Recovering the temperature  
distribution in a multilayer fractional  
inhomogeneous diffusion equation

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Abstract

In this paper, we consider the problem of the recovering the tempera­ture distribution in a multilayer fractional inhomogeneous diffusion equa­tion. We study that such problem is severely ill-posed and further apply a spectral regularization method to solve it based on the solution given by the Fourier method. Convergence estimates are presented under a pri­ori bound assumptions for the exact solution. The convergence analysis is performed in both the L2 and L^ settings. Finally, numerical implemen­tations are given to show the effectiveness of the proposed regularization methods.

1 Introduction

This paper aims at establishing a mathematical framework to recover the surface temperature on the inaccessible boundary in a two-layer domain as follows.

We consider a composite body consisting of two layers, D1 := {x|0 < x < l1} and D2 := {x|l1 < x < l2}. These two layers are in perfect thermal contact at the intersection point x = l1. Let k1,k2 > 0 be the thermal [==nductivities and a1, a2 > 0 the thermal diffusivities of the first and second layers, respectively. The temperature distributions in these layers, namely u^x,t) and u2(x,t), satisfy the following equations:

• on the first layer D1;

d7 U1 (x, t) = a10^u1(x, t) + S(x, t), x E D1, t > 0, U1 (11,t) = u2 (11,t),

(1.1)

(1.2)

(1.3)

*t >* 0*, t >* 0;

K1&XU1 (h,t) = K2*&x* U2 (h ,t) ,

• on the second layer D2,

d7 U2 (x,t) = a2dXu2(x,t), x E D2,t > 0, U2 (h,t)= g(t), t> 0,

dxU2 (I2, t) = f (t), t> 0;

subject to the homogeneous initial conditions

U1(x,0) = U2(x,0) = 0, x E (0,12) ,

The notation 57 in the problem [(1.1)](#bookmark34)-[(1.2)](#bookmark34) denotes the Caputo fractional derivative of order 7 (0 < 7 < 1) defined by

1

ur*(x,t*)

*dr,*

57U := 57u(x, t)

r(1 - Y) 0 (t - r)

and the functions f,g E L2(R) are the final datum.

Until now, many practical applications in many branches of engineering sci­ence are concerning about the problem of the fractional diffusion equation. It is very interesting to us that fractional differential equations can be used to model some anomalous diffusion phenomena in physics, for example, physical systems [4,8], physics and chemistry [10] and they describe anomalous diffu­sion [6,7], the predator-prey dynamic [9], subdiffusion processes, and viscoelastic materials [5]. In some mathematical models, several different definitions of the fractional derivatives including the Caputo, Riemann-Liouville, Caputo-Riesz, Atangana-Baleanu d==]vatives were proposed. All of them are related to each other and are defined as nonlocal operators in contrast to the integer-order deriva­tives that are local operators. Recently, the forward problem of fractional dif­ferential equation with many different fractional derivatives are well investigated in recent years. Beside that, the inverse problems for time or space-fractional diffusion equations, recovering initial data or source function or diffusion coeffi­cient and so on by some additional data from the final mearsurement, also attract much attention from scientists around the world.

The transient temperature distribution in a composite medium consisting of several layers in contact has several applications in engineering. One notable example can be observed in the field of aerospace engineering, in which this work serves as a means to measure both the internal and external temperature of the hulls. It offers a solution to detect the erosion during the process of thermal radiation within a nuclear reactor, given the fact that it is extremely challenging to record the temperature within the core baffle during operation. This is crucial to safety controls. Another example is the design problem of shielded thermocouple

which is a measurement device used for monitoring the temperature in the hostile environments [13]. A fractional diffusion equation model for cancer tumor [12].

Therefore, the objective is to reconstruct the whole structure in the inacces­sible first layer. The case 7 = 1 of the problem [(1.1)](#bookmark34)-[(1.3)](#bookmark34) has been studied in [11], where this problem is shown to be ill-posed in the sense of Hadamard. Due to the severe ill-posedness of the problem, Xiong et al. [11] have presented a spectral regularization method and derived the error estimate of the Holder type. Then the case 0 < 7 < 1 for the homogenous source has also been studied in [1]. Khieu et al. [1] have also proposed a Fourier spectral approach to achieve Holder approximations.

Following the previous works, we then move on to examine whether this prob­lem still holds true for the non-homogenous case, with varied thermal conductiv­ities and thermal diffusivities. To our knowledge, we have not found any works related to the nonhomogeneous case, so the consideration of the problem in this paper has practical significance. In this paper, we assume that k = for sim­plicity in the presentation, then we want to obtain an analytical solution [1] in multi-layer domain via Fourier transform techniques. To overcome the ill- posedness of the problem [(1.1)](#bookmark34)-[(1.3)](#bookmark34), we propose a simple but effective Fourier truncation method and derive convergence estimates of Holder type as the noise level tends to zero, in both L2(R) and Lc(R).

/(t)e \*tdt,

The paper is organized as follows. First, Section 2 shows the ill-posedness of the problem [(1.1)](#bookmark34)-[(1.3)](#bookmark34). In Section 3, the spectral regularization technique is adopted for the reconstruction of stable solution with proven error estimates. With three conditions introduced, three evaluations can be obtained in both L2(R) and Lc(R). Finally, numerical implementations are given to show the effectiveness of the proposed regularization methods in Section 4 and some con­clusions are presented in Section 5.

**2 The ill-posedness of the problem (**[**1.1**](#bookmark34)**)-(**[**1.3**](#bookmark34)**)**

We extend all the functions above to the whole line — ^ < t < by making them zero outside the original domains, if necessary. Let

1

/(£) =

v/2n \_c

denote the Fourier transform of a function / e L2(R). Applying the Fourier transform with respect to t to both sides of [(1.2)](#bookmark34) we obtain the solution in the second layer,

u2(x,0 = g(C)cosh (Vk2(0 (l2 - x)) /==sinh (Vk2(0 (l2 - x)) , (2.1)

k2 (0 V '

for ll < x < l2, where 17 \ 1/2

)

7

ier

*a*

*3*

27 a

(1 + i sign(C))7 = cos (4^ + isin (4^ sign(£) j

for j E {1, 2}. Thanks to the representation [(2.1)](#bookmark35), we can solve the problem in the first layer to obtain

ui(x,O=0(f)0i(x,£) - f(f)02(x,£) + /" S(y,C)sinh^//k^L^——dy

0 ki(C)

where

rh ^ sinh k1(C) (11 —

S (y,f) cosh ki(0(li — x

0

ysvf)

(2.2)

0i(x, £) = cosh I 'C'' (li — x) | cosh l FT (I2 — li)

ai

^2

+ Ah/Oi sinh | ,/hT

«2 Oi

(/i — x) I sinh

O2

(I2 — li)

02(x,f) =cosh I (/1 — x)

sinh((/2 — /i))

Oi

v/k2fe)

(/i — x)

+ *kjai*sinh *LIFT*

O2 Oi

cos^(/2 — /i))

and k = —2. Regularised solution Ki

llg5 — gllL2(R) + llf<$ — f IIl2(R) < ^. (2.3)

Lemma 2.1 For arbitrary z E C

sinh |&(z)| < | sinhz| < cosh!R(z) < e|K(z)|, sinh |&(z)| < | coshz| < cosh !R(z) < e|K(z)|.

Proof. The inequalities follow from the definitions by elementary calculations.

Lemma 2.2 Set

*Then*

t'(x)

/1 — x + I2 — /i

*'Zai*

n

cos

4 = ^(Q).

0i(x,C) 02(x,C)

<

<

1 + k J0 e|?|7/2(x)

O2

1 + fcT e!£P/2(»

O2

for Q < x < /i, for Q < x < /1.

(2.2.1)

(2.2.2)

Proof. Let ax{£) = ^ cos (f 7) {h — x), cix{£) = sin (47) agn(f) - x)

b(f) = Jlfcos (4y) (12 - 1i), b(f) = Jlfsin (47) sign(f) (12 - li), We have

©2(x,f)| < |cosh(ax(f)+ iax(f))sinh(b(f) + ib(f))|

k — sinh (ax(f) + i«x(f)) cosh(b(f) + ib(f))

+

a

By Lemma [(*2.1*)](#bookmark36)

@2(x,f) < |cosh(ax(f) + iax(f))| 1 sinh(b(f) + ib(f))|

+ k — |sinh (ax(£) + i«x(f))| I cosh(b(f) + ib(f))| a2

< 1 + k^ «)+b(« = 1 + k^ el«'7/2^(x).

a a

Similar for [*2.2.1*](#bookmark36), which completes the proof ■

Lemma 2.3 For 0 < x < l1, put

^B~i ln 2 )2/y f ^0) ln 2

2/7'

( ) '',(l1 — x)cos(n7/4)y ’ V(l2 — l1)cos(n7/4)

Then, for |f | > A(x)

1

Ao = A(0).

[ei(x,£) > Y 1 + k a e'£r'/2«x», 16 a](#bookmark38)

(2.3.1)

(2.3.2)

[e2(x,f) > 116 f1 + k./F) el£l’/2«x>. 16 a2](#bookmark39)

Proof. Similar to [ ], by direct computation and Lemma [(*2.1*)](#bookmark36), we have [(*2.3.1*)](#bookmark37) and [(*2.3.2*)](#bookmark37).

Theorem 2.1 The problem [(*1.1*)](#bookmark34)-[(*1.3*)](#bookmark34) is ill-posed in the sense of Hadamard.

Proof. We give an example to demonstrate that the problem [(1.1)](#bookmark34)-[(1.3)](#bookmark34) is ill- posed. For any n e n with n > A(x), where A is the same function as in Lemma [(2.3)](#bookmark37) define\*^1 := {f E R; n < f < n +1}. Let gn E L2(R) be the measured data such that

|  |  |  |
| --- | --- | --- |
| gn(f) = - | g(f) + 1/n | if f E ^n, |
| g(f) | if f E R\Q, |
| fn(f) = < | f(f) +1/n /(f) | if f E ^n,  if f E R\a |

n

n

Let u1 and u1n be solutions of [(1.1)](#bookmark34)-[(1.3)](#bookmark34) exact data f ,gs and /, g satisfy that.

u1(x,0=g(C)0 1(x,C) - /(C)02(x,C) + /\* S(y,C)sinh(^k^/=^——dy

0 k1(C)

rh , sinh x/MO (11- y) f N

S(y,C) cosh k1(C)(11 - x) dy,

and

**( \** 1/2

By Parseval’s identity ||gn - g\\L2(R) = (^ fQn n-2d£j = n-1 ^ 0, and ||/« - f ||

n II

/ 1/2

Q n-2d£ = n-1 ^ 0 as n ^ ro.

0

x/MC)

[U1n(x,C) =gn(C)01(x,C) - fn(C)02(x,C) + S^ C) —dy](#bookmark17)

do Vk1(C)

h, sinh VMO (11- y) f \

- S(y,0 cosh k1(C)(11 - x) dy.

[0 k1(C) v 7](#bookmark18)

By Parseval’s identity and Lemma[(2.3)](#bookmark37),

lim ||(u1n - U1) (x, -)||

L2(R)

: lim —

n——<^ n

0(x,C)

Qn

2 \ 1/2

d£

* lim — (1 + 1 )e|?|P/2^(x)|d£

n—~n Qn a2

* 1 lim 1 (7 fe^p/27(x))2 dCX1/2 8 n—~ n Qn

> lim

1 g *nY/2£(x)*

+ro.

8 n—^ n

This proves the ill-posedness of the problem.

3 The Fourier spectral method

The problem [(1.1)](#bookmark34)-[(1.3)](#bookmark34) is ill-posed. Let E^ = [—^, ^^] denote the regularisation domain, where ^ := ,d(£) is the regularisation parameter that will be chosen later.

(x,C) =

g(C)01(x,C) - /(C)02(x,C) + S(y, Cdy

0 k1(C)

Ime (C)

, sinh gARO (/, - y)

Ie, (C), (3.1)

5(y,C) t-osh k1<C) (11 - x) dy

\/M?)

0

where IEp denotes the characteristic function of the interval Ej

Ies«)={0 if 11 Ej, and Ej = RR| < B},

where Bs := B(0) is the regularisation parameter that will be chosen later Ba := B(0) —— ^ then £ — 0.

Put

Dp(x) = ||u1j(x, ■) - Ul(x, •)^LP(R) ,

the Lp -distance between the exact and regularised solutions. The rest of this section is devoted to estimating the distances D2(x) and D^(x).

Theorem 3.1 *(The* L2 *-distance) Assume that u1 is exact solution of the problem* [*(1.1)*](#bookmark34)*-*[*(1.3)*](#bookmark34) *and u\j is regularized solution. We suppose measurement data and exact data (fs,g*s) *and (f,g) satisfy that. The regularisation parameter B£ is given by*

2/7

E 2p.

ln — ln ln

*op o*

E " 2/7

(3.2)

where E = E + 0 ^ee8p/7 + e22°(A°)7/ j , then, for every x E [0,11), we obtain the

HA^lder convergence estimate

Put B(x) = min{£(x), —x, —(x + p)}. If u1 satisfies the conditions 3.1.1

sup \\ul(x, •)|HP (R) := sup (1 + (2)p |ui(x,^)|2 d( < Ei p> 0

xe(o,e{) £e(o,p) r

(3.3)

Put q = min j Yx), 1 j , then

D2(x) < ln —

**EX** 2**pq r**

1 1 + k —) EqS1-1^ + (2€o)^ Ei 8 02

*If* u1 *satisfies the conditions 3.1.2*

e2xgp/2

1/2

sup

c€(0/i) R\E^

e2x«'"-|U1(X,0|2d5) <E2 ump> 1,

(3.4)

Then

—2p l(%)

D2(x) <1 8 (1 + M) + E2°-1) ^ (ln IV V^

If U1 satisfies the conditions 3.1.3

1/2

e2(x+P)\tp/2 |u1(x,^)|2 d( < E3 with p> -, (3.5)

sup

c€(0gi) R\E^

Si(x)

where u^ is the regularised solution [3.1](#bookmark16) with respect to the exact data (f, g). Let us first evaluate Sr(x). By Lemma [2.2](#bookmark36), [(2.3)](#bookmark36) and [(3.2)](#bookmark41)

S2(x)

0 1 (x,£) (g(0 - g(C)) + 02(x

u *1/3 - u1/3*

Then

-2p l(x)

1 or e 7' i° i «(x)

D2(x) < - 1+ — + E38-1 E i° ln — 81-W

8^’ \ja2J 3 ) ^ 8

Proof. From Parseval’s identity and the triangle inequality

D2 (X) < || (U13 - Ujg) (x, •)^L2(R) + ||(U13 - •)|

L2(R),

(3.6)

Sj^x)

2

01(x,C) (g(£) - g(C)) + 02(x

2

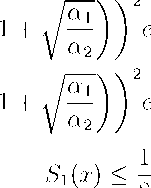
(C)dC

°1(x,0 |g (C) - g(£)|2 d£ + 2 / °2(x,C) f (0 - f(0

s

s

16 ‘(-6 (



*x*

2r(x) 2



- g\?L2{ R) + || fS - f ||

L2(R)

(3.7)

We are now in a position to estimate S2(x). By Lemma [2.3](#bookmark37)

S2(x) = || (u13 - ■)|L2(R)

1/2

|u13(x,e) - u1(x,e)|2 de

' R\Es

1/2

|u1(x,e)|2 de

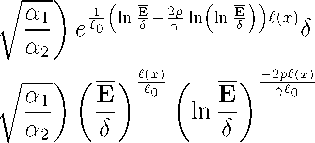
From [(3.2)](#bookmark41) we deduce

' R\Es

S1(x) < - 1 + k,/^ e(3-)7/2(x)8

8 02

8



8

—2pl(x)

1 a1 i(xl E 7i° 1 i(x)

< - 1 + k — E i° ln - 81-~i°

8 02 8

If u1 satisfies the conditions [(3.3)](#bookmark41)

1/2

S2 (x)

< (Ps)~P El.

2 *-p*

*2 p*

|ui(x,C)|2 dC

Since E > 5ee8p/7, it remains to prove that

\_ E 4p E ln ~ > — ln ln — 0 y 0

E \\2/Y

thatis > (2o (ln -

Put a = 8p/y,y = ln E/0 and h(y) = y — (a/2) ln(y). since a > 0, we claim that h(y) > 0 for y e (ea, +ro). This follows since h'(y) = 1 — a/2y > 1 — a/2ea > 0 for all y e (ea, +ro). Then h(y) > h (ea) = ea — 2a2 > 0. Thus, the estimate holds. Having disposed of this preliminary step, we can estimate S2 by

1

52<x> <1 i (ln -

*la 1*

i(x)

D2(x) < 1 + k 1 E ln

8 a2 0

— 2 **pC(x)**

E y^q

-2p

Y

Ei.

i(x)

***2p***

01 £q + (24) Y ( ln —

***—2p***

E Y

Ei.

Put q = min{g, i|

E x 2pq

I

1 1 + k — Eq01-^ + (24)2p El

8 a2

D2(x) <^( 1 + k —1 Eq lnE + (24) Y ln E

<lln -

a2 x -2pq r

-2 pq

E1

If u1 satisfies the conditions [(3.4)](#bookmark41)

1/2

S2 (x)

f R\E^

|u1(x,C)|2 dC

11/2

' R\E^

< e-x(T)Y/2e2 ,

e-2x|t|2e2x\^\2 |U1(x,£)|2 d£

where ps = (i)/7 (ln?— fln (lnif))

2/7

—xT (ln d — — ln(ln h4

E2

S2(x) < e £q s y s

—x

a E)'0 (-

***2p***

E £0 Y

7

E2

2p

\_ , E £0 Y jx.

< (E) £q ( ln E20 £q .

|ui(x,£)|2 d£

S2 (x) =

1

*I a* 1

1(x)

D2(x) < 1 + k - E £° ln

8 *a2 o*

Put £(x) = min^x), -x, - (x + p)}

—2p1(x)

E Y^°

**1(x) —x**

O- £° + E) £° ( ln —

E2O £°.

1

1(x)

D2(x) < 1 + k - E £° ln

8 *a*2 *o*

< (80+

If u1 satisfies the conditions [(3.5)](#bookmark41)

la-



|  |  |  |  |
| --- | --- | --- | --- |
| —2p 1(x) | ) |  |  |
| . Y 1° | 1 1(x)  O1— | i(x) | / ~ |
| ) | + E 1° | ln |
|  |  | — 2p 1(x) | \ |
| 1(x) | ' - \ | 0  On  1 | 1(x) |
| E 1° | ln -t | ' 1° |
|  | o |  |

-i

— 2**p** 1(x)

1/2

O

E2O i

i(x)

°

' R\E^

< e-(x+p)(^£)7/2e3,

e-2(x+p)|?|7/2e2(x+p)|?|7/2 |u-(x,f)|2 d£

1/2

where £e = (i)h (lnf - fln (lnf))

2/7

— (x+p) f ln E — — ln ln E

S2(x) < e £° S Y S E3

**—**(**x**+**p**) (**x**+**p**) 2p

E ^ E £° 7

* 1 ln 1 E

(**x**+**p**) **2p**

— (x+p) E £° Y (x+p)

* (E) £° ln E3O £° .

1

*I a-*

l(x)

D2(x) < 1 + k - E £° ln

8 a2 o

Put i(x) = min{P(x), —x, — (x + p)}

—2p1(x)

E y+

r-1 —

**1(x)** **—(x+p)**

°

+E)

°

ln o

(x+p) 2p

E £° Y \_ (x+p)

E.,O~

—2p 1(x) —2p 1(x)

1 a- dxt E Y £° 1 Ux) E Y £° i(x)

D2(x) < - 1 + k -1 E 1° ln - O1—+ E 1° ln - E3O^T

—2p 1(x)

a- ux — i(x)

<( 8^1 + k a- + E3O— 1 E 1° ln - O-—^

which is the desired conclusion. The theorem is proved. ■

Theorem 3.2 (The L^ -distance) Let u1 be the solution of problem [(*1.1*)](#bookmark34) and

-g be as in Theorem A Let the measured data g$ Put I(x) = min{P(x), —x, — (x + p)}

**u**

1 1 Y — 2

P(x) = P(x) + Y — 1 and E = E + O (ee 4(2p 1)/Y + e24(A(0))Y/2) . Y 2

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