

PHASE PORTRAITS, EQUILIBRIUM TYPES, TRACE-DETERMINANT PLANE, AND STABILITY.

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1. WHAT DO SOLUTIONS LOOK LIKE? PHASE PLANE PORTRAITS AND EQUILIBRIUM POINTS.

Given a system of equations $\vec{y}'(x) = f(x, \vec{y})$, a **phase portrait** is a collection of representative **solution curves** (also called **trajectories**, **orbits**) $\vec{y}(x)$ plotted parametrically in **phase space**. This helps visualize solutions to the system. We focus on planar systems $\vec{y}' = A\vec{y}$, where A is a 2×2 real matrix.

Suppose A is an $n \times n$ **constant real matrix**, and $\vec{y}_1(x)$ and $\vec{y}_2(x)$ are two solutions to $\vec{y}' = A\vec{y}$. Then the parametric plots of $\vec{y}_1(x)$ and $\vec{y}_2(x)$ in phase space \mathbb{R}^n are either disjoint or they coincide completely. In particular,

Theorem 1. *If A is a 2×2 real constant matrix, the phase portrait of $\vec{y}' = A\vec{y}$ consists of curves that do not cross each other.*

We first look at situations where the eigenvalues are distinct and nonzero, these are easier to analyze.

Example 2. (Nodal source)

The system $y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} y$ has fundamental solution set $\{e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e^x \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$. Its general solution is $\vec{y}(x) = C_1 e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

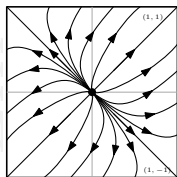
The solutions have two competing behaviors:

Growing as e^{3x} in the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ direction;

and grow as e^x in the $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ direction.

For $x \gg 0$, $e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ dominates, so if $x \gg 0$, the trajectory approaches in the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ direction (to ∞).

For $x \ll 0$, $e^x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ dominates, so if $x \ll 0$, the trajectory approaches in the $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ direction (to $\vec{0}$).



Its equilibrium solution is a constant solution $\vec{y} = \vec{c}$ such that $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{c} = \vec{0}$, which is just the kernel of the matrix. In this case, it is just $\vec{0}$.

Here $\vec{0}$ is an equilibrium point of type **nodal source**. This is characterized by the eigenvalues $0 < \lambda_1 < \lambda_2$. Since solutions near the equilibrium point all diverge away from the equilibrium point, we say it is **unstable**.

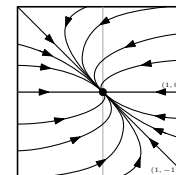
Example 3. (Nodal sink)

The system $y' = \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix} y$ has fundamental solution set $\{e^{-2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$. Its general solution is $\vec{y}(x) = C_1 e^{-2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The solutions have two competing behaviors:

For $x \gg 0$, $e^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ dominates, so if $x \gg 0$, the trajectory approaches in the $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ direction (to $\vec{0}$).

For $x \ll 0$, $e^{-2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ dominates, so if $x \ll 0$, the trajectory approaches in the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ direction (to ∞).



Its only equilibrium point is $\vec{0}$, and it is of type **nodal sink**, characterized by the eigenvalues $\lambda_1 < \lambda_2 < 0$. Solutions near the equilibrium point will converge towards it, so it is **asymptotically stable**.

Example 4. (Saddle)

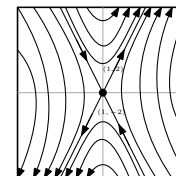
The system $y' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} y$ has fundamental solution set $\{e^{-x} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, e^{3x} \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$. Its general solution is $\vec{y}(x) = C_1 e^{-x} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3x} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Again, for $x \gg 0$, $e^{3x} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ dominates (to ∞); while for $x \ll 0$, $e^{-x} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ dominates (to ∞).

The only equilibrium point is $\vec{0}$, and it is of type **saddle**, characterized by eigenvalues $\lambda_1 < 0 < \lambda_2$.

Solutions near equilibrium will diverge away from it, so it is **unstable**.

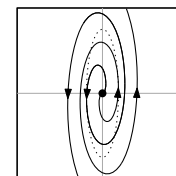
(Are these solution curves hyperbolas? They are actually not hyperbolas in general! Find out when they will be hyperbolas.)



Example 5. (Spiral source) The system $\vec{y}' = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \vec{y}$ has eigenvalues $1 \pm 2i$. This gives fundamental solution set of the form $e^x (\vec{v} \cos 2x - \vec{w} \sin 2x)$, $e^x (\vec{w} \cos 2x + \vec{v} \sin 2x)$.

The equilibrium point $\vec{0}$ has type **spiral source**, characterized by having complex conjugate pairs of eigenvalues with positive real part.

The handedness of spiral can be determined by computing \vec{y}' at a position, say when $\vec{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, this will be counterclockwise.



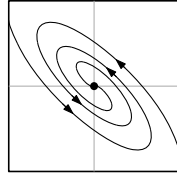
(Note if A is 2×2 with complex conjugate pairs of eigenvalues, the sign of the bottom left entry of the matrix A can determine the handedness of the spiral, how?). Solutions near equilibrium will diverge away from it, so it is **unstable**.

Example 6. (Spiral sink) The system $y' = \begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix} y$ has eigenvalues $\frac{-1 \pm i\sqrt{7}}{2}$. This is similar to above except the real part of the eigenvalue is negative. This give rise to an exponential term with negative power, so as $x \gg 0$, the magnitude goes to 0.

The equilibrium point $\vec{0}$ has type **spiral sink**, characterized by having complex conjugate pairs of eigenvalues with negative real part.

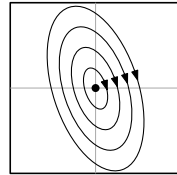
Solutions near the equilibrium point will converge towards it, so it is **asymptotically stable**.

Again we can decide the handedness of the spiral like above, by testing an initial vector \vec{y} and see what its derivative $\vec{y}' = A\vec{y}$ is.



Example 7. (Center) The system $\vec{y}' = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix} \vec{y}$ has eigenvalues $\pm 2i$. In this case, solution curves are closed curves because the fundamental solutions are each periodic.

One can further show they are all similar **ellipses**. We say that the equilibrium point $\vec{0}$ has type **center**, which is characterized by having purely imaginary complex conjugate eigenvalues. The handedness of the trajectory can again be determined by testing a vector.



Solutions near equilibrium neither converge nor diverge from it, but rather stay within some distance from it. One calls this **neutrally stable**.

2. CHARACTERIZATION OF PHASE PORTRAITS AND EQUILIBRIA: TRACE-DETERMINANT PLANE.

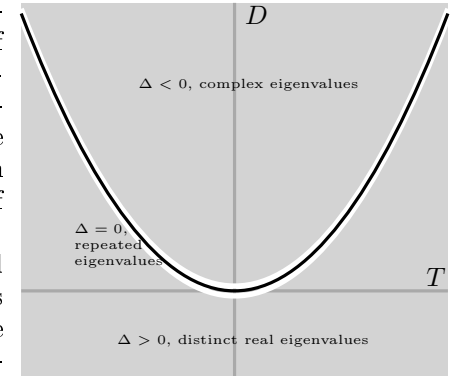
The behavior of solutions to $y' = Ay$ largely depend on two factors: (1) the eigenvalues of the matrix A and (2) the diagonalizability of the matrix A (namely, can we produce a genuine eigenbasis).

Let recall for a 2×2 real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its characteristic equation is $p(x) = x^2 - (a+d)x + (ad-bc)$, which is the same as $p(x) = x^2 - \text{tr}(A)x + \det(A)$. The roots of this polynomial, hence the eigenvalues of A , are

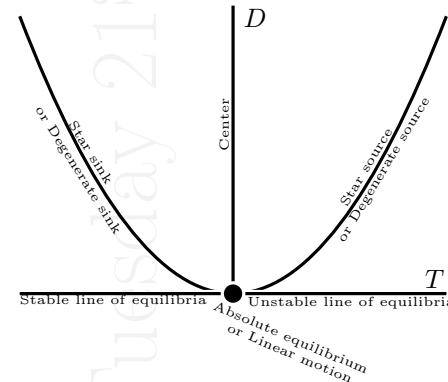
$$x = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}$$

by the quadratic formula.

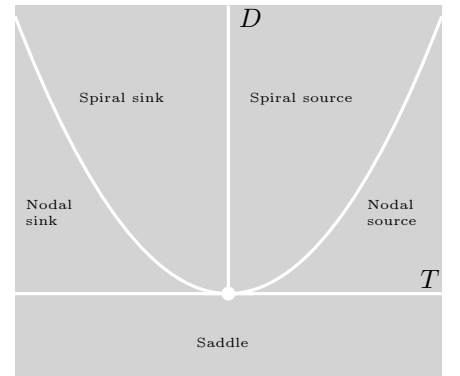
The roots are controlled by the two values $T = \text{tr}(A)$ and $D = \det(A)$ of the matrix A , in particular the **discriminant** $\Delta = T^2 - 4D$. If the discriminant Δ is positive, then we have two distinct real roots; if $\Delta = 0$, then we have repeated real roots; and if $\Delta < 0$, then we have complex roots. This informs us to look at the so-called **trace-determinant plane** (sometimes we write the (T, D) -plane). The curve $D = T^2/4$ on this plane provides a division of various behaviors.



Now the discriminant $\Delta = T^2 - 4D$ only tells us roughly what sort of eigenvalues we get. But that is not enough. For instance when $\Delta < 0$ we can have spirals or ellipses, and sources or sinks. In this case, the source or sink behavior is the sign of the real part of the eigenvalues, which is the sign of $T/2$. So to go further, we need the signs of D and T as well. Using the curve $D = T^2/4$ and the axes of the (T, D) -plane, this divides the (T, D) -plane into 11 parts: Five open regions, three rays, two half-parabolic curves, and a point. Each of these parts corresponds to one or two types of phase portraits and equilibria.



Non-generic equilibria (9 types)



Generic equilibria (5 types)

Now the 5 parts that are open regions are called **generic**, because most of the time, a point on the (T, D) -plane will be in these open regions. The other 6 parts are called **non-generic**, they corresponds to specific combinations of a trace and determinant values.

In the 5 generic parts, they corresponds to 5 kinds of phase portraits and equilibrium types mentioned before: Nodal source, nodal sink, saddle, spiral source, and spiral sink.

The non-generic part with positive D -axis corresponds to the phase portrait with a center as equilibrium type.

Along the $D = T^2/4$ curve, we have repeated eigenvalues. So for the non-generic parts here we have to account for diagonalizability: Whether the repeated eigenvalues give rise to two linearly independent eigenvectors or not. So along this curve, each part will have two different kinds of behavior. This will require a careful analysis.

And along the positive and negative T -axes, the determinant D is zero, which means one of the eigenvalue is 0, while the other eigenvalue $\lambda \neq 0$ is nonzero (recall magical formula for eigenvalues). Since we have two distinct eigenvalues, this 2×2 matrix is diagonalizable, so we have an eigenbasis. Say v_0 is an eigenvector corresponding to eigenvalue 0, and v_λ corresponds to the nonzero eigenvalue λ . The fundamental solution set is $\{v_0, e^{\lambda x}v_\lambda\}$. One can analyze this to show there is a line of equilibrium points, along the direction v_0 , and lines of solution curves parallel to v_λ .

In summary, there are the following 14 equilibrium types on the 11 parts of the (T, D) -plane: (1) **Nodal source**, (2) **nodal sink**, (3) **saddle**, (4) **spiral source**, (5) **spiral sink**, (6) **center**, (7) **degenerate nodal source** (or near-spiral source), (8) **degenerate nodal sink** (near-spiral sink), (9) **star source**, (10) **star sink**, (11) **linear motion**, (12) **absolute equilibria**, (13) **unstable line of equilibria**, and (14) **stable line of equilibria**.

Exercise 8. Find examples and sketch their corresponding phase portrait that illustrate each of the 14 types of equilibrium type. This completes the classification of planar systems $y' = Ay$, with A a 2×2 real matrix.

Notice that $y' = Ay$ will have exactly one equilibrium point if and only if A is invertible (if and only if $D = \det(A) \neq 0$). If $D = 0$, then the matrix A is not invertible and there will be infinitely many equilibrium points, which are precisely the points in the kernel of A . (Show this.)

We give a language to describe the stability of an equilibrium point. We say an equilibrium point is **unstable** if no matter how close are to the equilibrium point, there is a point near it that will eventually diverge away from the equilibrium point. Namely, there is a way to perturb from the equilibrium position slightly to be on a trajectory that moves away from the equilibrium position to infinity. Otherwise we say the equilibrium point is **stable**, namely a slight perturbation from the equilibrium position will not move into a trajectory that diverge away from the equilibrium point. Further, we classify two types of stability: We say it is **asymptotically stable** if it is stable and any slight perturbation will always move into a trajectory that converge to the equilibrium point. If it is stable but not asymptotically stable, we say it is **neutrally stable**. In a neutral stability case, a slight perturbation need not restore to the equilibrium point, but rather either stay in the new configuration or stay near

the original equilibrium point without diverging away to infinity. There may be other terminologies in other literature, but let us stick to this one: At least we are aware there are different kinds of stability.

(An analogy: Imagine placing a ball in a bowl, or on top of an upside-down bowl, or on a flat table.)

Exercise 9. Classify the stability of each equilibrium point in the 14 types of equilibria above.

3. OPTIONAL: GEOMETRY OF THE ELLIPTICAL AND SPIRAL ORBITS.

Assuming we know when the system $y' = Ay$ with A having purely imaginary conjugate pairs of eigenvalues $\pm i\beta$, the solutions have elliptical trajectories.

How do we then figure out the axes of symmetry of this ellipse?

Note that the elliptical trajectory y must have a tangent vector normal to its **axes of symmetry**. So if a trajectory $y(x)$ is such that $y(x_0) = u_1$ is a nonzero vector on an axis of symmetry for some x_0 , we have $y'(x_0) = Ay(x_0) = Au_1$ is normal to u_1 , namely $u_1 \cdot Au_1 = 0$.

So to find the axes, we solve the matrix equation

$$x \cdot Ax = 0,$$

where x some vector.

Since there are two axes and one of them must have the first coordinate nonzero, we can set $x = \begin{pmatrix} 1 \\ t \end{pmatrix}$ instead and solve for t . So solve

$$\begin{pmatrix} 1 \\ t \end{pmatrix} \cdot A \begin{pmatrix} 1 \\ t \end{pmatrix} = 0.$$

This will be a quadratic equation in t . Once we have one axis, the other will be perpendicular to it.

Next, we like to know the **aspect ratio** of this ellipse. Namely, suppose we found u_1 and u_2 , nonzero vectors on the axes of symmetry, what is the ratio of length of the ellipse crossing u_1 -axis to the length of the ellipse crossing the u_2 -axis?

Suppose \hat{u}_1 and \hat{u}_2 are the corresponding unit vectors on these axes. Then the matrix $Q = [\hat{u}_1 | \hat{u}_2]$ is an orthogonal transformation, which on the plane is just a rotation/reflection rigid motion. Now if we apply Q^{-1} to the elliptical trajectory y to get $Q^{-1}y$, we should get an up-right elliptical trajectory, namely one given by the matrix equation $z' = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} z$ where $a, b > 0$. Now this ellipse with trajectory $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ has $(z_1^2)' = 2z_1 z_1' = 2z_1 a z_2$ and $(z_2^2)' = 2z_2 z_2' = -2z_2 b z_1$, so $(\frac{z_1^2}{a} + \frac{z_2^2}{b})' = 0 \implies \frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = \text{constant}$.

This up-right ellipse has aspect ratio $\sqrt{a} : \sqrt{b}$, which is the same aspect ratio as the original ellipse. So we need to find a and b . Note $Q^{-1}y' = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} Q^{-1}y$ suggests that $A = Q \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} Q^{-1}$, or $AQ = Q \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$. But as $Q = [\hat{u}_1 | \hat{u}_2]$, we have

$A\hat{u}_1 = b\hat{u}_2$ and $A\hat{u}_2 = a\hat{u}_1$. Since \hat{u}_i are orthonormal, we get $a = \hat{u}_1 \cdot A\hat{u}_2$ and $b = \hat{u}_2 \cdot A\hat{u}_1$. Finally the aspect ratio $\sqrt{a} : \sqrt{b} = 1 : R$ has $R = \sqrt{\left| \frac{\hat{u}_2 \cdot A\hat{u}_1}{\hat{u}_1 \cdot A\hat{u}_2} \right|}$. If we re-scale \hat{u}_i by arbitrary nonzero constants, they cancel out in the expression of R . So we can just use the axes vectors u_1, u_2 we found earlier and get

$$R = \sqrt{\left| \frac{u_2 \cdot Au_1}{u_1 \cdot Au_2} \right|}.$$

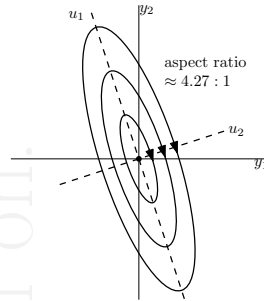
Remark 10. This can be applied to find the characterizing shape of the spiral as well, by making it zero trace to remove the real part in its eigenvalue. This is done by considering the matrix $\tilde{A} = A - \frac{\text{Tr}(A)}{2}I$, and perform the same computation above for \tilde{A} .

Example 11. Consider $y' = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix} y$, with characteristic equation $\lambda^2 + 4 = 0$ has eigenvalues $\lambda = \pm 2i$. So we know the phase portrait consists of ellipses and the equilibrium has type center. To find the axes of these ellipses, we solve $x \cdot Ax = 0$. We try x of the form $\begin{pmatrix} 1 \\ t \end{pmatrix}$, and get $\begin{pmatrix} 1 \\ t \end{pmatrix} \cdot \begin{pmatrix} 2+t \\ -8-2t \end{pmatrix} = (2+t) + t(-8-2t) = 0$, or $-2t^2 - 7t + 2 = 0$. This gives $t = \frac{7 \pm \sqrt{49+16}}{-4} = \frac{7 \pm \sqrt{65}}{-4}$. So $\begin{pmatrix} 1 \\ \frac{7+\sqrt{65}}{-4} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \frac{7-\sqrt{65}}{-4} \end{pmatrix}$ are vectors on the axes of these ellipses. We can rescale it to have $u_1 = \begin{pmatrix} -4 \\ 7+\sqrt{65} \end{pmatrix}$ and $u_2 = \begin{pmatrix} -4 \\ 7-\sqrt{65} \end{pmatrix}$. One can check that they are indeed perpendicular.

Now we compute the aspect ratio to be $1 : R$ with

$$R = \sqrt{\left| \frac{u_2 \cdot Au_1}{u_1 \cdot Au_2} \right|} = \sqrt{\left| \frac{260 - 36\sqrt{65}}{260 + 36\sqrt{65}} \right|} \approx 0.23$$

So the aspect ratio is $1 : 0.23 \approx 4.27 : 1$. Finally, by testing the vector $y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we see that $y' = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$, so the trajectory evolves clockwise.



4. OPTIONAL: ELLIPTICAL ORBITS IN THE CASE OF $\lambda = \pm i\beta$.

When 2×2 real matrix A have eigenvalues $\pm i\beta \neq 0$, the trajectories to $y' = Ay$ are **similar ellipses**. Let us show this fact.

First we use a factorization of 2×2 real matrix A having complex conjugate pairs of eigenvalues $\alpha \pm i\beta$.

Theorem 12. (Planar complex representation.) If A is 2×2 real matrix with eigenvalues $\alpha \pm i\beta$ with $\beta \neq 0$, then A has conjugate pairs of eigenvectors $v \pm iw$. And A is similar to the matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ by the matrix $P = [v|w]$, where $v + iw$ is an eigenvector with eigenvalue $\alpha + i\beta$. (In particular P is invertible.)

Proof. Suppose $v + iw$ is an eigenvector of A with eigenvalue $\alpha + i\beta$, with $\beta \neq 0$, then $A(v + iw) = (\alpha + i\beta)(v + iw)$. By taking complex conjugate to both sides and noting A is real, we have $A(v - iw) = (\alpha - i\beta)(v - iw)$. If $\beta \neq 0$, then $\alpha + i\beta \neq \alpha - i\beta$, whence making $v + iw$ and $v - iw$ linearly independent over \mathbb{C} . This implies the determinant $\det[v + iw | v - iw] \neq 0$. But expanding it gives $\det[v + iw | v - iw] = -2i \det[v | w] \neq 0$, so v and w are also linearly independent. This gives an invertible matrix $P = [v|w]$. Now note $AP = [Av|Aw]$ and $P \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = [\alpha v - \beta w | \beta v + \alpha w]$. And also note $A(v + iw) = (\alpha + i\beta)(v + iw)$, which implies $Av = \alpha v - \beta w$ and $i(Aw) = i(\beta v + \alpha w)$ by matching real and imaginary components. Whence $A = P \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} P^{-1}$ as claimed. \square

So if A has eigenvalues $\pm i\beta \neq 0$, we can write $A = P \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} P^{-1}$ for some invertible P . If y is an orbit to $y' = Ay$, then $y' = P \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} P^{-1}y$, or $(P^{-1}y)' = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} (P^{-1}y)$. If we write $z = P^{-1}y$, we see that z is an orbit of $z' = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} z$. What is the shape of the orbit for z ? Writing $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, we have $z_1' = \beta z_2$ and $z_2' = -\beta z_1$. By cross multiplying we see $z_1 z_1' + z_2 z_2' = 0$. But this means $(z_1^2 + z_2^2)' = 0$, implying $z_1^2 + z_2^2 = \text{constant}$. In other words, the orbit of z is a circle centered at the origin!

Now, $y = Pz$, so the orbit of y is the image of a circle centered at the origin under an invertible linear transformation P . Intuitively, this is an oval shape of some kind, but let us show it is an ellipse.

To show this, we use another factorization fact in linear algebra called **polar decomposition**.

Theorem 13. (Polar decomposition.) Every square $n \times n$ real matrix A can be factorized as $A = QS$, where Q is an orthogonal matrix and S is a positive semi-definite symmetric matrix. We also have left-polar decomposition, where $A = S'Q'$, where Q' orthogonal and S' positive semi-definite symmetric matrix. If A is invertible, then the factorization is unique.

Since P is invertible, its polar decomposition $P = QS$ has S a positive definite symmetric matrix. Now, by spectral theorem, symmetric matrices are orthogonally diagonalizable, and the action of a symmetric matrix is to scale along each of the orthogonal eigenbasis vectors. This means the action of S on the plane is to scale in two orthogonal directions, each by some scaling factor. And recall the action of an orthogonal matrix Q on the plane is a rotation with (possible) reflection, a **roto-reflection**.

Hence if z is a circular orbit centered at the origin, $Pz = QSz$, which first scales the circular orbit in two orthogonal directions each by some amount, giving an ellipse, then followed by a roto-reflection, remaining an ellipse.