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Relaxed Forward–Backward Splitting Methods for Solving Variational Inclusions and Applications

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Abstract
In this paper, we revisit the modified forward–backward splitting method (MFBSM) for solving a variational inclusion problem of the sum of two operators in Hilbert spaces. First, we introduce a relaxed version of the method (MFBSM) where it can be implemented more easily without the prior knowledge of the Lipschitz constant of component operators. The algorithm uses variable step-sizes which are updated at each iteration by a simple computation. Second, we establish the convergence and the linear rate of convergence of the proposed algorithm. Third, we propose and analyze the convergence of another relaxed algorithm which is a combination between the first one with the inertial method. Finally, we give several numerical experiments to illustrate the convergence of some new algorithms and also to compare them with others.

Keywords Variational inclusion · Modified forward–backward splitting method · Inertial method · Signal recovery · Convergence rate

Mathematics Subject Classification 65Y05 · 65K15

1 Introduction

Let $\mathscr{H}$ be a real Hilbert space, $\mathscr{A} : \mathscr{H} \to 2^{\mathscr{H}}$ be a multi-valued operator and $\mathscr{B} : \mathscr{H} \to \mathscr{H}$ be an operator.

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In this paper, we consider the following *variational inclusion* (VI) of the sum of two operators:

$$\text{Find } p^* \in \mathscr{H} \text{ such that } 0 \in (\mathscr{A} + \mathscr{B})(p^*). \tag{VI}$$

The solution set of the problem (VI) is denoted by $(\mathscr{A} + \mathscr{B})^{-1}(0)$. If $\mathscr{A} = N_C$ is the normal cone of some nonempty closed and convex subset C of $\mathscr{H}$, the problem (VI) becomes a classical variational inequality problem in [5,11]. A popular model, which can be formulated under the VI, is the following *optimization problem* of the sum of two functions:

$$\min_{x \in \mathscr{H}} (f(x) + g(x)),$$

where $f,\ g : \mathscr{H} \to \mathscr{R} \cup \{+\infty\}$ are two proper, lower semicontinuous and convex functions such that f is differentiable and g is subdifferentiable. In this case, the optimization problem is equivalent to the problem (VI) with $\mathscr{A} = \partial g$ and $\mathscr{B} = \nabla f$. It is well-known that the problem (VI) plays a central role in nonlinear analysis and unifies many important concepts in applied mathematics as optimization problems, variational inequality problems, fixed point problems, systems of operator equations and equilibrium problems [4,8,12].

The simplest method for solving the problem (VI) is the following *forward–backward splitting method* (FBSM) [13,16]:

$$x_{n+1} = J_{\lambda\mathscr{A}}(x_n - \lambda\mathscr{B}(x_n)),$$

where $\lambda > 0$ is a suitable parameter and $J_{\lambda\mathscr{A}} = (I + \lambda\mathscr{A})^{-1}$. The convergence of this method is ensured under the restrictive hypothesis of c-inverse strongly monotonicity (or the c-cocoercivity) of operator $\mathscr{B}$, i.e., there exists a number $c > 0$ such that

$$\langle \mathscr{B}(x) - \mathscr{B}(y), x - y\rangle \geq c||\mathscr{B}(x) - \mathscr{B}(y)||^2, \quad \forall x,\ y \in \mathscr{H}.$$

A popularly used method for solving the problem (VI) is the *modified forward–backward splitting method* (MFBSM) [19]. The method (MFBSM) generates the sequence $\{x_n\}$, from a starting point $x_0 \in \mathscr{H}$, defined by

$$\begin{cases} y_n = J_{\alpha_n\mathscr{A}}(x_n - \alpha_n\mathscr{B}(x_n)), \\ x_{n+1} = y_n - \alpha_n(\mathscr{B}(y_n) - \mathscr{B}(x_n)), \end{cases} \tag{MFBSM}$$

where $\{\alpha_n\} \subset (0, +\infty)$ is a suitable sequence. This sequence can be defined as follows:

If the Lipschitz constant L of $\mathscr{B}$ is known, then $\{\alpha_n\} \subset [a, b] \subset (0, 1/L)$ and, in the inverse case, it can be found by some linesearch procedure with a finite stopping criterion at each iteration. However, the Lipschitz constant of an operator is often unknown or difficult to estimate in nonlinear problems. Moreover, using a linesearch is not cheap and can be time-consuming because the linesearch often requires many extra-computations at each iteration.

In recent years, the method (MFBSM) has received a lot of attention by some authors who proposed modified versions of the method (MFBSM) in different ways. Several other methods for solving the problem (VI) can be found, for example, as the *Douglas–Rachford splitting method* (DRSM) [13], the *forward–reflected–backward splitting method* (FRBSM) [14] and others [6,7,9]. A general form of the problem (VI), which involves the sum of three operator, can be found in [3,10,14,18,20].

In this paper, motivated by the results of Tseng in [19], first, we propose an explicit relaxed version of the method (MFBSM), so-called the *Relaxed Forward–Backward Splitting Method* (RFBSM), for solving the problem (VI) in a Hilbert space. The algorithm originates from a discretization of a forward–backward dynamical system in time. The algorithm uses a simple step-size rule without the prior knowledge of Lipschitz constant of the operator and also without any linesearch procedure. The step-sizes are updated at each iteration by a cheap computation based on the previous iterates. The convergence as well as the rate of convergence of the method (RFBSM) are proved under some mild conditions. Moreover, inspired by the results in [1], we introduce the second method for solving the problem (VI) which can be considered as a combination between the first method and the inertial technique in [1]. It was known that applying the inertial method is to improve the convergence properties of algorithm [17]. Finally, we implement several numerical experiments to illustrate the computational performance of the new algorithms.

The paper is organized as follows: Sect. 2 recalls some definitions and preliminary results used further in the paper. Section 3 presents a relaxed version of the method (MFBSM) as well as the analyses of its convergence and convergence rate. Section 4 deals with the description of a new inertial relaxed forward–backward splitting method for the problem (VI). Finally, in Sect. 5, we implement some numerical experiments to illustrate the main results in this paper.

2 Preliminaries

Let $\mathscr{H}$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $||\cdot||$. Let $\{x_n\}$ be a sequence in $\mathscr{H}$. We write $x_n \rightharpoonup x$ (resp., $x \to x$) to stand for the weak (resp., strong) convergence of the sequence $\{x_n\}$ to $x \in \mathscr{H}$ as $n \to \infty$. Let $\mathscr{A} : \mathscr{H} \to 2^{\mathscr{H}}$ be a multi-valued operator. The *graph* of $\mathscr{A}$ is the set in $\mathscr{H} \times \mathscr{H}$ defined by

$$\text{Graph}(\mathscr{A}) := \{(x, u) : x \in \mathscr{H},\ u \in \mathscr{A}(x)\}.$$

An operator $\mathscr{A} : \mathscr{H} \to 2^{\mathscr{H}}$ is called:

(1) *monotone* if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall x,\ y \in \mathscr{H},\ u \in \mathscr{A}(x),\ v \in \mathscr{A}(y);$$

(2) *maximally monotone* if $\mathscr{A}$ is monotone and its graph is not a proper subset of the graph of any monotone operator;

(3) γ*-strongly monotone* if there exists a number $\gamma > 0$ such that

$$\langle u - v, x - y \rangle \geq \gamma ||x - y||^2, \quad \forall x,\ y \in \mathscr{H},\ u \in \mathscr{A}(x),\ v \in \mathscr{A}(y);$$

(4) *L-Lipschitz continuous* if there exists a number $L > 0$ such that

$$||u - v|| \leq L||x - y||, \quad \forall x,\ y \in \mathscr{H},\ u \in \mathscr{A}(x),\ v \in \mathscr{A}(y);$$

An important property of a maximal monotone operator $\mathscr{A} : \mathscr{H} \to 2^{\mathscr{H}}$ is that, if a pair $(x, u) \in \mathscr{H} \times \mathscr{H}$ and $\langle x - y, u - v \rangle \geq 0$ for all $(y, v) \in \text{Graph}(\mathscr{A})$, then $u \in \mathscr{A}(x)$.

The *resolvent* of a multi-valued operator $\mathscr{A} : \mathscr{H} \to 2^{\mathscr{H}}$ is a single-valued operator defined by

$$J_{\mathscr{A}}(x) = (I + \mathscr{A})^{-1}(x), \quad \forall x \in \mathscr{H},$$

where I is the identity mapping in $\mathscr{H}$. Remark that a point $x \in \mathscr{H}$ is a solution of the problem (VI) if and only if $x = J_{\lambda\mathscr{A}}(x - \lambda\mathscr{B}(x))$ for some $\lambda > 0$, i.e., $x \in Fix(J_{\lambda\mathscr{A}}(I - \lambda\mathscr{B}))$, the set of fixed points of the operator $J_{\lambda\mathscr{A}}(I - \lambda\mathscr{B})$.

Let us recall two basic concepts of convergence rate of a sequence (see [15, Chapter 9]). Let $\{x_n\}$ be a sequence in $\mathscr{H}$ and $x \in \mathscr{H}$. The sequence $\{x_n\}$ is called:

(1) *convergent R-linearly* to x if

$$\limsup_{n\to\infty} ||x_n - x||^{\frac{1}{n}} < 1;$$

(2) *convergent Q-linearly* to x if there exists two numbers $r \in (0, 1)$ and $n_0 \geq 1$ such that

$$||x_{n+1} - x|| \leq r||x_n - x||, \quad \forall n \geq n_0.$$

It is well-known that Q-linear convergence implies R-linear convergence [15, Section 9.3]. The inverse in general is not true.

We need the following results to prove the convergence of the new algorithms:

Lemma 1 [2, Corollary 2.14] *For all $x, y \in \mathscr{H}$ and $\alpha \in \mathscr{R}$, the following equality holds:*

$$||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2.$$

Lemma 2 [1] *Let $\{\Phi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be the sequences in $[0, +\infty)$ such that*

$$\Phi_{n+1} \leq \Phi_n + \alpha_n(\Phi_n - \Phi_{n-1}) + \delta_n, \quad \forall n \geq 1, \quad \sum_{n=1}^{+\infty} \delta_n < +\infty$$

and suppose that there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \geq 0$. Then the followings hold:

(1) $\sum_{n=1}^{+\infty}[\Phi_n - \Phi_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$*;*
(2) *There exists* $\Phi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \Phi_n = \Phi^*$.

3 Relaxed Forward–Backward Splitting Method

In this section, we first introduce a relaxed version, the relaxed forward–backward splitting method (RFBSM), of the method (MFBSM) in [19] with a new step-size rule. After that, we prove the convergence and establish the R-linear rate of convergence of the resulting algorithm. For the readable purpose and the presentation of the new algorithm, we associate our problem (VI) with a dynamical system of the following form:

$$y(t) = J_{\lambda(t)\mathscr{A}}(x(t) - \lambda(t)\mathscr{B}(x(t)), \tag{1}$$

$$\frac{dx(t)}{dt} = -\theta(t)\left[x(t) - y(t) - \lambda(t)\mathscr{B}(x(t)) + \lambda(t)\mathscr{B}(y(t))\right], \tag{2}$$

where $\theta(t) > 0$ and $\lambda(t) > 0$ for all $t > 0$. Applying an explicit discretization to the dynamical system (1)–(2) in time variable t, with step-size $h_n > 0$, we come to the following system:

$$y_n = J_{\lambda_n\mathscr{A}}(x_n - \lambda_n\mathscr{B}(x_n), \tag{3}$$

$$\frac{x_{n+1} - x_n}{h_n} = -\theta_n\left[x_n - y_n - \lambda_n\mathscr{B}(x_n) + \lambda_n\mathscr{B}(y_n)\right]. \tag{4}$$

Now, if we choose the time step-size $h_n = 1$ for each $n \geq 1$, then the system (3)–(4) can be rewritten as follows:

$$\begin{cases} y_n = J_{\lambda_n\mathscr{A}}(x_n - \lambda_n\mathscr{B}(x_n), \\ x_{n+1} = (1-\theta_n)x_n + \theta_n y_n + \theta_n\lambda_n\left(\mathscr{B}(x_n) - \mathscr{B}(y_n)\right). \end{cases} \tag{5}$$

If $\theta_n = 1$ for each $n \geq 1$, then the relaxed method (5) becomes the method (MFBSM) in [19]. As in [19], we can choose the sequence of step-sizes $\{\lambda_n\} \subset [a, b] \subset (0, 1/L)$ if the Lipschitz constant L of the operator $\mathscr{B}$ is known and, in the inverse case, a linesearch can be used. In that case, the convergence of (5) is also established. However, as aforementioned, the Lipschitz constant in general is not known or difficult to approximate. In addition, a linesearch requires many extra-computations and can be time-consuming. Then we present below a simple rule to find step-sizes $\{\lambda_n\}$. The step-size λ_n is computed at each iteration by a cheap computation.

For the sake of simplicity in presentation, we adopt the conventions

$$\frac{0}{0} = +\infty \quad \text{and} \quad \frac{a}{0} = +\infty \quad \text{with} \quad a > 0.$$

Our first algorithm is of the following form:

Algorithm 1 (The method (RFBSM) with new step-sizes for the problem (VI))

Initialization: Choose $x_0 \in \mathscr{H}$, a number $\lambda_0 > 0$, and two sequences $\{\theta_n\} \subset [a, b] \subset (0, 1]$, $\{\mu_n\} \subset [c, d] \subset (0, 1)$.

Iterative Steps: Assume that $x_n \in \mathscr{H}$ and λ_n are known, calculate x_{n+1} and λ_{n+1} as follows:

(C1) Compute

$$y_n = J_{\lambda_n \mathscr{A}}(x_n - \lambda_n \mathscr{B}(x_n)).$$

(C2) Compute

$$x_{n+1} = (1 - \theta_n)x_n + \theta_n y_n + \theta_n \lambda_n (\mathscr{B}(x_n) - \mathscr{B}(y_n))$$

(C3) Update

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu_n ||x_n - y_n||}{||\mathscr{B}(x_n) - \mathscr{B}(y_n)||}\right\}.$$

Observe that the step-size λ_n is updated at each iteration by a simple computation, based on the previous iterates x_n and y_n, and without any linesearch procedure. As in the theorem below, the convergence of Algorithm 1 is proved under the hypothesis of Lipschitz continuity of the operator $\mathscr{B}$. However, the Lipschitz constant is not necessary to be known.

If $y_n = x_n$ for some $n \geq 1$, i.e., $x_n = J_{\lambda_n \mathscr{A}}(x_n - \lambda_n \mathscr{B}(x_n))$, then $x_n \in (\mathscr{A} + \mathscr{B})^{-1}(0)$. Thus, if $y_n = x_n$, a solution of the problem (VI) can be found. In what follows, we suppose that Algorithm 1 does not terminate and then the sequence $\{x_n\}$ generated by Algorithm 1 is infinite. In that case, we have the following result:

Theorem 1 *Suppose that $\mathscr{A} : \mathscr{H} \to 2^{\mathscr{H}}$ is maximal monotone and $\mathscr{B} : \mathscr{H} \to \mathscr{H}$ is monotone and Lipschitz continuous. In addition, the solution set $(\mathscr{A}+\mathscr{B})^{-1}(0)$ of the problem* (VI) *is nonempty. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a solution of the problem* (VI).

Proof We divide the proof of Theorem 1 into several steps.

Claim 1 *We have the following estimate:*

$$||x_{n+1} - p||^2 \le ||x_n - p||^2 - \theta_n \left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) ||x_n - y_n||^2 \\ - \theta_n(1-\theta_n)||y_n - x_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n))||^2. \tag{6}$$

The proof of Claim 1. We have

$$\begin{aligned} &||y_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)) - p||^2 \\ &\quad = ||(y_n - x_n) + (x_n - p) + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n))||^2 \\ &\quad = ||y_n - x_n||^2 + ||x_n - p||^2 + \lambda_n^2||\mathscr{B}(x_n) - \mathscr{B}(y_n)||^2 + 2\langle y_n - x_n, x_n - p\rangle \\ &\qquad + 2\lambda_n \langle y_n - x_n, \mathscr{B}(x_n) - \mathscr{B}(y_n)\rangle + 2\lambda_n \langle x_n - p, \mathscr{B}(x_n) - \mathscr{B}(y_n)\rangle \\ &\quad = ||y_n - x_n||^2 + ||x_n - p||^2 + \lambda_n^2||\mathscr{B}(x_n) - \mathscr{B}(y_n)||^2 + 2\langle y_n - x_n, x_n - y_n\rangle \\ &\qquad + 2\langle y_n - x_n, y_n - p\rangle + 2\lambda_n \langle y_n - x_n, \mathscr{B}(x_n) - \mathscr{B}(y_n)\rangle \\ &\qquad + 2\lambda_n \langle x_n - p, \mathscr{B}(x_n) - \mathscr{B}(y_n)\rangle \\ &\quad = ||x_n - p||^2 - ||y_n - x_n||^2 - 2\langle x_n - y_n - \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)), y_n - p\rangle \\ &\qquad + \lambda_n^2||\mathscr{B}(x_n) - \mathscr{B}(y_n)||^2. \end{aligned} \tag{7}$$

It follows from the definition of y_n that $x_n - \lambda_n\mathscr{B}(x_n) \in (I + \lambda_n\mathscr{A})(y_n)$ and so

$$x_n - y_n - \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)) \in \lambda_n(\mathscr{A} + \mathscr{B})(y_n).$$

Moreover, we also have

$$0 \in \lambda_n(\mathscr{A} + \mathscr{B})(p).$$

Since $\lambda_n(\mathscr{A} + \mathscr{B})$ is monotone, we obtain

$$\langle x_n - y_n - \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)), y_n - p\rangle \ge 0. \tag{8}$$

Combining the relations (7), (8) and the definition of λ_{n+1}, we obtain

$$\begin{aligned} ||y_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)) - p||^2 &\le ||x_n - p||^2 - ||y_n - x_n||^2 + \lambda_n^2||\mathscr{B}(x_n) - \mathscr{B}(y_n)||^2 \\ &\le ||x_n - p||^2 - ||y_n - x_n||^2 + \frac{\lambda_n^2\mu_n^2}{\lambda_{n+1}^2}||x_n - y_n||^2 \\ &= ||x_n - p||^2 - \left(1 - \frac{\lambda_n^2\mu_n^2}{\lambda_{n+1}^2}\right)||x_n - y_n||^2. \end{aligned} \tag{9}$$

This together with the definition of x_{n+1} implies that

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||(1-\theta_n)x_n + \theta_n y_n + \theta_n\lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)) - p||^2 \\ &= ||(1-\theta_n)(x_n - p) + \theta_n(y_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)) - p)||^2 \\ &= (1-\theta_n)||x_n - p||^2 + \theta_n||y_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)) - p||^2 \\ &\quad - \theta_n(1-\theta_n)||y_n - x_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n))||^2 \\ &\le (1-\theta_n)||x_n - p||^2 + \theta_n\left(||x_n - p||^2 - \left(1 - \frac{\lambda_n^2\mu_n^2}{\lambda_{n+1}^2}\right)||x_n - y_n||^2\right) \\ &\quad - \theta_n(1-\theta_n)||y_n - x_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n))||^2 \end{aligned}$$

$$= ||x_n - p||^2 - \theta_n \left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) ||x_n - y_n||^2$$
$$- \theta_n(1 - \theta_n)||y_n - x_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n))||^2. \tag{10}$$

Claim 2 $\lim_{n\to\infty} \lambda_n = \lambda > 0$.

The proof of Claim 2. From the aforementioned conventions, if $\mathscr{B}(x_n) = \mathscr{B}(y_n)$, then $\lambda_{n+1} = \lambda_n$. Otherwise, from the Lipschitz continuity of $\mathscr{B}$, there exists a number $L > 0$ such that

$$||\mathscr{B}(x_n) - \mathscr{B}(y_n)|| \le L||x_n - y_n||$$

and so

$$\frac{\mu_n||x_n - y_n||}{||\mathscr{B}(x_n) - \mathscr{B}(y_n)||} \ge \frac{\mu_n||x_n - y_n||}{L||x_n - y_n||} \ge \frac{c}{L},$$

which, together with the definition of $\{\lambda_n\}$, implies that the sequence $\{\lambda_n\}$ is bounded from below by the constant $\min\left\{\lambda_0, \frac{c}{L}\right\} > 0$. Moreover, it is easy to see that the sequence $\{\lambda_n\}$ is non-increasing monotone. Hence we have

$$\lim_{n\to\infty} \lambda_n = \lambda > 0. \tag{11}$$

Claim 3 *The limit* $\lim_{n\to\infty} ||x_n - p||^2$ *exists for each* $p \in (\mathscr{A} + \mathscr{B})^{-1}(0)$ *and* $\lim_{n\to\infty} ||x_n - y_n||^2 = 0$.

The proof of Claim 3. It follows from Claim 2 that

$$\theta_n \left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) \ge a \left(1 - \frac{\lambda_n^2 d^2}{\lambda_{n+1}^2}\right) \longrightarrow a(1 - d^2) > 0. \tag{12}$$

Now, take a fixed number $\epsilon \in (0, a(1 - d^2))$. From the relation (12), there exists $n_0 \ge 1$ such that

$$\theta_n \left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) \ge \epsilon > 0, \quad \forall n \ge n_0. \tag{13}$$

Thus, from Claim 1, we obtain

$$||x_{n+1} - p||^2 \le ||x_n - p||^2 - \epsilon||x_n - y_n||^2, \quad \forall n \ge n_0. \tag{14}$$

Thus, the limit $\lim_{n\to\infty} ||x_n - p||^2$ exists and

$$||x_{n+1} - p||^2 \le ||x_n - p||^2 - \epsilon||x_n - y_n||^2, \quad \forall n \ge n_0. \tag{15}$$

Thus the sequence $\left\{||x_n - p||^2\right\}_{n\ge n_0}$ is non-increasing and bounded from below by zero. Consequently, there exists the limit $\lim_{n\to\infty} ||x_n - p||^2 \in \mathscr{R}$. Moreover, passing to the limit in (15) as $n \to \infty$, we obtain immediately $\lim_{n\to\infty} ||x_n - y_n||^2 = 0$

Claim 4 *Every weak cluster point of* $\{x_n\}$ *is in* $(\mathscr{A} + \mathscr{B})^{-1}(0)$.

The proof of Claim 4. By Claim 3, we see that the sequence $\{x_n\}$ is bounded. Assume that x is a weak cluster point of $\{x_n\}$, i.e., there exists a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to x. Since $||x_m - y_m||^2 \to 0$, we also have $y_m \rightharpoonup x$ as $m \to \infty$.

Let $(u, v) \in \mathrm{Graph}(\mathscr{A} + \mathscr{B})$, i.e., $v \in \mathscr{A}(u) + \mathscr{B}(u)$. Thus we have $v - \mathscr{B}(u) \in \mathscr{A}(u)$. Moreover, since $y_m = J_{\lambda_m \mathscr{A}}(x_m - \lambda_m \mathscr{B}(x_m))$, we have

$$\frac{1}{\lambda_m}(x_m - \lambda_m \mathscr{B}(x_m) - y_m) \in \mathscr{A}(y_m).$$

Thus, from the monotonicity of $\mathscr{A}$, we obtain

$$\left\langle v - \mathscr{B}(u) - \frac{1}{\lambda_m}(x_m - \lambda_m \mathscr{B}(x_m) - y_m), u - y_m \right\rangle \geq 0$$

and so

$$\begin{aligned}
\langle v, u - y_m \rangle &\geq \left\langle \mathscr{B}(u) + \frac{1}{\lambda_m}(x_m - \lambda_m \mathscr{B}(x_m) - y_m), u - y_m \right\rangle \\
&= \frac{1}{\lambda_m}\langle x_m - y_m, u - y_m \rangle + \langle \mathscr{B}(u) - \mathscr{B}(y_m), u - y_m \rangle + \langle \mathscr{B}(y_m) - \mathscr{B}(x_m), u - y_m \rangle \\
&\geq \frac{1}{\lambda_m}\langle x_m - y_m, u - y_m \rangle + \langle \mathscr{B}(y_m) - \mathscr{B}(x_m), u - y_m \rangle \\
&\geq -\frac{1}{\lambda_m}||x_m - y_m||||u - y_m|| - L||y_m - x_m||||u - y_m|| \\
&\geq -M_u||x_m - y_m||,
\end{aligned}$$

where $M_u = \sup_{m \geq 1}\left\{\frac{1}{\lambda_m}||u - y_m|| + L||u - y_m||\right\} < +\infty$. Passing to the limit in the last inequality and noting that $||x_m - y_m|| \to 0$ and $y_m \rightharpoonup x$, we obtain $\langle v, u - x \rangle \geq 0$ for all $(u, v) \in \mathrm{Graph}(\mathscr{A} + \mathscr{B})$. Since $\mathscr{A} + \mathscr{B}$ is maximal monotone, $x \in (\mathscr{A} + \mathscr{B})^{-1}(0)$.

Claim 5 *The whole sequence $\{x_n\}$ converges weakly to a point in $(\mathscr{A} + \mathscr{B})^{-1}(0)$.*

The proof of Claim 5. Assume that there exists another subsequence $\{x_k\}$ of $\{x_n\}$ converging weakly to x' and $x' \neq x$. By Claims 3 and 4, we have

$$x' \in (\mathscr{A} + \mathscr{B})^{-1}(0), \quad \lim_{n \to \infty}||x_n - x'||^2 \in \mathscr{R}.$$

Also, we have

$$2\left\langle x_n, x - x' \right\rangle = (||x||^2 - ||x'||^2) + (||x_n - x||^2 - ||x_n - x'||^2).$$

Noting that $\lim_{n \to \infty}||x_n - x||^2 \in \mathscr{R}$. Thus the limit of the sequence $\left\{\left\langle x_n, x - x' \right\rangle\right\}$ exists. Set

$$l = \lim_{n \to \infty}\left\langle x_n, x - x' \right\rangle. \tag{16}$$

Now, passing to the limit in (16) as $n = m \to \infty$ and, after that, $n = k \to \infty$, we obtain

$$\left\langle x, x - x' \right\rangle = \lim_{m \to \infty}\left\langle x_m, x - x' \right\rangle = l = \lim_{k \to \infty}\left\langle x_k, x - x' \right\rangle = \left\langle x', x - x' \right\rangle.$$

Thus $||x - x'||^2 = 0$ or $x' = x$. This means that the whole sequence $\{x_n\}$ converges weakly to a point in $(\mathscr{A} + \mathscr{B})^{-1}(0)$. This completes the proof. □

Now, we establish the convergence rate of Algorithm 1. In this purpose, we assume that the operator $\mathscr{B}$ is strongly monotone (SM), i.e., i.e., there exists $\gamma > 0$ such that

$$(\mathrm{SM}): \langle \mathscr{B}(y) - \mathscr{B}(x), y - x \rangle \geq \gamma||x - y||^2, \quad \forall x, \; y \in \mathscr{H}.$$

Now, we have the following result:

Theorem 2 *Under the assumption* (SM) *and the hypotheses of Theorem* 1*, the sequence* $\{x_n\}$ *generated by Algorithm* 1 *converges at least* R*-linearly to the unique solution* p *of the problem* (VI)*. Moreover, the following estimate hold: for all* $n \geq 0$*,*

$$||x_{n+1} - p||^2 \leq \left(1 - \frac{\theta_n \left[\left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) + (1-\theta_n)\left(1 - \frac{\mu_n \lambda_n}{\lambda_{n+1}}\right)^2\right]}{\left(1 + \frac{1+\frac{\mu \lambda_n}{\lambda_{n+1}}}{\lambda_n \gamma}\right)^2}\right) ||x_n - p||^2.$$

Proof We have

$$\begin{aligned} ||y_n - x_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n))|| &\geq ||y_n - x_n|| - \lambda_n ||\mathscr{B}(x_n) - \mathscr{B}(y_n)|| \\ &\geq ||y_n - x_n|| - \lambda_n \frac{\mu_n}{\lambda_{n+1}} ||x_n - y_n|| \\ &= \left(1 - \frac{\mu_n \lambda_n}{\lambda_{n+1}}\right) ||x_n - y_n||. \end{aligned} \tag{17}$$

Note that, since $\lambda_{n+1} \leq \lambda_n$ and $\mu_n \in (0, 1)$, $1 - \frac{\mu_n \lambda_n}{\lambda_{n+1}} > 0$. Thus, from the relation (17), we obtain

$$||y_n - x_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n))||^2 \geq \left(1 - \frac{\mu_n \lambda_n}{\lambda_{n+1}}\right)^2 ||x_n - y_n||^2,$$

which together with Claim 1 implies that

$$\begin{aligned} &||x_{n+1} - p||^2 \\ &\leq ||x_n - p||^2 - \theta_n \left[\left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) + (1-\theta_n)\left(1 - \frac{\mu_n \lambda_n}{\lambda_{n+1}}\right)^2\right] ||x_n - y_n||^2. \end{aligned} \tag{18}$$

Since $y_n = J_{\lambda_n \mathscr{A}}(x_n - \lambda_n \mathscr{B}(x_n))$, we have $x_n - \lambda_n \mathscr{B}(x_n) \in (I + \lambda_n \mathscr{A})(y_n)$ and so

$$x_n - y_n - \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)) \in \lambda_n(\mathscr{A} + \mathscr{B})(y_n).$$

Moreover, we also have that $0 \in \lambda_n(\mathscr{A} + \mathscr{B})(p)$. Since $\lambda_n(\mathscr{A} + \mathscr{B})$ is strongly monotone, we obtain

$$\langle x_n - y_n - \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)), y_n - p\rangle \geq \gamma \lambda_n ||y_n - p||^2$$

and thus

$$\begin{aligned} \gamma \lambda_n ||y_n - p||^2 &\leq \langle x_n - y_n, y_n - p\rangle - \lambda_n \langle \mathscr{B}(x_n) - \mathscr{B}(y_n), y_n - p\rangle \\ &\leq ||x_n - y_n|| ||y_n - p|| + \frac{\lambda_n \mu_n}{\lambda_{n+1}} ||x_n - y_n|| ||y_n - p|| \\ &= \left(1 + \frac{\lambda_n \mu_n}{\lambda_{n+1}}\right) ||x_n - y_n|| ||y_n - p|| \end{aligned}$$

or

$$||y_n - p|| \leq \frac{1 + \frac{\lambda_n \mu_n}{\lambda_{n+1}}}{\gamma \lambda_n} ||x_n - y_n||.$$

Hence we have

$$||x_n - p|| \leq ||x_n - y_n|| + ||y_n - p|| \leq \left[1 + \frac{1 + \frac{\mu_n \lambda_n}{\lambda_{n+1}}}{\lambda_n \gamma}\right] ||x_n - y_n||,$$

or, equivalently,

$$||x_n - y_n|| \geq \left[1 + \frac{1 + \frac{\mu_n \lambda_n}{\lambda_{n+1}}}{\lambda_n \gamma}\right]^{-1} ||x_n - p||. \tag{19}$$

Combining the relations (18) and (19), we obtain

$$||x_{n+1} - p||^2 \leq \left(1 - \frac{\theta_n \left[\left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) + (1 - \theta_n)\left(1 - \frac{\mu_n \lambda_n}{\lambda_{n+1}}\right)^2\right]}{\left(1 + \frac{1 + \frac{\mu_n \lambda_n}{\lambda_{n+1}}}{\lambda_n \gamma}\right)^2}\right) ||x_n - p||^2. \tag{20}$$

Set

$$r_n = 1 - \frac{\theta_n \left[\left(1 - \frac{\lambda_n^2 \mu_n^2}{\lambda_{n+1}^2}\right) + (1 - \theta_n)\left(1 - \frac{\mu_n \lambda_n}{\lambda_{n+1}}\right)^2\right]}{\left(1 + \frac{1 + \frac{\mu \lambda_n}{\lambda_{n+1}}}{\lambda_n \gamma}\right)^2}$$

and

$$r = 1 - \frac{a\left[(1 - d^2) + (1 - b)(1 - d)^2\right]}{\left(1 + \frac{1+d}{\lambda \gamma}\right)^2}.$$

Note that $r \in (0, 1)$. Moreover, since $\{\theta_n\} \subset [a, b] \subset (0, 1]$, $\{\mu_n\} \subset [c, d] \subset (0, 1)$ and the fact $\lambda_n \to \lambda$, we get

$$0 \leq r_n \leq 1 - \frac{a\left[\left(1 - \frac{\lambda_n^2 d^2}{\lambda_{n+1}^2}\right) + (1 - b)\left(1 - \frac{\lambda_n d}{\lambda_{n+1}}\right)^2\right]}{\left(1 + \frac{1 + \frac{d \lambda_n}{\lambda_{n+1}}}{\lambda_n \gamma}\right)^2} \longrightarrow r \in (0, 1) \quad \text{as } n \to \infty.$$

This together with the relation (20) implies that the sequence $\left\{||x_n - p||^2\right\}$ converges Q-linearly. Thus the sequence $\{x_n\}$ converges at least R-linearly. This completes the proof. $\square$

4 Inertial Relaxed Forward–Backward Splitting Method

Now, we introduce the inertial relaxed forward–backward splitting method (IRFBSM) for the problem (VI).

Algorithm 2 (The method (IRFBSM) with new step-sizes for the problem (VI))

Initialization: Choose $x_{-1},\ x_0 \in \mathscr{H}$ and $\lambda_0 > 0$. Take three numbers $\theta \in (0, 1]$, $\mu \in (0, 1)$, $\alpha \in [0, 1)$ such that

$$\frac{\theta\left(1-\mu^2\right)}{(2-\theta+\mu\theta)^2}+\frac{1-\theta}{\theta}>\frac{\alpha(1+\alpha)}{(1-\alpha)^2}. \tag{21}$$

Iterative Steps: Assume that $x_{n-1},\ x_n \in \mathscr{H}$ and λ_n are known, calculate x_{n+1} and λ_{n+1} as follows:

(D1) Set $w_n = x_n + \alpha(x_n - x_{n-1})$ and compute

$$y_n = J_{\lambda_n A}(w_n - \lambda_n \mathscr{B}(w_n)).$$

(D2) Compute

$$x_{n+1} = (1-\theta)w_n + \theta y_n + \theta\lambda_n(\mathscr{B}(w_n) - \mathscr{B}(y_n))$$

(D3) Update

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu||w_n - y_n||}{||\mathscr{B}(w_n) - \mathscr{B}(y_n)||}\right\}.$$

Remark that, if set

$$K = \frac{\theta\left(1-\mu^2\right)}{(2-\theta+\mu\theta)^2}+\frac{1-\theta}{\theta} > 0,$$

then the condition (21) can be rewritten equivalently to the following one:

$$0 \le \alpha < \frac{\sqrt{8K^2+8K+1}-2K-1}{2(K+1)}.$$

Theorem 3 *Under the hypotheses of Theorem 1, the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to a point in $(\mathscr{A}+\mathscr{B})^{-1}(0)$.*

Proof By arguing similarly as in the proof of Claim 1, we obtain

$$\begin{aligned} ||x_{n+1}-p||^2 \le ||w_n - p||^2 - \theta\left(1-\frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)||w_n - y_n||^2 \\ -\theta(1-\theta)||y_n - w_n + \lambda_n(\mathscr{B}(w_n) - \mathscr{B}(y_n))||^2. \end{aligned} \tag{22}$$

From the definition of x_{n+1}, x_{n+1} can be rewritten as

$$x_{n+1} = w_n + \theta\left[y_n - w_n + \lambda_n(\mathscr{B}(w_n) - \mathscr{B}(y_n))\right]$$

and so

$$||y_n - w_n + \lambda_n(\mathscr{B}(w_n) - \mathscr{B}(y_n))||^2 = \frac{1}{\theta^2}||x_{n+1} - w_n||^2. \tag{23}$$

On the other hand, from the definition of x_{n+1}, we obtain

$$||x_{n+1} - y_n|| = ||(1-\theta)(w_n - y_n) + \theta\lambda_n(\mathscr{B}(w_n) - \mathscr{B}(y_n))||$$

$$
\begin{aligned}
&\leq (1-\theta)||w_n - y_n|| + \theta\lambda_n||\mathscr{B}(w_n) - \mathscr{B}(y_n))|| \\
&\leq (1-\theta)||w_n - y_n|| + \theta\lambda_n \frac{\mu}{\lambda_{n+1}}||w_n - y_n|| \\
&\leq \left(1-\theta + \frac{\mu\theta\lambda_n}{\lambda_{n+1}}\right)||w_n - y_n||
\end{aligned}
$$

and so

$$
||x_{n+1} - w_n|| \leq ||x_{n+1} - y_n|| + ||y_n - w_n|| \leq \left(2-\theta + \frac{\mu\theta\lambda_n}{\lambda_{n+1}}\right)||w_n - y_n||
$$

or

$$
||w_n - y_n|| \geq \left(2-\theta + \frac{\mu\theta\lambda_n}{\lambda_{n+1}}\right)^{-1}||x_{n+1} - w_n||. \tag{24}
$$

Combining the relations (22)–(24), we obtain

$$
||x_{n+1} - p||^2 \leq ||w_n - p||^2 - \left[\frac{\theta\left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)}{\left(2-\theta + \frac{\mu\theta\lambda_n}{\lambda_{n+1}}\right)^2} + \frac{1-\theta}{\theta}\right]||w_n - y_n||^2. \tag{25}
$$

Set

$$
\mathscr{K}_n = \frac{\theta\left(1 - \frac{\lambda_n^2\mu^2}{\lambda_{n+1}^2}\right)}{\left(2-\theta + \frac{\mu\theta\lambda_n}{\lambda_{n+1}}\right)^2} + \frac{1-\theta}{\theta}. \tag{26}
$$

Thus the relation (25) can be rewritten as follows:

$$
||x_{n+1} - p|| \leq ||w_n - p||^2 - \mathscr{K}_n||w_n - y_n||^2. \tag{27}
$$

We have

$$
\begin{aligned}
||w_n - p||^2 &= ||x_n + \alpha(x_n - x_{n-1}) - p||^2 \\
&= ||(1+\alpha)(x_n - p) - \alpha(x_{n-1} - p)||^2 \\
&= (1+\alpha)||x_n - p||^2 - \alpha||x_{n-1} - p||^2 + \alpha(1+\alpha)||x_n - x_{n-1}||^2
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
||x_{n+1} - w_n||^2 &= ||(x_{n+1} - x_n) - \alpha(x_n - x_{n-1})||^2 \\
&= ||x_{n+1} - x_n||^2 + \alpha^2||x_n - x_{n-1}||^2 - 2\alpha\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\
&\geq ||x_{n+1} - x_n||^2 + \alpha^2||x_n - x_{n-1}||^2 - 2\alpha||x_{n+1} - x_n||||x_n - x_{n-1}|| \\
&\geq ||x_{n+1} - x_n||^2 + \alpha^2||x_n - x_{n-1}||^2 - \alpha||x_{n+1} - x_n||^2 - \alpha||x_n - x_{n-1}||^2 \\
&= (1-\alpha)||x_{n+1} - x_n||^2 - \alpha(1-\alpha)||x_n - x_{n-1}||^2.
\end{aligned} \tag{29}
$$

Thus, from the relations (27)–(29), we obtain

$$
\begin{aligned}
||x_{n+1} - p||^2 \leq\ & (1+\alpha)||x_n - p||^2 - \alpha||x_{n-1} - p||^2 \\
& + \alpha(1+\alpha+\mathscr{K}_n(1-\alpha))||x_n - x_{n-1}||^2 \\
& - \mathscr{K}_n(1-\alpha)||x_{n+1} - x_n||^2
\end{aligned} \tag{30}
$$

and so

$$\begin{aligned}
&||x_{n+1} - p||^2 - \alpha||x_n - p||^2 + \alpha(1 + \alpha + \mathscr{K}_{n+1}(1 - \alpha))||x_{n+1} - x_n||^2 \\
&\leq ||x_n - p||^2 - \alpha||x_{n-1} - p||^2 + \alpha(1 + \alpha + \mathscr{K}_n(1 - \alpha))||x_n - x_{n-1}||^2 \\
&\quad - \left[\mathscr{K}_n(1 - \alpha) - \alpha(1 + \alpha + \mathscr{K}_{n+1}(1 - \alpha))\right] ||x_{n+1} - x_n||^2
\end{aligned}$$

or

$$\Xi_{n+1} \leq \Xi_n - \Delta_n||x_{n+1} - x_n||^2, \tag{31}$$

where

$$\Delta_n := \mathscr{K}_n(1 - \alpha) - \alpha(1 + \alpha + \mathscr{K}_{n+1}(1 - \alpha))$$

and

$$\Xi_n := ||x_n - p||^2 - \alpha||x_{n-1} - p||^2 + \alpha(1 + \alpha + \mathscr{K}_n(1 - \alpha))||x_n - x_{n-1}||^2.$$

It follows from the relation (31) and the fact $\Delta_n \geq 0$ that the sequence $\{\Xi_n\}$ is non-increasing. Moreover, from the definition of Ξ_n, we obtain $\Xi_n \geq ||x_n - p||^2 - \alpha||x_{n-1} - p||^2$. Thus, from the fact $\Xi_n \leq \Xi_{n-1} \leq \cdots \leq \Xi_2 \leq \Xi_1$, we get

$$||x_n - p||^2 \leq \Xi_n + \alpha||x_{n-1} - p||^2 \leq \Xi_1 + \alpha||x_{n-1} - p||^2,$$

which, by induction, implies that

$$\begin{aligned}
||x_n - p||^2 &\leq (1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1})\Xi_1 + \alpha^n||x_0 - p||^2 \\
&\leq \frac{\Xi_1}{1 - \alpha} + \alpha^n||x_0 - p||^2.
\end{aligned} \tag{32}$$

Since $\alpha \in [0, 1)$, the sequence $\left\{||x_n - p||^2\right\}$ is bounded and thus $\{x_n\}$ are bounded. Now, let $N \geq 1$. Then, from the relation (31), we obtain

$$\sum_{n=1}^{N} \Delta_n||x_{n+1} - x_n||^2 \leq \Xi_1 - \Xi_{N+1}. \tag{33}$$

By the definition of Ξ_{N+1}, we see that $\Xi_{N+1} \geq -\alpha||x_N - p||^2$, which, together with the relations (31) and (33), implies that

$$\begin{aligned}
\sum_{n=1}^{N} \Delta_n||x_{n+1} - x_n||^2 &\leq \Xi_1 + \alpha||x_N - p||^2 \\
&\leq \Xi_1 + \alpha\left(\frac{\Xi_1}{1 - \alpha} + \alpha^N||x_0 - p||^2\right) \\
&= \frac{\Xi_1}{1 - \alpha} + \alpha^{N+1}||x_0 - p||^2.
\end{aligned}$$

Thus, since $\alpha \in [0, 1)$, we obtain

$$\sum_{n=1}^{\infty} \Delta_n||x_{n+1} - x_n||^2 < +\infty. \tag{34}$$

This implies that

$$\lim_{n\to\infty} \Delta_n||x_{n+1} - x_n||^2 = 0. \tag{35}$$

Since $\lambda_n \to \lambda > 0$, it follows from (26) that

$$\lim_{n\to\infty} \mathscr{K}_n = K := \frac{\theta\left(1-\mu^2\right)}{(2-\theta+\mu\theta)^2} + \frac{1-\theta}{\theta} > 0. \tag{36}$$

Thus, from the definition of Δ_n and the condition (21), we obtain

$$\lim_{n\to\infty} \Delta_n = K(1-\alpha) - \alpha(1+\alpha+K(1-\alpha)) = K(1-\alpha)^2 - \alpha(1+\alpha) > 0. \tag{37}$$

Combining the relations (35) and (37), we have the following limit:

$$\lim_{n\to\infty} ||x_{n+1} - x_n||^2 = 0. \tag{38}$$

Since $w_n = x_n + \theta(x_n - x_{n-1})$, we have

$$||x_{n+1} - w_n|| \leq ||x_{n+1} - x_n|| + \theta||x_n - x_{n-1}||$$

and

$$||x_n - w_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - w_n|| \leq 2||x_n - x_{n+1}|| + \theta||x_n - x_{n-1}||.$$

Thus we have

$$\lim_{n\to\infty} ||x_{n+1} - w_n||^2 = \lim_{n\to\infty} ||x_n - w_n||^2 = 0. \tag{39}$$

It follows from the relation (30) that

$$\begin{aligned} &||x_{n+1} - p||^2 \\ &\quad \leq (1+\alpha)||x_n - p||^2 - \alpha||x_{n-1} - p||^2 + \alpha(1+\alpha+\mathscr{K}_n(1-\alpha))||x_n - x_{n-1}||^2 \\ &\quad = ||x_n - p||^2 + \alpha(||x_n - p||^2 - ||x_{n-1} - p||^2) \\ &\qquad + \alpha(1+\alpha+\mathscr{K}_n(1-\alpha))||x_n - x_{n-1}||^2. \end{aligned} \tag{40}$$

From the relations (34) and (37), we see that

$$\sum_{n=1}^{\infty} ||x_{n+1} - x_n||^2 < +\infty.$$

Thus, since $\{\mathscr{K}_n\}$ is bounded, we have

$$\sum_{n=1}^{\infty} \alpha(1+\alpha+\mathscr{K}_n(1-\alpha))||x_n - x_{n-1}||^2 < +\infty.$$

This together with the relation (40) and Lemma (2) implies that $\lim_{n\to\infty} ||x_n - p||^2 \in \mathscr{R}$. Thus it follows from (39) that

$$\lim_{n\to\infty} ||x_{n+1} - p||^2 = \lim_{n\to\infty} ||x_n - p||^2 = \lim_{n\to\infty} ||w_n - p||^2 \in \mathscr{R}. \tag{41}$$

Now, passing to the limit in (27) as $n \to \infty$ and using the relation (41) and the fact $\lim_{n\to\infty} \mathscr{K}_n > 0$, we obtain

$$\lim_{n\to\infty} ||y_n - w_n||^2 = 0. \tag{42}$$

The rest of proof is similar to the one of Theorem 1. This completes the proof. □

In Theorem 3, if $\theta = 1$, then we obtain the following corollary:

Corollary 1 *Suppose that the assumptions of Theorem 1 are satisfied. Take* $\lambda_0 > 0$ *and two numbers* $\mu \in (0, 1)$ *and* $\alpha \in [0, 1)$ *such that*

$$\frac{\alpha(1+\alpha)}{(1-\alpha)^2} < \frac{1-\mu^2}{(1+\mu)^2}.$$

Let $\{x_n\}$ *be the sequence generated by the following manner: choose* $x_{-1},\ x_0 \in \mathscr{H}$ *and compute*

$$\begin{cases} w_n = x_n + \alpha(x_n - x_{n-1}), \\ y_n = J_{\lambda_n A}(w_n - \lambda_n \mathscr{B}(w_n)), \\ x_{n+1} = y_n + \lambda_n(\mathscr{B}(w_n) - \mathscr{B}(y_n)), \\ \lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu||w_n - y_n||}{||\mathscr{B}(w_n) - \mathscr{B}(y_n)||}\right\}. \end{cases}$$

Then the sequence $\{x_n\}$ *converges weakly to a point in* $(\mathscr{A} + \mathscr{B})^{-1}(0)$.

The following corollary follows directly from Theorem 3 with $\theta = 1$ and $\alpha = 0$:

Corollary 2 *Suppose that the assumptions of Theorem 1 are satisfied. Take* $\lambda_0 > 0$ *and* $\mu \in (0, 1)$. *Let* $\{x_n\}$ *be the sequence generated by the following manner: choose* $x_0 \in \mathscr{H}$ *and compute*

$$\begin{cases} y_n = J_{\lambda_n A}(x_n - \lambda_n \mathscr{B}(x_n)), \\ x_{n+1} = y_n + \lambda_n(\mathscr{B}(x_n) - \mathscr{B}(y_n)), \\ \lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu||x_n - y_n||}{||\mathscr{B}(x_n) - \mathscr{B}(y_n)||}\right\}. \end{cases}$$

Then the sequence $\{x_n\}$ *converges weakly to a point in* $(\mathscr{A} + \mathscr{B})^{-1}(0)$.

5 Signal Recovery

In this section, we give some numerical examples to the signal recovery in compressed sensing. We provide a comparison among the forward–backward method (FBSM), the method (MFBSM), Algorithms 1 and 2. Compressed sensing can be modelled as the following *underdeterminated linear equation system*:

$$y = Ax + \varepsilon, \tag{43}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^N$ is the observed or measured data with noisy ε, and $A : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear operator. It is known that to solve (43) can be seen as solving the LASSO problem:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1, \tag{44}$$

where $\lambda > 0$. So we can apply our method for solving the problem (44) in case $f(x) = \frac{1}{2}\|y - Ax\|_2^2$ and $g(x) = \lambda\|x\|_1$.

In our experiment, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $A \in \mathbb{R}^{M \times N}$ is generated from a normal distribution with mean zero and one variance. The observation y is generated by Gaussian noise white signal-to-noise ratio SNR=40. The initial point x_{-1} is chosen randomly and x^0 is zero. The restoration accuracy is measured by the mean squared error as follows:

$$E_n = \|x_{n+1} - x_n\| < \kappa. \tag{45}$$

The step size λ_n in the methods (FBSM) and (MFBSM) is $\frac{0.2}{\|A\|^2}$. Set $\lambda_0 = 1$, $\theta = 1$, $\mu = 0.9$ and $\alpha = 0.1$.

We test four cases as follows:

Case 1: $N = 512$, $M = 256$ and $m = 10$;

Case 2: $N = 1024$, $M = 512$ and $m = 30$.

We denote Iter by the number of iterations and CPU by CPU time. The numerical results are reported in Table 1.

From Table 1, we see that Algorithm 2 has a better convergence rate than other algorithms in term of Iter and CPU.

Next, we show the graphs of original signal and recovered signal by the methods (FBSM), (MFBSM), Algorithms 1 and 2 when $N = 512$, $M = 256$, $m = 10$ and $\kappa = 10^{-6}$. The number of iterations and CPU time are reported in Fig. 1.

We next show the graphs of signal recovery by the methods (FBSM), (MFBSM), Algorithms 1 and 2 when $N = 1024$, $M = 512$, $m = 30$ and $\kappa = 10^{-6}$. The number of iterations and CPU time are reported in Fig. 2.

From Figs. 1 and 2, it is shown that, in both cases, our proposed methods (Algorithms 1 and 2) can be used to recover signal in a good performance.

We next show the error plotting of the methods (FBSM), (MFBSM), Algorithms 1 and 2 in Case 1 (Fig. 3) and Case 2 (Fig. 4).

From Figs. 3 and 4, we observe that our algorithms converge faster than the methods (FBSM) and (MFBSM).

6 Image Debluring

In this section, we present an application to image restoration problems using our main result. We provide some comparisons to other algorithms.

For a Grey scale image of M pixels wide by N pixels height, each pixel value is known to range from 0 to 255. Let $D = M \times N$. Then the underlying real Hilbert space is $\mathbb{R}^D$ equipped with the standard Euclidean norm $\|\cdot\|_2$. Let $C = [0, 255]^D$. In order to estimate an approximation of the vector x, which represents the image of the original image scene, we consider the *convex minimization model*:

Table 1 Computational results to recover the signal

Methods	$\kappa = 10^{-5}$		$\kappa = 10^{-6}$	
	Iter	CPU	Iter	CPU
Case 1				
The method (FBSM)	2106	32.6352	3237	51.1075
The method (MFBSM)	2108	33.5675	3241	52.7706
Algorithm 1	1469	0.2750	1328	0.2300
Algorithm 2	1339	0.2282	1277	0.1595
Case 2				
The method (FBSM)	6880	605.5408	7467	676.5340
The method (MFBSM)	6882	627.9596	7471	677.3778
Algorithm 1	2592	2.6161	2831	2.9538
Algorithm 2	2498	2.5715	2787	2.9009

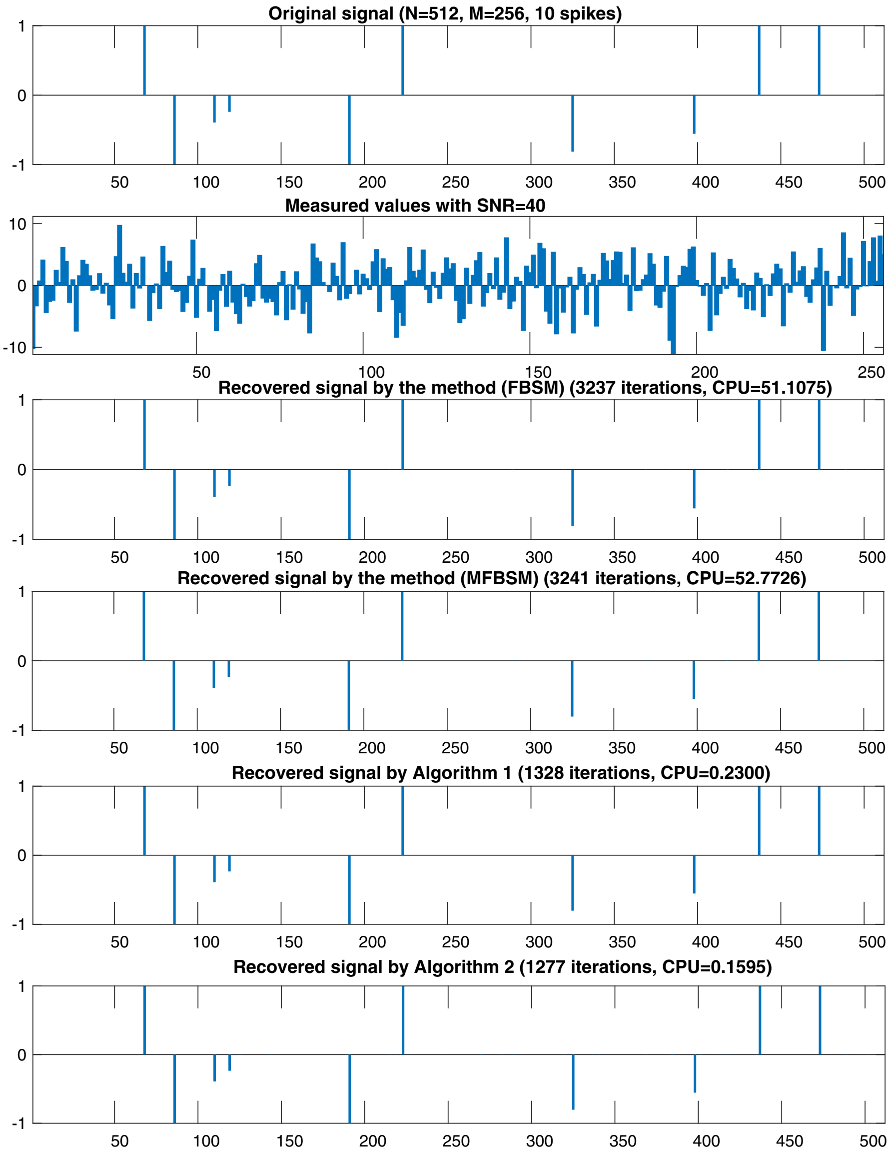


Fig. 1 From top to bottom: original signal, observation data, recovered signal by the methods (FBSM), (MFBSM), Algorithms 1 and 2 in Case 1, respectively

$$\min_{x \in C} \|Ax - y\|_2. \tag{46}$$

By choosing $Q = \{y\}$, the problem (46) can be seen as the *split feasibility problem* (SFP). Therefore, we can apply our algorithm to solve the image restoration problem.

In this numerical experiment, we use Matlab R2018b to write all codes. To determine the efficiency of algorithms, we need an image quality measure of restored images. We define

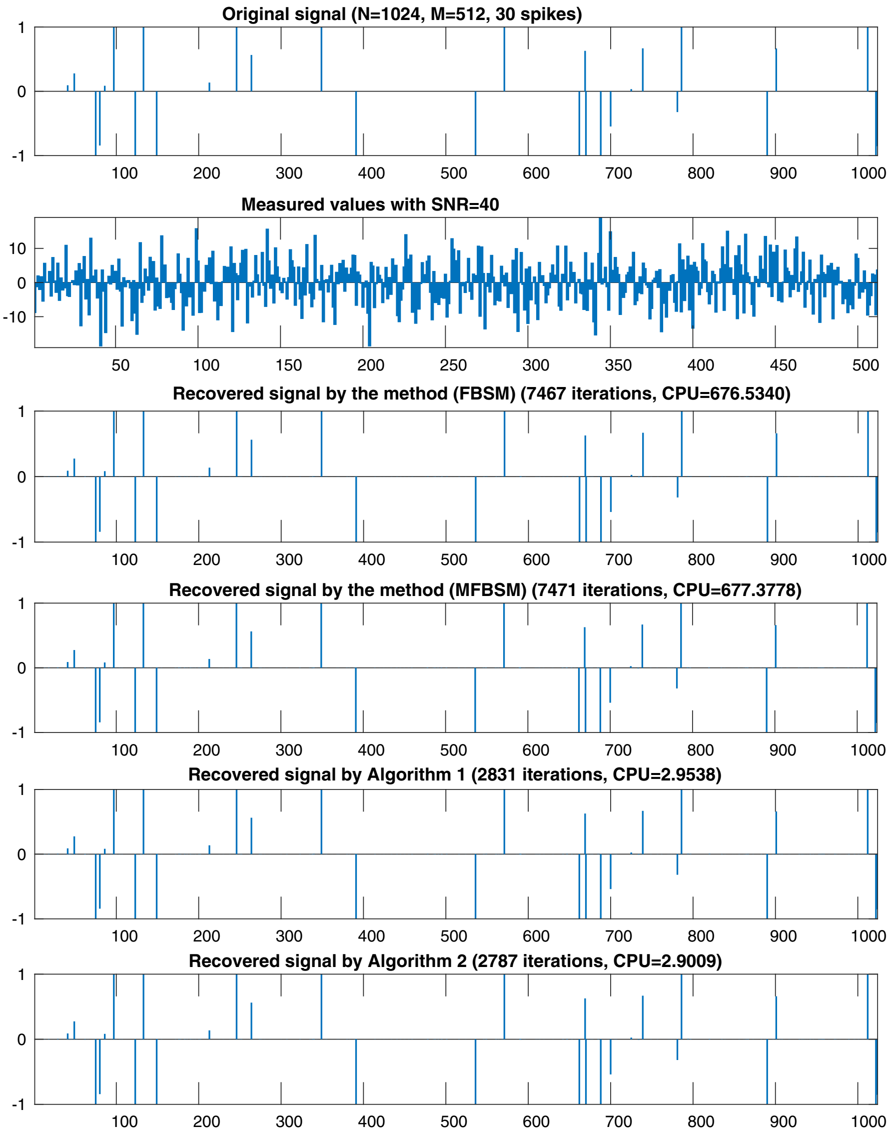


Fig. 2 From top to bottom: original signal, observation data, recovered signal by the methods (FBSM), (MFBSM), Algorithms 1 and 2 in Case 2, respectively

the Peak signal to noise ratio (PSNR) in decibel (dB) as follows:

$$PSNR = 20 \log_{10} \frac{\|\bar{x}\|_2}{\|x - \bar{x}\|_2}, \tag{47}$$

where $\bar{x}$ is an original image and x is a restored image. It can be observed that the larger PSNR values, the better restored images. The step size λ_n in the methods (FB) and (MFBSM) is $\frac{0.1}{\|A\|_F}$. To begin, set x_{-1}be $1 \in \mathbb{R}^D$, x_0 to be $0 \in \mathbb{R}^D$. Set all parameters by $\mu = 0.7$, $\lambda_0 = 1$, $\theta = 0.5$ and α according to the condition of Algorithm 2. Each image is degraded by a motion

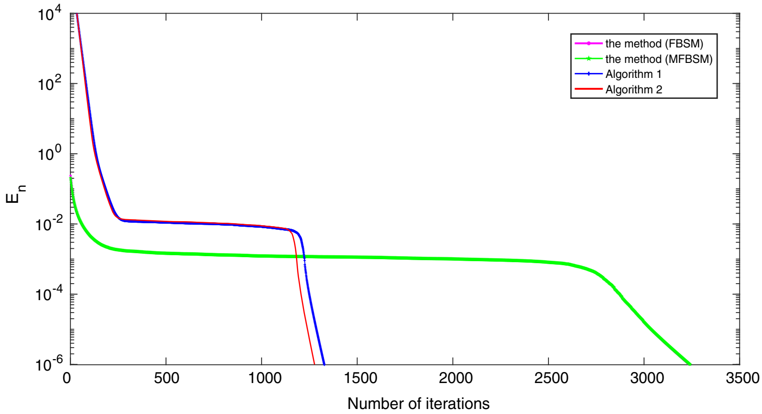


Fig. 3 E_n versus number of iterations in Case 1

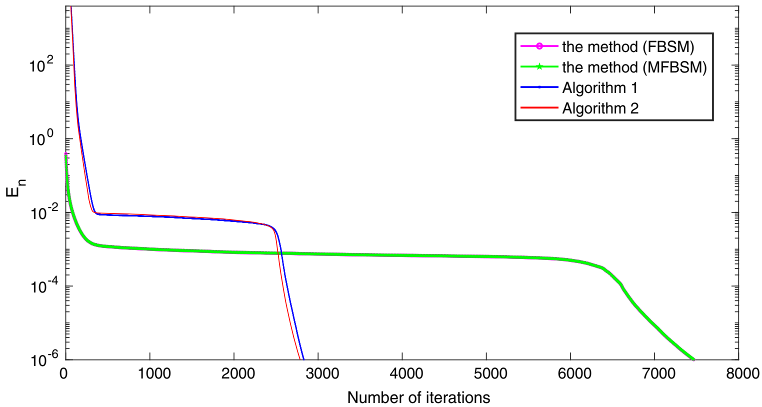
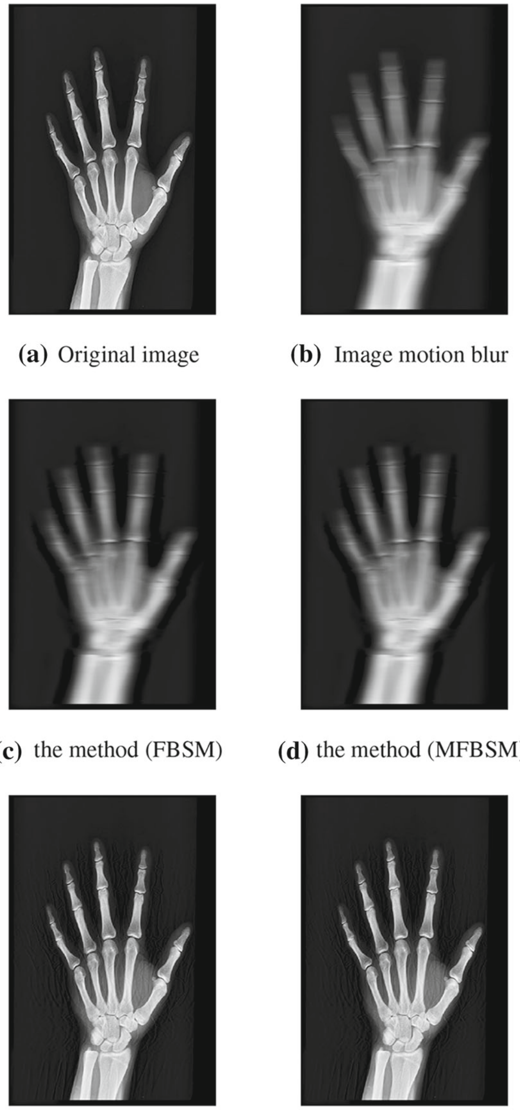


Fig. 4 E_n versus number of iterations in Case 2

Table 2 Numerical comparison for the methods (FBSM), (MFBSM), Algorithms 1 and 2

Image	PSNR (dB)				
	Iter	The method (FBSM)	The method (MFBSM)	Algorithm 1	Algorithm 2
Hand X-ray	2500	9.5317	9.5305	22.6503	23.0560
Size = 640 × 532	3000	9.8928	9.8921	23.2600	23.6636
PSNR (dB) = 9.1323	4000	10.3799	10.3796	24.2191	24.6250
	5000	10.7353	10.7351	24.9676	25.3816
Woman	2500	13.0474	13.0461	20.2199	20.5826
Size = 512 × 512	2000	13.2994	13.2989	20.6579	21.0330
PSNR (dB) = 12.6164	3000	13.5803	13.5802	21.3770	21.7634
	4000	13.7711	13.7710	21.9475	22.3357

Fig. 5 Comparison of recovered images by using different algorithms when the number of iterations is 5000 of Hand X-ray image



(a) Original image **(b)** Image motion blur

(c) the method (FBSM) **(d)** the method (MFBSM)

(e) Algorithm 1 **(f)** Algorithm 2

blur with a motion length 45 and an angle 180. Then the numerical results are reported in Table 2.

From Table 2, we observe that our proposed Algorithm 2 has a better convergence behavior than the methods (FBSM), (MFBSM) and Algorithm 1 in term of PSNR.

Next, we show original image, blurred image and recovered images by using the methods (FBSM), (MFBSM), Algorithms 1 and 2 (see, Figs. 5 and 6).

We next provide the PSNR plotting of the methods (FBSM), (MFBSM), Algorithms 1 and 2 (see, Figs. 7 and 8).

Fig. 6 Comparison of recovered images by using different algorithms when the number of iterations is 5000 of Woman image

(a) Original image

(b) Image motion blur

(c) the method (FBSM)

(d) the method (MFBSM)

(e) Algorithm 1

(f) Algorithm 2

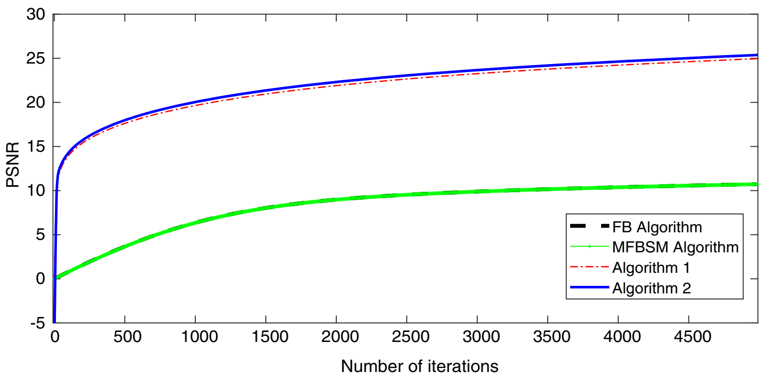


Fig. 7 Graphs of PSNR for the methods (FBSM), (MFBSM), Algorithms 1 and 2 of Hand X-ray image

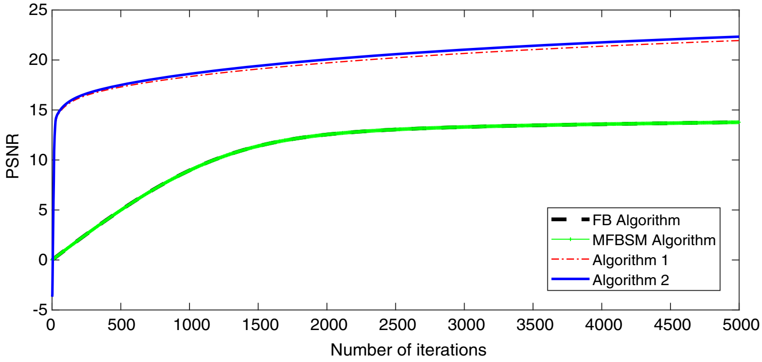


Fig. 8 Graphs of PSNR for the methods (FBSM), (MFBSM), Algorithms 1 and 2 of Woman image

From Figs. 5, 6, 7 and 8, we see that Algorithms 1 and 2 have a better convergence than other algorithms.

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Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

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