Phi_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$*;*
(2) *There exists* $\Phi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty}\Phi_n = \Phi^*$.

3 An Inertial Splitting Method with Linesearch

In this section, we introduce an inertial splitting method with a linesearch procedure for solving the VI in a Hilbert space. We consider the VI with the following assumptions imposed on the operators $\mathcal{A}$ and $\mathcal{B}$.

Assumption A

A1. The operator $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone.
A2. The operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is monotone and Lipschitz continuous.
A3. The solution set $(\mathcal{A} + \mathcal{B})^{-1}(0)$ of problem VI is nonempty.

The algorithm is described as follows:

Algorithm 3.1 [Inertial splitting method with linesearch for VIs]

Initialization: Choose $x_{-1},\ x_0 \in \mathcal{H}$. Take numbers $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$, $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$.

Iterative steps: Assume that $x_{n-1},\ x_n \in \mathcal{H}$, calculate x_{n+1} as follows: 1. Compute

$$p_n = J_{\lambda_n \mathcal{A}}\left(x_n - \lambda_n \mathcal{B}(x_n) + \alpha_1(x_n - x_{n-1})\right).$$

where λ_n is the smallest number in $\left\{\gamma,\ \gamma l,\ \gamma l^2,\ \ldots\right\}$ such that

$$\lambda_n ||\mathcal{B}(x_n) - \mathcal{B}(p_n)|| \leq \mu ||x_n - p_n||. \tag{3.1}$$

2. Compute

$$x_{n+1} = p_n + \lambda_n(\mathcal{B}(x_n) - \mathcal{B}(p_n)) + \alpha_2(x_n - x_{n-1})$$

We have the following remark.

Remark 3.2 (i) The (linesearch) rule (3.1) is well-defined, and
(ii) $\min\left\{\gamma, \frac{\mu l}{L}\right\} \leq \lambda_n \leq \gamma$ where L is the Lipschitz constant of $\mathcal{B}$.

Indeed, we have $||\mathcal{B}(x) - \mathcal{B}(y)|| \leq L||x - y||,\ \forall x,\ y \in \mathcal{H}$. For each $n \geq 1$, set

$$p_n(\lambda) = J_{\lambda_n \mathcal{A}}\left(x_n - \lambda \mathcal{B}(x_n) + \alpha_1(x_n - x_{n-1})\right)$$

for all $\lambda > 0$. Hence $||\mathcal{B}(x_n) - \mathcal{B}(p_n(\lambda))|| \leq L||x_n - p_n(\lambda)||$, or equivalently

$$\frac{\mu}{L}||\mathcal{B}(x_n) - \mathcal{B}(p_n(\lambda))|| \leq \mu||x_n - p_n(\lambda)||.$$

This ensures that the condition (3.1) holds for all $\lambda \leq \frac{\mu}{L}$. Therefore, λ_n is well-defined. The conclusion (i) is proved. Moreover, it is obvious that $\lambda_n \leq \gamma$. If $\lambda_n = \gamma$ then conclusion (ii) is true. Otherwise, if $\lambda_n < \gamma$. In this case, we find that $\frac{\lambda_n}{l}$ does not satisfy the condition (3.1), i.e.,

$$\mu||x_n - p_n(\lambda_n/l)|| < \frac{\lambda_n}{l}||\mathcal{B}(x_n) - \mathcal{B}(p_n(\lambda_n/l))||.$$

Thus, since $||\mathcal{B}(x_n) - \mathcal{B}(p_n(\lambda_n/l))|| \leq L||x_n - p_n(\lambda_n/l)||$, we obtain that $\mu < \frac{\lambda_n L}{l}$ or $\lambda_n > \frac{l\mu}{L}$. The conclusion (ii) is proved.

In order to get the convergence of Algorithm 3.1 we consider the following condition imposed on the parameters α_1, α_2 and μ.

Condition B

$$\frac{(1-\alpha_1-\alpha_2-2\mu^2)(1-2\alpha_2^2)}{2(1+\mu)^2} - 2(\alpha_1+\alpha_2+\alpha_2^2) > 0.$$

Set $\bar{\alpha} = \alpha_1 + \alpha_2$, and

$$\mathcal{M} = \frac{1-\bar{\alpha}-2\mu^2}{2(1+\mu)^2} \quad \text{and} \quad \mathcal{N} = 2(\bar{\alpha}+\alpha_2^2) + \frac{\alpha_2^2(1-\bar{\alpha}-2\mu^2)}{(1+\mu)^2}.$$

We begin the convergence analysis of Algorithm 3.1 with the following lemma.

Lemma 3.3 *For each $x^* \in (\mathcal{A}+\mathcal{B})^{-1}(0)$ and $n \geq 1$,*

$$\begin{aligned}&||x_{n+1}-x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1}-x^*||^2\\ &\quad\leq -\mathcal{M}||x_{n+1}-x_n||^2 + \mathcal{N}||x_n - x_{n-1}||^2.\end{aligned}$$

Proof Following the definition of p_n, we have

$$x_n - p_n - \lambda_n\mathcal{B}(x_n) + \alpha_1(x_n - x_{n-1}) \in \lambda_n\mathcal{A}(p_n). \tag{3.2}$$

By the definition of x_{n+1}, we obtain

$$\lambda_n\mathcal{B}(x_n) = x_{n+1} - p_n + \lambda_n\mathcal{B}(p_n) - \alpha_2(x_n - x_{n-1}). \tag{3.3}$$

It follows from relations (3.2) and (3.3) that

$$x_n - x_{n+1} - \lambda_n\mathcal{B}(p_n) + (\alpha_1+\alpha_2)(x_n - x_{n-1}) \in \lambda_n\mathcal{A}(p_n). \tag{3.4}$$

Moreover, since $x^* \in (\mathcal{A}+\mathcal{B})^{-1}(0)$, we obtain $-\lambda_n\mathcal{B}(x^*) \in \lambda_n\mathcal{A}(x^*)$. Thus, by relation (3.4) and the monotonicity of $\lambda_n\mathcal{A}$, we find

$$\left\langle x_n - x_{n+1} - \lambda_n(\mathcal{B}(p_n) - \mathcal{B}(x^*)) + (\alpha_1+\alpha_2)(x_n - x_{n-1}), p_n - x^*\right\rangle \geq 0.$$

This together with the monotonicity of $\mathcal{B}$ implies that

$$\left\langle x_n - x_{n+1} + (\alpha_1 + \alpha_2)(x_n - x_{n-1}),\, p_n - x^* \right\rangle \geq 0.$$

Thus

$$2\left\langle x_n - x_{n+1},\, p_n - x^* \right\rangle + 2\bar{\alpha}\left\langle x_n - x_{n-1},\, p_n - x^* \right\rangle \geq 0, \tag{3.5}$$

where $\bar{\alpha} = \alpha_1 + \alpha_2$. Relation (3.5) can be rewritten as

$$\begin{aligned} &2\left\langle x_n - x_{n+1},\, p_n - x_{n+1} \right\rangle + 2\left\langle x_n - x_{n+1}, x_{n+1} - x^* \right\rangle \\ &+2\bar{\alpha}\left\langle x_n - x_{n-1},\, p_n - x_n \right\rangle + 2\bar{\alpha}\left\langle x_n - x_{n-1}, x_n - x^* \right\rangle \geq 0. \end{aligned} \tag{3.6}$$

Applying inequalities $||a \pm b||^2 = ||a||^2 \pm 2\langle a, b\rangle + ||b||^2$ to inequality (3.6) and reorganizing the obtained inequality, we come to the following one

$$\begin{aligned} &||x_{n+1} - x^*||^2 - (1 + \bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1} - x^*||^2 \\ &\leq ||x_{n+1} - p_n||^2 - (1 + \bar{\alpha})||x_n - p_n||^2 + \bar{\alpha}||x_{n-1} - p_n||^2. \end{aligned} \tag{3.7}$$

Following from the definitions of x_{n+1}, the rule (3.1), and inequality $(a + b)^2 \leq 2a^2 + 2b^2$ that

$$\begin{aligned} ||x_{n+1} - p_n||^2 &= ||\lambda_n(\mathcal{B}(x_n) - \mathcal{B}(p_n)) + \alpha_2(x_n - x_{n-1})||^2 \\ &\leq (\lambda_n||\mathcal{B}(x_n) - \mathcal{B}(p_n)|| + \alpha_2||x_n - x_{n-1}||)^2 \\ &\leq (\mu||x_n - p_n|| + \alpha_2||x_n - x_{n-1}||)^2 \\ &\leq 2\mu^2||x_n - p_n||^2 + 2\alpha_2^2||x_n - x_{n-1}||^2. \end{aligned} \tag{3.8}$$

We also have that

$$\begin{aligned} &||x_{n-1} - p_n||^2 \leq (||x_{n-1} - x_n|| + ||x - p_n||)^2 \\ &\leq 2||x_{n-1} - x_n||^2 + 2||x - p_n||^2. \end{aligned} \tag{3.9}$$

Combining relations (3.7), (3.8), and (3.9), we obtain

$$\begin{aligned} &||x_{n+1} - x^*||^2 - (1 + \bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1} - x^*||^2 \\ &\leq -(1 - \bar{\alpha} - 2\mu^2)||x_n - p_n||^2 + 2(\bar{\alpha} + \alpha_2^2)||x_n - x_{n-1}||^2. \end{aligned} \tag{3.10}$$

On the other hand, from the definition of x_{n+1} and the rule (3.1), we also obtain

$$\begin{aligned} ||x_{n+1} - x_n||^2 &= ||p_n - x_n + \lambda_n(\mathcal{B}(x_n) - \mathcal{B}(p_n)) + \alpha_2(x_n - x_{n-1})||^2 \\ &\leq (||p_n - x_n|| + \lambda_n||\mathcal{B}(x_n) - \mathcal{B}(p_n)|| + \alpha_2||x_n - x_{n-1}||)^2 \\ &\leq ((1 + \mu)||x_n - p_n|| + \alpha_2||x_n - x_{n-1}||)^2 \\ &\leq 2(1 + \mu)^2||x_n - p_n||^2 + 2\alpha_2^2||x_n - x_{n-1}||^2. \end{aligned} \tag{3.11}$$

Thus

$$||x_n - p_n||^2 \geq \frac{1}{2(1+\mu)^2}||x_{n+1} - x_n||^2 - \frac{\alpha_2^2}{(1+\mu)^2}||x_n - x_{n-1}||^2. \quad (3.12)$$

Moreover, since $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$, we have $1 - 2\alpha_2^2 > 0$. Thus, from Condition B, we obtain $1 - \alpha_1 - \alpha_2 - 2\mu^2 > 0$ or

$$1 - \bar{\alpha} - 2\mu^2 > 0. \quad (3.13)$$

Combining relations (3.10), (3.12) and (3.13), we derive

$$\begin{aligned}
&||x_{n+1} - x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1} - x^*||^2 \\
&\leq -\frac{1 - \bar{\alpha} - 2\mu^2}{2(1+\mu)^2}||x_{n+1} - x_n||^2 \\
&+ \left(2(\bar{\alpha} + \alpha_2^2) + \frac{\alpha_2^2(1 - \bar{\alpha} - 2\mu^2)}{(1+\mu)^2}\right)||x_n - x_{n-1}||^2, \quad \forall n \geq 1, \quad (3.14)
\end{aligned}$$

which, together with the definitions $\mathcal{M}$ and $\mathcal{N}$, implies the desired conclusion. □

Now we prove the main theorem.

Theorem 3.4 *Suppose that Assumption* A *and Condition* B *hold. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges weakly to some point in* $(\mathcal{A} + \mathcal{B})^{-1}(0)$.

Proof Set

$$\zeta_n = ||x_n - x^*||^2 - \bar{\alpha}||x_{n-1} - x^*||^2 + \mathcal{N}||x_n - x_{n-1}||^2.$$

Following Lemma 3.3, we have

$$\begin{aligned}
\zeta_{n+1} - \zeta_n &= ||x_{n+1} - x^*||^2 - \bar{\alpha}||x_n - x^*||^2 + \mathcal{N}||x_{n+1} - x_n||^2 \\
&\quad - \left(||x_n - x^*||^2 - \bar{\alpha}||x_{n-1} - x^*||^2 + \mathcal{N}||x_n - x_{n-1}||^2\right) \\
&= \left(||x_{n+1} - x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1} - x^*||^2\right) \\
&\quad + \mathcal{N}||x_{n+1} - x_n||^2 - \mathcal{N}||x_n - x_{n-1}||^2 \\
&\leq -\mathcal{M}||x_{n+1} - x_n||^2 + \mathcal{N}||x_n - x_{n-1}||^2 \\
&\quad + \mathcal{N}||x_{n+1} - x_n||^2 - \mathcal{N}||x_n - x_{n-1}||^2 \\
&= -(\mathcal{M} - \mathcal{N})||x_{n+1} - x_n||^2. \quad (3.15)
\end{aligned}$$

From the definitions of $\mathcal{M}$ and $\mathcal{N}$ and Condition B, we obtain

$$\mathcal{M} - \mathcal{N} = \frac{1 - \bar{\alpha} - 2\mu^2}{2(1+\mu)^2} - 2(\bar{\alpha} + \alpha_2^2) - \frac{\alpha_2^2(1 - \bar{\alpha} - 2\mu^2)}{(1+\mu)^2}$$

$$= \frac{(1-\bar{\alpha}-2\mu^2)(1-2\alpha_2^2)}{2(1+\mu)^2} - 2(\bar{\alpha}+\alpha_2^2) > 0.$$

Set $\epsilon = \mathcal{M} - \mathcal{N} > 0$ and relation (3.15) can be rewritten as

$$\zeta_{n+1} - \zeta_n \leq -\epsilon ||x_{n+1} - x_n||^2, \ \forall n \geq 0. \tag{3.16}$$

Thus, the sequence $\{\zeta_n\}_{n\geq 0}$ is non-increasing. From the definition of ζ_n, we obtain $\zeta_n \geq ||x_n - x^*||^2 - \bar{\alpha}||x_{n-1} - x^*||^2$ for all $n \geq 0$. Thus $||x_n - x^*||^2 \leq \zeta_n + \bar{\alpha}||x_{n-1} - x^*||^2 \leq \zeta_0 + \bar{\alpha}||x_{n-1} - x^*||^2 \leq |\zeta_0| + \bar{\alpha}||x_{n-1} - x^*||^2$ for all $n \geq 0$. Thus, by induction, we obtain

$$\begin{aligned} ||x_n - x^*||^2 &\leq (1 + \bar{\alpha} + \cdots + \bar{\alpha}^{n-1})|\zeta_0| + \bar{\alpha}^n ||x - x^*||^2 \\ &\leq \frac{|\zeta_0|}{1-\bar{\alpha}} + \bar{\alpha}^n ||x_0 - x^*||^2. \end{aligned}$$

Therefore, the sequence $\left\{||x_n - x^*||^2\right\}$ and thus $\{x_n\}$ are bounded. From the definition of ζ_n, we obtain $\zeta_n \geq -\bar{\alpha}||x_{n-1} - x^*||^2$ or $-\zeta_n \leq \bar{\alpha}||x_{n-1} - x^*||^2$ for all $n \geq 0$. By relation (3.16), we obtain for all $N \geq 1$,

$$\begin{aligned} \epsilon \sum_{n=0}^{N} ||x_{n+1} - x_n||^2 \leq \zeta_0 - \zeta_{N+1} &\leq \zeta_0 + \bar{\alpha}||x_N - x^*||^2 \\ &\leq |\zeta_0| + \bar{\alpha}\left(\frac{|\zeta_0|}{1-\bar{\alpha}} + \bar{\alpha}^N ||x_{n_1} - x^*||^2\right) \\ &= \frac{|\zeta_0|}{1-\bar{\alpha}} + \bar{\alpha}^{N+1} ||x_0 - x^*||^2. \end{aligned}$$

Passing to the limit in the last inequality as $N \to \infty$, we obtain

$$\sum_{n=0}^{\infty} ||x_{n+1} - x_n||^2 \leq \frac{|\zeta_0|}{\epsilon(1-\bar{\alpha})} < +\infty. \tag{3.17}$$

Thus

$$\lim_{n\to\infty} ||x_{n+1} - x_n||^2 = 0. \tag{3.18}$$

From Lemma 3.3, we obtain

$$||x_{n+1} - x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1} - x^*||^2 \leq \mathcal{N}_n ||x_n - x_{n-1}||^2.$$

Thus

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + \bar{\alpha}(||x_n - x^*||^2 - ||x_{n-1} - x^*||^2) + \mathcal{N}_n ||x_n - x_{n-1}||^2.$$

This together with Lemma 2.2, relation (3.17), the boundedness of $\{\mathcal{N}_n\}$ implies that the limit of $\left\{||x_n - x^*||^2\right\}$ exists, and we denote it by l. Thus

$$\lim_{n\to\infty} ||x_{n+1} - x^*||^2 = \lim_{n\to\infty} ||x_n - x^*||^2 = \lim_{n\to\infty} ||x_{n-1} - x^*||^2 = l. \quad (3.19)$$

Now, passing to the limit in (3.10) as $n \to \infty$ and using relations (3.13), (3.18), and (3.19), we obtain

$$\lim_{n\to\infty} ||x_n - p_n||^2 = 0. \quad (3.20)$$

Assume that x is a weak cluster point of $\{x_n\}$, i.e., there exists a subsequence $\{x_k\}$ of $\{x_n\}$ converging weakly to x. From relation (3.20), we also have $p_k \rightharpoonup x$ as $k \to \infty$. Take any pair $(u, v) \in \text{Graph}(\mathcal{A} + \mathcal{B})$, i.e., $v - \mathcal{B}(u) \in \mathcal{A}(u)$. It follows from the definition of p_k that

$$\frac{x_k - p_k}{\lambda_k} - \mathcal{B}(x_k) + \frac{\alpha_1}{\lambda_k}(x_k - x_{k-1}) \in \mathcal{A}(p_k).$$

Thus, from the monotonicity of $\mathcal{A}$, we obtain

$$\left\langle v - \mathcal{B}(u) - \frac{x_k - p_k}{\lambda_k} + \mathcal{B}(x_k) - \frac{\alpha_1}{\lambda_k}(x_k - x_{k-1}), u - p_k \right\rangle \geq 0$$

Thus, from the monotonicity of $\mathcal{B}$, we obtain

$$\begin{aligned}
\langle v, u - p_k\rangle &\geq \left\langle \mathcal{B}(u) + \frac{x_k - p_k}{\lambda_k} - \mathcal{B}(x_k) + \frac{\alpha_1}{\lambda_k}(x_k - x_{k-1}), u - p_k \right\rangle \\
&= \langle \mathcal{B}(u) - \mathcal{B}(p_k), u - p_k\rangle + \langle \mathcal{B}(p_k) - \mathcal{B}(x_k), u - p_k\rangle \\
&\quad + \left\langle \frac{x_k - p_k}{\lambda_k} + \frac{\alpha_1}{\lambda_k}(x_k - x_{k-1}), u - p_k \right\rangle \\
&\geq \langle \mathcal{B}(p_k) - \mathcal{B}(x_k), u - p_k\rangle + \left\langle \frac{x_k - p_k}{\lambda_k} + \frac{\alpha_1}{\lambda_k}(x_k - x_{k-1}), u - p_k \right\rangle \\
&\geq -\left(||\mathcal{B}(p_k) - \mathcal{B}(x_k|| + \frac{||x_k - p_k||}{\lambda_k} + \frac{\alpha_1}{\lambda_k}||x_k - x_{k-1}|| \right) ||u - p_k||.
\end{aligned}$$

Note that from (3.20) and the Lipschitzian continuity of $\mathcal{B}$, we get $||\mathcal{B}(p_k) - \mathcal{B}(x_k|| \to 0$. Thus, passing to the limit in the last inequality and using relations (3.18), (3.20) and the fact $\lambda_n \geq \min\left\{\gamma, \frac{\mu l}{L}\right\}$, we obtain $\langle v, u - x\rangle \geq 0$ for each pair $(u, v) \in \text{Graph}(\mathcal{A} + \mathcal{B})$. Since $\mathcal{A} + \mathcal{B}$ is maximally monotone, we can conclude that $x \in (\mathcal{A} + \mathcal{B})^{-1}(0)$.

Finally, we prove that the whole sequence $\{x_n\}$ converges weakly to x. Indeed, assume that there exists another subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to x' and $x' \neq x$. We have

$$2\left\langle x_n, x - x'\right\rangle = ||x||^2 - ||x'||^2 + ||x_n - x||^2 - ||x_n - x'||^2. \quad (3.21)$$

Since $x,\ x' \in (\mathcal{A}+\mathcal{B})^{-1}(0)$, as relation (3.19), we see that the limits $\lim_{n\to\infty} ||x_n - x||^2$ and $\lim_{n\to\infty} ||x_n - x'||^2$ exist. Thus, from (3.21), the limit $\lim_{n\to\infty} \langle x_n, x - x'\rangle$ exists, i.e.,

$$\lim_{n\to\infty} \langle x_n, x - x'\rangle = \xi \in \mathcal{R}. \tag{3.22}$$

Taking the limit in (3.22) as $n = k \to 0$ and after that $n = m \to \infty$, we obtain

$$\langle x, x - x'\rangle = \lim_{k\to\infty} \langle x_k, x - x'\rangle = \xi = \lim_{m\to\infty} \langle x_m, x - x'\rangle = \langle x', x - x'\rangle.$$

Thus $||x - x'||^2 = 0$ or $x = x'$. This finishes the proof. □

4 An Inertial Splitting Method with Simpler Stepsize Rule

In this section, we present an inertial splitting method without a linesearch. Instead of that, we consider a simpler stepsize rule with a cheap computation. For the sake of presentation, we adopt the convention $a/0 = +\infty$ for each $a \geq 0$. The following is the algorithm in details.

Algorithm 4.1 [Inertial splitting method without linesearch for VIs]

Initialization: Choose $x_{-1},\ x_0 \in \mathcal{H}$. Take numbers $\lambda_0 > 0$, $\mu \in (0, 1)$, $\alpha_1 \in [0, 1]$, $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$, and a sequence $\{\kappa_n\} \subset [0, +\infty)$ such that $\sum_{n=0}^{\infty} \kappa_n < +\infty$.

Iterative steps: Assume that $x_{n-1},\ x_n \in \mathcal{H}$ and λ_n are known, calculate x_{n+1} and λ_{n+1} as follows: 1. Compute

$$p_n = J_{\lambda_n \mathcal{A}}\left(x_n - \lambda_n \mathcal{B}(x_n) + \alpha_1(x_n - x_{n-1})\right).$$

2. Compute

$$x_{n+1} = p_n + \lambda_n(\mathcal{B}(x_n) - \mathcal{B}(p_n)) + \alpha_2(x_n - x_{n-1})$$

3. Update λ_n by

$$\lambda_{n+1} = \min\left\{\lambda_n + \kappa_n, \frac{\mu||x_n - p_n||}{||\mathcal{B}(x_n) - \mathcal{B}(p_n)||}\right\}.$$

The stepsize $\{\lambda_n\}$ generated by Algorithm 4.1 does not depend on the Lipschitz constant of $\mathcal{B}$. The computation in the third step of Algorithm 4.1 is simple. Actually, λ_{n+1} can be expressed as follows:

$$\lambda_{n+1} = \begin{cases} \lambda_n + \kappa_n & \text{if } \mathcal{B}(x_n) = \mathcal{B}(p_n), \\ \min\left\{\lambda_n + \kappa_n, \frac{\mu||x_n - p_n||}{||\mathcal{B}(x_n) - \mathcal{B}(p_n)||}\right\} & \text{if } \mathcal{B}(x_n) \neq \mathcal{B}(p_n). \end{cases}$$

An important property of the sequence $\{\lambda_n\}$ generated by Algorithm 4.1 is that

$$\lim_{n\to\infty} \lambda_n = \lambda > 0.$$

The proof of this property is similar to which can be found, for example, in [17, Lemma 5]. A special case when $\kappa_n = 0$ was studied in [15, Lemma 3].

We set

$$\bar{\mathcal{M}}_n = \frac{1-\bar{\alpha}-\frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2}}{2\left(1+\frac{\mu\lambda_n}{\lambda_{n+1}}\right)^2} \quad \text{and} \quad \bar{\mathcal{N}}_n = 2\left(\bar{\alpha}+\alpha_2^2\right) + \frac{\alpha_2^2\left(1-\bar{\alpha}-\frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2}\right)}{\left(1+\frac{\mu\lambda_n}{\lambda_{n+1}}\right)^2},$$

where $\bar{\alpha} = \alpha_1 + \alpha_2$. We have the following lemma.

Lemma 4.2 *There exists $n_0 \geq 1$ such that, for each $x^* \in (\mathcal{A}+\mathcal{B})^{-1}(0)$ and $n \geq n_0$,*

$$||x_{n+1}-x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1}-x^*||^2 \leq -\bar{\mathcal{M}}_n||x_{n+1}-x_n||^2 + \bar{\mathcal{N}}_n||x_n - x_{n-1}||^2.$$

Proof As relation (3.7), we have

$$\begin{aligned} &||x_{n+1}-x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1}-x^*||^2 \\ &\leq ||x_{n+1}-p_n||^2 - (1+\bar{\alpha})||x_n-p_n||^2 + \bar{\alpha}||x_{n-1}-p_n||^2. \end{aligned} \tag{4.1}$$

Using the definitions of x_{n+1}, λ_{n+1}, and inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we derive

$$\begin{aligned} ||x_{n+1}-p_n||^2 &= ||\lambda_n(\mathcal{B}(x_n)-\mathcal{B}(p_n)) + \alpha_2(x_n - x_{n-1})||^2 \\ &\leq (\lambda_n||\mathcal{B}(x_n)-\mathcal{B}(p_n)|| + \alpha_2||x_n - x_{n-1}||)^2 \\ &\leq \left(\frac{\mu\lambda_n}{\lambda_{n+1}}||x_n - p_n|| + \alpha_2||x_n - x_{n-1}||\right)^2 \\ &\leq \frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2}||x_n-p_n||^2 + 2\alpha_2^2||x_n - x_{n-1}||^2. \end{aligned} \tag{4.2}$$

This together with (4.1) and the inequality $||x_{n-1}-p_n||^2 \leq 2||x_{n-1}-x_n||^2 + 2||x - p_n||^2$, we obtain

$$\begin{aligned} &||x_{n+1}-x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1}-x^*||^2 \\ &\leq -\left(1-\bar{\alpha}-\frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2}\right)||x_n-p_n||^2 \\ &+2(\bar{\alpha}+\alpha_2^2)||x_n-x_{n-1}||^2. \end{aligned} \tag{4.3}$$

Also, by the definition of x_{n+1}, λ_{n+1} and the triangle inequality, we find

$$\begin{aligned}
||x_{n+1}-x_n||^2 &= ||p_n - x_n + \lambda_n(\mathcal{B}(x_n)-\mathcal{B}(p_n)) + \alpha_2(x_n - x_{n-1})||^2\\
&\le (||p_n - x_n|| + \lambda_n||\mathcal{B}(x_n)-\mathcal{B}(p_n)|| + \alpha_2||x_n - x_{n-1}||)^2\\
&\le \left(\left(1+\frac{\mu\lambda_n}{\lambda_{n+1}}\right)||x_n - p_n|| + \alpha_2||x_n - x_{n-1}||\right)^2\\
&\le 2\left(1+\frac{\mu\lambda_n}{\lambda_{n+1}}\right)^2||x_n - p_n||^2 + 2\alpha_2^2||x_n - x_{n-1}||^2. \qquad (4.4)
\end{aligned}$$

Hence

$$||x_n - p_n||^2 \ge \frac{1}{2\left(1+\frac{\mu\lambda_n}{\lambda_{n+1}}\right)^2}||x_{n+1}-x_n||^2 - \frac{\alpha_2^2}{\left(1+\frac{\mu\lambda_n}{\lambda_{n+1}}\right)^2}||x_n - x_{n-1}||^2. \quad (4.5)$$

Moreover, since $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$, we have $1-2\alpha_2^2 > 0$. Thus, from Condition B, we obtain $1-\alpha_1-\alpha_2-2\mu^2 > 0$ or

$$1-\bar{\alpha}-2\mu^2 > 0.$$

Thus, since $\lambda_n \to \lambda > 0$, we get

$$\lim_{n\to\infty}\left(1-\bar{\alpha}-\frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2}\right) = 1-\bar{\alpha}-2\mu^2 > 0,$$

Therefore, there exists $n_0 \ge 1$ such that

$$1-\bar{\alpha}-\frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \forall n \ge n_0. \qquad (4.6)$$

Combining relations (4.3), (4.5) and (4.6), we derive

$$\begin{aligned}
||x_{n+1}-x^*||^2 - &\ (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1}-x^*||^2\\
&\le -\frac{1-\bar{\alpha}-\frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2}}{2\left(1+\frac{\mu\lambda_n}{\lambda_{n+1}}\right)^2}||x_{n+1}-x_n||^2\\
&\quad + \left(2(\bar{\alpha}+\alpha_2^2) + \frac{\alpha_2^2(1-\bar{\alpha}-\frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2})}{(1+\frac{\mu\lambda_n}{\lambda_{n+1}})^2}\right)||x_n - x_{n-1}||^2\\
&= -\bar{\mathcal{M}}_n||x_{n+1}-x_n||^2 - \bar{\mathcal{N}}_n||x_n - x_{n-1}||^2, \quad \forall n \ge n_0. \qquad (4.7)
\end{aligned}$$

Lemma 4.2 is proved. □

Now we prove the following theorem.

Theorem 4.3 *Suppose that Assumption* A *and Condition* B *hold. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 4.1 *converges weakly to some point in* $(\mathcal{A}+\mathcal{B})^{-1}(0)$.

Proof Set

$$\bar{\zeta}_n = ||x_n - x^*||^2 - \bar{\alpha}||x_{n-1} - x^*||^2 + \mathcal{N}_n||x_n - x_{n-1}||^2.$$

Lemma 4.2 ensures that

$$\begin{aligned}
\bar{\zeta}_{n+1} - \bar{\zeta}_n &= ||x_{n+1} - x^*||^2 - \bar{\alpha}||x_n - x^*||^2 + \bar{\mathcal{N}}_{n+1}||x_{n+1} - x_n||^2 \\
&\quad - \left(||x_n - x^*||^2 - \bar{\alpha}||x_{n-1} - x^*||^2 + \bar{\mathcal{N}}_n||x_n - x_{n-1}||^2\right) \\
&= \left(||x_{n+1} - x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1} - x^*||^2\right) \\
&\quad + \bar{\mathcal{N}}_{n+1}||x_{n+1} - x_n||^2 - \bar{\mathcal{N}}_n||x_n - x_{n-1}||^2 \\
&\le -\bar{\mathcal{M}}_n||x_{n+1} - x_n||^2 + \bar{\mathcal{N}}_n||x_n - x_{n-1}||^2 + \bar{\mathcal{N}}_{n+1}||x_{n+1} - x_n||^2 \\
&\quad - \bar{\mathcal{N}}_n||x_n - x_{n-1}||^2 = -(\bar{\mathcal{M}}_n - \bar{\mathcal{N}}_{n+1})||x_{n+1} - x_n||^2. \qquad (4.8)
\end{aligned}$$

From the definitions of $\bar{\mathcal{M}}_n$ and $\bar{\mathcal{N}}_n$, we have

$$\bar{\mathcal{M}}_n - \bar{\mathcal{N}}_{n+1} = \frac{1 - \bar{\alpha} - \frac{2\mu^2\lambda_n^2}{\lambda_{n+1}^2}}{2\left(1 + \frac{\mu\lambda_n}{\lambda_{n+1}}\right)^2} - 2\left(\bar{\alpha} + \alpha_2^2\right) - \frac{\alpha_2^2\left(1 - \bar{\alpha} - \frac{2\mu^2\lambda_{n+1}^2}{\lambda_{n+2}^2}\right)}{\left(1 + \frac{\mu\lambda_{n+1}}{\lambda_{n+2}}\right)^2}.$$

Since $\lim_{n\to\infty}\lambda_n = \lambda > 0$ and Condition B, we obtain

$$\begin{aligned}
\lim_{n\to\infty}(\bar{\mathcal{M}}_n - \bar{\mathcal{N}}_{n+1}) &= \frac{1 - \bar{\alpha} - 2\mu^2}{2(1+\mu)^2} - 2(\bar{\alpha} + \alpha_2^2) - \frac{\alpha_2^2(1 - \bar{\alpha} - 2\mu^2)}{(1+\mu)^2} \\
&= \frac{(1 - \bar{\alpha} - 2\mu^2)(1 - 2\alpha_2^2)}{2(1+\mu)^2} - 2(\bar{\alpha} + \alpha_2^2) > 0.
\end{aligned}$$

Take a number ϵ such that

$$0 < \epsilon < \frac{(1 - \bar{\alpha} - 2\mu^2)(1 - 2\alpha_2^2)}{2(1+\mu)^2} - 2(\bar{\alpha} + \alpha_2^2).$$

Therefore, there exists a number $n_1 \ge n_0 \ge 1$ such that

$$\bar{\mathcal{M}}_n - \bar{\mathcal{N}}_{n+1} \ge \epsilon > 0, \quad \forall n \ge n_1. \qquad (4.9)$$

This together with relation (4.8) implies that

$$\zeta_{n+1} - \zeta_n \le -\epsilon||x_{n+1} - x_n||^2, \quad \forall n \ge n_1. \qquad (4.10)$$

Thus, the sequence $\{\zeta_n\}_{n\geq n_1}$ is non-increasing. From the definition of ζ_n, we obtain $\zeta_n \geq ||x_n - x^*||^2 - \bar{\alpha}||x_{n-1} - x^*||^2$ for all $n \geq n_1$. Thus $||x_n - x^*||^2 \leq \zeta_n + \bar{\alpha}||x_{n-1} - x^*||^2 \leq \zeta_{n_1} + \bar{\alpha}||x_{n-1} - x^*||^2 \leq |\zeta_{n_1}| + \bar{\alpha}||x_{n-1} - x^*||^2$ for all $n \geq n_1$. Thus, by induction, we obtain

$$\begin{aligned}||x_n - x^*||^2 &\leq \left(1 + \bar{\alpha} + \cdots + \bar{\alpha}^{n-n_1-1}\right)|\zeta_{n_1}| + \bar{\alpha}^{n-n_1}||x_{n_1} - x^*||^2 \\ &\leq \frac{|\zeta_{n_1}|}{1-\bar{\alpha}} + \bar{\alpha}^{n-n_1}||x_{n_1} - x^*||^2.\end{aligned}$$

Therefore, the sequence $\left\{||x_n - x^*||^2\right\}$ and thus $\{x_n\}$ are bounded. From the definition of ζ_n, we obtain $\zeta_n \geq -\bar{\alpha}||x_{n-1} - x^*||^2$ or $-\zeta_n \leq \bar{\alpha}||x_{n-1} - x^*||^2$ for all $n \geq n_1$. By relation (4.10), we obtain for all $N \geq n_1$,

$$\begin{aligned}\epsilon\sum_{n=n_1}^{N}||x_{n+1} - x_n||^2 \leq \zeta_{n_1} - \zeta_{N+1} &\leq \zeta_{n_1} + \bar{\alpha}||x_N - x^*||^2 \\ &\leq |\zeta_{n_1}| + \bar{\alpha}\left(\frac{|\zeta_{n_1}|}{1-\bar{\alpha}} + \bar{\alpha}^{N-n_1}||x_{n_1} - x^*||^2\right) \\ &= \frac{|\zeta_{n_1}|}{1-\bar{\alpha}} + \bar{\alpha}^{N-n_1+1}||x_{n_1} - x^*||^2.\end{aligned}$$

Passing to the limit in the last inequality as $N \to \infty$, we obtain

$$\sum_{n=n_1}^{\infty}||x_{n+1} - x_n||^2 \leq \frac{|\zeta_{n_1}|}{\epsilon(1-\bar{\alpha})} < +\infty. \tag{4.11}$$

Thus

$$\lim_{n\to\infty}||x_{n+1} - x_n||^2 = 0. \tag{4.12}$$

From Lemma 3.3, we obtain

$$||x_{n+1} - x^*||^2 - (1+\bar{\alpha})||x_n - x^*||^2 + \bar{\alpha}||x_{n-1} - x^*||^2 \leq \mathcal{N}_n||x_n - x_{n-1}||^2.$$

Thus

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + \bar{\alpha}(||x_n - x^*||^2 - ||x_{n-1} - x^*||^2) + \mathcal{N}_n||x_n - x_{n-1}||^2.$$

This together with Lemma 2.2, relation (4.11), the boundedness of $\{\mathcal{N}_n\}$ implies that the limit of $\left\{||x_n - x^*||^2\right\}$ exists, and we denote it by l. Thus

$$\lim_{n\to\infty}||x_{n+1} - x^*||^2 = \lim_{n\to\infty}||x_n - x^*||^2 = \lim_{n\to\infty}||x_{n-1} - x^*||^2 = l. \tag{4.13}$$

Now, passing to the limit in (4.3) as $n \to \infty$ and using relations (4.6), (4.12), and (4.13), we obtain

$$\lim_{n\to\infty} ||x_n - p_n||^2 = 0.$$

The rest of the proof is similar to the one of Theorem 3.4. Theorem 4.3 is proved. $\square$

In the case when $\alpha_2 = 0$, we obtain the following result.

Corollary 4.4 *Assume that Condition A holds. Choose $x_{-1},\ x_0 \in \mathcal{H}$ and take $\lambda_0 > 0$, $\mu \in (0, 1)$, $\alpha_1 \in [0, 1]$ such that*

$$\frac{1 - \alpha_1 - 2\mu^2}{2(1 + \mu)^2} - 2\alpha_1 > 0.$$

Take a sequence $\{\kappa_n\} \subset [0, +\infty)$ such that $\sum_{n=0}^{\infty} \kappa_n < +\infty$. Let $\{x_n\}$ be the sequence generated by the following manner:

$$\begin{cases} p_n = J_{\lambda_n \mathcal{A}}\left(x_n - \lambda_n \mathcal{B}(x_n) + \alpha_1(x_n - x_{n-1})\right), \\ x_{n+1} = p_n + \lambda_n(\mathcal{B}(x_n) - \mathcal{B}(p_n)), \\ \lambda_{n+1} = \min\left\{\lambda_n + \kappa_n, \frac{\mu||x_n - p_n||}{||\mathcal{B}(x_n) - \mathcal{B}(p_n)||}\right\}. \end{cases}$$

Then the sequence $\{x_n\}$ converges weakly to some point in $(\mathcal{A} + \mathcal{B})^{-1}(0)$.

In the case where $\mathcal{B} = 0$, from the definition of λ_{n+1} and the aforementioned convention in the beginning of Sect. 4, we see that $\lambda_{n+1} = \lambda_n + \kappa_n$ for all $n \geq 0$. Take $\lambda_0 = \lambda > 0$ and $\kappa_n = 0$, then we have that $\lambda_n = \lambda$ for all $n \geq 0$. The iterative steps of Algorithm 3.1 can be rewritten as

$$x_{n+1} = J_{\lambda\mathcal{A}}\left(x_n + \alpha_1(x_n - x_{n-1})\right) + \alpha_2(x_n - x_{n-1}).$$

In this case, we can consider μ small enough, and Condition B could be replaced by $(1 - \alpha_1 - \alpha_2)(1 - 2\alpha_2^2) - 4(\alpha_1 + \alpha_2 - \alpha_2^2) > 0$ or

$$1 + 2\alpha_2^2 - (\alpha_1 + \alpha_2)(5 - 2\alpha_2^2) > 0.$$

We obtain the following corollary.

Corollary 4.5 *Assume that the operator $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone. Let $\{x_n\}$ be the sequence in $\mathcal{H}$, defined by*

$$x_{n+1} = J_{\lambda\mathcal{A}}\left(x_n + \alpha_1(x_n - x_{n-1})\right) + \alpha_2(x_n - x_{n-1}),$$

where $x_0 \in \mathcal{H}$, $\lambda > 0$, and $\alpha_1 \in [0, 1),\ \alpha_2 \in [0, 1/\sqrt{2})$ such that $1 + 2\alpha_2^2 - (\alpha_1 + \alpha_2)(5 - 2\alpha_2^2) > 0$. Then, the sequence $\{x_n\}$ converges weakly to some point in $\mathcal{A}^{-1}(0)$.

5 Applications

In this section, we present the applications of the results of Sect. 4 to some mathematical models as convex optimization problem, split feasibility problem, and composite monotone inclusion. The results of Sect. 3 is also applied similarly and we do not present them in this section.

5.1 Two-function sum optimization

Let $f : \mathcal{H} \to \mathcal{R} \cup \{\pm\infty\}$ be a function with the *effective domain*

$$\mathrm{dom}(f) = \{x \in \mathcal{H} : f(x) < +\infty\}.$$

The function f is called: (i) *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in \mathcal{H}$; (ii) *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $t \in [0, 1]$ and $x,\ y \in \mathcal{H}$; (iii) β- *strongly convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\beta}{2}||x - y||^2$ for all $t \in [0, 1]$ and $x,\ y \in \mathcal{H}$. We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous, convex functions. The *subdifferential* of f at $x \in \mathcal{H}$ is the set

$$\partial f(x) = \{w \in \mathcal{H} : f(y) - f(x) \geq \langle w, y - x\rangle,\ \forall y \in \mathcal{H}\}.$$

If $\partial f(x) \neq \emptyset$ then f is called subdifferentiable at x. In the case if f is differentiable then $\partial f(x) = \{\nabla f(x)\}$ where $\nabla f(x)$ is the gradient of f at $x \in \mathcal{H}$. The proximal mapping of f is defined by

$$\mathrm{prox}_f(x) = \arg\min\left\{f(y) + \frac{1}{2}||y - x||^2 : y \in \mathcal{H}\right\},\ \ x \in \mathcal{H}.$$

In this subsection, we apply Algorithm 4.1 to solve an optimization problem of the following two-function sum.

Problem 5.1 Let $f,\ g$ be two functions in $\Gamma_0(\mathcal{H})$ such that f is differentiable with its gradient being Lipschitz continuous and g is subdifferentiable. Consider the convex optimization problem of the following two-operator sum,

$$\min_{x \in \mathcal{H}} (f(x) + g(x)) \qquad \text{(OP)}$$

We assume that the solution set $\mathrm{Arg}\min(f+g)$ of the (OP) is nonempty. This problem can be equivalently reformulated to our variational inclusion problem with $\mathcal{A} = \partial g$ and $\mathcal{B} = \nabla f$. Note that ∂g is maximally monotone, ∇f is monotone and Lipschitz continuous (assumed), and $J_{\partial g}(x) = (I + \partial g)^{-1}(x) = \mathrm{prox}_g(x)$.

The following result follows directly from Theorem 4.3.

Theorem 5.2 *Let $f,\ g : \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ be two proper, lower semicontinuous, and convex functions such that f is differentiable with its gradient being Lipschitz continuous and g is subdifferentiable. In addition, suppose that the solution set* $\mathrm{Arg}\min(f+g)$

is nonempty. Take two starting points x_{-1}, $x_0 \in \mathcal{H}$, four real numbers $\lambda_0 > 0$, $\mu \in (0, 1)$, $\alpha_1 \in [0, 1]$, $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$ such that Condition B holds, and a sequence $\{\kappa_n\} \subset [0, +\infty)$ such that $\sum_{n=0}^{\infty} \kappa_n < +\infty$. Let $\{x_n\}$ be the sequence in $\mathcal{H}$ generated by the following manner,

$$\begin{cases} p_n = \operatorname{prox}_{\lambda_n g}\left[x_n - \lambda_n \nabla f(x_n) + \alpha_1(x_n - x_{n-1})\right], \\ x_{n+1} = p_n + \lambda_n(\nabla f(x_n) - \nabla f(p_n)) + \alpha_2(x_n - x_{n-1}), \\ \lambda_{n+1} = \min\left\{\lambda_n + \kappa_n, \frac{\mu||x_n - p_n||}{||\nabla f(x_n) - \nabla f(p_n)||}\right\}. \end{cases}$$

Then, the sequence $\{x_n\}$ converges weakly to some point in $\operatorname{Arg\,min}(f + g)$.

5.2 Split Feasibility Problem

The split feasibility problem (SFP) [9] consists of finding a point in a nonempty closed convex subset of a space such that its image under a bounded linear operator belongs to a nonempty closed convex subset of another space. This problem is applied to solve many mathematical models in signal processing, specifically in phase retrieval and other image restoration problems [19,37]. After that it is found that the (SFP) can be applied to model the intensity modulated radiation therapy [8] and other fields [5–7]. The (SFP) in Hilbert space is of the following form.

Problem 5.3 Let $\mathcal{H}_i$ $(i = 1, 2)$ be two Hilbert spaces and C, Q be nonempty closed convex subsets in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\mathcal{T} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded and linear operator with its adjoint $\mathcal{T}^*$. The split feasibility problem (SFP) is stated as follows:

$$\text{find } x^* \in C \text{ such that } \mathcal{T}x^* \in Q. \qquad q(\text{SFP})$$

We assume that the solution set of the (SFP) is nonempty. Set

$$f(x) = \frac{1}{2}||\mathcal{T}x - P_Q\mathcal{T}x||^2_{\mathcal{H}_2}, \ x \in \mathcal{H}_1.$$

Note that $f(x)$ is differentiable and its gradient $\nabla f = \mathcal{T}^*(I - P_Q)\mathcal{T}$ is Lipschitz continuous and monotone. It is easy to see that x^* is a solution of the (SFP) if and only if x^* is a solution of the following optimization problem with zero optimal value,

$$\min_{x \in C} f(x) = \frac{1}{2}||\mathcal{T}x - P_Q\mathcal{T}x||^2_{\mathcal{H}_2}.$$

This optimization problem is equivalent to the variational inclusion problem,

$$\text{find } x^* \in \mathcal{H}_1 \text{ such that } 0 \in \nabla f(x^*) + N_C(x^*),$$

where N_C is the normal cone of C, defined by $N_C(x) = \{w \in \mathcal{H}_1 : \langle w, y - x\rangle \leq 0, \ \forall y \in \mathcal{H}_1\}$. It is well-known that the normal cone N_C of the convex set C is maximally monotone. Thus, the (SFP) can be equivalently reformulated to our variational inclusion problem with $\mathcal{B}(x) = \mathcal{T}^*(I - P_Q)\mathcal{T}(x)$ and $\mathcal{A} = N_C$. The following result follows directly from Theorem 4.3.

Theorem 5.4 *Suppose that the solution set of problem (SFP) is nonempty. Take two starting points $x_{-1},\ x_0 \in \mathcal{H}_1$, four real numbers $\lambda_0 > 0$, $\mu \in (0, 1)$, $\alpha_1 \in [0, 1]$, $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$ such that Condition B holds, and a sequence $\{\kappa_n\} \subset [0, +\infty)$ such that $\sum_{n=0}^{\infty} \kappa_n < +\infty$. Let $\{x_n\}$ be the sequence in $\mathcal{H}_1$ generated by the following manner,*

$$\begin{cases} p_n = P_C\left[x_n - \lambda_n \mathcal{T}^*(I - P_Q)\mathcal{T}(x_n) + \alpha_1(x_n - x_{n-1})\right], \\ x_{n+1} = p_n + \lambda_n(\mathcal{T}^*(I - P_Q)\mathcal{T}(x_n) - \mathcal{T}^*(I - P_Q)\mathcal{T}(p_n)) + \alpha_2(x_n - x_{n-1}), \\ \lambda_{n+1} = \min\left\{\lambda_n + \kappa_n, \frac{\mu||x_n - p_n||}{||\mathcal{T}^*(I-P_Q)\mathcal{T}(x_n) - \mathcal{T}^*(I-P_Q)\mathcal{T}(p_n)||}\right\}. \end{cases}$$

Then, the sequence $\{x_n\}$ converges weakly to some solution of problem (SFP).

5.3 Composite Monotone Inclusion

The goal of this part is to present how Algorithm 3.1 can be used to solve a composite monotone inclusion [12]. The problem here is described as follows:

Problem 5.5 Let z be a given point in a real Hilbert space $\mathcal{H}$. Let $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ be a monotone and Lipschitz continuous operator, and $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone multi-valued operator. Let $m \in \mathbb{N}_*$ and $(\mathcal{G}_i)_{1\le i\le m}$ be real Hilbert spaces. For each $i \in \{1, 2, \dots, m\}$, let $r_i \in \mathcal{G}_i$, let $\mathcal{B}_i : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let $\mathcal{D}_i : \mathcal{H} \to 2^{\mathcal{H}}$ be monotone such that $\mathcal{D}_i^{-1}$ is Lipschitz continuous, and $\mathcal{L}_i : \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator. The problem is to solve the primal inclusion,

$$\text{find } \bar{x} \in \mathcal{H} \text{ such that } z \in \mathcal{A}\bar{x} + \sum_{i=1}^{m} \mathcal{L}_i^*\left((\mathcal{B}_i \Box \mathcal{D}_i)(\mathcal{L}_i\bar{x} - r_i)\right) + \mathcal{C}\bar{x}, \tag{5.1}$$

together with the dual inclusion

$$\text{find } \bar{v}_1 \in \mathcal{G}_1, \dots,\ \bar{v}_m \in \mathcal{G}_m \text{ such that } \exists x \in \mathcal{H} : \begin{cases} z - \sum_{i=1}^{m} \mathcal{L}_i^*\bar{v}_i \in \mathcal{A}x + \mathcal{C}x \\ \bar{v}_i \in (\mathcal{B}_i \Box \mathcal{D}_i)(\mathcal{L}_i x - r_i), \\ i = 1, \dots, m, \end{cases} \tag{5.2}$$

where $A \Box B : \mathcal{H} \to 2^{\mathcal{H}}$ is the parallel sum of two operators $A,\ B : \mathcal{H} \to 2^{\mathcal{H}}$, defined by $A \Box B = (A^{-1} + B^{-1})^{-1}$.

Problem 5.5 covers a large class of monotone inclusion problems, and some interesting instances of it can be found in [12]. In order to solve Problem 5.5 we assume that

$$z \in \operatorname{ran}\left(\mathcal{A} + \sum_{i=1}^{m} \mathcal{L}_i^*\left((\mathcal{B}_i \Box \mathcal{D}_i)(\mathcal{L}_i - r_i)\right) + \mathcal{C}\right). \tag{5.3}$$

We say that $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is a primal-dual solution to Problem 5.5 if

$$z - \sum_{i=1}^{m} \mathcal{L}_i^* \bar{v}_i \in \mathcal{A}\bar{x} + \mathcal{C}\bar{x} \text{ and } \bar{v}_i \in (\mathcal{B}_i \square \mathcal{D}_i)(\mathcal{L}_i \bar{x} - r_i),\ i = 1, \ldots, m.$$

It is well-known that if $\bar{x} \in \mathcal{H}$ is a solution of (5.1) then there exists $(\bar{v}_1, \ldots, \bar{v}_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ such that $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is a primal-dual solution of Problem 5.5, and otherwise if $(\bar{v}_1, \ldots, \bar{v}_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is a solution of (5.2) then there exists $\bar{x} \in \mathcal{H}$ such that $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \ldots \times \mathcal{G}_m$ is also a primal-dual solution of Problem 5.5. Moreover, if $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is a solution of Problem 5.5 then $\bar{x}$ is a solution of (5.1) and $(\bar{v}_1, \ldots, \bar{v}_m)$ is a solution of (5.2).

We endow the product space $\mathbb{H} = \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ with the inner product $\langle ., . \rangle_{\mathbb{H}}$ and the induced norm $||.||_{\mathbb{H}}$ as follows: for each $\mathbb{X} = (x, v_1, \ldots, v_m) \in \mathbb{H}$ and $\mathbb{Y} = (x, w_1, \ldots, w_m) \in \mathbb{H}$,

$$\langle \mathbb{X}, \mathbb{Y} \rangle_{\mathbb{H}} = \langle x, y \rangle_{\mathcal{H}} + \sum_{i=1}^{m} \langle v_i, w_i \rangle_{\mathcal{G}_i}$$

and

$$||\mathbb{X}||_{\mathbb{H}} = \left(||x||_{\mathcal{H}}^2 + \sum_{i=1}^{m} ||v_i||_{\mathcal{G}_i}^2 \right)^{1/2}.$$

We introduce the operators $\mathbb{A} : \mathbb{H} \to 2^{\mathbb{H}}$,

$$\mathbb{A}(\mathbb{X}) = (-z + \mathcal{A}x) \times (r_1 + \mathcal{B}_1^{-1} v_1) \times \cdots \times (r_m + \mathcal{B}_m^{-1} v_m)$$

and $\mathbb{B} : \mathbb{H} \to \mathbb{H}$,

$$\mathbb{B}(\mathbb{X}) = \left(\mathcal{C}x + \sum_{i=1}^{m} \mathcal{L}_i^* v_i, -\mathcal{L}_1 x + \mathcal{D}_1^{-1} v_1, \ldots, -\mathcal{L}_m x + \mathcal{D}_m^{-1} v_m \right).$$

Note that the assumption (5.3) is equivalent to

$$\operatorname{zer}(\mathbb{A} + \mathbb{B}) \neq \emptyset.$$

Moreover, $(x, v_1, \ldots, v_m) \in \operatorname{zer}(\mathbb{A} + \mathbb{B})$ if and only if $(x, v_1, \ldots, v_m)$ is a primal-dual solution of Problem 5.5. Since $\mathcal{A}$ and $\mathcal{B}_i$ are maximally monotone, $\mathbb{A}$ is maximally monotone. Moreover, the operator $\mathbb{B}$ is monotone and Lipschitz continuous (see, [12, Theorem 3.1]). It is easy to show that in the product space $\mathbb{H}$, the resolvent $\mathbb{J}_{\lambda\mathbb{A}}$ of the operator $\lambda\mathbb{A}$ is given by

$$\mathbb{J}_{\lambda \mathbb{A}}(\mathbb{X}) = \left(J_{\lambda \mathcal{A}}(x + \lambda z), J_{\lambda B_1^{-1}}(v_1 - \lambda r_1), \ldots, J_{\lambda B_m^{-1}}(v_m - \lambda r_m) \right),$$

for each $\mathbb{X} = (x, v_1, \ldots, v_m) \in \mathbb{H}$ and $\lambda > 0$ (see, [2, Proposition 23.16]). We have the following result which follows from Theorem 4.3.

Theorem 5.6 *In Problem 5.5, suppose that hypothesis (5.3) holds. Choose $\mathbb{X}_{-1}, \mathbb{X}_0 \in \mathbb{H}$, take numbers $\lambda_0 > 0$, $\mu \in (0, 1)$, $\alpha_1 \in [0, 1]$, $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$ such that Condition B holds, and a sequence $\{\kappa_n\} \subset [0, +\infty)$ such that $\sum_{n=0}^{\infty} \kappa_n < +\infty$. Let $\{\mathbb{X}_n\}$ be the sequence in $\mathbb{H}$ generated by the following manner,*

$$\begin{cases} \mathbb{P}_n = \mathbb{J}_{\lambda \mathbb{A}}\left[\mathbb{X}_n - \lambda_n \mathbb{B}(\mathbb{X}_n) + \alpha_1(\mathbb{X}_n - \mathbb{X}_{n-1})\right], \\ \mathbb{X}_{n+1} = \mathbb{P}_n + \lambda_n(\mathbb{B}(\mathbb{X}_n) - \mathbb{B}(\mathbb{P}_n)) + \alpha_2(\mathbb{X}_n - \mathbb{X}_{n-1}), \\ \lambda_{n+1} = \min\left\{\lambda_n + \kappa_n, \frac{\mu||\mathbb{X}_n - \mathbb{P}_n||_{\mathbb{H}}}{||\mathbb{B}(\mathbb{X}_n) - \mathbb{B}(\mathbb{P}_n)||_{\mathbb{H}}}\right\}. \end{cases} \tag{5.4}$$

Then, the sequence $\{\mathbb{X}_n\}$ converges weakly to a point $\bar{\mathbb{X}} = (\bar{x}, \bar{v}_1, \ldots, \bar{v}_m) \in \mathbb{H}$ which is a primal-dual solution to Problem 5.5, i.e., $\bar{x}$ solves (5.1) and $(\bar{v}_1, \ldots, \bar{v}_m)$ solves (5.2).

Now, for each $n \geq 0$, we set

$$\mathbb{X}_n = (x_n, v_{1,n}, ., v_{m,n}) \quad \text{and} \quad \mathbb{P}_n = (p_n, p_{1,n}, \ldots, p_{m,n}).$$

Then, algorithm (5.4) can be rewritten as follows:

$$\begin{cases} p_n = J_{\lambda_n \mathcal{A}}\left[x_n - \lambda_n\left(\mathcal{C}x_n + \sum_{i=1}^m \mathcal{L}_i^* v_{i,n} - z\right) + \alpha_1(x_n - x_{n-1})\right], \\ p_{i,n} = J_{\lambda_n \mathcal{B}_i^{-1}}\left[v_{i,n} + \lambda_n\left(\mathcal{L}_i x_n - \mathcal{D}_i^{-1} v_{i,n} - r_i\right) + \alpha_1\left(v_{i,n} - v_{i,n-1}\right)\right], \\ \quad i = 1, 2, \ldots, m \\ v_{i,n+1} = \lambda_n \mathcal{L}_i(p_n - x_n) + \lambda_n(\mathcal{D}_i^{-1} v_{i,n} - \mathcal{D}_i^{-1} p_{i,n}) + p_{i,n} + \alpha_2(v_{i,n} - v_{i,n-1}), \\ \quad i = 1, 2, \ldots, m \\ x_{n+1} = \lambda_n \sum_{i=1}^m \mathcal{L}_i^*(v_{i,n} - p_{i,n}) + \lambda_n(\mathcal{C}x_n - \mathcal{C}p_n) + p_n + \alpha_2(x_n - x_{n-1}), \\ \lambda_{n+1} = \min\left\{\lambda_n + \kappa_n, \frac{\mu\left(||x_n - p_n||_{\mathcal{H}}^2 + \sum\limits_{i=1}^m ||v_{i,n} - p_{i,n}||_{\mathcal{G}_i}^2\right)^{1/2}}{\left(||\nabla h x_n - \nabla h p_n + \sum\limits_{i=1}^m \mathcal{L}_i^*(v_{i,n} - p_{i,n})||_{\mathcal{H}}^2 + \sum\limits_{i=1}^m ||\nabla l_i^* v_{i,n} - \nabla l_i^* p_{i,n} - \mathcal{L}_i(x_n - p_n)||_{\mathcal{G}_i}^2\right)^{1/2}}\right\}. \end{cases}$$

We obtain that $x_n \rightharpoonup \bar{x}$ solves (5.1) and $\left(\bar{v}_{1,n}, \ldots, \bar{v}_{m,n}\right) \rightharpoonup (\bar{v}_1, \ldots, \bar{v}_m)$ solves (5.2).

5.4 Minimization Problem

This part focuses on an application of Algorithm 4.1 to the following convex minimization problem [12]. Let $f \in \Gamma_0(\mathcal{H})$. The *conjugate function* f^* is defined by

$$f^*(u) = \sup_{x \in \mathcal{H}} \{\langle u, x \rangle - f(x)\}.$$

It is well-known that if $f \in \Gamma_0(\mathcal{H})$ then ∂f is maximally monotone (see, [29]) and $(\partial f)^{-1} = \partial f^*$. Let $f, g : \mathcal{H} \to \mathcal{R} \cup \{\pm\infty\}$. The *infimal convolution* of two functions f, g is a function $f \Box g : \mathcal{H} \to \mathcal{R} \cup \{\pm\infty\}$, which is defined by

Table 1 Computational results for solving the LASSO problem when $M = 1024$, $N = 512$

m-sparse signal	Method	MSE $< 10^{-3}$		MSE $< 10^{-5}$	
		CPU(s)	Iter	CPU	Iter
$m = 30$	Algorithm IFBFS	66.4655	7281	57.3266	6752
	Algorithm 3.1	60.2155	4112	53.0988	3819
	Algorithm 4.1	56.8967	5717	50.5920	5386
$m = 40$	Algorithm IFBFS	67.0469	7145	67.2738	7403
	Algorithm 3.1	62.8973	4048	58.4584	4181
	Algorithm 4.1	54.5094	5604	55.6984	5728
$m = 50$	Algorithm IFBFS	77.5548	8053	81.5876	8414
	Algorithm 3.1	67.9690	4522	68.8494	4766
	Algorithm 4.1	67.7050	6369	72.7261	6869
$m = 60$	Algorithm IFBFS	93.2134	9043	111.6275	9787
	Algorithm 3.1	74.2481	5100	96.3695	5545
	Algorithm 4.1	71.9125	6857	96.0897	7708
$m = 70$	Algorithm IFBFS	98.7533	9349	107.6999	9811
	Algorithm 3.1	78.1546	5289	82.3055	5520
	Algorithm 4.1	76.9389	7165	88.4114	7749

$$(f \square g)(x) = \inf_{y \in \mathcal{H}} \{f(y) + g(x - y)\}, \ x \in \mathcal{H}.$$

Problem 5.7 Let $\mathcal{H}$ be a real Hilbert space, let $z \in \mathcal{H}$, let $m \in \mathbb{N}_*$, let $f \in \Gamma_0(\mathcal{H})$, and let $h : \mathcal{H} \to \mathbb{R}$ be convex and differentiable with its gradient being Lipschitz continuous. For each $i \in \{1, 2, \ldots, m\}$, let $\mathcal{G}_i$ be a real Hilbert space, let $r_i \in \mathcal{G}_i$, let $g_i \in \Gamma_0(\mathcal{G}_i)$, $l_i \in \Gamma_0(\mathcal{G}_i)$ be strongly convex, and suppose that $\mathcal{L}_i : \mathcal{H} \to \mathcal{G}_i$ is a nonzero bounded linear operator. Consider the problem

$$\min_{x \in \mathcal{H}} \ f(x) + \sum_{i=1}^{m} (g_i \square l_i)(\mathcal{L}_i x - r_i) + h(x) - \langle x, z \rangle, \tag{5.5}$$

and the dual problem

$$\min_{v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m} \ (f^* \square h^*) \left(z - \sum_{i=1}^{m} \mathcal{L}_i^* v_i \right) + \sum_{i=1}^{m} \left(g_i^*(v_i) + l_i^*(v_i) + \langle v_i, r_i \rangle \right). \tag{5.6}$$

In order to solve Problem 5.7, we consider the following assumption,

$$z \in \operatorname{ran} \left(\partial f + \sum_{i=1}^{m} \mathcal{L}_i^* (\partial g_i \square \partial l_i)(L_i \cdot - r_i) + \nabla h \right). \tag{5.7}$$

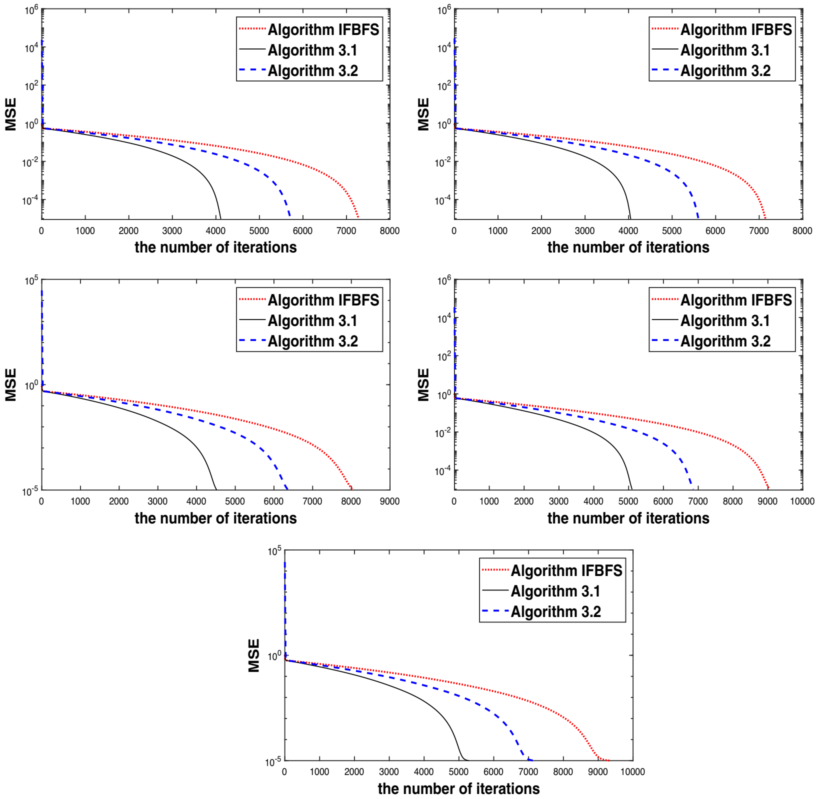

Fig. 1 The MSE value and number of iterations when $N = 1024$, $M = 512$ and $\mathrm{MSE} < 10^{-3}$

It is easy to establish a connection between Problems 5.5 and 5.7 with

$$\mathcal{A} = \partial f,\ \mathcal{C} = \nabla h,\ \mathcal{B}_i = \partial g_i,\ \mathcal{D}_i = \partial l_i,\quad \forall i = 1, 2, \ldots, m.$$

We obtain the following result.

Theorem 5.8 *In Problem 5.7, suppose that the condition (5.7) is satisfied. Choose $x_{-1},\ x_0 \in \mathcal{H}$ and $v_{i,-1},\ v_{i,0} \in \mathcal{G}_i$ for each $i \in \{1, 2, \ldots, m\}$. Take $\lambda_0 > 0$, $\mu \in (0, 1)$, $\alpha_1 \in [0, 1]$, $\alpha_2 \in [0, \frac{1}{\sqrt{2}})$ such that Condition B holds, and a sequence $\{\kappa_n\} \subset [0, +\infty)$ such that $\sum_{n=0}^{\infty} \kappa_n < +\infty$. Let $\{x_n\} \subset \mathcal{H}$ and $\left\{(v_{1,n}, v_{2,n}, \ldots, v_{m,n})\right\} \subset \mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_m$ be the sequences generated by the following manner,*

$$\begin{cases} p_n = \mathrm{prox}_{\lambda_n f}\left[x_n - \lambda_n\left(\nabla h x_n + \sum_{i=1}^{m} \mathcal{L}_i^* v_{i,n} - z\right) + \alpha_1\left(x_n - x_{n-1}\right)\right], \\ p_{i,n} = \mathrm{prox}_{\lambda_n g_i^*}\left[v_{i,n} + \lambda_n\left(\mathcal{L}_i x_n - \nabla l_i^* v_{i,n} - r_i\right) + \alpha_1\left(v_{i,n} - v_{i,n-1}\right)\right],\ i = 1, 2, \ldots, m, \\ v_{i,n+1} = \lambda_n \mathcal{L}_i(p_n - x_n) + \lambda_n(\nabla l_i^* v_{i,n} - \nabla l_i^* p_{i,n}) + p_{i,n} + \alpha_2(v_{i,n} - v_{i,n-1}),\ i = 1, 2, \ldots, m, \\ x_{n+1} = \lambda_n \sum_{i=1}^{m} \mathcal{L}_i^*(v_{i,n} - p_{i,n}) + \lambda_n(\nabla h x_n - \nabla h p_n) + p_n + \alpha_2(x_n - x_{n-1}), \\ \lambda_{n+1} = \min\left\{\lambda_n + \kappa_n, \dfrac{\mu\left(\|x_n - p_n\|_{\mathcal{H}}^2 + \sum\limits_{i=1}^{m} \|v_{i,n} - p_{i,n}\|_{\mathcal{G}_i}^2\right)^{1/2}}{\left(\|\nabla h x_n - \nabla h p_n + \sum\limits_{i=1}^{m} \mathcal{L}_i^*(v_{i,n} - p_{i,n})\|_{\mathcal{H}}^2 + \sum\limits_{i=1}^{m} \|\nabla l_i^* v_{i,n} - \nabla l_i^* p_{i,n} - \mathcal{L}_i(x_n - p_n)\|_{\mathcal{G}_i}^2\right)^{1/2}}\right\}. \end{cases}$$

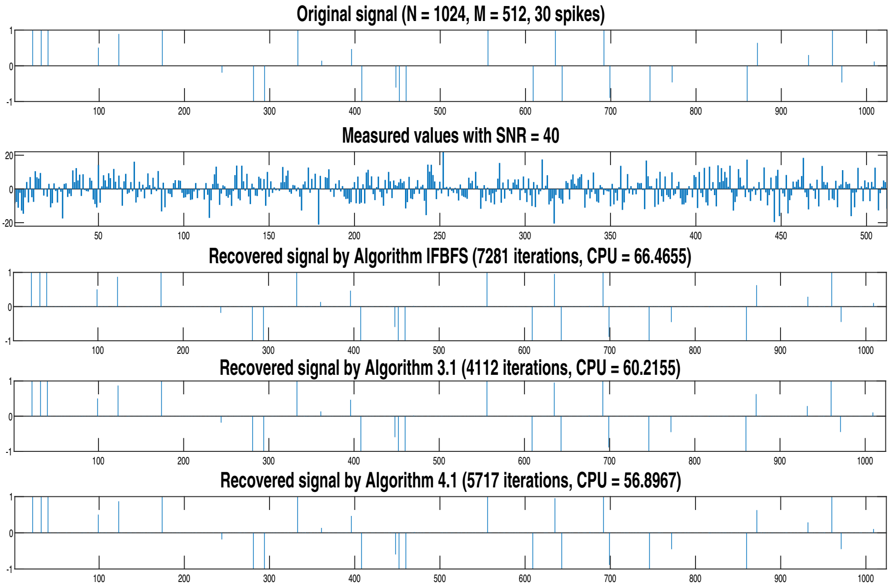

Fig. 2 From top to bottom: original signal, observation data, recovered signal by Algorithm IFBFS, Algorithm 3.1 and Algorithm 4.1 with MSE$< 10^{-3}$

Then, $x_n \rightharpoonup \bar{x}$ *solves* (5.5) *and* $\left(\bar{v}_{1,n}, \ldots, \bar{v}_{m,n}\right) \rightharpoonup (\bar{v}_1, \ldots, \bar{v}_m)$ *solves* (5.6).

6 Numerical Illustrations

Next, we apply our result to the signal recovery in compressive sensing. We show the performance of our proposed Algorithm 3.1, the inertial splitting method with linesearch, Algorithm 4.1, the inertial splitting method without linesearch, and provide a comparison among our algorithms and Algorithm IFBFS in [3]. In this case, we set $J_{\lambda_n \mathcal{A}}(x) = \text{prox}_{\lambda_n g}(x)$ and $\mathcal{B}(x) = \nabla f(x)$. This problem can be modeled as:

$$y = Ax + \epsilon, \tag{6.1}$$

where $y \in \mathbb{R}^M$ is the observed data, ϵ is the noise, $A : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded and linear operator and $x \in \mathbb{R}^N$ is a recovered vector containing m nonzero components. It is known that (6.1) can be modeled as the LASSO problem:

$$\min_{x \in \mathbb{R}^N} \left(\frac{1}{2} \|Ax - y\|_2^2 + \lambda \|x\|_1 \right), \tag{6.2}$$

where $\lambda > 0$. So, we can apply the proposed method to solve convex minimization problem when $f(x) = \frac{1}{2}\|y - Ax\|_2^2$ and $g(x) = \lambda \|x\|_1$.

In experiment y is generated by the Gaussian noise with SNR = 40, A is generated by the normal distribution with mean zero and variance one and $x \in \mathbb{R}^N$ is generated

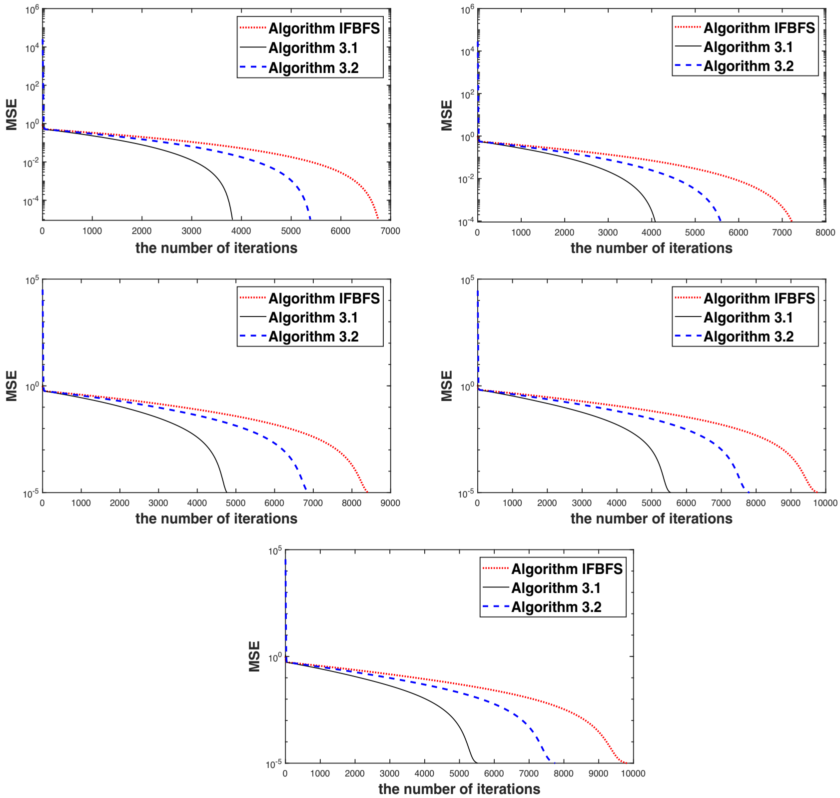

Fig. 3 The MSE value and number of iterations when $N = 1024$, $M = 512$ and MSE $< 10^{-5}$

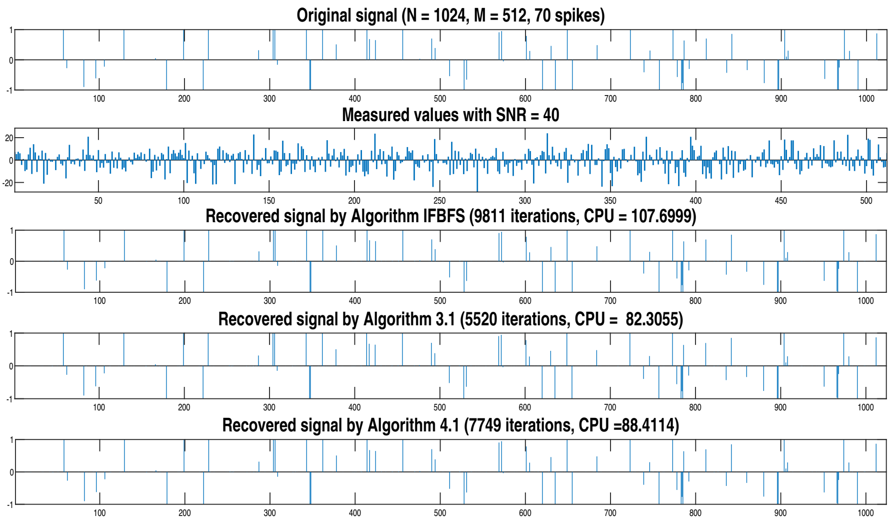

Fig. 4 From top to bottom: original signal, observation data, recovered signal by Algorithm IFBFS, Algorithm 3.1 and Algorithm 4.1 with MSE $< 10^{-5}$

by uniform distribution in $[-2, 2]$. We use the stopping criterion by

$$\mathrm{MSE} = \frac{1}{N}\|x_n - x^*\|^2,$$

where x_n is an estimated signal of x_*.

In following, the initial point x_0 is ones([N,1]) and x_1 is chosen randomly, $\alpha_1 = 0.01$, $\alpha_2 = 0.02$. Set $\kappa_n = 0$, $\sigma = 0.01$, $\mu = 0.5$, let $\theta = 0.4$ in Algorithm 3.1 and $\lambda_0 = 0.01$ in Algorithm 4.1.

We denote by CPU the time of CPU and Iter by the number of iterations. All codes were written in Matlab 2019b and run on Asus Core i3 laptop. The numerical results are shown as follows.

From Table 1, we see that the experiment result of our proposed algorithms are better than Algorithms IFBFS of Bot and Csetnek [3] in terms of CPU time and number of iterations in each cases.

Next, we provide Figs, 1 and 3 to show the convergence of each algorithms via the graph of the MSE value and number of iterations and Figs. 2 and 4 to show signal recovery in compressed sensing when $N = 1024$ and $M = 512$.

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Declarations

Conflict of interest There are no conflicts of interest to this work

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