# How does gradient descent work?

# The loss “landscape”

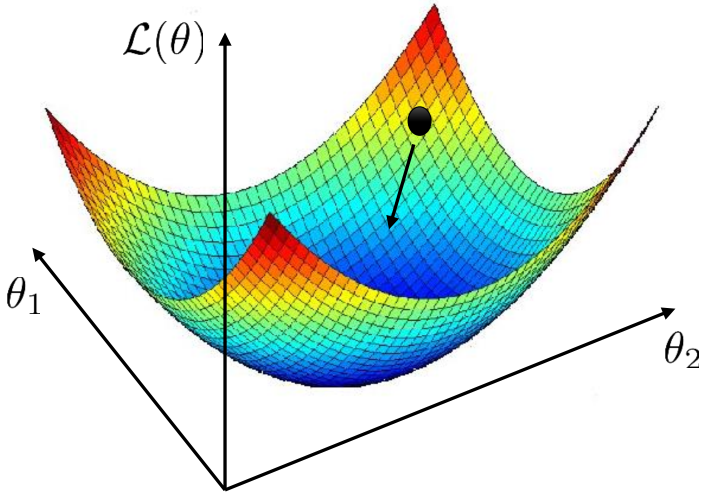
$$\theta^\star \leftarrow \arg\min_\theta \underbrace{-\sum_i \log p_\theta(y_i|x_i)}_{\mathcal{L}(\theta)}$$

let’s say $\theta$ is 2D

An algorithm:

1. Find a *direction* $v$ where $\mathcal{L}(\theta)$ decreases
2. $\theta \leftarrow \theta + \alpha v$

some small constant called “learning rate” or “step size”

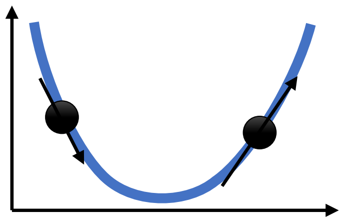


# Gradient descent

An algorithm:

1. Find a *direction* $v$ where $\mathcal{L}(\theta)$ decreases
2. $\theta \leftarrow \theta + \alpha v$

Which way does $\mathcal{L}(\theta)$ decrease?



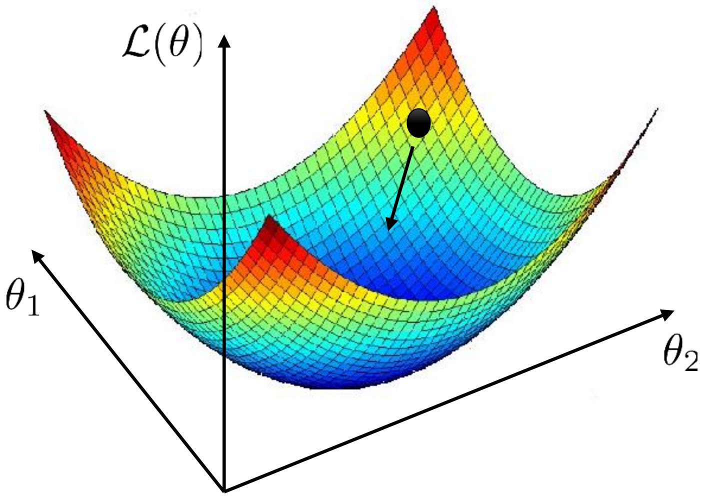
negative slope = go to the right

positive slope = go to the left

in general:

for each dimension, go in the direction opposite the slope **along that dimension**

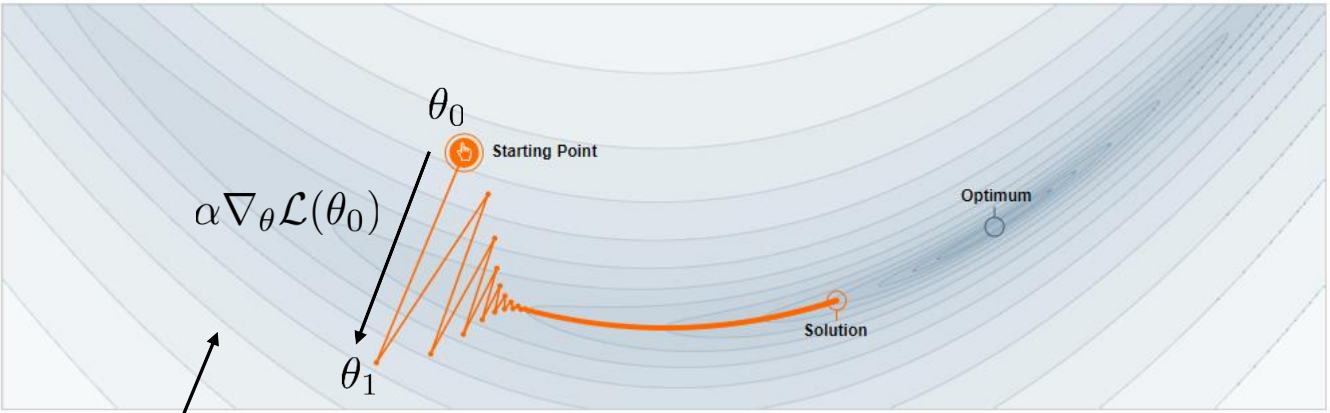
$$v_1 = -\frac{d\mathcal{L}(\theta)}{d\theta_1} \qquad v_2 = -\frac{d\mathcal{L}(\theta)}{d\theta_2} \qquad \text{etc.}$$



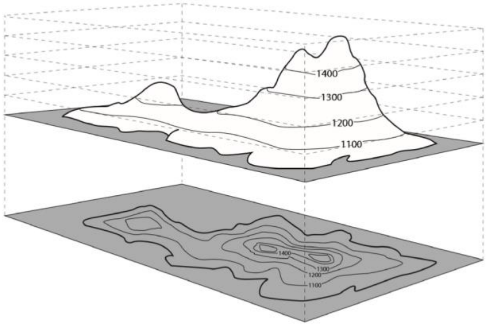
gradient:

$$\nabla_\theta \mathcal{L}(\theta) = \begin{pmatrix} \frac{d\mathcal{L}(\theta)}{d\theta_1} \\ \frac{d\mathcal{L}(\theta)}{d\theta_2} \\ \vdots \\ \frac{d\mathcal{L}(\theta)}{d\theta_n} \end{pmatrix}$$

# Visualizing gradient descent

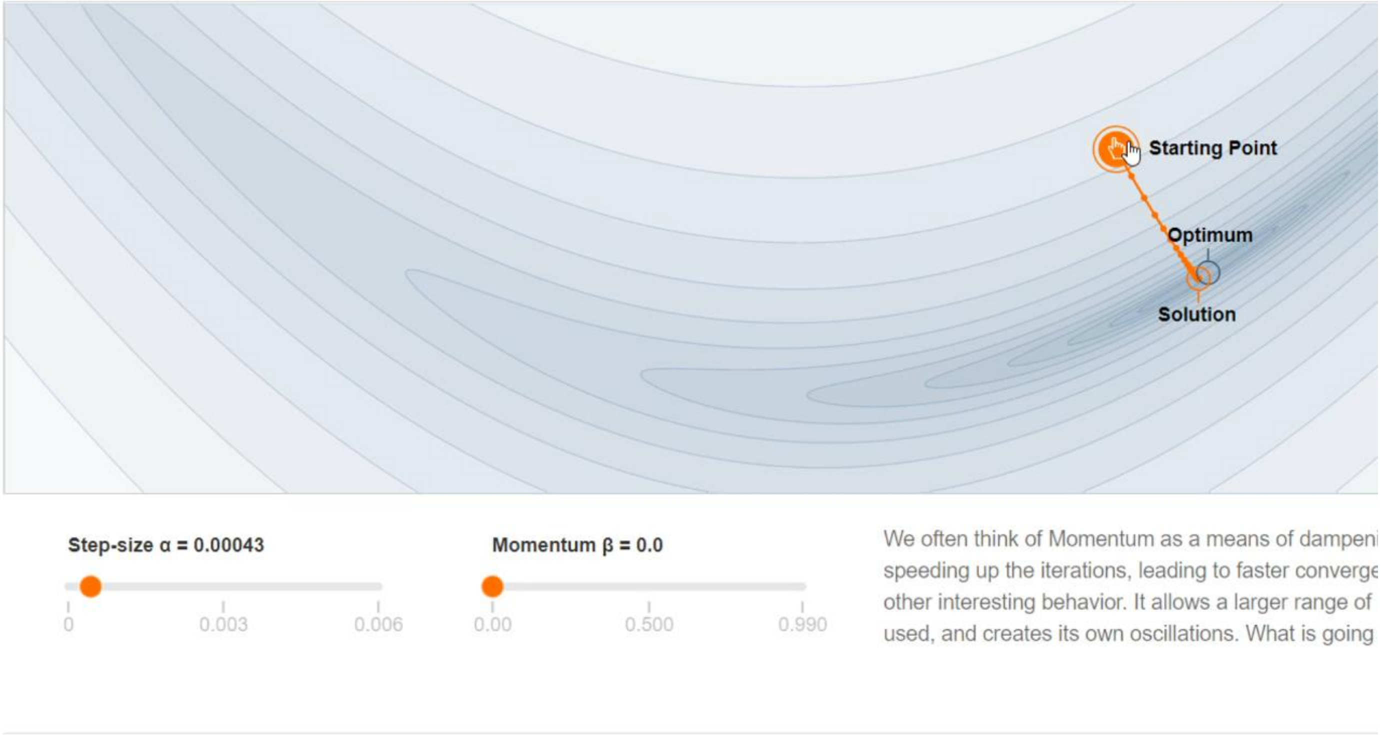


level set contours
for all $\theta$ values along a line
$\mathcal{L}(\theta)$ takes on the same value



visualizations based on Gabriel Goh’s distill.pub article: https://distill.pub/2017/momentum/

# Demo time!



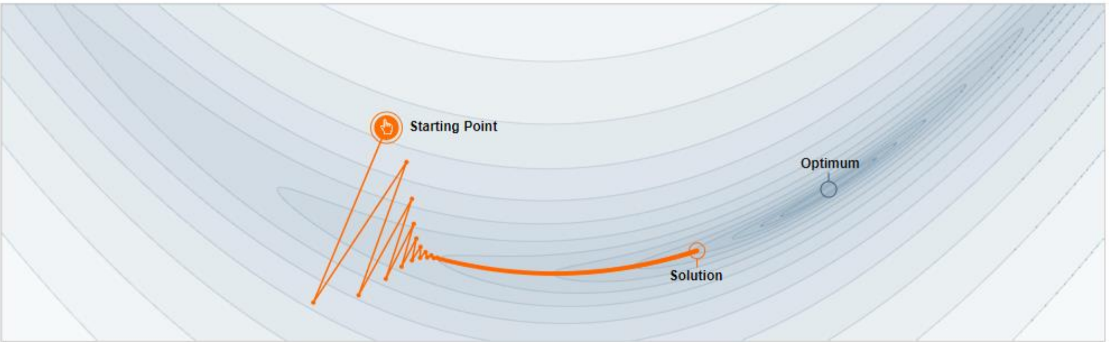
Step-size α = 0.00043

Momentum β = 0.0

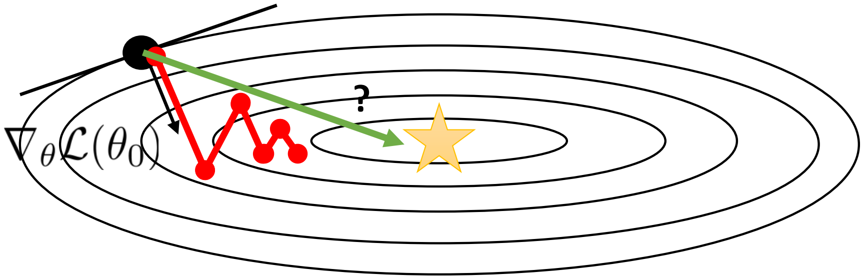
We often think of Momentum as a means of dampeni
speeding up the iterations, leading to faster converge
other interesting behavior. It allows a larger range of
used, and creates its own oscillations. What is going

visualizations based on Gabriel Goh’s distill.pub article: https://distill.pub/2017/momentum/

# What's going on?



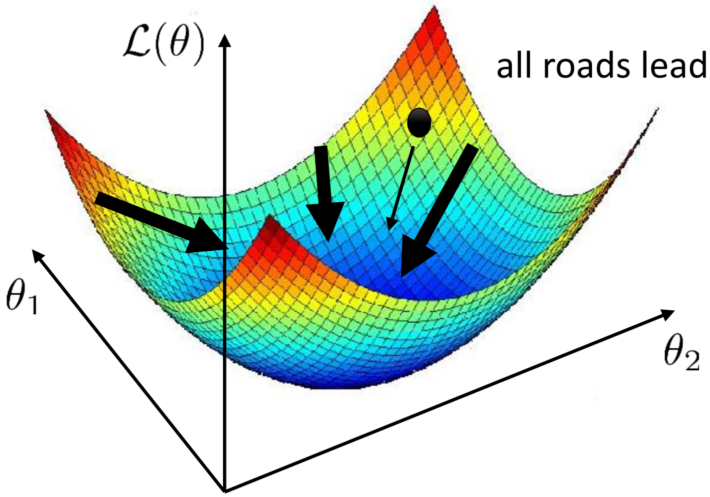
we don't always move toward the optimum!



the steepest direction is not always best!

more on this later…

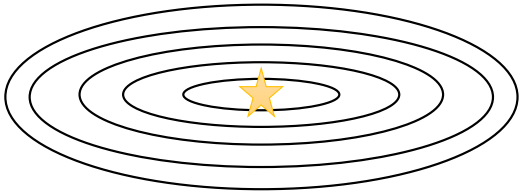
# The loss surface



all roads lead to Rome

This is a *very* nice loss surface    Why?

Is our loss actually this nice?



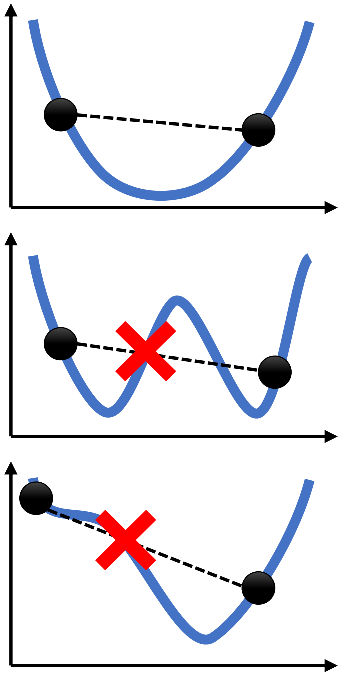
Logistic regression:

$$p_\theta(y = i|x) = \frac{\exp(x^T\theta_i)}{\sum_{j=1}^{m}\exp(x^T\theta_j)}$$

Negative likelihood loss for **logistic regression** is guaranteed to be **convex**

(this is **not** an obvious or trivial statement!)

Convexity:

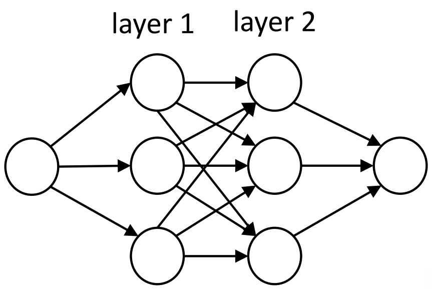


a function is convex if a line segment between any two points lies entirely “above” the graph

convex functions are “nice” in the sense that simple algorithms like gradient descent have strong guarantees

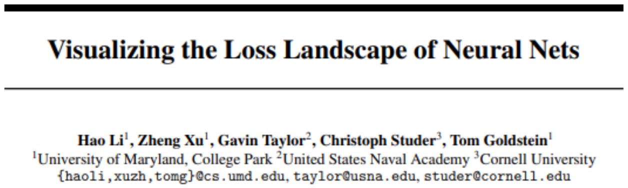
the **doesn’t** mean that gradient descent works well for all convex functions!

# The loss surface…
# …of a neural network



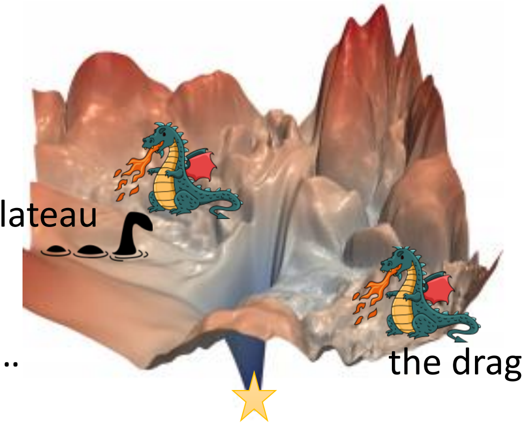
pretty hard to visualize, because neural networks have very large numbers of parameters

but let's give it a try!



**Visualizing the Loss Landscape of Neural Nets**

Hao Li[1], Zheng Xu[1], Gavin Taylor[2], Christoph Studer[3], Tom Goldstein[1]
[1]University of Maryland, College Park [2]United States Naval Academy [3]Cornell University
{haoli,xuzh,tomg}@cs.umd.edu, taylor@usna.edu, studer@cornell.edu

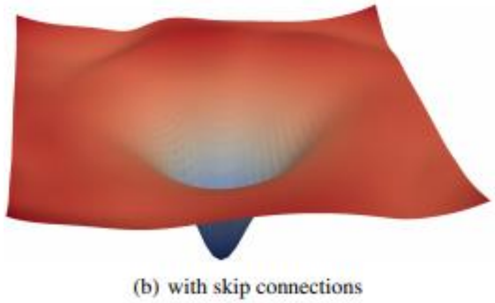


the monster of the plateau

Oh no…

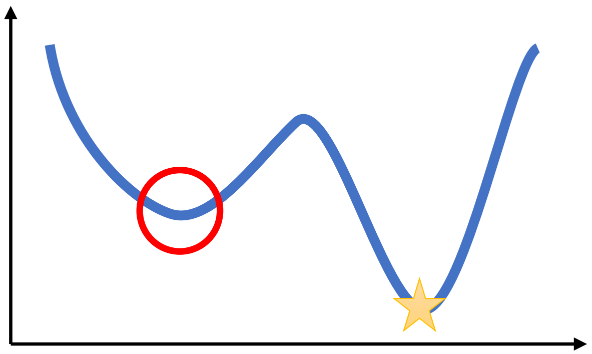
the dragon of local optima

…though some networks are better!

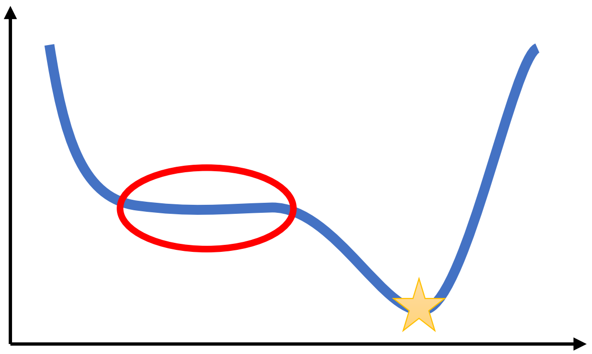


(b) with skip connections

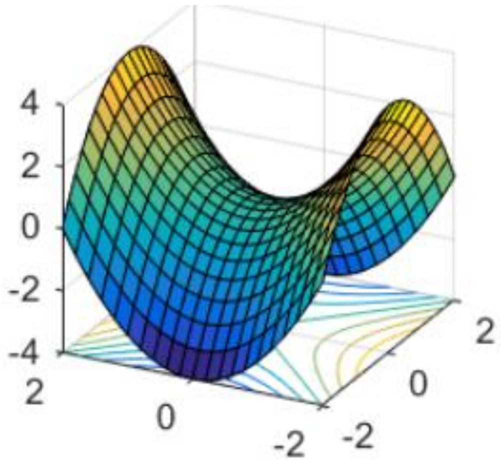
# The geography of a loss landscape



the local optimum

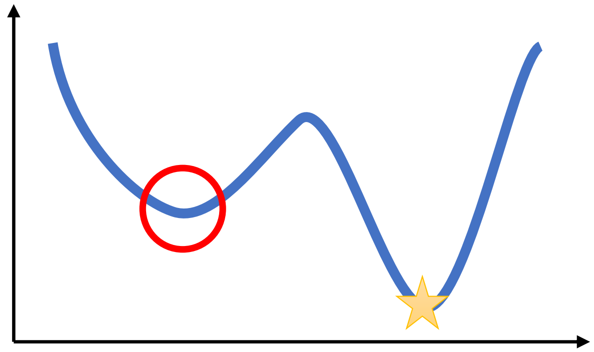


the plateau



the saddle point

# Local optima



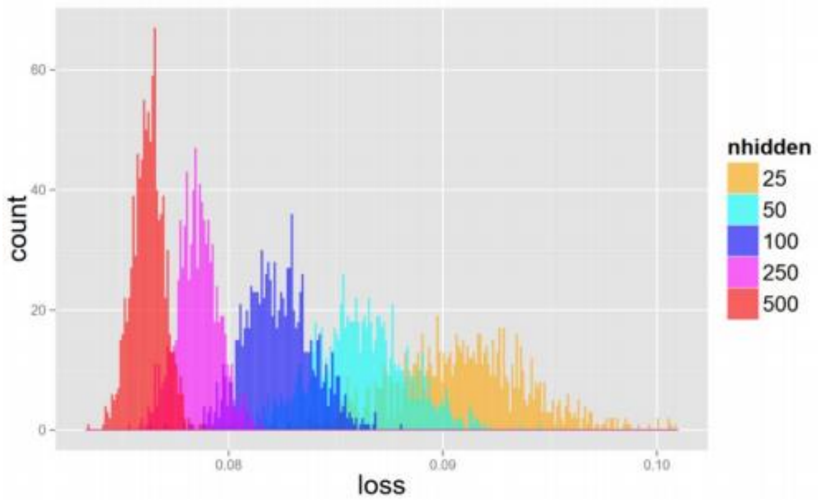
the most obvious issue with non-convex loss landscapes

one of the big reasons people used to worry about neural networks!

**very** scary in principle, since gradient descent could converge to a solution that is arbitrarily worse than the global optimum!

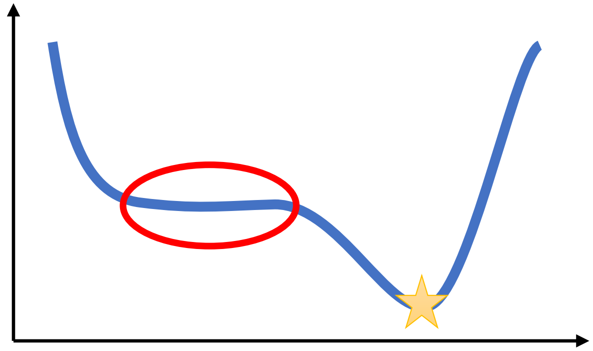
a bit surprisingly, this becomes less of an issue as the number of parameters increases!

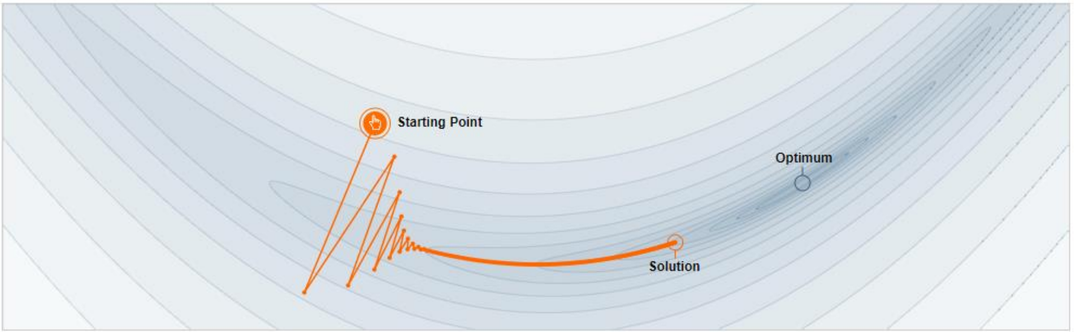
for big networks, local optima exist, but tend to be not much worse than global optima



Choromanska, Henaff, Mathieu, Ben Arous, LeCun.
**The Loss Surface of Multilayer Networks.**

# Plateaus



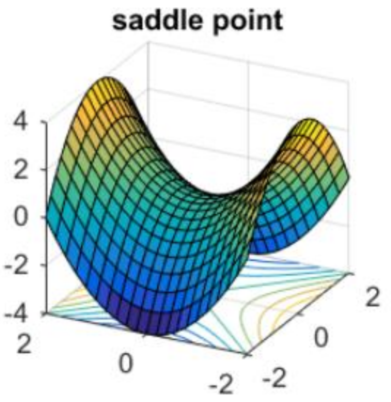


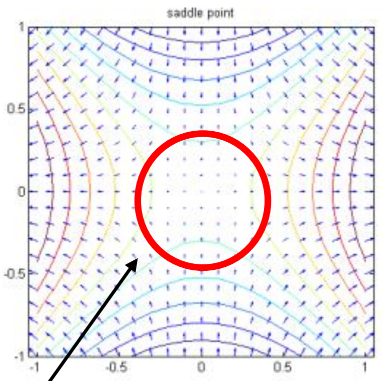
Can’t just choose tiny learning rates to prevent oscillation!

Need learning rates to be large enough not to get stuck in a plateau

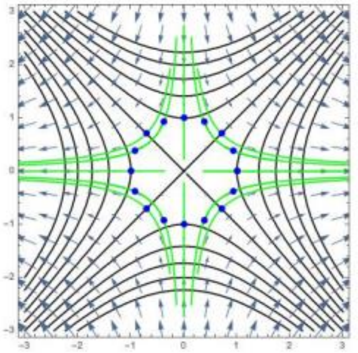
We’ll learn about **momentum**, which really helps with this

# Saddle points





Gradient vectors



Gradient flows (green)

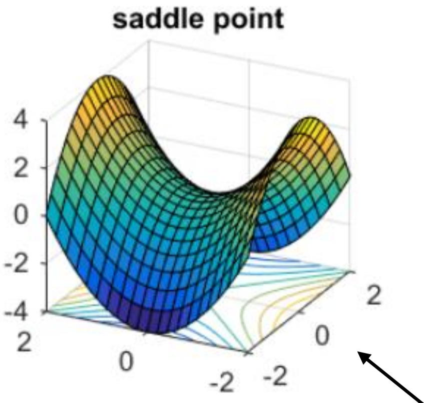
the gradient here is very small

it takes a long time to get out of saddle points

this seems like a **very** special structure, does it really happen **that** often?

**Yes!** in fact, most critical points in neural net loss landscapes are saddle points

# Saddle points



Critical points:

any point where $\nabla_\theta \mathcal{L}(\theta) = 0$

is it a **maximum**, **minimum**, or **saddle**?

In higher dimensions:

Hessian matrix:

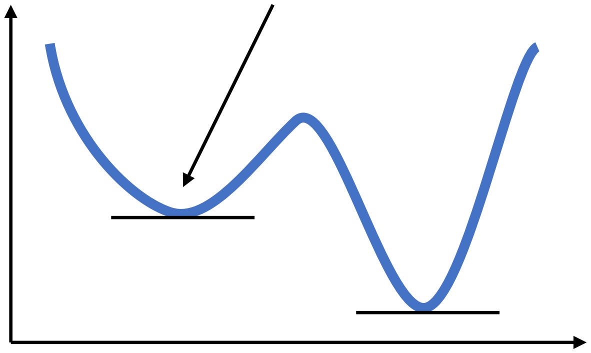
$$\begin{bmatrix} \frac{d^2\mathcal{L}}{d\theta_1 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_1 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_1 d\theta_3} \\ \frac{d^2\mathcal{L}}{d\theta_2 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_2 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_2 d\theta_3} \\ \frac{d^2\mathcal{L}}{d\theta_3 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_3 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_3 d\theta_3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

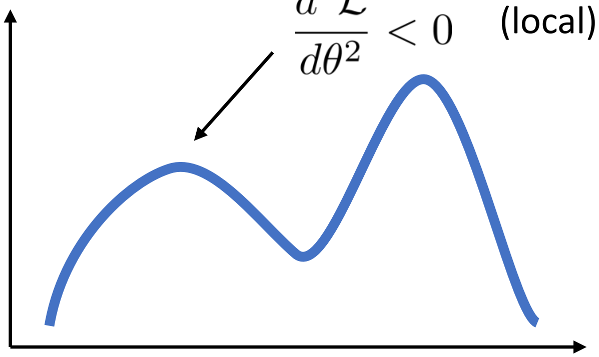
only maximum or minimum if all diagonal entries are positive or negative!

how often is that the case?

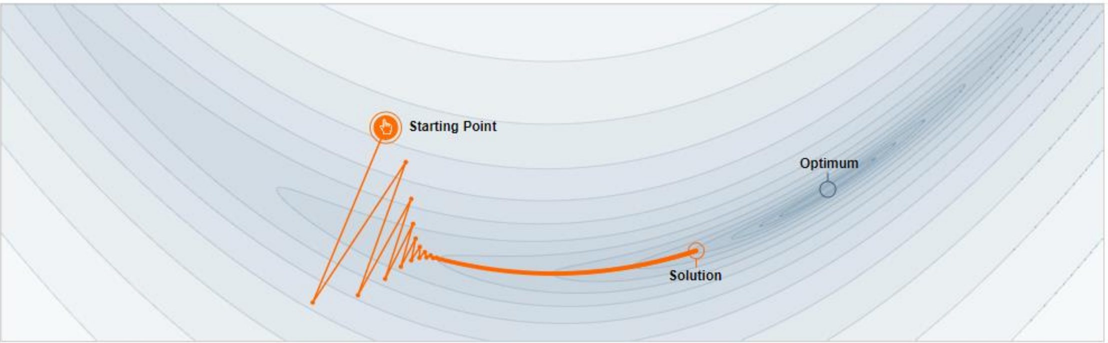
$\frac{d^2\mathcal{L}}{d\theta^2} > 0$ (local) minimum



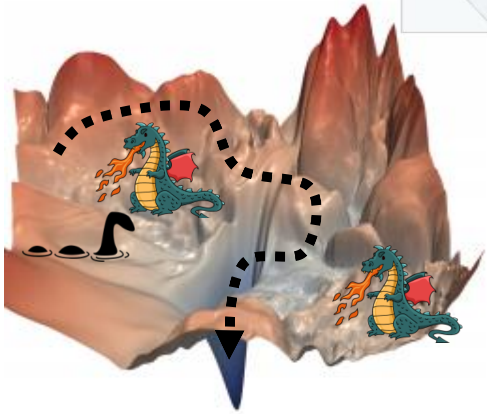
$\frac{d^2\mathcal{L}}{d\theta^2} < 0$ (local) maximum

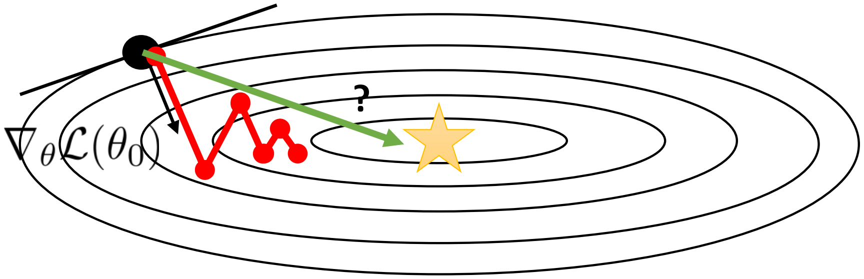


# Which way do we go?



we don’t always move toward the optimum!



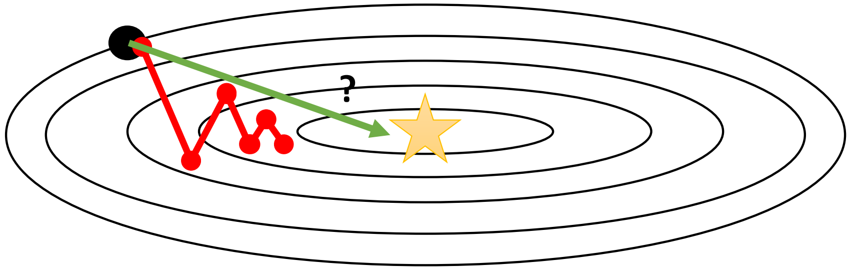


the steepest direction is not always best!

more on this later…

# Improvement directions

# A better direction…



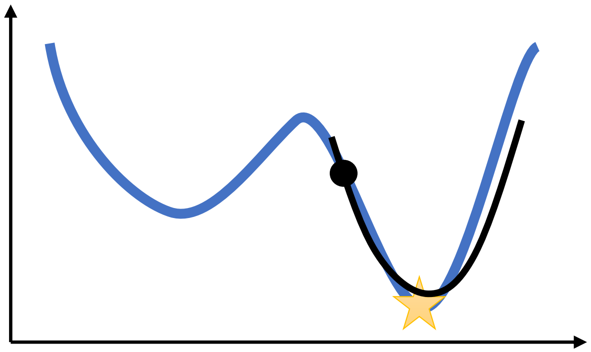
can we find this direction?

yes, with Newton's method!

we won't use Newton's method (can't afford it)

but it's an "ideal" to aspire to

# Newton's method

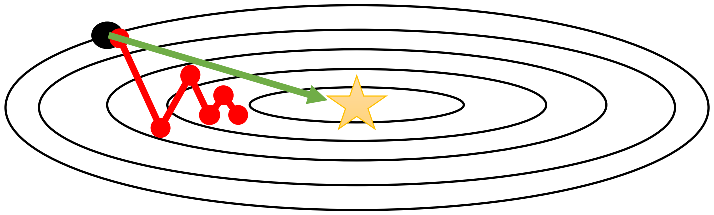


Taylor expansion:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

multivariate case:

$$\mathcal{L}(\theta) \approx \mathcal{L}(\theta_0) + \underbrace{\nabla_\theta \mathcal{L}(\theta_0)}_{\text{gradient}}(\theta - \theta_0) + \frac{1}{2}(\theta - \theta_0)^T \underbrace{\nabla_\theta^2 \mathcal{L}(\theta_0)}_{\text{Hessian}}(\theta - \theta_0)$$



$$\begin{bmatrix} \frac{d^2\mathcal{L}}{d\theta_1 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_1 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_1 d\theta_3} \\ \frac{d^2\mathcal{L}}{d\theta_2 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_2 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_2 d\theta_3} \\ \frac{d^2\mathcal{L}}{d\theta_3 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_3 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_3 d\theta_3} \end{bmatrix}$$

can optimize this analytically!

set derivative to zero and solve:

$$\theta^\star \leftarrow \theta_0 - (\nabla_\theta^2 \mathcal{L}(\theta_0))^{-1} \nabla_\theta \mathcal{L}(\theta_0)$$

# Tractable acceleration

Why is Newton's method not a viable way to improve neural network optimization?

$$\text{gradient descent: } \theta_{k+1} \leftarrow \theta_k - \alpha \nabla_\theta \mathcal{L}(\theta_k) \qquad \text{runtime?} \quad \mathcal{O}(n)$$

$$\nabla_\theta \mathcal{L}(\theta) = \left.\begin{pmatrix} \frac{d\mathcal{L}(\theta)}{d\theta_1} \\ \frac{d\mathcal{L}(\theta)}{d\theta_2} \\ \vdots \\ \frac{d\mathcal{L}(\theta)}{d\theta_n} \end{pmatrix}\right\updownarrow n$$

Hessian

$$\underbrace{\left.\begin{bmatrix} \frac{d^2\mathcal{L}}{d\theta_1 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_1 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_1 d\theta_3} \\ \frac{d^2\mathcal{L}}{d\theta_2 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_2 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_2 d\theta_3} \\ \frac{d^2\mathcal{L}}{d\theta_3 d\theta_1} & \frac{d^2\mathcal{L}}{d\theta_3 d\theta_2} & \frac{d^2\mathcal{L}}{d\theta_3 d\theta_3} \end{bmatrix}\right\updownarrow n}_{n}$$

$$\theta^\star \leftarrow \theta_0 - (\nabla_\theta^2 \mathcal{L}(\theta_0))^{-1} \nabla_\theta \mathcal{L}(\theta_0)$$

$\text{runtime?}$

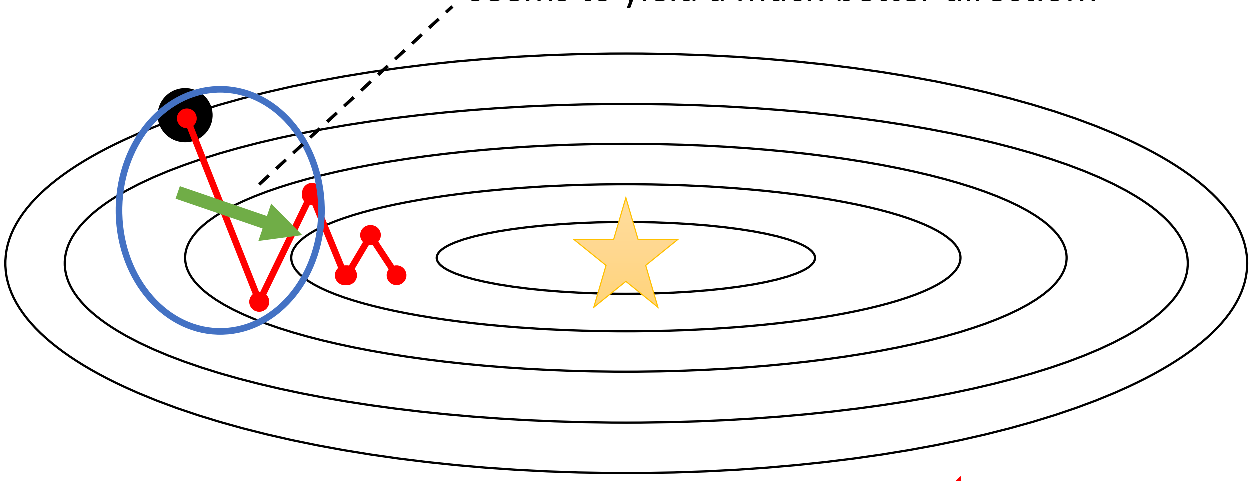
$\mathcal{O}(n^3)$

if using naïve approach, though fancy methods can be much faster if they avoid forming the Hessian explicitly

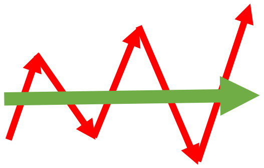
because of this, we would really prefer methods that don't require second derivatives, but somehow "accelerate" gradient descent instead

# Momentum

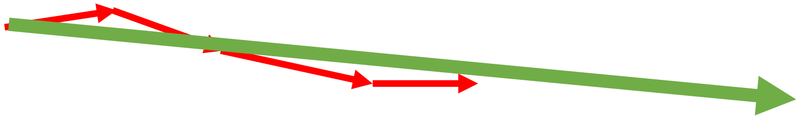
averaging together successive gradients
seems to yield a much better direction!



**Intuition:** if successive gradient steps point in **different** directions, we should **cancel off** the directions that disagree



if successive gradient steps point in **similar** directions, we should **go faster** in that direction



# Momentum

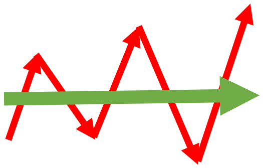
update rule:

$\theta_{k+1} = \theta_k - \alpha g_k$

before: $g_k = \nabla_\theta \mathcal{L}(\theta_k)$

now: $g_k = \nabla_\theta \mathcal{L}(\theta_k) + \underbrace{\mu g_{k-1}}$

"blend in" previous direction



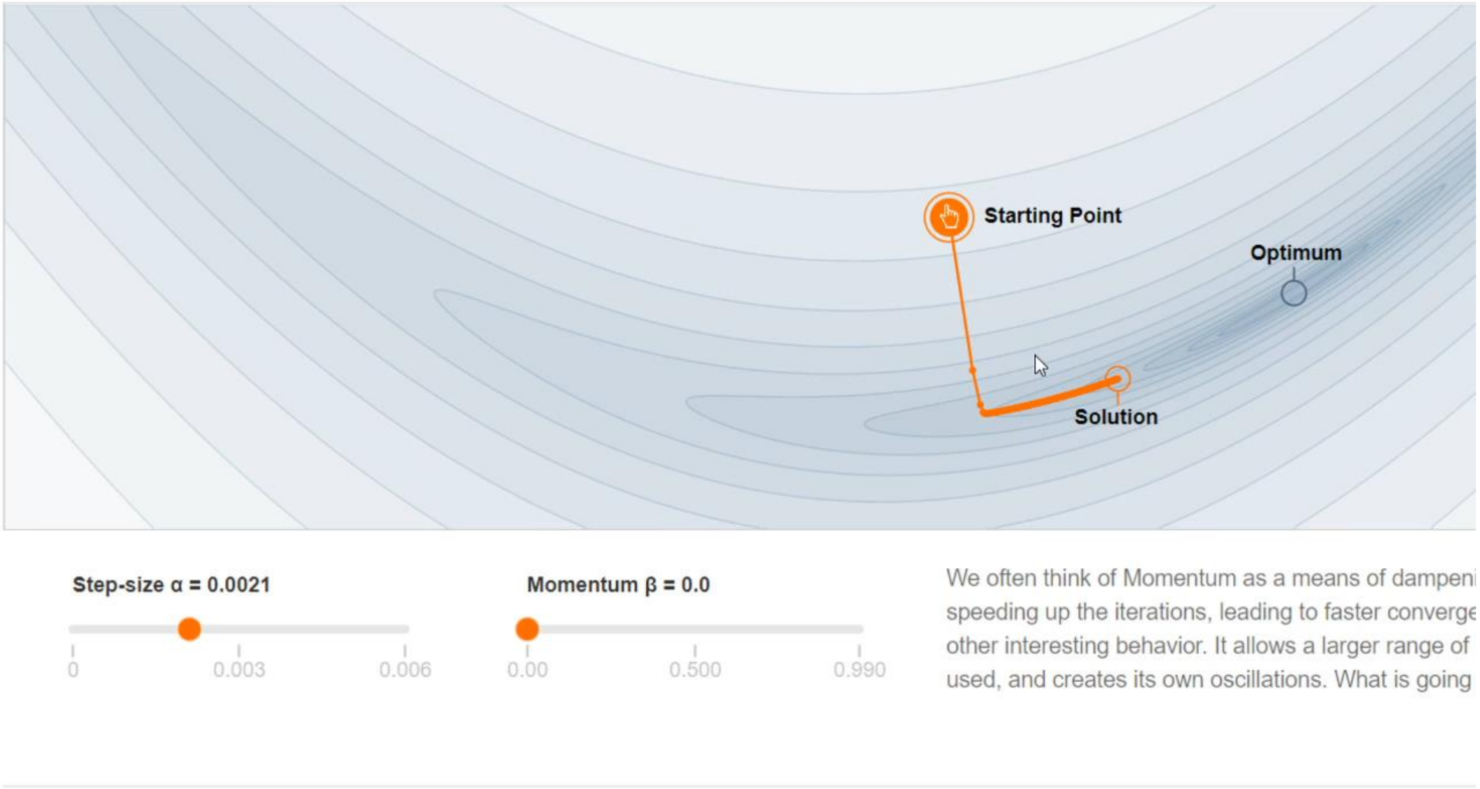
this is a **very** simple update rule

in practice, it brings some of the benefits of Newton's method, at virtually no cost

this kind of momentum method has few guarantees

a closely related idea is "Nesterov accelerated gradient," which **does** carry very appealing guarantees (in practice we usually just momentum)

# Momentum Demo



Step-size α = 0.0021

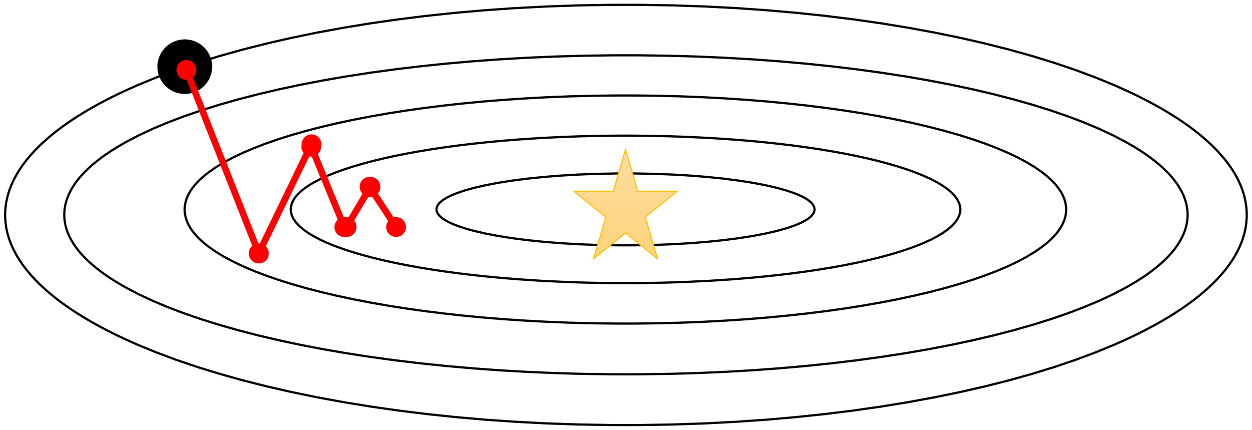
Momentum β = 0.0

We often think of Momentum as a means of dampeni
speeding up the iterations, leading to faster converge
other interesting behavior. It allows a larger range of
used, and creates its own oscillations. What is going

visualizations based on Gabriel Goh's distill.pub article: https://distill.pub/2017/momentum/

# Gradient scale

$$\nabla_\theta \mathcal{L}(\theta) = \begin{pmatrix} \frac{d\mathcal{L}(\theta)}{d\theta_1} \\ \frac{d\mathcal{L}(\theta)}{d\theta_2} \\ \vdots \\ \frac{d\mathcal{L}(\theta)}{d\theta_n} \end{pmatrix}$$



$$\mathcal{L}(\theta) = ||f_\theta(x) - y||^2$$

$$\nabla_\theta \mathcal{L}(\theta) = \underbrace{(f_\theta(x) - y)^T}_{\text{huge when far from optimum}} \frac{df}{d\theta}$$

**Intuition**: the **sign** of the gradient tells us which way to go along each dimension, but the magnitude is not so great

**Even worse**: overall magnitude of the gradient can change drastically over the course of optimization, making learning rates hard to tune

**Idea**: “normalize” out the magnitude of the gradient **along each dimension**

# Algorithm: RMSProp

Estimate per-dimension magnitude (running average):

$$s_k \leftarrow \beta s_{k-1} + (1 - \beta)(\nabla_\theta \mathcal{L}(\theta_k))^2$$

this is *roughly* the squared length of each dimension

$$\theta_{k+1} = \theta_k - \alpha \frac{\nabla_\theta \mathcal{L}(\theta_k)}{\sqrt{s_k}}$$

each dimension is divided by its magnitude

# Algorithm: AdaGrad

Estimate per-dimension cumulative magnitude:

$$s_k \leftarrow s_{k-1} + (\nabla_\theta \mathcal{L}(\theta_k))^2$$

$$\theta_{k+1} = \theta_k - \alpha \frac{\nabla_\theta \mathcal{L}(\theta_k)}{\sqrt{s_k}}$$

RMSProp:

$$s_k \leftarrow \beta s_{k-1} + (1 - \beta)(\nabla_\theta \mathcal{L}(\theta_k))^2$$

How does AdaGrad and RMSProp compare?

AdaGrad has some appealing guarantees for **convex** problems

Learning rate effectively “decreases” over time, which is good for convex problems

But this only works if we find the optimum quickly before the rate decays too much

RMSProp tends to be much better for deep learning (and most non-convex problems)

# Algorithm: Adam

**Basic idea:** combine **momentum** and **RMSProp**

$$m_k = (1 - \beta_1)\nabla_\theta \mathcal{L}(\theta_k) + \beta_1 m_{k-1}$$

first moment estimate ("momentum-like")

$$v_k = (1 - \beta_2)(\nabla_\theta \mathcal{L}(\theta_k))^2 + \beta_2 v_{k-1}$$

second moment estimate

$$\hat{m}_k = \frac{m_k}{1 - \beta_1^k}$$

why? $m_0 = 0$ $v_0 = 0$

so early on these values will be small, and this correction "blows them up" a bit for small *k*

$$\hat{v}_k = \frac{v_k}{1 - \beta_2^k}$$

good default settings:

$\alpha = 0.001$

$\beta_1 = 0.9$

$\beta_2 = 0.999$

$$\theta_{k+1} = \theta_k - \alpha \frac{\hat{m}_k}{\sqrt{\hat{v}_k} + \epsilon}$$

small number to prevent division by zero

$$\epsilon = 10^{-8}$$

# Stochastic optimization

# Why is gradient descent expensive?

$$\mathcal{L}(\theta) = -\underbrace{\frac{1}{N}\sum_{i=1}^{N} \log p_\theta(y_i|x_i)}_{\text{requires summing over \textbf{all} datapoints in the dataset}} \approx -E_{p_{\text{data}}(x,y)}[\log p_\theta(y_i|x_i)] \approx -\frac{1}{B}\sum_{j=1}^{B} \log p_\theta(y_{i_j}|x_{i_j})$$

could simply use **fewer** samples, and still have a correct (unbiased) estimator

$B << N$

**ILSVRC (ImageNet), 2009:** 1.5 **million** images

# Stochastic gradient descent

with minibatches

1. Sample $\mathcal{B} \subset \mathcal{D}$

draw **B** datapoints at random from dataset of size **N**

2. Estimate $g_k \leftarrow -\nabla_\theta \frac{1}{B} \sum_{i=1}^{B} \log p(y_i|x_i, \theta) \approx \nabla_\theta \mathcal{L}(\theta)$

(where sum is over elements in $\mathcal{B}$)

3. $\theta_{k+1} \leftarrow \theta_k - \alpha g_k$

can also use momentum, ADAM, etc.

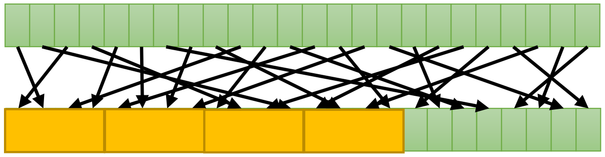
each iteration samples a different **minibatch**

Stochastic gradient descent **in practice:**

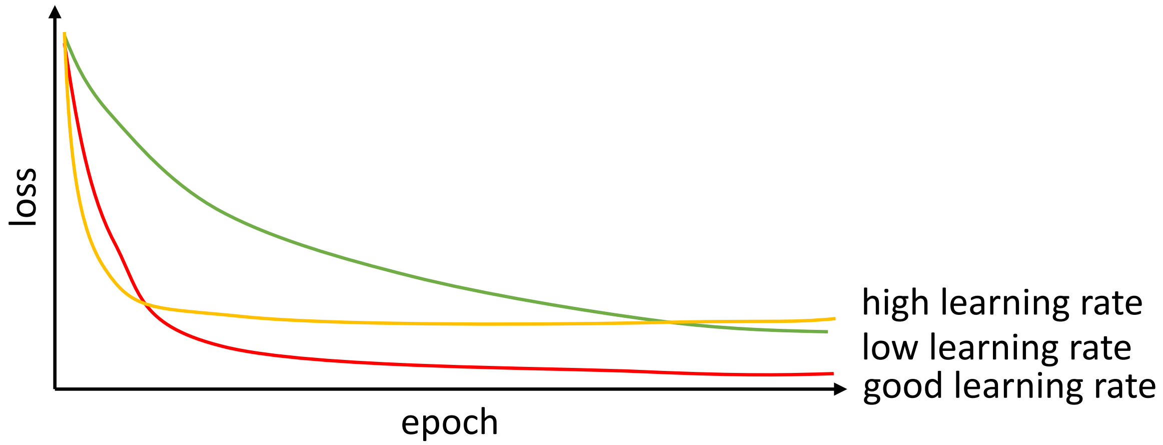
sampling randomly is slow due to random memory access

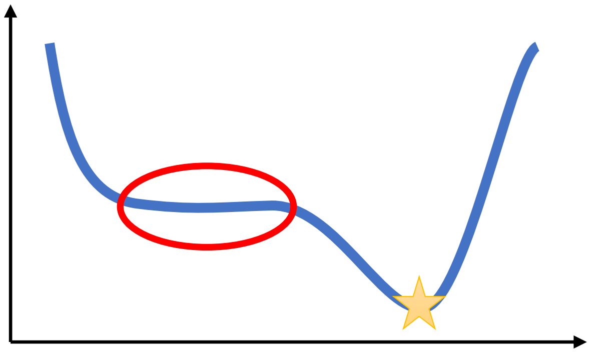
instead, shuffle the dataset (like a deck of cards…) once, in advance

then just construct batches out of consecutive groups of **B** datapoints

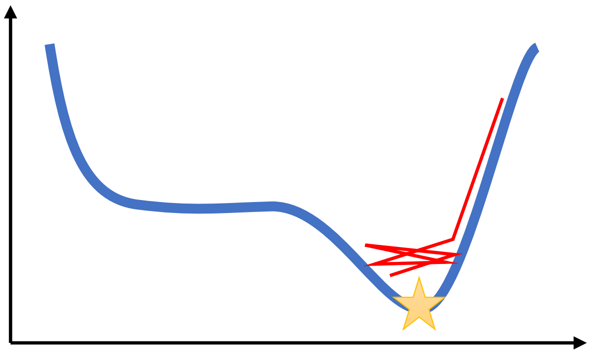


# Learning rates



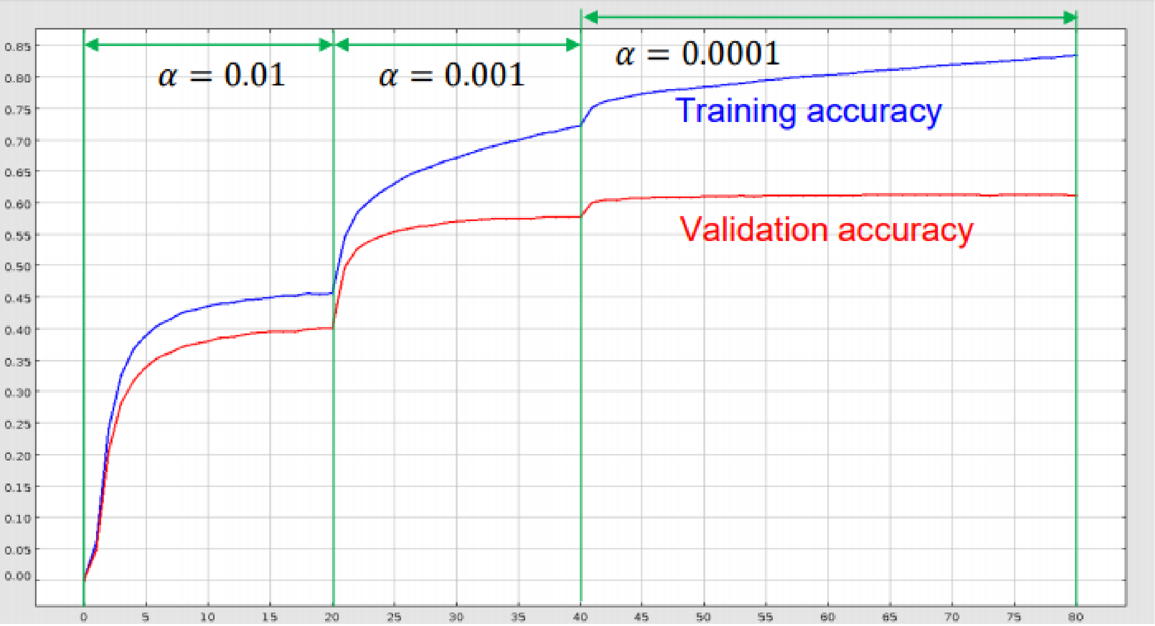


Low learning rates **can** result in convergence to worse values! This is a bit counter-intuitive



# Decaying learning rates

AlexNet trained on ImageNet



Learning rate decay schedules usually needed for best performance with SGD (+momentum)

Often not needed with ADAM

Opinions differ, some people think SGD + momentum is better than ADAM if you want the very best performance (but ADAM is easier to tune)

# Tuning (stochastic) gradient descent

**Hyperparameters:**

batch size: $B$ — larger batches = less noisy gradients, usually “safer” but more expensive

learning rate: $\alpha$ — best to use the biggest rate that still works, decay over time

momentum: $\mu$

0.99 is good

Adam parameters: $\beta_1, \beta_2$

keep the defaults (usually)

**What to tune hyperparameters on?**

Technically we want to tune this on the **training** loss, since it is a parameter of the optimization

Often tuned on **validation** loss

Relationship between stochastic gradient and regularization is
complex – some people consider it to be a good regularizer!
(this suggests we should use validation loss)