If the regularisation parameter β_ε *is selected by*

$$\beta_\varepsilon = \left(\frac{1}{\widetilde{\ell}_0}\right)^{2/\gamma}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta} - \frac{2p-1}{\gamma}\ln\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)\right)^{2/\gamma}. \tag{3.8}$$

then, for every $x \in [0, l_1)$

If u_1 ***satisfies the conditions 3.2.1***

$$\sup_{x\in(0,\ell_1)} \|u_1(x,\cdot)\|_{\mathrm{H}^p(\mathbb{R})} := \sup_{x\in(0,\ell_1)}\left(\int_{\mathbb{R}}\left(1+\xi^2\right)^p |\hat{u}_1(x,\xi)|^2\, d\xi\right)^{1/2} \le \mathbf{E}_1, \quad p>0. \tag{3.9}$$

Put $q = \min\left\{\frac{\tilde{\ell}_x}{\gamma\tilde{\ell}_0}, \frac{1}{\gamma}\right\}$ *then*

$$\mathbf{D}_\infty(x) \le \left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{(1-2p)q}\left[2\sqrt{\frac{2}{\gamma\ell\left(l_1\right)}}\left(1+k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}} + \sqrt{\frac{2}{2p-1}}\left(2\ell_0\right)^{\frac{2p-1}{\gamma}}\mathbf{E}_1\right].$$

If u_1 ***satisfies the conditions 3.2.2***

$$\sup_{x\in(0,\ell_1)}\left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \mathrm{e}^{2x|\xi|^{\gamma/2}}|\hat{u}_1(x,\xi)|^2\, d\xi\right)^{1/2} \le \mathbf{E}_2 \text{ with } p > \frac{1}{2}, \tag{3.10}$$

Then

$$\mathbf{D}_\infty(x) \le \left[2\sqrt{\frac{2}{\gamma\ell\left(l_1\right)}}\left(1+k\sqrt{\frac{\alpha_1}{\alpha_2}}\right) + \sqrt{\frac{2}{(x+p)\gamma}}\mathbf{E}_2\delta^{-1}\right]\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{(1-2p)\tilde{\ell}_x}{\gamma\tilde{\ell}_0}}.$$

If u_1 ***satisfies the conditions 3.2.3***

$$\sup_{x\in(0,\ell_1)}\left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \mathrm{e}^{2(x+p)|\xi|^{\gamma/2}}|\hat{u}_1(x,\xi)|^2\, d\xi\right)^{1/2} \le \mathbf{E}_3 \text{ with } p > \frac{1}{2}, \tag{3.11}$$

Then

$$\mathbf{D}_\infty(x) \le \left[2\sqrt{\frac{2}{\gamma\ell\left(l_1\right)}}\left(1+k\sqrt{\frac{\alpha_1}{\alpha_2}}\right) + \sqrt{\frac{2}{(x+p)\gamma}}\mathbf{E}_3\delta^{-1}\right]\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{(1-2p)\tilde{\ell}_x}{\gamma\tilde{\ell}_0}}.$$

Proof.

$$\mathbf{D}_\infty(x) = \left\|u_{1\beta}^\delta(x,\cdot) - u_1(x,\cdot)\right\|_{L^\infty(\mathbb{R})}$$
$$\mathbf{D}_\infty(x) \le \underbrace{\left\|\left(\hat{u}_{1\beta} - \hat{u}_{1\beta}^\delta\right)(x,\cdot)\right\|_{L^\infty(\mathbb{R})}}_{\mathcal{T}_1(x)} + \underbrace{\left\|\left(\hat{u}_{1\beta} - \hat{u}_1\right)(x,\cdot)\right\|_{L^\infty(\mathbb{R})}}_{\mathcal{T}_2(x)}$$

$$\hat{u}_{1\beta} - \hat{u}_{1\beta}^\delta = \left[\hat{\Theta}_1(x,\xi)\left(\hat{g}_\delta(\xi) - \hat{g}(\xi)\right) + \hat{\Theta}_2(x,\xi)\left(\hat{f}_\delta(\xi) - \hat{f}(\xi)\right)\right]\mathcal{I}_{\mathrm{E}_\beta}(\xi)$$

$$
\begin{aligned}
\mathcal{T}_1(x) &= \int_{\mathbb{R}} \left| \left[\hat{\Theta}_1(x,\xi)\left(\hat{g}_\delta(\xi) - \hat{g}(\xi)\right) - \hat{\Theta}_2(x,\xi)\left(\hat{f}_\delta(\xi) - \hat{f}(\xi)\right)\right] \mathcal{I}_{\mathrm{E}_\beta}(\xi)\right| d\xi \\
&\le \int_{\mathbb{E}_\beta} \left|\hat{\Theta}_1(x,\xi)\left(\hat{g}_\delta(\xi) - \hat{g}(\xi)\right)\right| d\xi + \int_{\mathbb{E}_\beta} \left|\hat{\Theta}_2(x,\xi)\left(\hat{f}_\delta(\xi) - \hat{f}(\xi)\right)\right| d\xi \\
&\le 2\sqrt{2}\left(1 + k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)\left(\int_0^{\beta_\delta} e^{2\xi^{\gamma/2}\ell(x)} d\xi\right)^{1/2} \delta \\
&= 2\sqrt{2}\left(1 + k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)\left(\int_0^{\beta_\delta} \frac{\xi^{1-\gamma/2}}{\gamma\ell(x)} d\left(e^{2\xi^{/2}(x)}\right)\right)^{1/2} \delta \\
&\le 2\sqrt{2}\left(1 + k\sqrt{\frac{\alpha_1}{\alpha_2}}\right) \frac{(\beta_\varepsilon)^{\frac{2-\gamma}{4}}}{\sqrt{\gamma\ell\left(l_1\right)}} e^{(\beta_\varepsilon)^{\gamma/2}\ell(x)}\delta \\
&\le 2\sqrt{\frac{2}{\gamma\ell\left(l_1\right)}}\left(1 + k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)(\beta_\varepsilon)^{\frac{2-\gamma}{4}} e^{(\beta_\varepsilon)^{\gamma/2}\ell(x)}\delta
\end{aligned}
$$

$$
\mathcal{T}_1(x) \le 2\sqrt{\frac{2}{\gamma\ell\left(l_1\right)}}\left(1 + k\sqrt{\frac{\alpha_1}{\alpha_2}}\right) \tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}} \delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}} \left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{(1-2p)\tilde{\ell}_x}{\gamma\tilde{\ell}_0}}. \tag{3.12}
$$

To reach the conclusion, it is necessary to estimate $\mathcal{T}_2$. Again, in view of HÃ¶lder inequality and the parameter choice (3.8)

$$
\begin{aligned}
\mathcal{T}_2(x) &= \int_{\mathbb{R}\setminus\mathbb{E}_\beta} \left|\hat{u}_{1\beta}(x,\xi) - \hat{u}_1(x,\xi)\right| d\xi \\
&= \int_{\mathbb{R}\setminus\mathbb{E}_\beta} \left|\hat{u}_1(x,\xi)\right| d\xi.
\end{aligned}
$$

If u_1 satisfies the conditions (3.9)

$$
\begin{aligned}
\mathcal{T}_2(x) &= \int_{\mathbb{R}\setminus\mathbb{E}_\beta} \left(1+\xi^2\right)^{-p/2}\left(1+\xi^2\right)^{p/2}\left|\hat{u}_1(x,\xi)\right| d\xi \\
&\le \left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \left(1+\xi^2\right)^{-p} d\xi\right)^{1/2}\left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \left(1+\xi^2\right)^{p}\left|\hat{u}_1(x,\xi)\right|^2 d\xi\right)^{1/2} \\
&\le \sqrt{2}\left(\int_{\beta_\varepsilon}^{\infty} \xi^{-2p} d(\xi)\right)^{1/2} \mathbf{E}_1 \\
&\le \sqrt{\frac{2}{2p-1}}\left(\beta_\varepsilon\right)^{\frac{1-2p}{2}} \mathbf{E}_1.
\end{aligned}
$$

$$
\widetilde{\mathbf{E}} = \mathbf{E} + \delta\left(e^{e^{4(2p-1)/\gamma}} + e^{2\widetilde{\ell}_0(\Lambda(0))^{\gamma/2}}\right) \quad \text{that is} \quad \tilde{\mathbf{E}} > \delta e^{e^{4(2p-1)/\gamma}}
$$

$$
\ln\frac{\tilde{\mathbf{E}}}{\delta} \ge \frac{4(2p-1)}{\gamma}\ln\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right), \quad \text{that is, } \beta_\varepsilon \ge \left(\frac{1}{2\ell_0}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)\right)^{2/\gamma}
$$

$$(\beta_\delta)^{\frac{2p-1}{2}} \geq \left(\frac{1}{2\ell_0}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)\right)^{\frac{2p-1}{\gamma}}$$

$$\frac{1}{(\beta_\delta)^{\frac{2p-1}{2}}} \leq (2\ell_0)^{\frac{2p-1}{\gamma}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}}$$

Then

$$\mathcal{T}_2(x) \leq \sqrt{\frac{2}{2p-1}}\,(2\ell_0)^{\frac{2p-1}{\gamma}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}}\mathbf{E}_1$$

$$\mathbf{D}_\infty(x) \leq 2\sqrt{\frac{2}{\gamma\ell(l_1)}}\left(1+k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{(1-2p)\tilde{\ell}_x}{\gamma\tilde{\ell}_0}} + \sqrt{\frac{2}{2p-1}}\,(2\ell_0)^{\frac{2p-1}{\gamma}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}}\mathbf{E}_1.$$

Put $q = \min\left\{\frac{\tilde{\ell}_x}{\gamma\tilde{\ell}_0}, \frac{1}{\gamma}\right\}$

$$\mathbf{D}_\infty(x) \leq \left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{(1-2p)q}\left[2\sqrt{\frac{2}{\gamma\ell(l_1)}}\left(1+k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}} + \sqrt{\frac{2}{2p-1}}\,(2\ell_0)^{\frac{2p-1}{\gamma}}\mathbf{E}_1\right].$$

If u_1 satisfies the conditions (3.10)

$$\begin{aligned}
\mathcal{T}_2(x) &= \int_{\mathbb{R}\setminus\mathbb{E}_\beta} \mathrm{e}^{-x|\xi|^{\gamma/2}}\mathrm{e}^{x|\xi|^{\gamma/2}}\,|\hat{u}_1(x,\xi)|\,d\xi \\
&\leq \left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta}\mathrm{e}^{-2x|\xi|^{\gamma/2}}d\xi\right)^{1/2}\left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta}\mathrm{e}^{2x|\xi|^{\gamma/2}}\,|\hat{u}_1(x,\xi)|^2\,d\xi\right)^{1/2} \\
&\leq \sqrt{2}\left(\int_{\beta_\varepsilon}^{\infty}\frac{\xi^{1-\gamma/2}}{-x\gamma}d\left(e^{-2x\xi^{\gamma/2}}\right)\right)^{1/2}\mathbf{E}_2 \\
&\leq \sqrt{\frac{2}{(x+p)\gamma}}\,(\beta_\varepsilon)^{\frac{2-\gamma}{4}}\,e^{-x(\beta_\varepsilon)^{\gamma/2}}\mathbf{E}_2,
\end{aligned}$$

where $\beta_\varepsilon = \left(\frac{1}{\tilde{\ell}_0}\right)^{2/\gamma}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta} - \frac{2p+1}{\gamma}\ln\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)\right)^{2/\gamma}$,

$$\mathcal{T}_2(x) \leq \sqrt{\frac{2}{(x+p)\gamma}}\left[\frac{1}{\tilde{\ell}_0}\ln\frac{\tilde{\mathbf{E}}}{\delta} - \frac{2p-1}{\gamma\tilde{\ell}_0}\ln\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)\right]^{\frac{2-\gamma}{2\gamma}}\left[\left(\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{-x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}\frac{(-x)}{\tilde{\ell}_0}}\right]\mathbf{E}_2$$

$$\begin{aligned}
\mathcal{T}_2(x) &\leq \sqrt{\frac{2}{(x+p)\gamma}}\left(\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1}{\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}\right)}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}\right)}(\tilde{\mathbf{E}})^{\frac{-x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}\frac{-x}{\tilde{\ell}_0}}\mathbf{E}_2\delta^{\frac{x}{\tilde{\ell}_0}} \\
&\leq \sqrt{\frac{2}{(x+p)\gamma}}\tilde{\mathbf{E}}^{\frac{1}{\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}-x\right)}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}-x\right)}\mathbf{E}_2\delta^{-\frac{1}{\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}-x\right)} \\
&\leq \sqrt{\frac{2}{(x+p)\gamma}}\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\mathbf{E}_2\delta^{-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}.
\end{aligned}$$

(3.12) then

$$\mathbf{D}_\infty(x) \le \left[2\sqrt{\frac{2}{\gamma\ell\left(l_1\right)}}\left(1+k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)+\sqrt{\frac{2}{(x+p)\gamma}}\mathbf{E}_2\delta^{-1}\right]\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{(1-2p)\tilde{\ell}_x}{\gamma\tilde{\ell}_0}}.$$

If u_1 satisfies the conditions (3.11)

$$\begin{aligned}
\mathcal{T}_2(x) &= \int_{\mathbb{R}\setminus\mathbb{E}_\beta} \mathrm{e}^{-(x+p)|\xi|^{\gamma/2}}\mathrm{e}^{(x+p)|\xi|^{\gamma/2}}\left|\hat{u}_1(x,\xi)\right| d\xi \\
&\le \left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \mathrm{e}^{-2(x+p)|\xi|^{\gamma/2}} d\xi\right)^{1/2}\left(\int_{\mathbb{R}\setminus\mathbb{E}_\beta} \mathrm{e}^{2(x+p)|\xi|^{\gamma/2}}\left|\hat{u}_1(x,\xi)\right|^2 d\xi\right)^{1/2} \\
&\le \sqrt{2}\left(\int_{\beta_\varepsilon}^{\infty}\frac{\xi^{1-\gamma/2}}{-(x+p)\gamma}d\left(e^{-2(x+p)\xi^{\gamma/2}}\right)\right)^{1/2}\mathbf{E}_3 \\
&\le \sqrt{\frac{2}{(x+p)\gamma}}\left(\beta_\varepsilon\right)^{\frac{2-\gamma}{4}}e^{-(x+p)(\beta_\varepsilon)^{\gamma/2}}\mathbf{E}_3.
\end{aligned}$$

where $\beta_\varepsilon = \left(\frac{1}{\tilde{\ell}_0}\right)^{2/\gamma}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}-\frac{2p+1}{\gamma}\ln\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)\right)^{2/\gamma}$,

$$\mathcal{T}_2(x) \le \sqrt{\frac{2}{(x+p)\gamma}}\left[\frac{1}{\tilde{\ell}_0}\ln\frac{\tilde{\mathbf{E}}}{\delta}-\frac{2p-1}{\gamma\tilde{\ell}_0}\ln\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)\right]^{\frac{2-\gamma}{2\gamma}}\left[\left(\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{-(x+p)}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}\frac{(-(x+p))}{\tilde{\ell}_0}}\right]\mathbf{E}_3$$

$$\begin{aligned}
\mathcal{T}_2(x) &\le \sqrt{\frac{2}{(x+p)\gamma}}\left(\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1}{\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}\right)}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}\right)}(\tilde{\mathbf{E}})^{\frac{-(x+p)}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}\frac{-(x+p)}{\tilde{\ell}_0}}\mathbf{E}_3\delta^{\frac{(x+p)}{\tilde{\ell}_0}} \\
&\le \sqrt{\frac{2}{(x+p)\gamma}}\tilde{\mathbf{E}}^{\frac{1}{\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}-(x+p)\right)}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}-(x+p)\right)}\mathbf{E}_3\delta^{-\frac{1}{\tilde{\ell}_0}\left(\frac{1}{\gamma}-\frac{1}{2}-(x+p)\right)} \\
&\le \sqrt{\frac{2}{(x+p)\gamma}}\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{1-2p}{\gamma}\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\mathbf{E}_3\delta^{-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}},
\end{aligned}$$

(3.12) then

$$\mathbf{D}_\infty(x) \le \left[2\sqrt{\frac{2}{\gamma\ell\left(l_1\right)}}\left(1+k\sqrt{\frac{\alpha_1}{\alpha_2}}\right)+\sqrt{\frac{2}{(x+p)\gamma}}\mathbf{E}_3\delta^{-1}\right]\tilde{\mathbf{E}}^{\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\delta^{1-\frac{\tilde{\ell}_x}{\tilde{\ell}_0}}\left(\ln\frac{\tilde{\mathbf{E}}}{\delta}\right)^{\frac{(1-2p)\tilde{\ell}_x}{\gamma\tilde{\ell}_0}}.$$

which is the desired conclusion. The theorem is proved. ■

4 Numerical experiment

In general, for an ill-posed problem, we can only obtain the worst-case error for regularized methods but in practical computation, the errors in numerical computation of regularization methods are far less than the worst-case errors. This phenomenon has been observed in many literatures, e.g. [3]. It is difficult to compute the formula 3.5 because of a computation of the priori condition on the exact solution **E**. Moreover, in practice the test of an inversion process avoiding the "inverse crime" can be done using a model for the numerically simulated data and a different one to invert the data. To overcome this difficultily, we recall a result in [2] calculating continuous Fourier transform by using the discrete Fast Fourier Transform (FFT) and inverse discrete Fast Fourier Transform according to the formulas () and () in the above section. Based on the theoretical analysis derived in the above section, we construct in the following numerical example to verify the convergence of the proposed method. Errors between the exact and its regularized solutions are estimated by the relative error estimation defined by

$$E\left(x\right) = \frac{\left(\sum_{j=1}^{N_t}\left|u_{1\beta}^{\delta}(x,t_j)-u_1(x,t_j)\right|^2\right)^{1/2}}{\left(\sum_{j=1}^{N_t}\left|u_1(x,t_j)\right|^2\right)^{1/2}},$$

where $t_j = j\Delta t, \Delta t = \frac{T}{N_t}, j = \overline{0, N_t}$. In our numerical experiment, for simplicity, we always fix $T = 5$. The following numerical implementation is performed by using Matlab and the computations are done on a computer equipped with I7-Core CPU 2.5 GHz and having 8.0 GB total RAM. More detail, we solve the following problem

- On the first layer $\mathbb{D}_1 := \{x|0 \le x \le 1\}$

$$\begin{array}{ll} \partial_t^{1/2} u_1\left(x,t\right) = \alpha_1 \partial_x^2 u_1\left(x,t\right) + S\left(x,t\right), & x \in \mathbb{D}_1, t > 0, \\ u_1\left(1,t\right) = u_2\left(1,t\right), & t > 0, \\ \kappa_1 \partial_x u_1\left(1,t\right) = \kappa_2 \partial_x u_2\left(1,t\right), & t > 0, \end{array}$$

- On the second layer $\mathbb{D}_2 := \{x|1 \le x \le 2\}$

$$\begin{array}{ll} \partial_t^{1/2} u_2\left(x,t\right) = \alpha_2 \partial_x^2 u_2\left(x,t\right), & x \in \mathbb{D}_2, t > 0, \\ u_2\left(2,t\right) = g\left(t\right), & t > 0, \\ \partial_x u_2\left(2,t\right) = f\left(t\right), & t > 0. \end{array}$$

The numerical experiments are composed of four steps:

Step 1. First, In the numerical test, we take $\alpha_1 = 2, \alpha_2 = 3, \kappa_1 = 9, \kappa_2 = 6$ and choose N_x, N_t to generate spatial and temporal discretizations as follows

$$\begin{aligned} x_k &= k\Delta x, \quad \Delta x = \frac{1}{N_x}, \quad k = \overline{0, N_x}, \\ t_j &= j\Delta t, \quad \Delta t = \frac{T}{N_t}, \quad j = \overline{0, N_t}. \end{aligned}$$

In this experiment, we choose $N_x = N_t = 100$.

Step 2. We consider the exact data and the source function as follows

$$\begin{aligned} f(t) &= \exp(-t^2)\cos 2, \;\; g(t) = \exp(-t^2)\sin 2, \\ S(x,t) &= \exp(-t^2)\cos\left(x^3\right) + \frac{\sin\left(x^2\right)}{t^2+5}. \end{aligned}$$

Suppose that vector

$$[F,G] = \begin{bmatrix} f(t_1) & g(t_1) \\ f(t_2) & g(t_2) \\ \vdots & \vdots \\ f(t_{N_t}) & g(t_{N_t}) \end{bmatrix},$$

represents the discrete form of functions f and g. As in practical problems, the data (f,g) is obtained by measurement and thus inevitably is contaminated by measurement errors, some uniformly distributed random noises δ are added to $[F,G]$ in our test example, i.e.,

$$\left[F^\delta, G^\delta\right] = [F,G] + f_{\max} * \delta\left[\mathrm{rand}\left(size\left(F\right)\right), \mathrm{rand}\left(size\left(G\right)\right)\right].$$

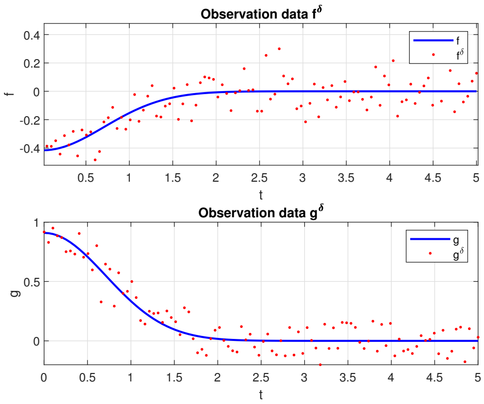

Figure 1: Graphs of Cauchy data in the case of deterministic and random noise.

Then we use the discrete Fast Fourier Transform (FFT) technique to obtain the Fourier Transform $\left(\widehat{f},\widehat{g}\right), \left(\widehat{f^\delta},\widehat{g^\delta}\right), \widehat{S}$ of the exact data, the measured data and the source function, respectively.

Step 3 By applying the formulas () and (), we construct the Fourier transform of the exact solution and the regularized solutions at the certain point $x = x_0$ in various cases of the noise level $\delta = \{0.1, 0.05, 0.01\}$ in which the integral terms

$$\int_0^{x_0} \widehat{S}(y,\xi) \frac{\sinh\left(\sqrt{k_1(\xi)}(x_0 - y)\right)}{\sqrt{k_1(\xi)}} dy,$$

$$\int_0^{1} \widehat{S}(y,\xi) \frac{\sinh\left(\sqrt{k_1(\xi)}(1 - y)\right)}{\sqrt{k_1(\xi)}} \cosh\left(\sqrt{k_1(\xi)}(1-x_0)\right) dy,$$

are approximated by the Simpson's rule. Moreover, because of a difficult computation of the priori condition on the exact solution $\mathbf{E}$, the regularized parameter β_δ is chosen as follows

$$\beta_\delta = \frac{1}{l_0^4}\left(\ln \frac{10^{12}}{\delta} - 8 \ln\left(\ln \frac{10^{12}}{\delta}\right)\right)^4. \tag{4.1}$$

Then, we apply the inverse Fast Fourier transform (IFFT) to recover the discrete exact solution and the discrete regularized solutions by constructing the following matrices of size $\mathbb{R}^{N_x+1} \times \mathbb{R}^{N_t+1}$

$$\mathfrak{U} = \begin{bmatrix} u_1(x_0,t_0) & u_1(x_0,t_1) & \cdots & u_1(x_0,t_{N_t}) \\ u_1(x_1,t_0) & u_1(x_1,t_1) & \cdots & u_1(x_1,t_{N_t}) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(x_{N_x},t_0) & u_1(x_{N_x},t_1) & \cdots & u_1(x_{N_x},t_{N_t}) \end{bmatrix},$$

and

$$\mathfrak{U}_{1,\beta_i}^{\delta_i} = \begin{bmatrix} u_{1,\beta_i}^{\delta_i}(x_0,t_0) & u_{1,\beta_i}^{\delta_i}(x_0,t_1) & \cdots & u_{1,\beta_i}^{\delta_i}(x_0,t_{N_t}) \\ u_{1,\beta_i}^{\delta_i}(x_1,t_0) & u_{1,\beta_i}^{\delta_i}(x_1,t_1) & \cdots & u_{1,\beta_i}^{\delta_i}(x_1,t_{N_t}) \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,\beta_i}^{\delta_i}(x_{N_x},t_0) & u_{1,\beta_i}^{\delta_i}(x_{N_x},t_1) & \cdots & u_{1,\beta_i}^{\delta_i}(x_{N_x},t_{N_t}) \end{bmatrix},$$

in which $\delta_1 = 0.1, \delta_2 = 0.05, \delta_3 = 0.01$ and β_i is defined as (4.1) corresponding to δ_i.

Step 4 For $r = \overline{0, N_x}$,the relative error estimation at certain point x_r is computed by

$$E(x_r) = \frac{\left(\sum_{j=1}^{N_t} \left|u_{1\beta}^{\delta}(x_r,t_j) - u_1(x_r,t_j)\right|^2\right)^{1/2}}{\left(\sum_{j=1}^{N_t} |u_1(x_r,t_j)|^2\right)^{1/2}}.$$

Then the result is shown in Table 4. From this computation, we observe the following important facts: The regularization method given in this paper works well for even acceptable error levels. The regularized solution converges to the exact solution with different values of δ. However, the numerical accuracy becomes worse as x tends to 0.

	$E(x_r)$		
z	$\delta_1 = 0.1$	$\delta_2 = 0.05$	$\delta_3 = 0.01$
0	1.2927	0.6492	0.1324
0.1	1.1223	0.5636	0.1149
0.2	0.9537	0.4789	0.0977
0.3	0.7868	0.3951	0.0807
0.4	0.6214	0.3122	0.0640
0.5	0.4578	0.2303	0.0476
0.6	0.2967	0.1500	0.0317
0.7	0.1446	0.0751	0.0176
0.8	0.0872	0.0485	0.0128
0.9	0.2191	0.1122	0.0238

Table 4: The relative errors between u_1 and $u^{\delta_i}_{1,\beta_i}$ with $\delta_1 = 0.1, \delta_2 = 0.05, \delta_3 = 0.01$.

Next, Figure 2 - Figure 5 help us to show the comparisions between the exact solution and its computed approximations corresponding to different noise level $\delta_i, i = \overline{1.3}$. Then, we get Figure 6 which consider the solutions at the fixed point $x = 0.1$

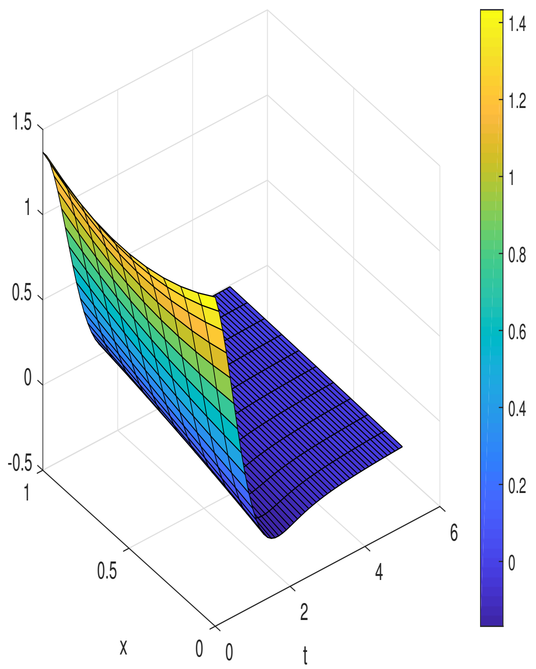

Figure 2: The exact solution u_1.

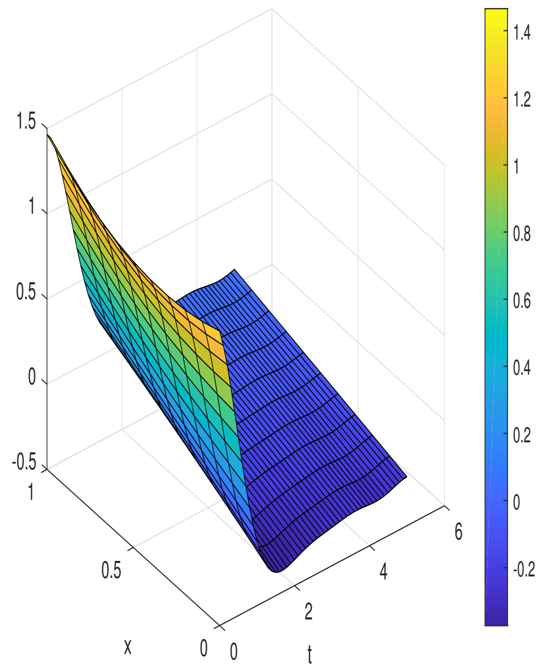

Figure 3: The regularized solution $u^{\delta_1}_{1,\beta_1}$.

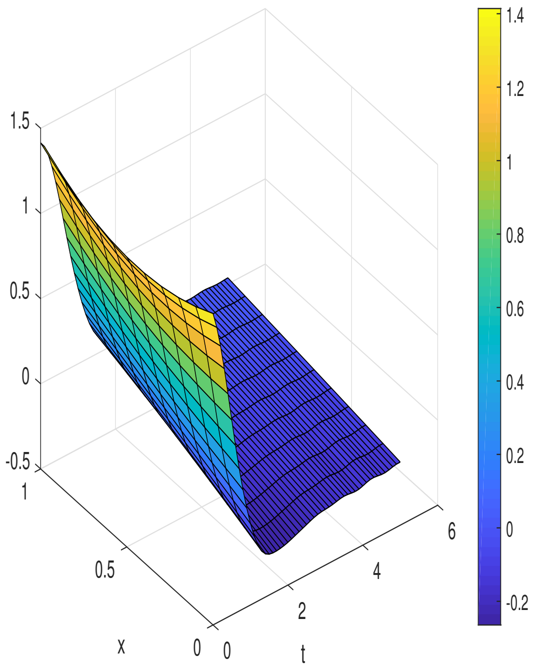

Figure 4: The regularized solution $u_{1,\beta_2}^{\delta_2}$.

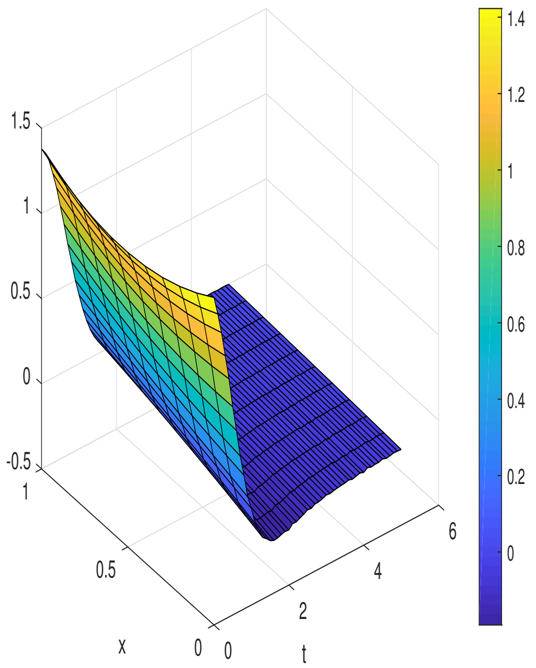

Figure 5: The regularized solution $u_{1,\beta_3}^{\delta_3}$.

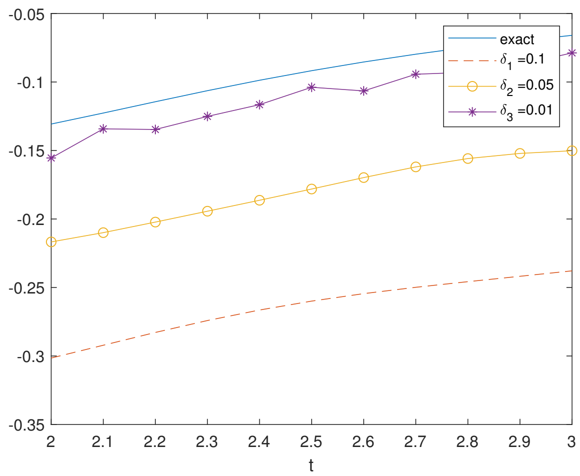

Figure 6: The exact solution u_1 and the regularized solution $u_{1,\beta_i}^{\delta_i}$, respectively in the case $x = 0.1$.

5 Conclusions

Although there are several regularization methods for stabilizing the inverse heat conduction problem in a single-layer body by using an a priori information on the exact solution, the regularization error estimates for the fractional inverse heat conduction problem with the nonhomogenuous source in a multi-layer body are still very rare. This is due to the complexity of the forward operators as shown in (2.23) and (2.24). Therefore, the direct extension of the existing methods for solving the FIHCP in single-layer domain is unavailable. However, the idea for stabilizing the FIHCP in single-layer domain can be used. In this paper, we found that the Fourier truncation is efficient in solving the FIHCP in two layer domain. Furthermore, we obtain the error estimates for our method for solving the FIHCP in two-layer domain. In theoretical aspect, the order of the error estimates is Hölder type. The constructed numerical examples also verify that the proposed regularization method is effective for solving the FIHCP in the two-layer domain.

Acknowledgement

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