Auctioning the Digital Dividend: A Model for Spectrum Auctions

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Abstract

We model a spectrum auction as a multi-unit auction where participants use the goods purchased to participate in a constrained, multi-product downstream market. We use dynamic programming techniques to numerically solve for the optimal bidding strategy for firms in a clock auction. Firms often value constraining competitor market power highly, and incumbents will often bid aggressively to shut out entrants. We find that inefficient firms may hold up the market when competing with efficient firms, in which case the auction becomes inefficient and results in zero revenue. Meanwhile, if information asymmetries exist, a risk-averse Regulator concerned with both auction revenue and total surplus in the downstream market may impose a loose cap on a potentially inefficient firm and a tighter cap on an efficient firm in order to hedge his/her dual objectives.

Keywords: Clock Auction, Spectrum Auction, Telecommunications Market, Equilibrium Bidding, Capacity Constraints

JEL Classification Numbers: C61, C73, D44, L13, L96

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1 Introduction

In most modern economies, the "Digital Dividend" represents a substantial technological and financial windfall. Developments in television broadcasting allow the transmission of digital video and sound that require only one-sixth the bandwidth of inferior quality analog transmissions. Therefore, replacing analog transmissions with digital transmissions frees up a substantial quantity of spectrum frequencies, which can be easily employed to transmit wireless data. This data is used by mobile phones, laptop computers, and other devices.

These unallocated new spectra offer opportunities for consumers, producers and government. First, wireless telecommunications companies can expand their services. Firms that have access to the new spectra can effectively provide superior data services relative to firms who do not. Second, consumers can enjoy a wider range of services, potentially delivered in a more timely manner. Third, the government benefits from the unallocated spectra in two ways. On the one hand, the government, as an auctioneer, can earn revenue from the auction. On the other hand, the government, as a regulator, can affect the degree of competition in the telecommunications market in order to increase total surplus. A common policy tool adopted by governments to achieve social optimum is to conduct an auction with caps on firms to limit their winnings. In this way, the regulator can prevent one firm from winning all units and becoming a monopolist.

Spectrum auctions are potentially an efficient way to allocate the new spectra across firms and do not represent new phenomena in most countries.[1] A spectrum auction is an example of a *multi-unit auction*. The auctioneer is selling a collection of relatively homogeneous goods to multiple firms.[2] If the number of firms in the auction is small, then this can result

[1] The United States began its auctions for spectrum licenses during the 1990s, and has since assessed their efficiency. For example, see Cramton (1997), Cramton (1998), Kwerel and Rosston (2000), and Bush (2010). Meanwhile, the GSM (second generation mobile telecommunication) and UMTS (third generation) auctions in Europe from 1999 to 2001 attracted a lot of attention from the public for their interesting outcomes. For example, see van Damme (2002), Klemperer (2005), and Grimm, Riedel, and Wolfstetter (2003). Economists have also been surprised by the huge revenues the British government realized from the sale of its 3G telecom licenses (Binmore and Klemperer (2002)).

[2] In reality, spectrum frequencies are not truly homogeneous because technology-dependent synergies to having access to adjacent frequencies exist. In addition, regional standards, technological limitations, and

in participants winning multiple units and obtaining market power. Our aim in this paper is to examine the equilibrium properties of a Digital Dividend auction where participants compete for market power in the telecommunications market. Our model consists of a downstream market, in which firms play a Cournot game with capacity constraints, similar to Laye and Laye (2008). Specifically, firms produce two goods: low and high data use plans, and are constrained in their ability to produce data plans by the amount of spectra they have available. We assume that new (Digital Dividend) spectra are required to produce high data use plans. Having projected their potential profits in the downstream market, firms enter into a simultaneous uniform-price clock auction to increase their production capacity. It is worth noting that although we consider the spectrum auction as the motivating example, our model can be generalized to investigate other problems where there is interaction between a downstream market and an upstream auction/competition.[3]

We solve the auction problem numerically using dynamic programming techniques that allow firms to bid strategically. We find many instances of firms following mixed strategies in their bidding. In these cases, our model generates *distributions* of allocations, profits, levels of social welfare, and revenue for the auctioneer. We consider several scenarios for market structure and investigate the equilibrium outcomes using these measures. We find that when there is only one inefficient firm in the market, efficient firms can successfully obtain market power by winning enough units. However, two inefficient firms can hold up the market by threatening to raise prices (which will be harmful to efficient firms who seek to win larger number of units). Furthermore, our computational procedure allows us to explore the way in which an auctioneer might impose caps on how many spectrum units individual participants can win. In particular, we assume that the auctioneer cannot observe the type of firms due to informational asymmetries. Our findings show that (counter-intuitively) a risk-averse Regulator concerned with both revenue and total surplus may impose looser caps on a firm with

device manufacturers' decisions may make certain segments of a given band more or less desirable.

[3]Examples include procurement auctions (to obtain market access) for medical drugs in third-world countries and competition by airlines for landing slots; see Eső, Nocke, and White (2010).

high probability of being inefficient than on firms more likely to be efficient. Conversely, concerns about hold-up behaviour may encourage the regulator to constrain multiple inefficient firms in a market. Our work contributes to the literature in several ways. First, at multi-unit auctions, bidders are often assumed to have non-increasing marginal valuation (NIMV) for the goods in question.[4] This is a classic assumption in economic theory to make the models tractable. Unfortunately, this assumption is not necessarily valid in our case because marginal value of a spectrum unit may increase with a bidder's market power. In our model, units sold in the auction affect the production capacity. Therefore, winning more units not only increases a firm's production capacity, but also limits its competitors' capacities, which gives it market power. In such a case, the marginal value of an additional spectrum unit *may increase* as a firm gets closer to being a monopoly. This effect of market power on marginal valuations is also analyzed by Eső, Nocke, and White (2010). Assuming complete information, they model a downstream industry where firms compete to buy capacity in an upstream Vickrey-Clarke-Groves auction. Our approach is similar to theirs, but we allow firms to compete in a multi-product industry and adopt the simultaneous clock auction as the mechanism for spectrum allocation. Although the clock auction is more applicable in practice, it can generate multiple outcomes. Therefore, we investigate the distribution of these outcomes in our paper.

Second, potential market power in the downstream market may be of concern to a benevolent government who also happens to be the auctioneer. Similar to the literature on license auctions, a relationship exists between auction outcomes and market prices.[5] Offerman and Potter (2000) illustrate this relationship between auctioning of entry licenses and market prices using experiments and conclude that bidders' valuations depend not only on their licensed units but also on whether the remaining units are obtained by someone else and who it will be. Similar to their work, we incorporate the downstream market into our

[4] For example, see Kastl (2011); Hortaçsu and McAdams (2010); Mishra and Parkes (2009); Blume, Heidhues, Lafky, and Münster (2009); Riedel and Wolfstetter (2006); Katzman (1999).

[5] For example, see Janssen and Karamychev (2007); Hoppe, Jehiel, and Moldovanu (2006); Janssen (2006); Jehiel and Moldovanu (2003).

auction model. However, it is important to note that while licences may be complements (due to their geographical position), spctrum units (in our case) are substitutes. Furthermore, firms in our model compete in the auction not only to gain access to the market, but also to increase their production capacity.

Finally, we model the government as both an auctioneer and a regulator, and examine how the government's objective function affects the optimal cap structure in the auction. A trade-off may exist between the total surplus and the auction revenue. In fact, even if an auction is efficient, it may result in an inefficient downstream market, where one firm wins most of the units. In this case, the government may act as a regulator and aim to maximize some combination of total surplus in the downstream market as well as the auction revenue. Dana and Spier (1994) also focus on a risk-neutral government's problem to choose who produces in a downstream market. We consider a more general objective function, and allow the government to be risk-averse.[6] We compute the optimal caps by using the means and covariance matrix of the total surplus and revenue, for each possible level of caps.

The layout of the remainder of this paper is as follows. Section 2 outlines our model for the downstream market and the auction itself. This section also discusses our solution method. Section 3 presents our results for uncapped auctions, whereas Section 4 explores the optimal capping decision for a regulator organizing a spectrum auction. Lastly, Section 5 concludes.

2 Model

Downstream Market

We begin by describing the telecommunications market, which we will refer to as the downstream market throughout this paper. Suppose M firms operate in the market, which

[6] This multiple objective situation is not peculiar to the government. An airport, for example, may tradeoff revenues from gate auctions against passenger throughput that will affect retail floorspace rentals.

is composed of two products: high and low data use plans (henceforth, products). Let q_{hi} and q_{li} denote firm i's output for the high and low products. Cost of production may differ across firms as well as products. Thus, let $C_{hi}(q)$ and $C_{li}(q)$ represent firm i's total cost of producing q units of the respective products.

Inverse demand for the two products is described as follows:

$$\text{Low Product: } P_l = P_l\left(\mathbf{q_l}; \boldsymbol{\beta}_l\right) \tag{1}$$

$$\text{where } \mathbf{q_l} = (q_{l1}, q_{l2}, \ldots, q_{lM})$$

$$\text{High Product: } P_h = P_h\left(\mathbf{q_h}; \boldsymbol{\beta}_h\right) \tag{2}$$

$$\text{where } \mathbf{q_h} = (q_{h1}, q_{h2}, \ldots, q_{hM}).$$

In equations (1) and (2), $\mathbf{q_l}$ and $\mathbf{q_h}$ are the vector of outputs for each product by all firms, whereas $\boldsymbol{\beta}_l$ and $\boldsymbol{\beta}_h$ are the corresponding demand parameters. Note that if all firms' outputs for a particular product are perfect substitutes, then one can simplify the inverse demand functions so that price depends only on the aggregate production.[7]

We assume that firms are endowed with an initial allocation of legacy spectra, which we refer to as old spectra, and denote as $\mathbf{B_l}$. Meanwhile, depending on the auction outcome, firms may win new spectrum units ($\mathbf{B_a}$):

$$\text{Old Spectra: } \mathbf{B_l} = (B_{l1}, B_{l2}, \ldots, B_{lM}) \tag{3}$$

$$\text{New Spectra: } \mathbf{B_a} = (B_{a1}, B_{a2}, \ldots, B_{aM}). \tag{4}$$

[7]Alternatively, one could argue that the product prices may depend on both vectors of outputs, as the two products may be complements or substitutes. This relationship is easily incorporated into the demand functions.

Using the old and new spectra, firm i faces the following capacity constraints:

$$q_{hi} \leq \theta_{ai}\ B_{ai} \tag{5a}$$

$$q_{hi} + q_{li} \leq \theta_{ai}\ B_{ai} + \theta_{li}\ B_{li} \tag{5b}$$

$$q_{hi} \geq 0 \tag{5c}$$

$$q_{li} \geq 0 \tag{5d}$$

where firm i can use new spectra to increase production of the high product, the low product, or both. In contrast, legacy spectra only allow the firm to produce the low product. In equations (5), θ_{ai} and θ_{li} represent the marginal increase in capacity from old and new spectra, respectively. Finally, the last two inequalities ensure non-negative production for the two products.

We assume that firms' initial endowments are already paid for or can be regarded as sunk costs. Given the spectrum allocations $\{\mathbf{B_l}, \mathbf{B_a}\}$, one can write down firm i's profit in the downstream market as follows:

$$\begin{aligned}
\Pi_i^D(q_{hi}, q_{li}|\mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, B_{ai}, \mathbf{B_{a,-i}}, \mathbf{B_l}) =& P_h\left(\mathbf{q_h}; \boldsymbol{\beta}_h\right)\ q_{hi} - C_{hi}(q_{hi}) && \text{(6a)}\\
&+ P_l\left(\mathbf{q_l}; \boldsymbol{\beta}_l\right)\ q_{li} - C_{li}(q_{li})\\
\Pi_i(q_{hi}, q_{li}|\mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, B_{ai}, \mathbf{B_{a,-i}}, \mathbf{B_l}, P_a) =& \Pi_i^D(q_{hi}, q_{li}|\mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, B_{ai}, \mathbf{B_{a,-i}}, \mathbf{B_l}) && \text{(6b)}\\
&- C_a(B_{ai}, P_a)
\end{aligned}$$

where $\{\mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}\}$ represents the vector of other firms' supply, and $\mathbf{B_{a,-i}}$ denotes the new spectra allocated to other firms. Equation (6a) is the firm's profit in the downstream market, whereas the firm's profit net of the cost of new spectra is given in equation (6b). The right-hand side in equation (6a) is the sum of the profits from the high and low products, and the last term in equation (6b) is the cost of new spectra, which depends on the auction price P_a.

The assumption below describes the information structure in the downstream market.

Assumption. *There is complete information in the downstream market. All the information regarding the cost of production, the spectrum allocations, and the outputs by each firm, as well as the demand parameters, are known by all firms.*

Assuming complete information, firms can first solve for their profits in the downstream market, given the spectrum allocations. Then, they can enter the auction for the new spectra using information on marginal valuations to formulate bidding strategies.

Profit Maximization in Downstream Market

Given the spectrum allocations, we assume that the firms compete in a Cournot game by deciding how much to produce for both high and low products. Because firm outputs for the high and low products are limited by the capacity constraints described in equations (5), the distribution of the new and old spectra across firms can result in market power. Before we write down the profit maximization problem, we first describe the feasible set for firm i:

$$\mathcal{A}_i = \{(q_{hi}, q_{li}) \ : \ (q_{hi}, q_{li}) \text{ satisfy capacity constraints in (5) given } \mathbf{B_a}\} \tag{7}$$

Using the feasible set defined above, firm i's profit maximization problem is as follows:

$$\max_{(q_{hi}, q_{li}) \in \mathcal{A}_i} \ \Pi_i^D(q_{hi}, q_{li} | \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, B_{ai}, \mathbf{B_{a,-i}}, \mathbf{B_l}) \tag{8}$$

where firm i chooses the outputs (q_{hi}, q_{li}) from the feasible set $\mathcal{A}_i$, conditional on the spectrum allocations. Therefore, we are interested in firms' production decisions assuming $\mathbf{B_a}$ is the result of the auction. In this way, we can derive the payoffs for firms, which we then use to solve for the auction equilibrium. We first define the equilibrium in the downstream market.

Definition 2.1. *A Nash Equilibrium in pure strategies in the Cournot game for the down-*

stream market is such that given $(\mathbf{B_l}, \mathbf{B_a})$*:*

$$\{q^*_{hi}, q^*_{li}\}_{i=1}^{M} \;\; \textit{solves } (8); \forall\ i = 1, \ldots, M.$$

In other words, given the equilibrium behavior of all other firms $\left(\mathbf{q^*_{h,-i}}, \mathbf{q^*_{l,-i}}\right)$, firm i does not find it profitable to change its production decision. Furthermore, if we assume that demand and cost functions are differentiable, then we can solve for the Nash equilibrium using the following complementarity problem from (8):

$$\frac{\partial \Pi_i^D(q_{hi}, q_{li}, \cdot)}{\partial q_{hi}} + \eta_i - (\lambda_i + \mu_i) \geq 0 \perp q_{hi} \geq 0; \forall i = 1, \ldots, M \tag{9a}$$

$$\frac{\partial \Pi_i^D(q_{hi}, q_{li}, \cdot)}{\partial q_{li}} + \xi_i - \lambda_i \geq 0 \perp q_{li} \geq 0; \forall i = 1, \ldots, M \tag{9b}$$

$$\theta_{ai}\ B_{ai} - q_{hi} \geq 0 \perp \mu_i \geq 0; \forall i = 1, \ldots, M \tag{9c}$$

$$\theta_{ai}\ B_{ai} + \theta_{li}\ B_{li} - (q_{hi} + q_{li}) \geq 0 \perp \lambda_i \geq 0; \forall i = 1, \ldots, M \tag{9d}$$

$$q_{hi} \geq 0 \perp \eta_i \geq 0; \forall i = 1, \ldots, M \tag{9e}$$

$$q_{li} \geq 0 \perp \xi_i \geq 0; \forall i = 1, \ldots, M \tag{9f}$$

where $\{\mu_i, \lambda_i, \eta_i, \xi_i\}$ are the Lagrange multipliers on the capacity constraints given in equations (5). The complementarity problem (9) is a square problem with $6 \times M$ equations and $6 \times M$ unknowns $\{q_{hi}, q_{li}, \mu_i, \lambda_i, \eta_i, \xi_i\}_{i=1}^{M}$.

Remark. *The equilibrium for the capacity-constraint Cournot model in the downstream market is unique if the demands for both high and low products are linear and the cost functions are positive convex.*

As shown by Monderer and Shapley (1996), the equilibrium in these potential games may not be unique. However, Laye and Laye (2008) demonstrate the uniqueness of the Cournot-Nash equilibrium in a multi-product market with linear demands, positive convex production costs and closed convex sets of capacity constraints. For the remainder of the paper, we assume a unique equilibrium in the downstream market.

Clock Auction for the New Spectra

We adopt a clock auction in this paper. Suppose that there are N items to be sold. The auctioneer's goal is to find a price such that there is no excess demand for the goods. We assume that the goods are homogeneous so firms' bids consist of a quantity of spectra at the current auction price, rather than binary bids for individual spectrum units. Let P_a represent the current price in the auction. A clock auction works in the following way:

1. The auctioneer starts P_a at zero.

2. At the current price P_a, firms submit new bids, which cannot exceed their existing bids:

$$\mathbf{B_a}(P_a) = (B_{a1}(P_a), \ldots, B_{aM}(P_a))$$
$$B_{ai}(P) \leq B_{ai}(P'); \text{ for } P \geq P', \forall i = 1, 2, \ldots, M.$$

   Firms may respond to each others' bids at the ongoing price.

3. If there is excess demand at the current price, then the auctioneer increases the price:

$$\sum_{i=1}^{M} B_{ai}(P_a) > N \ \Rightarrow P_a \text{ increases to } P_a + \Delta P$$

4. The previous two steps are repeated until the auctioneer increases the price to P_a^* such that:

$$P_a^* = \inf \left\{ P_a \ : \ \sum_{i=1}^{M} B_{ai}(P_a) \leq N \right\}.$$

When the auction ends, the equilibrium vector of spectrum allocation is $\mathbf{B}_\mathbf{a}^*$. Firm i pays the auction price P_a^* for each unit it wins. Thus, the total cost of spectra to firm i is $P_a^* \ B_{ai}^*$.

The total revenue collected by the auctioneer equals $P_a^* \sum_{i=1}^{M} B_{ai}^*$, which may be less than $P_a^* N$, if the auction ends with an excess supply.

Solving the Auction

We now demonstrate how to solve for the equilibrium in both markets, incorporating the rival nature of spectra and the fact that firms cannot increase their bids, which makes the dynamics of the auction important. We assume that the firms can only submit their bids in discrete amounts. We list the rules for the firms' bidding *within a round* as follows:

1. In each round, firms submit their bids in discrete units:

$$\mathbf{B_a} = (B_{a1}, \ldots, B_{aM}) \in \{0, 1, 2, 3, \ldots, N\}^M$$

   where M denotes the number of firms.

2. At the end of each round of activity, if demand for spectrum units exceeds the supply, the price of the units is increased by an increment ΔP.

3. Within each round, firms must decide whether to lower their reported demand or not. If multiple firms wish to lower their demand simultaneously, only one firm will succeed, and firms are considered equally likely to succeed. We also allow sequential dropping: if a firm drops demand by one unit, then the firm has the option to drop its demand further before other firms react. Note that once the firm stops dropping its demand, the other firms can react without experiencing an increase in the price.

4. A round of bidding only ends when all firms have chosen not to act.

Given our solution to the downstream market and profit structure given in equations (6), we can calculate the payoffs that any firm receives at the end of the auction. Suppose that the vector $\mathbf{B_a}$ is the firms' winnings at the end of the auction. Then, firm i's profit net of spectrum costs equals:

$$\begin{aligned}\overline{V}_i(\mathbf{B_a}, P_a) &= \Pi_i(q^*_{hi}, q^*_{li}, B_{ai}, \mathbf{q^*_{h,-i}}, \mathbf{q^*_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}, P_a) \\ &= \Pi^D_i(q^*_{hi}, q^*_{li}, B_{ai}, \mathbf{q^*_{h,-i}}, \mathbf{q^*_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) - P_a\, B_{ai}\end{aligned} \tag{10}$$

where the first term of equation (10) is the downstream profits, whereas the second term is the cost of the spectrum units to firm i. We will refer to the value function in equation (10) as the *terminal value* for firm i.

With this structure in mind, we solve the auction as a dynamic programming problem using backward induction. We begin by considering a high level of price, and presume that at this price, all firms would wish to exit the auction. The market will thus clear at this stage. This results in all firms purchasing zero new spectrum units (and earning payoffs due to producing using their existing spectrum holdings).

Equilibrium in the auction is described by two sets of numbers. The first is the equilibrium probability that each firm drops its demand, which we collectively denote by $\boldsymbol{\pi}^*(\mathbf{B_a}, P_a, \phi)$, given a particular combination of competitors' demands, its own demand, and the current price. The parameter ϕ indicates which firm, if any, has just lowered demand. The second is the expected payoff for a firm as a function of the current set of firms' bids, the probability of dropping demand, and the current price:

$$\{\pi^*_i(\mathbf{B_a}, P_a, \phi), V^*_i(\mathbf{B_a}, P_a, \phi)\}^M_{i=1}\,.$$

To calculate V^*_i and π^*_i, we proceed by backward induction, working back from the terminal (high) price of the auction. For each price level, we work through the different combinations of $\mathbf{B_a}$ sequentially, beginning with cases where $\sum_{i=1}^M B_{ai} = 0$, before proceeding to cases where $\sum_{i=1}^M B_{ai} = 1$, then $\sum_{i=1}^M B_{ai} = 2$, etc. First, we define the *continuation value* $(\widehat{V}_i(B_a, P_a))$, which is attained if no firm reduces demand at the current price:

$$\widehat{V}_i(\mathbf{B_a}, P_a) = \begin{cases} V_i^*(\mathbf{B_a}, P_a + \Delta P, 0) & \text{if } \sum_{i=1}^M B_{ai} > N \\ \overline{V}_i(\mathbf{B_a}, P_a) & \text{if } \sum_{i=1}^M B_{ai} \leq N. \end{cases} \tag{11}$$

In equation (11), the continuation value equals the terminal value (see equation (10)), if there is no excess demand at the current price (in which case the auction ends). Otherwise, the auction continues with a higher price. Before we define the value function for firm i during the auction, it is important to introduce one more notation: the change in firm i's expected payoff due to firm j dropping its demand by one unit $\Delta V_{ij}(\mathbf{B_a}, P_a)$ is:

$$\Delta V_{ij}(B_{aj}, \mathbf{B_{a,-j}}, P_a) = V_i^*(B_{aj} - 1, \mathbf{B_{a,-j}}, P_a, j) - \widehat{V}_i(\mathbf{B_a}, P_a) \tag{12}$$

where $\Delta V_{ij}(\cdot)$ will only be well-defined for cases where $B_{aj} \geq 1$. Note that any reduction in demand results in the auction moving to a new level of demand, and that the price remains at the same level, which allows further reductions to take place.

Suppose that each firm chooses to drop its demand with probability π_i, with the vector of probabilities given by $\boldsymbol{\pi}$. Then firm i's expected payoff is given by:

$$\begin{aligned} V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, 0) &= \widehat{V}_i(\mathbf{B_a}, P_a) \\ &+ \sum_{\substack{\delta_1,\ldots,\delta_M \in \{0,1\}^M \\ \delta_1+\ldots+\delta_M \neq 0 \\ \delta_1 \leq B_{a1},\ldots,\delta_M \leq B_{aM}}} \left\{ \left[\prod_{k=1}^M \pi_k^{\delta_k} (1-\pi_k)^{1-\delta_k} \right] \left[\frac{\sum_{j=1}^M \Delta V_{ij}(\mathbf{B_a}, P_a)\ \delta_j}{\sum_{j'=1}^M \delta_{j'}} \right] \right\} \end{aligned} \tag{13}$$

where δ_i is a binary variable that represents whether firm i drops its demand or not. The case $\phi = 0$ implies that no firm has just lowered demand, so each firm that aims to drop demand is equally likely to do so successfully. In equation (13), the first term is the case where no firm drops demand, whereas the second term (starting with the summation) is the case where at least one firm drops demand. In particular, the first component in the summation is the probability of seeing a given combination of firms *trying* to drop their demand, while the final fraction weights the change in firm i's expected payoff from each demand reduction by the probability that each succeeds, conditional upon trying.

If firm m has just dropped its demand by 1 (and therefore can drop further units before its competitors can react), then firm i's current valuation is:

$$\begin{aligned} V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, m) =& \widehat{V}_i(\mathbf{B_a}, P_a) + \pi_m \ \Delta V_{im}(\mathbf{B_a}, P_a) \\ &+ (1-\pi_m) \ \left(V_i^*(\mathbf{B_a}, P_a, 0) - \widehat{V}_i(\mathbf{B_a}, P_a)\right); \forall m = 1, \ldots, M \end{aligned} \tag{14}$$

Given the current valuation, if firm i follows a mixed strategy, then it is indifferent between lowering its own demand and maintaining it at the current level:

$$V_i(\mathbf{B_a}, P_a, \pi_i = 1, \boldsymbol{\pi}_{-i}, \phi) = V_i(\mathbf{B_a}, P_a, \pi_i = 0, \boldsymbol{\pi}_{-i}, \phi). \tag{15}$$

Next, we define the mixed-strategy Nash equilibrium using the system of equations (15).

Definition 2.2. *Nash Equilibrium in mixed strategies for the downstream market and the auction with discrete bidding is such that given* $(\mathbf{B_a}, P_a, \phi)$*, for each firm* i*, there exists a probability* $\pi_i^*(\mathbf{B_a}, P_a, \phi)$ *that maximizes firm* i*'s expected payoff:*

$$V_i(\mathbf{B_a}, P_a, \pi_i^*(\mathbf{B_a}, P_a, \phi), \boldsymbol{\pi}^*_{-i}(\mathbf{B_a}, P_a, \phi), \phi) \tag{16}$$

$$\geq V_i(\mathbf{B_a}, P_a, \rho_i, \boldsymbol{\pi}^*_{-i}(\mathbf{B_a}, P_a, \phi), \phi); \forall \rho_i \in [0,1]; \forall i = 1, \ldots, M. \tag{17}$$

The value function for each firm is given by

$$V_i^*(\mathbf{B_a}, P_a, \phi) = V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}^*(\mathbf{B_a}, P_a, \phi), \phi).$$

Solving for $\boldsymbol{\pi}^*(\mathbf{B_a}, P_a, \phi)$ implies solving a system of equations, each of which is represented by (16) with the inequality binding, for each of the firms who is following a mixed strategy.[8] We explain our solution procedure in Appendix A.

3 Results

As a numerical example, we focus on a scenario whose parameters are given in Table 1. Some comments regarding this choice of numbers are in order. We have chosen to consider an auction with $M = 3$ firms over $N = 9$ units. Three firms are sufficient to avoid the singular feature of the two firm case: that spectra not won by a firm are necessarily either won by its one competitor or unused. With three firms, a firm who does not buy a particular spectrum unit may be uncertain as to which of its competitors will win the unit. The nine-unit auction is also interesting, because it avoids possible problems where the number of units is not divisible by the number of firms.

With three symmetric firms, it is possible for each firm to win three units of spectrum. With three firms and 9 units, there are $1,000$ bid combinations at a given price. We assume \$0.1 price increments from \$0 to \$7; this generates $70,000$ combinations and $280,000$ finite state games if we condition on which firm dropped its bid last.[9]

With our set of base case parameters, we consider nine scenarios that provide some insight into the workings of a spectrum auction. Each scenario is a slight variation of the benchmark model, which assumes parameter values given in Table 1. These scenarios are summarized in Table 2. Scenarios 0–3 focus on the effect of production inefficiencies among a triopoly market. Scenario 4 (5) focuses on a duopoly that faces entry by a single firm; this entrant is an (in)efficient producer in the high product market. Finally, Scenarios 6–8 investigate the

[8] We solve this system of equations using Newton's method. It is helpful to note that

$$\frac{\partial V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, 0)}{\partial \pi_n} = \sum_{\substack{\delta_1,\ldots,\delta_M \in \{0,1\}^M \\ \delta_1+\ldots+\delta_M \neq 0 \\ \delta_1 \leq B_{a1},\ldots,\delta_M \leq B_{aM}}} (2\delta_n - 1) \left[\prod_{k \neq n} \pi_k^{\delta_k} (1-\pi_k)^{1-\delta_k} \right] \left[\frac{\sum_{j=1}^{M} \Delta V_{ij}(\mathbf{B_a}, P_a)\ \delta_j}{\sum_{j'=1}^{M} \delta'_j} \right].$$

[9] We choose an upper bound of \$7, since this is sufficiently high that even if its competitors were bidding 0 units, a firm would not buy a single spectrum unit.

effect of two potential entrants (efficient or inefficient) into a monopoly market.

Scenario 0 presents our base case: all firms are incumbents and symmetric. Table 3 displays the equilibrium outcomes, which include the price and allocation of the spectrum at the end of the auction, the price and output produced in the high product market, as well as the total surplus and auction revenue. The average spectrum obtained by any firm is around 1.35 units, while the auction clears at an average price of $1.54 per unit. If sold, the spectrum units are fully utilized, so the total output in the high product market is generally 4 units, although there is some probability of the auction clearing at $3, with only 3 units being sold. Notice that since the firms are symmetric, the average spectrum, and hence the profit, is identical across the three firms. Although the solution is symmetric in expectation, each auction outcome results in an asymmetric allocation of spectrum, with one firm obtaining 2 units and holding market power.

In Scenario 1, firm 1 is less efficient than firms 2 and 3. According to Table 3, the auction results in two possible outcomes, in both of which one of the two efficient firms (i.e. firms 2 and 3) obtains 2 units and holds market power, while firm 1 and the other efficient firm win only one spectrum unit. Consequently, firms 2 and 3 produce more, on average, than firm 1. Less competition between firms (along with a less efficient industry) leads to lower auction revenue and lower total surplus relative to Scenario 0.

In Scenario 2, where firms 1 and 2 are inefficient producers in the high product market, we see that inefficient producers can hold up the market, preventing the efficient firm from obtaining more units (see Table 3). In fact, the auction is neither effective nor efficient, as it clears at zero price, and the efficient firm (i.e. firm 3) wins only one unit. The main reason for this outcome is the trade-off for the efficient firm: on the one hand, it can increase the price to obtain more units and force the inefficient firms to either exit the market or settle with less units. On the other hand, doing so increase the spectrum price, which offsets its profits. In Scenario 1, we saw that if there are two efficient firms, they can limit the inefficient firm's winnings at a low price (auction clears at $0.9 in Scenario 1), and capture market power. In

Scenario 2, though, firm 3 is the only efficient firm, and driving out two inefficient firms is too costly. As a result, the inefficient firms obtain 3.5 units on average, but utilize only 1.5 units. The efficient firm has higher profits (than the inefficient firms) due to efficiency, but not due to market power.

Scenario 3 has 3 symmetric but inefficient producers (see Table 3). In this case, the auction collapses and the firms share the spectrum at zero price. Each firm obtains 2.5 units on average. Total surplus is the lowest of all the cases so far (reflecting the inefficient industry), and the regulator collects no auction revenue.

Scenarios 4–5 cover the case where one firm (firm 1) is an entrant, so it has no legacy spectrum. Since firm 1 could use spectrum both in the low and high product markets, this provides an incentive for firms 2 and 3 to bid aggressively in the auction. In Table 4, the incumbent firms 2 and 3 increase the auction price to discourage firm 1 from obtaining more than one unit, which firm 1 then utilizes solely in the high product market. As a result, firms 2 and 3 may end up winning more spectrum than needed, even though up to 2.5 units are needed for production by any firm. Despite being an entrant, firm 1 is still efficient in production, therefore the total surplus is higher (i.e. 31.9 compared to 31.4 in Scenario 0), while the auction revenue is lower. If firm 1 is an inefficient entrant (Scenarios 5; Table 4), then there is some probability firm 1 will not win *any* spectrum. This leads to higher total surplus in the market but lower revenue.

Scenarios 6–8 (see Table 5) focus on the cases where there are two potential entrants (firms 1 and 2). If both entrants are efficient producers (Scenario 6; Table 5), then firm 3 can effectively prevent either from obtaining a second unit, thus dominating the market with two units. If only one entrant is efficient (Scenario 7; Table 5), the incumbent can obtain as many units as it would like at zero price as long as each entrant is allowed one unit. Firm 1 is inefficient, so only partially utilizes the spectrum unit (firm 1 output is 0.33 units). Firm 2 (efficient) fully utilizes its unit. When both entrants are inefficient (Scenario 8; Table 5), as in Scenario 2, we see the inefficient entrants hold up the market so that all three firms win 2 units. Compared to Scenario 2, the auction price stays low, but this improves welfare as firms 1 and 2 have greater need for the spectrum unit (i.e. to produce in the low product market, increasing competition there). Firms 1 and 2 utilize part of the spectrum (1.2 units) in the high product market and the rest (0.8 units) in the low product market.

4 Regulation through Caps

Although an auctioneer generally aims to maximize revenue in an auction, a Regulator tasked with running a spectrum auction may also take into account total surplus in the downstream market generated through the allocation of spectra. In particular, the Regulator may face a trade-off between the auction revenue and the total surplus. For example, suppose that one of the firms is substantially more efficient; then this firm may increase the price high enough to win all the units. While selling all units at a high price is quite desirable as an auctioneer, the Regulator may find that the firm becomes a monopolist in the downstream market, which decreases the total surplus. As a result, the Regulator may decide to intervene in the auction by setting *caps*, which stipulate the maximum number of units that each firm can win in the auction.

Given that our solution technique works recursively backwards from high prices and low demand to low prices and high demand, we model caps by starting the auction with participants bidding demands that do not equal the total number of units available in the market but rather equal the cap set by the government for each firm. For each combination of starting bids, it is possible to evaluate the distribution of auction clearing bids. These allow us to calculate the distribution of Total Surplus and Revenue if demand and supply parameters are known with certainty.

If the Regulator faces *uncertainty* regarding demand and supply parameters, we can evaluate the moments of Total Surplus and Revenue as follows:

$$\mathbb{E}(TS|\overline{\mathbf{B}}_{\mathbf{a}}) = \sum_{\omega\in\Omega} \psi_\omega \mathbb{E}\left(TS_\omega|\overline{\mathbf{B}}_{\mathbf{a}}\right) \tag{18}$$

$$\mathbb{E}(R|\overline{\mathbf{B}}_{\mathbf{a}}) = \sum_{\omega\in\Omega} \psi_\omega \mathbb{E}\left(R_\omega|\overline{\mathbf{B}}_{\mathbf{a}}\right) \tag{19}$$

$$\mathbb{V}(TS|\overline{\mathbf{B}}_{\mathbf{a}}) = \sum_{\omega\in\Omega} \psi_\omega (\mathbb{V}\left(TS_\omega|\overline{\mathbf{B}}_{\mathbf{a}}\right) + \mathbb{E}\left(TS_\omega|\overline{\mathbf{B}}_{\mathbf{a}})^2\right) - \mathbb{E}(TS|\overline{\mathbf{B}}_{\mathbf{a}})^2 \tag{20}$$

$$\mathbb{V}(R|\overline{\mathbf{B}}_{\mathbf{a}}) = \sum_{\omega\in\Omega} \psi_\omega (\mathbb{V}\left(R_\omega|\overline{\mathbf{B}}_{\mathbf{a}}\right) + \mathbb{E}\left(R_\omega|\overline{\mathbf{B}}_{\mathbf{a}})^2\right) - \mathbb{E}(R|\overline{\mathbf{B}}_{\mathbf{a}})^2 \tag{21}$$

$$\begin{aligned}\mathrm{Cov}(TS, R|\overline{\mathbf{B}}_{\mathbf{a}}) &= \sum_{\omega\in\Omega} \psi_\omega (\mathrm{Cov}\left(TS_\omega, R_\omega|\overline{\mathbf{B}}_{\mathbf{a}}\right) + \mathbb{E}\left(TS_\omega|\overline{\mathbf{B}}_{\mathbf{a}}\right)\mathbb{E}\left(R_\omega|\overline{\mathbf{B}}_{\mathbf{a}}\right) \\ &\quad -\mathbb{E}(TS|\overline{\mathbf{B}}_{\mathbf{a}})\mathbb{E}(R|\overline{\mathbf{B}}_{\mathbf{a}}),\end{aligned} \tag{22}$$

where $\mathbb{E}(\cdot)$ and $\mathbb{V}(\cdot)$ represent the mean and the variance of the relevant variables, and Ω denotes the set of possible states that demand and supply parameters may take, and ψ_ω is the probability that state $\omega \in \Omega$ occurs. The terms TS and R represent the total surplus in the downstream market and the auction revenue respectively. $\overline{\mathbf{B}}_{\mathbf{a}}$ denotes the vector of caps set by the Regulator for the firms.

The Regulator optimally chooses a cap structure for the auction depending on his/her objective function, which we define as follows:

$$\max_{\overline{\mathbf{B}}_{\mathbf{a}}\geq 0} \mathbb{E}\left(\alpha_1\ TS + \alpha_2\ R \mid \overline{\mathbf{B}}_{\mathbf{a}}\right) + \alpha_3\ \mathbb{V}\left[\alpha_1\ TS + \alpha_2\ R \mid \overline{\mathbf{B}}_{\mathbf{a}}\right] \tag{23}$$

The expectation and variance are evaluated using (18-22). The first term in equation (23) is the weighted average of the expected total surplus and revenue, and the last term is the variance of this weighted sum. This objective function generalizes the one used in Dana and Spier (1994), and allows the Regulator to be risk-neutral (when $\alpha_3 = 0$) or risk-averse (when $\alpha_3 < 0$).

For our numerical example, we focus on a Regulator who is risk averse, and concerned with a combination of Revenue maximisation and Total Surplus maximisation: $\alpha_1 = 0.5$,

$\alpha_2 = 0.5$, $\alpha_3 = -1$. We consider firms to differ in terms of marginal cost. As in Section 3, efficient firms have unit marginal cost, while inefficient firms have marginal cost of three. The Regulator does not know which firms are efficient, but does know the probabilities of firms being efficient.

We assume that the Regulator faces uncertainty as to which firms can efficiently exploit the spectrum, even though firms know each other's type in the downstream market. For example, each firm may have a different technology available to use in the high product market. Using our results from Section 3, we explore three scenarios. In the first scenario (see Table 6), three incumbent firms compete in the market. When the firms are relatively efficient (all have 50% or 90% probability of being efficient), the Regulator chooses to assign three units of spectrum of each firm. This generates zero revenue, but ensures a competitive market, albeit at the expense of inefficient firms potentially being assigned spectrum that they may not use.

In contrast, when some firms have a high probability of being inefficient (probabilities of 10% of being efficient) the Regulator favours them in its capping policy. For example, when firm 1 has a 10% probability of being efficient, and firms 2 and 3 have a 50% probability of being efficient, the Regulator assigns a cap of 6 to firm 1, and a cap of 2 to the other firms. This is likely to result in lower total surplus, but because an inefficient firm 1 will want to hold up the market, results in higher Revenue. Further, because an *efficient* firm 1 will result in higher Total Surplus, the Regulator faces negative covariance between revenue and total surplus. Effectively, by favouring firm 1, the Regulator has hedged his/her risks between Total Surplus and Revenue. Similar behaviour is exhibited when both firms 2 and 3 have 10% probability of being efficient.

In the second scenario (see Table 7), two duopolists are present in the market. The Regulator knows that they can make efficient use of Spectra from the auction. However, a new entrant to the market (who currently lacks legacy Spectra) has uncertain costs. If the new entrant is efficient, it may be desirable for the Regulator to use caps to encourage it. However, if the entrant is inefficient, the Regulator may prefer to see efficient use of the Spectra available, at the expense of firms exercising market power. Here if the entrant is relatively weak (probability 10% or 50% of being efficient) the Regulator assigns a tight cap on the entrant. This achieves two things: first, it encourages the incumbents to run up the price, and secondly, it avoids spectrum being assigned to a potentially inefficient firm. In contrast, when firm 1 has 90% probability of being efficient, the Regulator is more generous, and assigns a cap to firm 1 that is at least as large as he/she assigns to the incumbents.

Lastly, we consider a case of a monopoly market facing two entrants. Again, the Regulator knows the incumbent is efficient, but is uncertain as to the marginal costs of the entrants (see Table 8). Similar to the previous case, a weak pair of entrants will be given low caps, while strong entrants (90% probability of being efficient) will be rewarded with more favourable caps to promote competition in the market. Interestingly, when one entrant is strong (90% probability of being efficient) and the other is weak (50% probability of being efficient) the Regulator raises both of their caps relative to the case where both are weak. With the incumbent and one strong entrant, prices are liable to be high, and it is unlikely if firm 2 proves to be inefficient that it will win many units of Spectrum. In contrast, if firm 2 is efficient, then the Regulator may achieve a triopoly in the market, leading to high Total Surplus, along with high Revenue from the auction.

5 Conclusion

In this paper, we explore a multi-unit auction in which market participants use the auction units to provide services in a downstream market. Winning large quantities in the auction allows a participant to wield considerable power in the downstream market. Although our model is couched in the application of spectra, our approach to solving the auction is quite general, and depends only upon bidders receiving a payoff contingent upon the auction's allocation of items. From this perspective, the model could be used to examine other auction situations such as bidding for gates at an airport, auctioning of physical commodities or bidding for licences.

Our findings for un-capped auctions suggest that while an auction dominated by efficient firms may result in efficient firms winning spectrum at the expense of less efficient firms, when efficient firms are in the minority, hold-up problems may occur. In addition, although market entrants may place a high marginal valuation on winning spectrum, incumbents are incentivised to bid aggressively to keep them out, due to potential losses in their existing revenue base.

Lastly, we show how the model can be generalised to consider a Regulator managing the auction, who is concerned by efficiency of the downstream market as well as auction revenue. We show that the regulator can use innovative capping structures to deal with uncertainty regarding auction participant efficiency. We note that while we use the term "Regulator" here, the auctioneer could in many (alternative) cases be a profit maximising firm. For example, in the case of auctioning of aiport gates, the airport may impose a capping policy, and may be concerned about both auction revenue and airport passenger throughput, since this will affect retail rental revenues.

Appendix

A Solving the Finite State Game

At each state, which is described by an auction price and a vector of current bids, each firm decides whether to drop its demand by one unit or not, given others' actions. Let F denote the set of players (firms) and each player has only two actions (drop and not drop). We describe the pseudo-algorithm below to solve the finite state games:

i) Check for active players: let F_0 denote the set of players with zero current bids. The set of active players is: $F_1 = F \backslash F_0$

ii) For each player in F_1, check if there is a dominant strategy. Let $\bar{F}$ denote the set of such players with a dominant strategy. The set of players who could play a mixed strategy is $F_2 = F_1 \backslash \bar{F}$.

iii) For the players in F_2, solve for the mixed-strategy Nash equilibrium (MSNE) in two ways:

   a) Root-finding method: Compute the roots of the system of non-linear equations (15), and check if the root (i.e. probability of dropping $\{\pi_i\}_{i \in F_2}$) satisfies the following condition: $\pi_i \in (0, 1), \forall i \in F_2$

   b) Non-linear Optimization Method: If the root-finding method does not reveal a solution that satisfies the condition, then solve the nonlinear Nash equilibrium constrained optimization problem, with a predefined set of initial conditions.

iv) If the two methods in iii) do not yield an MSNE, look for pure-strategy Nash equilibrium (PSNE).

   a) If there is a single PSNE, then accept it as the solution to the game;

b) If there are multiple PSNEs, then we rank them according to their risk measure (Carlsson and van Damme (1993)): $R_j = \prod_{i=1}^{F_2}(P_{s_{ij}} - P_{s'_{ij}})$ where s_{ij} is the strategy played in PSNE j and s'_{ij} is the one-step deviation from s_{ij} for player i. The risk measure R_j for PSNE j is the product of the difference between equilibrium payoff ($P_{s_{ij}}$) and the corresponding one-step deviations ($P_{s'_{ij}}$). The PSNE with the highest risk measure is the solution to the game. If multiple PSNEs have identical risk measures, we assume they occur with equal probability.

v) If no MSNE or PSNE are found in iii) and iv), then look for a semi-mixed solution where some players are constrained to play pure strategies and others are allowed to play mixed strategies. For each combination, we check whether the outcome is a Nash equilibrium, and assume that each occurs with equal probability.

Tables

Symbol	Value(s)	Explanation
N	9	Number of units auctioned
M	3	Number of firms
a_h	9	Choke price in the high-product market
a_l	5	Choke price in the low-product market
b_h	1	slope of the demand curve for the high product
b_l	1	slope of the demand curve for the low product
$\mathbf{c}_h$	$(1,1,1)$	linear part of the cost of production of the high product
$\mathbf{c}_l$	$(1,1,1)$	linear part of the cost of production of the low product
$\mathbf{d}_h$	$(0,0,0)$	quadratic part of the cost of production of the high product
$\mathbf{d}_l$	$(0,0,0)$	quadratic part of the cost of production of the low product
$\boldsymbol{\theta}_a$	$(1,1,1)$	marginal increase in capacity for the new spectra
$\boldsymbol{\theta}_l$	$(1,1,1)$	marginal increase in capacity for the legacy spectra
$\mathbf{B}_l$	$(2,2,2)$	endowment of legacy spectra
ΔP	0.1	price increment in auction

Table 1: Parameters for the benchmark model (see Scenario 0 in Table 3)

Note: All other scenarios are perturbations of these parameter values.

Scenario	Characteristics
0	Benchmark Model: All firms are incumbents and efficient.
1	Firm 1 is inefficient: $c_{h1} = 3$.
2	Firms 1 and 2 are inefficient: $c_{h1} = c_{h2} = 3$.
3	All three firms are inefficient: $c_{h1} = c_{h2} = c_{h3} = 3$.
4	Firm 1 is an entrant: $B_{l1} = 0$.
5	Firm 1 is an inefficient entrant: $B_{l1} = 0, c_{h1} = 3$.
6	Firms 1 and 2 are entrants: $B_{l1} = B_{l2} = 0$.
7	Firms 1 and 2 are entrants and Firm 1 is inefficient: $B_{l1} = B_{l2} = 0, c_{h1} = 3$.
8	Firms 1 and 2 are inefficient entrants: $B_{l1} = B_{l2} = 0, c_{h1} = c_{h2} = 3$.

Table 2: Scenarios

Note: Perturbations to the parameters outlined in Table 1 to generate the scenarios for Section 3. The parameter c_{hi} is firm i's linear part of the marginal cost of producing high product. The endowment of legacy spectrum is denoted by $\mathbf{B}_l$.

Scenario	Probability	B_{a1}	B_{a2}	B_{a3}	P_a	Q_{h1}	Q_{h2}	Q_{h3}	P_h	TS	R
0	0.2032	1	1	2	1.1	1	1	2	5	31.5	4.4
	0.2032	1	2	1	1.1	1	2	1	5	31.5	4.4
	0.2032	2	1	1	1.1	2	1	1	5	31.5	4.4
	0.0708	1	2	1	2	1	2	1	5	31.5	8
	0.0708	1	1	2	2	1	1	2	5	31.5	8
	0.0708	2	1	1	2	2	1	1	5	31.5	8
	0.0246	1	1	1	3	1	1	1	6	27	9
	Average	1.34	1.34	1.34	1.54	1.33	1.33	1.33	5.01	31.40	6.14
1	0.5000	1	2	1	0.9	1	2	1	5	29.5	3.6
	0.5000	1	1	2	0.9	1	1	2	5	29.5	3.6
	Average	1	1.5	1.5	0.9	1	1.5	1.5	5	29.5	3.6
2	0.2500	7	1	1	0	2	1	1	5	25.5	0
	0.2500	1	7	1	0	1	2	1	5	25.5	0
	0.1250	6	1	1	0	2	1	1	5	25.5	0
	0.1250	1	6	1	0	1	2	1	5	25.5	0
	0.0625	5	1	1	0	2	1	1	5	25.5	0
	0.0625	1	5	1	0	1	2	1	5	25.5	0
	0.0313	4	1	1	0	2	1	1	5	25.5	0
	0.0313	1	4	1	0	1	2	1	5	25.5	0
	0.0156	2	1	1	0	2	1	1	5	25.5	0
	0.0156	3	1	1	0	2	1	1	5	25.5	0
	0.0156	1	2	1	0	1	2	1	5	25.5	0
	0.0156	1	3	1	0	1	2	1	5	25.5	0
	Average	3.52	3.52	1	0	1.5	1.5	1	5	25.5	0
3	0.2047	2	2	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0682	2	2	3	0	1.5	1.5	1.5	4.5	24.38	0
	0.0682	2	3	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0682	3	2	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0662	2	2	4	0	1.5	1.5	1.5	4.5	24.38	0
	0.0662	4	2	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0662	2	4	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0516	2	2	5	0	1.5	1.5	1.5	4.5	24.38	0
	0.0516	5	2	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0516	2	5	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0269	2	4	3	0	1.5	1.5	1.5	4.5	24.38	0
	0.0269	2	3	4	0	1.5	1.5	1.5	4.5	24.38	0
	0.0269	3	2	4	0	1.5	1.5	1.5	4.5	24.38	0
	0.0269	4	2	3	0	1.5	1.5	1.5	4.5	24.38	0
	0.0269	4	3	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0269	3	4	2	0	1.5	1.5	1.5	4.5	24.38	0
	0.0234	2	3	3	0	1.5	1.5	1.5	4.5	24.38	0
	0.0234	3	2	3	0	1.5	1.5	1.5	4.5	24.38	0
	0.0234	3	3	2	0	1.5	1.5	1.5	4.5	24.38	0
	Average	2.57	2.57	2.57	0.00	1.5	1.5	1.5	4.5	24.38	0.00

Table 3: Scenario 0–3: All firms are incumbents. In Scenario 0, all firms are efficient. In Scenario 1, Firm 1 is inefficient $c_{h1} = 3 > c_{h2} = c_{h3} = 1$. In Scenario 2, Firms 1 and 2 are inefficient $c_{h1} = c_{h2} = 3 > c_{h3} = 1$. In Scenario 3, all firms are inefficient $c_{h1} = c_{h2} = c_{h3} = 3$. All other parameters are as given in the base case (Table 1).

Note: Each row represents a possible outcome for this auction. Columns represent (in order) probability of the outcome occurring, allocations to the three firms, price for the outcome in question, quantity produced and price in the high product market, total surplus, and finally revenue. Outcomes with probability of less than 10^{-2} are not displayed in the table for Scenarios 0 and 3.

Scenario	Probability	B_{a1}	B_{a2}	B_{a3}	P_a	Q_{h1}	Q_{h2}	Q_{h3}	P_h	TS	R
4	0.4863	1	2	1	1.1	1	2	1	5	31.11	4.4
	0.4863	1	1	2	1.1	1	1	2	5	31.11	4.4
	0.0048	1	2	1	1.2	1	2	1	5	31.11	4.8
	0.0048	1	1	2	1.2	1	1	2	5	31.11	4.8
	0.0046	1	2	3	1.1	1	2	2.5	3.5	35.99	6.6
	0.0046	1	3	2	1.1	1	2.5	2	3.5	35.99	6.6
	0.0029	1	2	2	1.1	1	2	2	4	34.61	5.5
	0.0016	1	2	4	1.1	1	2	2.5	3.5	35.99	7.7
	0.0016	1	4	2	1.1	1	2.5	2	3.5	35.99	7.7
	Average	1	1.52	1.52	1.10	1	1.51	1.51	4.98	31.19	4.45
5	0.1756	1	6	2	0	0.35	2.82	2	3.82	34.68	0
	0.1756	1	2	6	0	0.35	2	2.82	3.82	34.68	0
	0.0878	1	5	2	0	0.35	2.82	2	3.82	34.68	0
	0.0878	1	2	5	0	0.35	2	2.82	3.82	34.68	0
	0.0696	0	2	1	2.6	0	2	1	6	26.61	7.8
	0.0696	0	1	2	2.6	0	1	2	6	26.61	7.8
	0.0508	1	1	1	2.6	1	1	1	6	24.61	7.8
	0.0439	1	3	2	0	0.35	2.82	2	3.82	34.68	0
	0.0439	1	4	2	0	0.35	2.82	2	3.82	34.68	0
	0.0439	1	2	3	0	0.35	2	2.82	3.82	34.68	0
	0.0439	1	2	4	0	0.35	2	2.82	3.82	34.68	0
	0.0208	0	3	3	0.1	0	2.67	2.67	3.67	35.56	0.6
	0.0162	0	2	2	1.1	0	2	2	5	31.11	4.4
	0.0116	0	3	3	0.2	0	2.67	2.67	3.67	35.56	1.2
	Average	0.77	3.08	3.08	0.55	0.31	2.22	2.22	4.25	33.02	1.75

Table 4: Scenarios 4 and 5: Firm 1 is an entrant and Firms 2 and 3 are incumbents $B_{l1} = 0 < B_{l2} = B_{l3} = 2$. In Scenario 4, all three firms are efficient $c_{h1} = c_{h2} = c_{h3} = 1$. In Scenario 5, Firm 1 is inefficient $c_{h1} = 3 > c_{h2} = c_{h3} = 1$. All other parameters are as given in the base case (Table 1).

Note: Each row represents a possible outcome for this auction. Columns represent (in order) probability of the outcome occurring, allocations to the three firms, price for the outcome in question, quantity produced and price in the high product market, total surplus, and finally revenue. Outcomes with probability of less than 10^{-2} are not displayed in the table for Scenario 5.

Scenario	Probability	B_{a1}	B_{a2}	B_{a3}	P_a	Q_{h1}	Q_{h2}	Q_{h3}	P_h	TS	R
6	1	1	1	2	1.1	1	1	2.00	5.00	30.00	4.4
7	0.500	1	1	7	0	0.33	1	3.33	4.33	32.39	0
	0.250	1	1	6	0	0.33	1	3.33	4.33	32.39	0
	0.125	1	1	4	0	0.33	1	3.33	4.33	32.39	0
	0.125	1	1	5	0	0.33	1	3.33	4.33	32.39	0
	Average	1	1	6.125	0.00	0.33	1	3.33	4.33	32.39	0.00
8	1	2	2	2	0.3	1.2	1.2	2	4.6	28	1.8

Table 5: Scenarios 6–8: Firms 1 and 2 are entrants and Firm 3 is an incumbent $B_{l1} = B_{l2} = 0 < B_{l3} = 2$. In Scenario 6, all three firms are efficient $c_{h1} = c_{h2} = c_{h3} = 1$. In Scenario 7, Firm 1 is inefficient $c_{h1} = 3 > c_{h2} = c_{h3} = 1$. In Scenario 8, Firms 1 and 2 are both inefficient $c_{h1} = c_{h2} = 3 > c_{h3} = 1$. All other parameters are as given in the base case (Table 1).

Note: Each row represents a possible outcome for this auction. Columns represent (in order) probability of the outcome occurring, allocations to the three firms, price for the outcome in question, quantity produced and price in the high product market, total surplus, and finally revenue.

	(0.5,0.5,0.5)	(0.1,0.5,0.5)	(0.9,0.5,0.5)	(0.5,0.1,0.1)		(0.5,0.9,0.9)
$\bar{B}_1$	3	6	3	2	2	3
$\bar{B}_2$	3	2	3	6	2	3
$\bar{B}_3$	3	2	3	2	6	3
$\mathbb{E}(B_{a1})$	3	3.06	3	1.69	1.69	3
$\mathbb{E}(B_{a2})$	3	1.63	3	3.3932	1.6549	3
$\mathbb{E}(B_{a3})$	3	1.63	3	1.6549	3.3932	3
$\sigma(B_{a1})$	0	2.66	0.00	0.22	0.22	0.00
$\sigma(B_{a2})$	0	0.23	0.00	2.16	0.23	0.00
$\sigma(B_{a3})$	0	0.23	0.00	0.23	2.16	0.00
$\mathbb{E}(TS)$	32.63	27.61	34.38	27.08	27.08	35.77
$\sigma(TS)$	15.3047	7.9097	8.30	8.84	8.84	4.10
$\mathbb{E}(R)$	0	1.59	0	0.91	0.91	0
$\sigma(R)$	0	5.29	0	2.33	2.33	0
$Cov(TS,R)$	0	-2.57	0	-2.26	-2.26	0

Table 6: Regulation of spectrum auction in the presence of uncertainty regarding firm marginal costs. Column headings display probabilities of each firm having low marginal cost ($c_{hi} = 1$) versus high ($c_{hi} = 3$) marginal cost. B_1, B_2, and B_3 are the caps that maximise regulator utility.

Note: Where multiple columns appear, each column represents a set of caps that yields equivalent utility to the regulator. In each case we report the mean and standard deviation for each firm's allocation, the market's total surplus, and the revenue generated by the auction. Lastly, we report the covariance of revenue and total surplus.

	(0.1,1,1)	(0.5,1,1)	(0.9,1,1)			
$\bar{B}_1$	2	2	4	5	4	5
$\bar{B}_2$	6	6	4	4	3	3
$\bar{B}_3$	6	6	3	3	4	4
$\mathbb{E}(B_{a1})$	0.15	0.09	2.34	2.34	2.34	2.34
$\mathbb{E}(B_{a2})$	2.94	3.04	2.11	2.11	2.1	2.1
$\mathbb{E}(B_{a3})$	2.94	3.04	2.1	2.1	2.11	2.11
$\sigma(B_{a1})$	0.13	0.08	0.22	0.23	0.23	0.23
$\sigma(B_{a2})$	0.14	0.35	0.13	0.11	0.09	0.09
$\sigma(B_{a3})$	0.14	0.35	0.09	0.09	0.13	0.11
$\mathbb{E}(TS)$	35.15	35.15	36.63	36.63	36.63	36.63
$\sigma(TS)$	1.64	1.63	0.42	0.42	0.42	0.42
$\mathbb{E}(R)$	1.35	1.24	0.93	0.93	0.93	0.93
$\sigma(R)$	1.25	1.42	0.15	0.15	0.15	0.15
$Cov(TS,R)$	-1.24	-1.28	0.02	0.02	0.02	0.02

Table 7: Regulation of spectrum auction, where an entrant (Firm 1) into a duopolistic market has uncertain marginal cost while the two incumbents (Firms 2 and 3) have known low marginal costs. Column headings display probabilities of each firm having low marginal cost ($c_{hi} = 1$) versus high ($c_{hi} = 3$) marginal cost. B_1, B_2, and B_3 are the caps that maximise regulator utility.

Note: Where multiple columns appear, each column represents a set of caps that yields equivalent utility to the regulator. In each case we report the mean and standard deviation for each firm's allocation, the market's total surplus, and the revenue generated by the auction. Lastly, we report the covariance of revenue and total surplus.

	(0.5,0.5,1)	(0.1,0.5,1)	(0.9,0.5,1)	(0.1,0.1,1)	(0.9,0.9,1)	
$\bar{B}_1$	2	2	4	2	4	3
$\bar{B}_2$	2	2	3	2	3	4
$\bar{B}_3$	6	6	4	6	3	3
$\mathbb{E}(B_{a1})$	1.34	1.20	3.10	1.35	2.99	2.11
$\mathbb{E}(B_{a2})$	1.34	1.60	1.33	1.35	2.11	2.99
$\mathbb{E}(B_{a3})$	3.98	4.26	2.62	3.80	2.19	2.19
$\sigma(B_{a1})$	0.23	0.16	0.22	0.23	0.09	0.10
$\sigma(B_{a2})$	0.23	0.24	0.35	0.23	0.10	0.09
$\sigma(B_{a3})$	2.34	1.71	0.38	1.16	0.15	0.15
$\mathbb{E}(TS)$	32.07	32.00	34.73	30.76	35.99	35.99
$\sigma(TS)$	3.89	3.91	0.83	2.64	0.50	0.50
$\mathbb{E}(R)$	1.89	1.47	1.64	2.26	1.17	1.17
$\sigma(R)$	5.15	3.86	1.12	3.80	0.27	0.27
$Cov(TS, R)$	-3.24	-3.06	0.14	-2.23	0.29	0.29

Table 8: Regulation of spectrum auction, where two entrants (Firms 1 and 2) into a monopolistic market have uncertain marginal cost while the incumbent (Firm 3) has known low marginal costs. Column headings display probabilities of each firm having low marginal cost ($c_{hi} = 1$) versus high ($c_{hi} = 3$) marginal cost. B_1, B_2, and B_3 are the caps that maximise regulator utility.

Note: Where multiple columns appear, each column represents a set of caps that yields equivalent utility to the regulator. In each case we report the mean and standard deviation for each firm's allocation, the market's total surplus, and the revenue generated by the auction. Lastly, we report the covariance of revenue and total surplus.

Figures

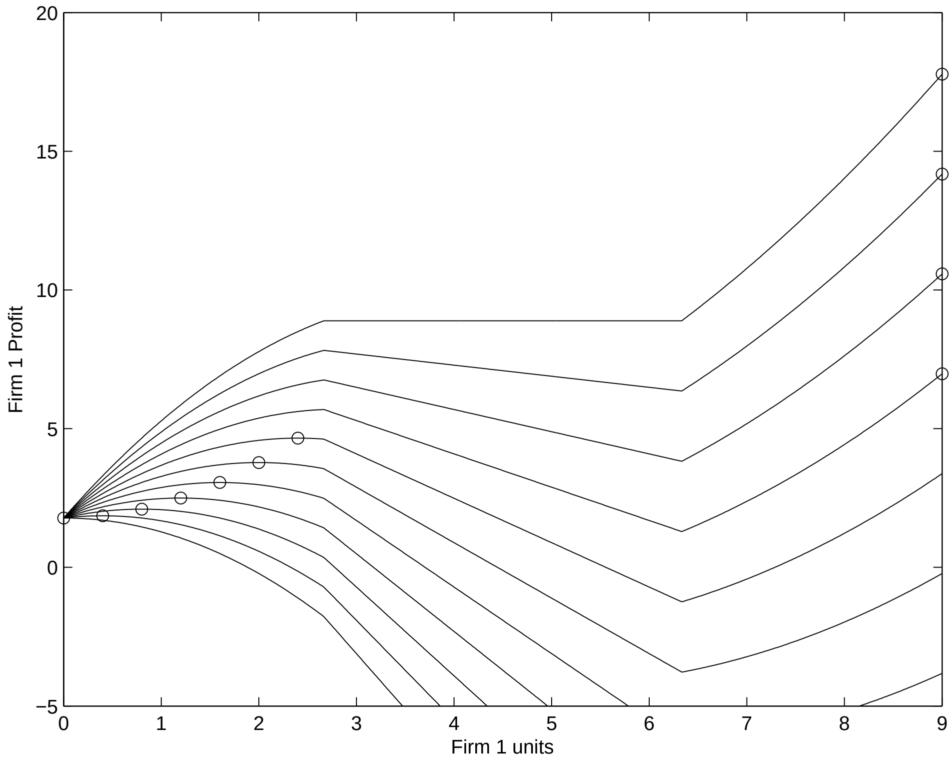

Figure 1: Duopoly Case: Profits vs. Auction Winnings

Note: These results assume market parameters as given in Table 1, with the exception of M (the number of firms) which is assumed to be two. The horizontal axis displays number of units firm 1 wins. Firm 2 is assumed to win the remaining units. Vertical axis displays firm 1's profit.

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