Review

February 20, 2023

1 Foundations (Section 2.1 - 2.4)

1. Ordinary differential Equations (ODEs)

ODEs are differential equations with respect to one variable.

2. Initial conditions

Suppose $E \subset R^n$, $(x_0, \mu_0) \in E$. For any closed ball $B \subset E$, $\exists a > 0, \delta > 0$, there is an initial condition $(t_0, y_0, \mu_0) \in [-a, a] \times B(x_0, \delta) \times B(x_0, \delta)$.

3. Initial value problems (IVP)

An IVP consists of a differential equation together with an initial condition $(t_0, x_0) \in R \times E$, such that:

$$\dot{x} = f(x)$$
$$x(t_0) = x_0$$

We say that $x(t)$ is a solution of this IVP on the interval I that satisfies:
$t_0 \in I$
x(t$_0$) $= x_0$
$x(t)$ is a solution of this IVP on I

4. Continuity and the derivative for function $f : R^n \to R^n$

a. Definition of a continuous function

Suppose $f : E \to R^n$, $x \in E$. We say that f is continuous at x if the following condition holds: $\forall \epsilon > 0$, $\exists \delta > 0$ if $y \in E$ and $|x - y| < \delta$, then:

$$|f(x) - f(y)| < \epsilon$$

Furthermore, f is continuous on E if f is continuous at each point of E

b. Differential function

Suppose $E \in R^n$, $f : E \to R^n$ is continuous. We say f is differential at x_0 if there is an linear transformation $Df(x_0)$ such that:

$$\lim_{|h| \to 0} = \frac{||f(x_0 + h) - f(x_0) - Df(x_0)h||}{||h||} = 0$$

In this case, $Df(x_0)$ is the derivative of f at x_0. Furthermore, if f is differentiable at each point of E, then f is differentiable on E.
The derivative Df is given by $n \times n$ Jacobian matrix

$$Df = \left[\frac{\partial f_j}{\partial x_j}\right]$$

5. $C^0(E), C^1(E)$
a. $C^0(E)$
For any open set $E \subset R^n$, we let:

$$C(E) = \{f : E \to R^n, \text{f is continuous on E}\}$$

b. $C^1(E)$
For any open set $E \subset R^n$, we let:

$$C^1(E) = \{f : E \to R^n, \text{f is differentiable on E }, \frac{\partial f_i}{\partial x_j} \text{are continuous}\}$$

6. Locally Lipschitz
Suppose $E \subset R^n$, $f \in C^1(E)$. For any closed ball $B \subset E$, there exits $K \geq 0$, $\forall x, y \in B$, we have:

$$|f(x) - f(y)| \leq K|x - y|$$

7. Peano Theorem
Suppose $E \subset R^n$, $f : E \to R^n$ is continuous. For any initial condition $(t_0, x_0) \in R \times E$, the IVP:

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_0) &= x_0 \end{aligned}$$

has at least one solution $x(t)$ on the interval I containing t_0.

8. Fundamental Theorem of Existence and Uniqueness
Suppose $E \subset R^n$, $f \in C^1(E)$. For any initial condition $(t_0, x_0) \in R^n \times E$, there exits $a > 0$, the IVP:

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_0) &= x_0 \end{aligned}$$

has a unique solution x(t) on the interval of the form $[t_0 - a, t_0 + a]$.

9. Integral Equations
Suppose $E \subset R^n$, $f : E \to R^n$ is continuous. Consider IVP:

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_0) &= x_0 \end{aligned}$$

has a solution $x(t)$ on the interval containing t_0 if and only if it satisfies:

$$x(t) = x_0 + \int_0^t f(x(s))ds$$

10. Successive approximations:
Apply the formula

$$u_{k+1} = x_0 + \int_0^t f(u_k(s))ds$$

to solve the IVP:

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_0) &= x_0 \end{aligned}$$

11. Dependence on initial conditions and parameters:
a. Dependence on initial conditions:
Suppose $E \subset R^n$, $f \in C^1(E)$, $x_0 \in E$. Then $\exists a > 0, \epsilon > 0$. $\forall y \in B(x_0, \epsilon)$, the IVP:

$$\dot{x} = f(x)$$
$$x(t_0) = y$$

has a unique solution $u(t, y)$ on the interval containing t_0.
$u(t, y) \in C^1(G)$ where $G = [-a, a] \times B(x_0, \epsilon)$.
$\forall y \in B$, $u(t, y)$ has twice continuously differentiable in t.

b. Dependence on parameters:
Suppose $E \subset R^{n+m}$, $f \in C^1(E)$, $(x_0, \mu_0) \in E$. Then $\exists a > 0, \epsilon > 0$, $\forall y \in B(x_0, \epsilon)$ and $\forall \mu \in B(\mu_0, \epsilon)$, the IVP:

$$\dot{x} = f(x)$$
$$x(t_0) = y$$

has a unique solution $u(t, y, \mu)$ and $u(t, y, \mu) \in C^1(G)$ where $G = [-a, a] \times B(x_0, \epsilon) \times B(\mu_0, \epsilon)$

12. Gronwall's Inequality:
Suppose $g(t)$ is continuous, $g(t) \geq 0$ $\forall t \in [0, a]$. It satisfies:

$$g(t) \leq C + K \int_0^t g(s) ds$$

We have:

$$g(t) \leq Ce^{Kt}$$

Proof:
Define $G(t) = C + K \int_0^t g(s) ds$
$G(t) \geq g(t)$ and $G(t) > 0$ $\forall t \in [0, a]$.
By FTC, we have:

$$G'(t) = Kg(t)$$

Then,

$$\frac{d}{dt}(\ln G(t)) = \frac{G'(t)}{G(t)} = \frac{Kg(t)}{G(t)} \leq \frac{KG(t)}{G(t)} = K$$

Integrate both sides from 0 to t, we have:

$$\ln G(t) - \ln G(0) \leq Kt$$

then, $\ln G(t) \leq \ln G(0) + Kt$
$\Rightarrow$g(t) $\leq G(t) \leq Ce^{Kt}$

13. Maximal interval of existence
a. Definition:
An IVP has a maximal interval of existence if there exits a solution $x(t)$ this IVP on the interval J such that if $y(t)$ is another solution on the interval I, $I \subset J$, and $x(t) = y(t), \forall t \in I$.

b. Theorem:
Suppose $E \subset R^n$, $f \in C^1(E)$, $x_0 \in E$. The IVP:

$$\dot{x} = f(x)$$
$$x(t_0) = y$$

has a maximal interval of existence J, J is open.

14. Escape from the compact set
Suppose $E \subset R^n$, $f \in C^1(E)$, $x_0 \in E$. Let (α, β) is the maximal interval of existence of the IVP:

$$\dot{x} = f(x)$$
$$x(t_0) = y$$

If $\beta < \infty$, there is any compact set $K \subset E$, $\forall t \in (\alpha, \beta)$, $x(t) \notin K$.
Corollary 1: If $\beta < \infty$, limit when $t \to \beta-$ exits, $x(t) \in K$(compact set), then the limit point $x_1 \in \partial E$.
Corollary 2: K is compact, for all $t \in (\alpha, \beta), x(t) \in K$, then $\beta = \infty$

2 Linear systems (Sections 1.1 - 1.5)

1. Linear system of equations
Let A be a nn matrix, $x_0 \in R^n$:

$$\dot{x} = Ax$$
$$x(0) = x_0$$

Since $f(x) = Ax$ is linear, then the IVP is called a linear system of equations.

2. Fundamentals Theorem for Linear Systems
Let A be a $n \times n$ matrix, $x_0 \in R^n$:

$$\dot{x} = Ax$$
$$x(0) = x_0$$

has a unique solution $x(t) = e^{At}x_0, \forall t \in R$