

Quantisation

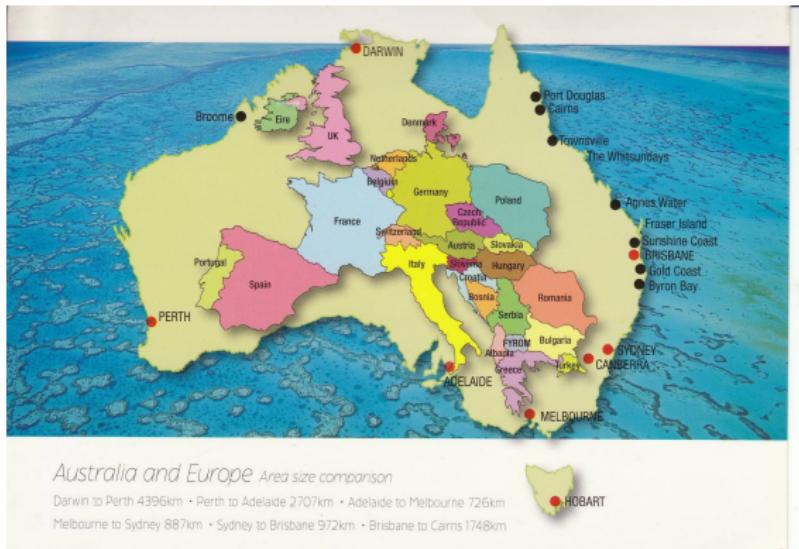
Efficient implementation of convolutional neural networks

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Australia



- Australia 25M, Guangdong 100M

Computer Engineering Lab

- Focuses on applying parallelism to solve demanding problems
 - Architectures, applications and design techniques using FPGAs, VLSI and high performance computers
 - Research: reconfigurable computing, machine learning, nanoscale interfaces



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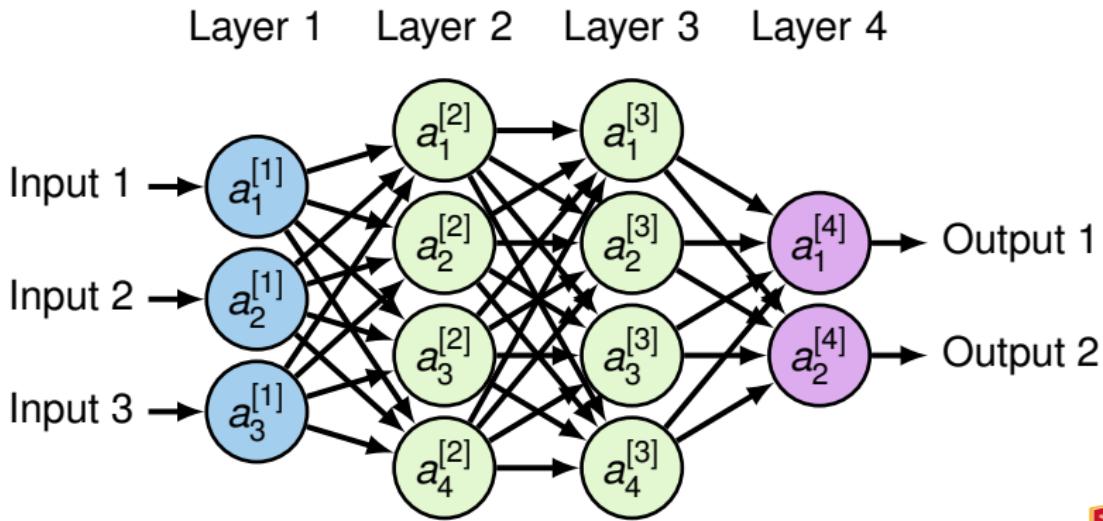
⑤ Tutorial

Introduction

- There are several degrees of freedom to explore when optimising DNNs
 - NN architecture (SqueezeNet, MobileNet, EfficientNet [TL19])
 - Compression (SVD, Deep Compression, Circulant [Din+17])
 - Quantization (FP16, TF-Lite, FINN, DoReFa-Net)
- This talk: quantisation

Backpropagation Algorithm [HH18]

- Forward Pass $z^{[i]} = W^{[i]} a^{[i-1]}$, $a^{[i]} = \sigma(z^{[i]})$
- dActivations $\delta^{[i]} = \text{diag}(\sigma'(z^{[i]}))(W^{[i+1]})^T \delta^{[i+1]}$
- dWeights $\Delta W^{[i]} = \delta^{[i]}(a^{[i-1]})^T$

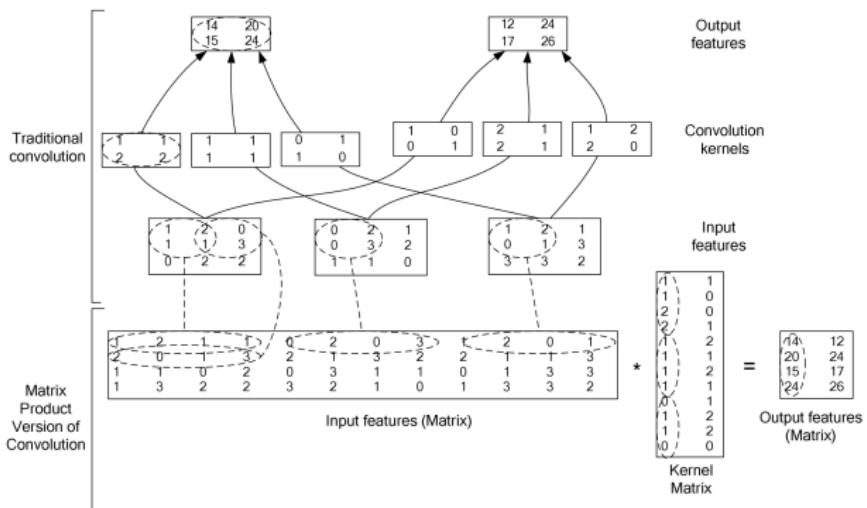


Backpropagation Algorithm [HH18]

```
1: function BACKPROP( $x, y$ )
2:    $a^{[1]} = x$ 
3:   for  $i = 2, \dots, L$  do                                 $\triangleright$  forward pass
4:      $z^{[i]} = W^{[i]} a^{[i-1]}$ 
5:      $a^{[i]} = \sigma(z^{[i]})$ 
6:   end for
7:    $\delta^{[L]} = diag(\sigma'(z^{[L]}))(a^{[L]} - y)$ 
8:   for  $i = L-1, \dots, 2$  do                       $\triangleright$  backward pass  $\delta$ 
9:      $\delta^{[i]} = diag(\sigma'(z^{[i]}))(W^{[i+1]})^T \delta^{[i+1]}$ 
10:  end for
11:  for  $i = L, \dots, 2$  do                       $\triangleright$  backward pass  $\Delta W$ 
12:     $\Delta W^{[i]} = \delta^{[i]}(a^{[i-1]})^T$ 
13:  end for
14:  return  $\Delta W$ 
15: end function
```

Convolution Layer as MM

- Convolution layers converted to GEMM [CPS06]
- Efficient BLAS libraries can be exploited



DNN Computation

Computational problem in DNNs is to compute a number of dot products

$$a = \sigma(\mathbf{w}^T \mathbf{x}) \quad (1)$$

where

- σ is an element-wise nonlinear activation function
- $\mathbf{x} \in \mathbb{R}^{i.w.h}$ is the input vector
- $\mathbf{w} \in \mathbb{R}^{i.w.h}$ is the weight vector

Arithmetic Intensity

- Computation of a DNN layer is MV multiplication
- For MV multiply is $O(1)$, for MM is $O(b)$ where b is block size
- Efficient CPU/GPU implementations use batch size $\gg 1$ (process a number of inputs together)
- For latency-critical applications (e.g. object detection for self-driving car), we want a batch size of 1
- **Make sure comparisons are at the same batch size!**

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Unsigned Numbers

$$\begin{aligned}U &= (u_{W-1} u_{W-2} \dots u_0), u_i \in \{0, 1\} \\&= \sum_{i=0}^{W-1} u_i 2^i\end{aligned}$$

- U is a W -bit unsigned integer
- Range $[0, 2^W)$
- Question: what value does 1010 represent?

Two's Complement Numbers

$$\begin{aligned} X &= (x_{W-1}x_{W-2}\dots x_0), x_i \in \{0, 1\} \\ &= -x_{W-1}2^{W-1} + \sum_{i=0}^{W-2} x_i 2^i \end{aligned}$$

- X is a W -bit signed integer
- Range $[-2^{W-1}, 2^{W-1})$
- Question: what value does 1010 represent?

Two's Complement Fractions

$$\begin{aligned} Y &= (\overbrace{y_{W-1} \dots y_F}^{\text{I-bit integer}} \overbrace{y_{F-1} \dots x_0}^{\text{F-bit fraction}}), y_i \in \{0, 1\} \\ &= 2^{-F} \times (-x_{W-1} 2^{W-1} + \sum_{i=0}^{W-2} x_i 2^i) \end{aligned}$$

- Y is a W -bit signed fraction with F -bit fraction
- Are two's complement numbers scaled by 2^{-F}
- Notation used: (I,F) (with $I + F = W$)
 - $(W,0)$ same as two's complement integers
 - $(1,W-1)$ has range $[-1,1]$ and multiplication never overflows
- Question: what does 1010 represent as a $(1,3)$ two's complement fraction?

Block Floating Point [CBD14]

$$D = (-1)^S \cdot 2^{-F} \sum_{i=0}^{W-2} x_i 2^i$$

- D is block floating point number with sign bit S , fractional length F , W is word length
- Sign-magnitude fraction with F being shared within a group
- In a tensor, can be shared for the entire tensor or along a dimension
- Allows range to be adapted to different network segments e.g. layer inputs, weights and outputs can have different F

Operations on Two's Complement Fractions

- Addition and subtraction same as two's complement
- Multiplication
 - An (I,F) multiplication gives a (2I,2F) result, need to discard F bits
 - For (1,3)

$$\begin{aligned}0.75 \times 0.75 &= 0.110 \times 0.110 \\&= 00.100100 \quad \text{in (2I,2F) format} \\&\approx 0.100 \quad \text{in (I,F) format (truncated)}\end{aligned}$$

- Integer part controls range
- Fractional part controls spacing between numbers

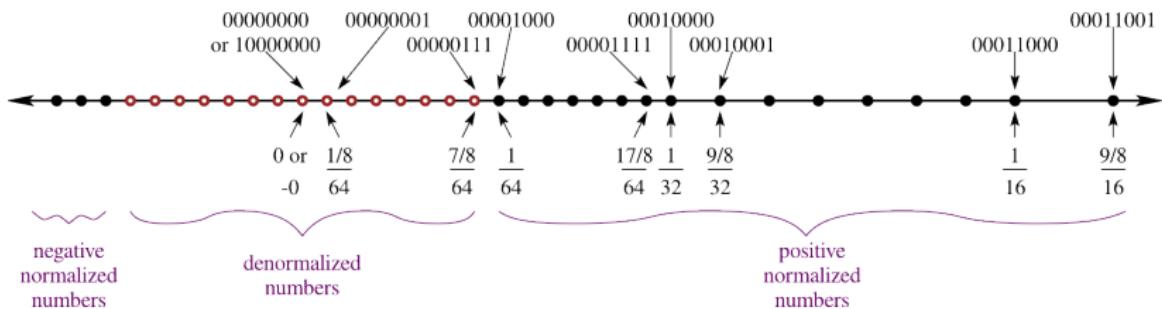
Floating Point 1

$$Z = (\overbrace{a_0}^A \overbrace{b_{J-1} \dots b_0}^B \overbrace{c_{F-1} \dots c_0}^C), (a_i, b_i c_i) \in \{0, 1\}$$

- Treating A, B and C as unsigned integers
 - The sign bit is $S = \begin{cases} +1 & \text{if } a_0 = 0 \\ -1, & \text{otherwise} \end{cases}$
 - The exponent is stored in a biased representation with $E = B - (2^{J-1} - 1)$
 - For normalised numbers, $B \neq 0$, and M is a positive (1,F) two's complement fraction $M = 1 + C2^{-F}$
 - For denormalised numbers $B = 0$ and there is no implicit 1 in the positive (0,F) two's complement fraction $M = C2^{-F}$

Floating Point 2

$$Z = \begin{cases} S \times 2^E \times M & \text{if } (0 < B < 2^J - 1) \\ S \times 2^E \times (M - 1) & \text{if } (B = 0) \\ S \times \infty & \text{if } (B = 2^J - 1 \text{ and } C = 0) \\ \text{NaN} & \text{if } (B = 2^J - 1 \text{ and } C \neq 0) \end{cases}$$



Operations on Floating Point Numbers

- Much larger resource utilisation
- Longer latency
- We will focus on fixed point

Stochastic Rounding

- Suppose we want to round 10.4 to an integer
- Stochastic rounding

$$r(x) = \begin{cases} \lfloor x \rfloor & \text{with probability } 1 - (x - \lfloor x \rfloor) \\ \lfloor x \rfloor + 1 & \text{with probability } x - \lfloor x \rfloor \end{cases}$$

- Implement by adding a random number $\in [0, 1)$ and truncating

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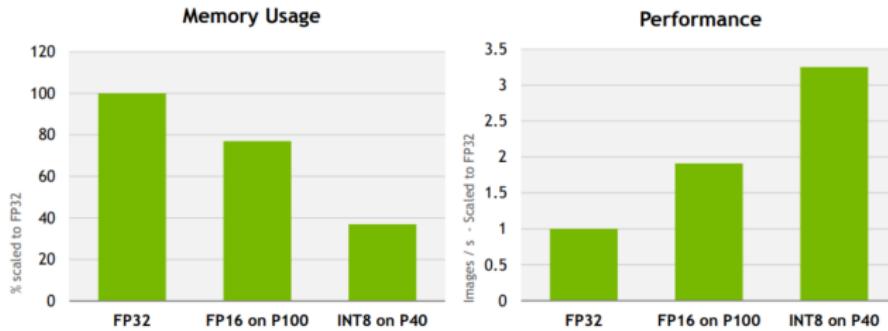
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Role of Wordlength on Performance

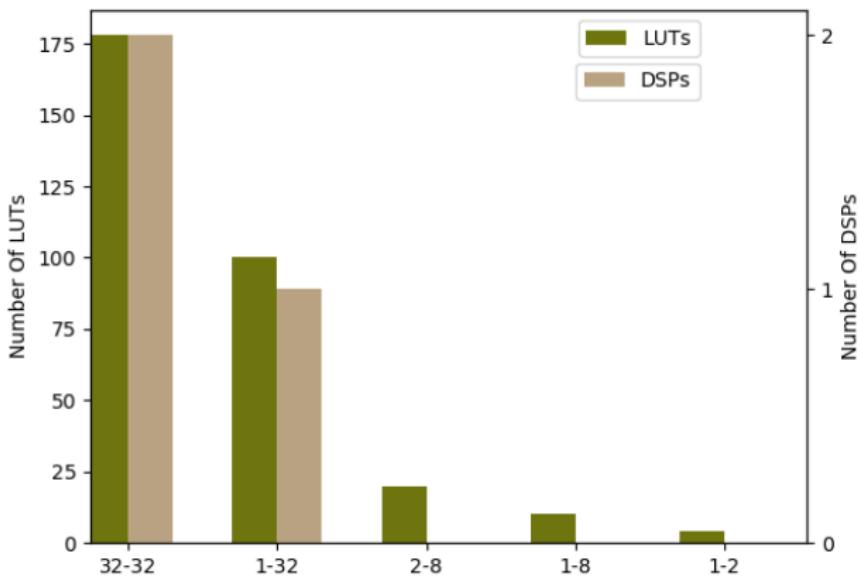
- CPU/GPU
 - Floating point performance comparable to fixed
 - Integer data types usually vectorisable hence faster
 - Nvidia offers FP64, FP32 and FP16 (> Tegra X1 and Pascal)
- FPGA
 - Datapath is flexible
 - No floating point unit so fixed point normally preferred



ResNet50 Model, Batch Size = 128, TensorRT 2.1 RC pre-release

Role of Wordlength on Resources

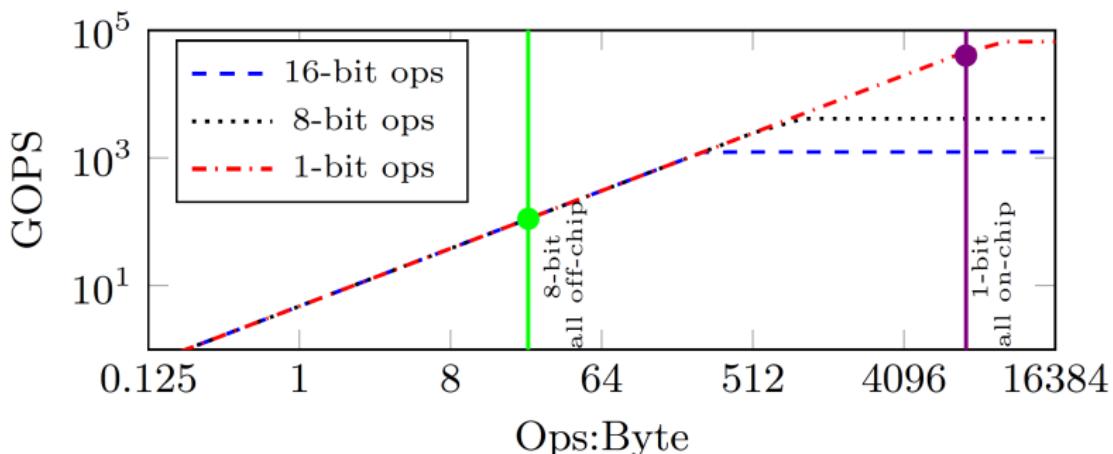
- X axis is bitwidth (weight-activation) and Y axis Number of LUTs/DSPs for MAC
- For k -bits, area is $\mathcal{O}(k^2)$



Roofline Model

Roofline model for Xilinx ZU19EG

- X axis is computational intensity (ops to perform / byte fetch), Y axis is performance
- Diagonal parts show memory-bandwidth limited space
- Horizontal parts show computation limited space
- Actually this is a better metric to optimise than say GOPs/s
- **Low precision extremely advantageous for performance**



Integer Quantization [Jac+18]

A way to map numbers $r \in \mathbb{R}$ to unsigned integers $q \in \mathbb{U}+$ is via an affine transformation

$$r = S(q - Z) \tag{2}$$

- $\mathbb{U}+$ is the set of unsigned W-bit integers
- S, Z are the quantisation parameters
 - $S \in \mathbb{R}+$ represents a scaling constant
 - $Z \in \mathbb{U}+$ represents a zero-point

Integer MM [Jac+18]

- $N \times N$ MM defined as

$$r_3^{(i,k)} = \sum_{j=1}^N r_1^{(i,j)} r_2^{(j,k)}, \quad (3)$$

substituting $r = S(q - Z)$ (2) and rewriting¹ we get

$$q_3^{(i,k)} = Z_3 + M \left(NZ_1 Z_2 - Z_1 a_2^{(k)} - Z_2 \bar{a}_1^{(i)} + \sum_{j=1}^N q_1^{(i,j)} q_2^{(j,k)} \right) \quad (4)$$

- Multiplication with $M = \frac{S_1 S_2}{S_3}$ is implemented in (high-precision) two's complement fixed point
- $a_2^{(k)}$ and $\bar{a}_1^{(i)}$ together only take $2N^2$ additions
- Sum in (4) takes $2N^3$ and is a standard integer MAC
- CPU implementation uses uint8, accumulated as int32

¹A good exercise is to derive Equation (4) from (3) and (2)

Quantisation Range [Jac+18]

For each layer, quantisation parameterised by (a,b,n):

$$\text{clamp}(r; a, b) = \min(\max(x, a), b)$$

$$s(a, b, n) = \frac{b - a}{n - 1}$$

$$q(r; a, b, n) = \text{rnd}\left(\frac{\text{clamp}(r; a, b) - a}{s(a, b, n)}\right)s(a, b, n) + a \quad (5)$$

where $r \in \mathbb{R}$ is number to be quantised, $[a, b]$ is quantisation range, n is number of quantisation levels and $\text{rnd}()$ rounds to nearest integer

Figure from [Jac+18] (with permission)

Training Algorithm [Jac+18]

- 1 Create training graph of the floating-point model
- 2 Insert quantisation operations for integer computation in inference path using (5)
- 3 Train with quantised inference but floating-point backpropagation until convergence
- 4 Use weights thus obtained for low-precision inference

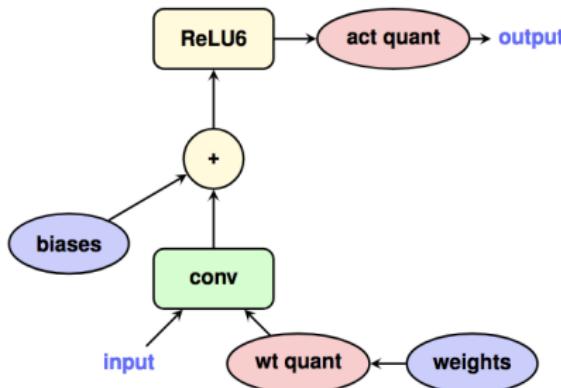


Figure from [Jac+18] (with permission)

Accuracy vs Precision [Jac+18]

ResNet50 on ImageNet, comparison with other approaches

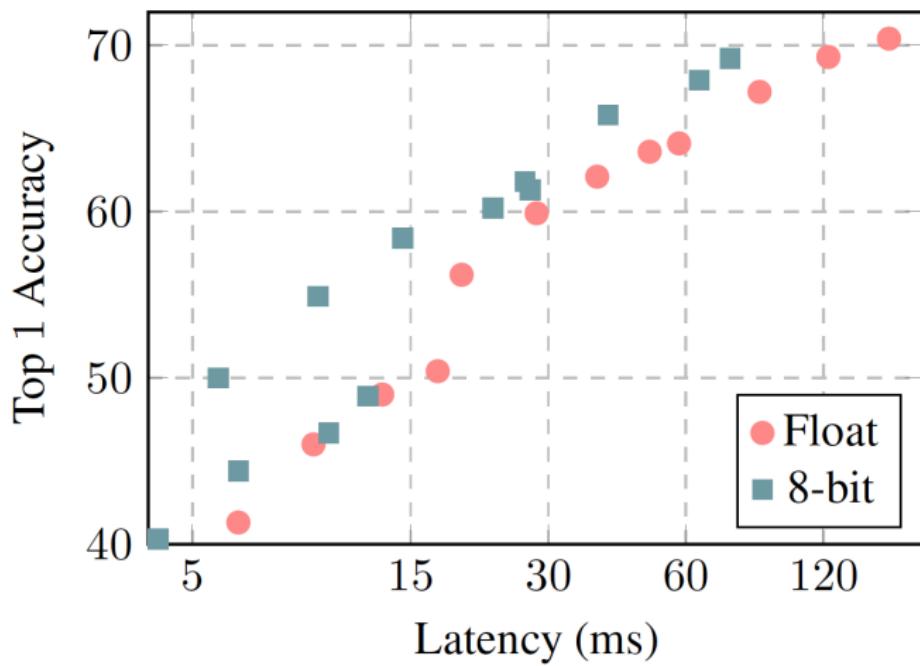
Scheme	BWN	TWN	INQ	FGQ	Ours
Weight bits	1	2	5	2	8
Activation bits	float32	float32	float32	8	8
Accuracy	68.7%	72.5%	74.8%	70.8%	74.9%

Table 4.2: ResNet on ImageNet: Accuracy under various quantization schemes, including binary weight networks (BWN [21, 15]), ternary weight networks (TWN [21, 22]), incremental network quantization (INQ [33]) and fine-grained quantization (FGQ [26])

Figure from [Jac+18] (with permission)

Accuracy vs Latency [Jac+18]

ImageNet classifier on Google Pixel 2 (Qualcomm Snapdragon 835 big cores)



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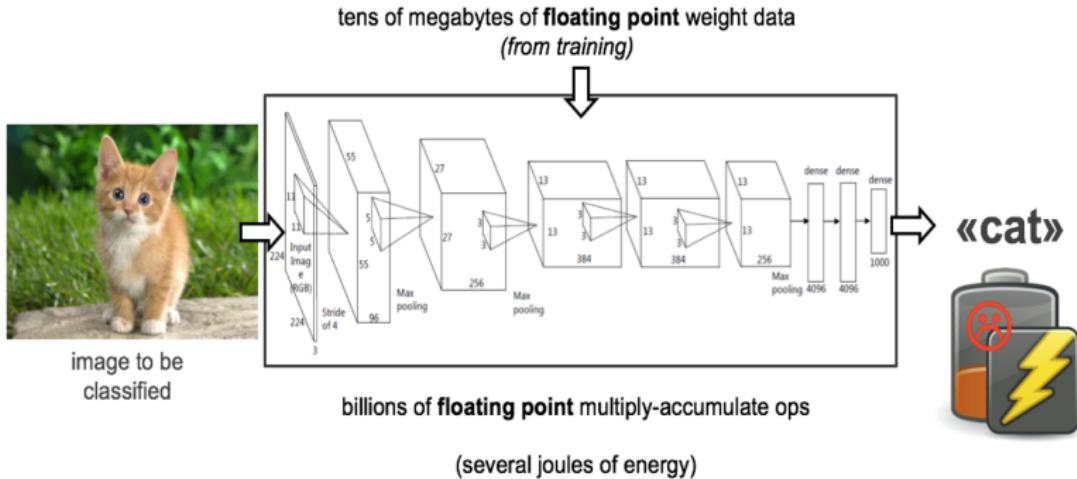
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Inference with Convolutional Neural Networks

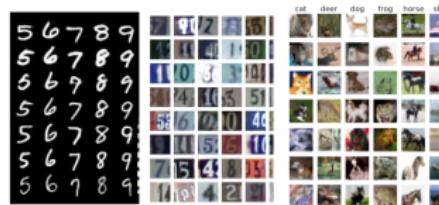
Slides from Yaman Umuroglu et. al., "FINN: A framework for fast, scalable binarized neural network inference," FPGA'17



Binarized Neural Networks

- › The extreme case of quantization
 - Permit only two values: +1 and -1
 - Binary weights, binary activations
 - Trained from scratch, not truncated FP

- › Courbariaux and Hubara et al. (NIPS 2016)
 - Competitive results on three smaller benchmarks
 - Open source training flow
 - Standard “deep learning” layers
 - Convolutions, max pooling, batch norm, fully connected...



	MNIST	SVHN	CIFAR-10
Binary weights & activations	0.96%	2.53%	10.15%
FP weights & activations	0.94%	1.69%	7.62%
BNN accuracy loss	-0.2%	-0.84%	-2.53%

% classification error (lower is better)

Advantages of BNNs

Vivado HLS estimates on Xilinx UltraScale+ MPSoC ZU19EG

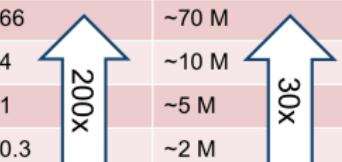
› Much smaller datapaths

- Multiply becomes XNOR, addition becomes popcount
- No DSPs needed, everything in LUTs
- Lower cost per op = more ops every cycle

› Much smaller weights

- Large networks can fit entirely into on-chip memory (OCM)
- More bandwidth, less energy compared to off-chip

Precision	Peak TOPS	On-chip weights
1b	~66	~70 M
8b	~4	~10 M
16b	~1	~5 M
32b	~0.3	~2 M



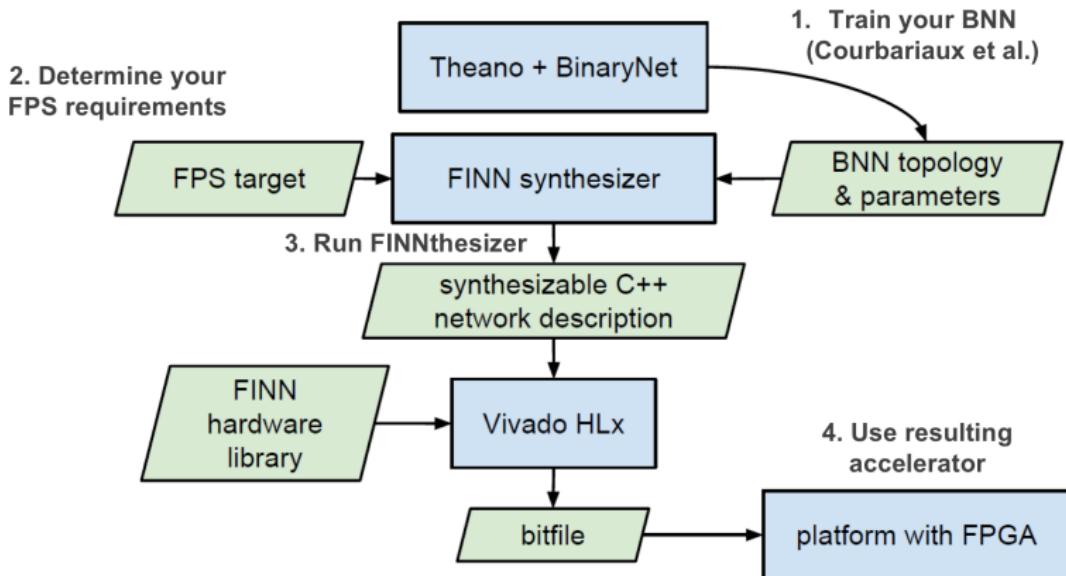
The table shows performance metrics for different precision levels. As precision decreases, peak TOPS increase significantly (200x for 16b vs 1b) while on-chip weight requirements decrease (30x for 16b vs 32b).

› fast inference with large BNNs

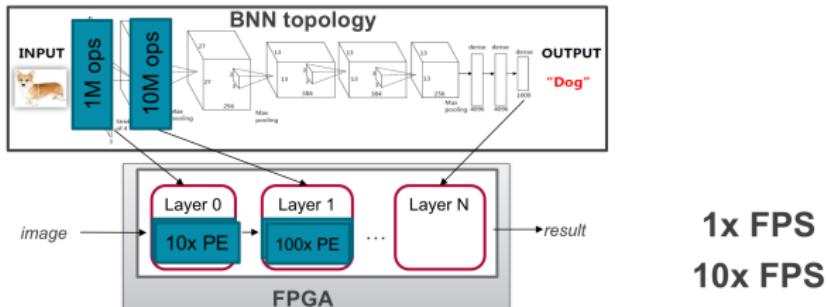
Design Flow

- One size does not fit all - Generate tailored hardware for network and use-case
- Stay on-chip - Higher energy efficiency and bandwidth
- Support portability and rapid exploration - Vivado HLS (High-Level Synthesis)
- Simplify with BNN-specific optimizations - Exploit compile time optimizations to simplify hardware, e.g. batchnorm and activation => thresholding

Design Flow



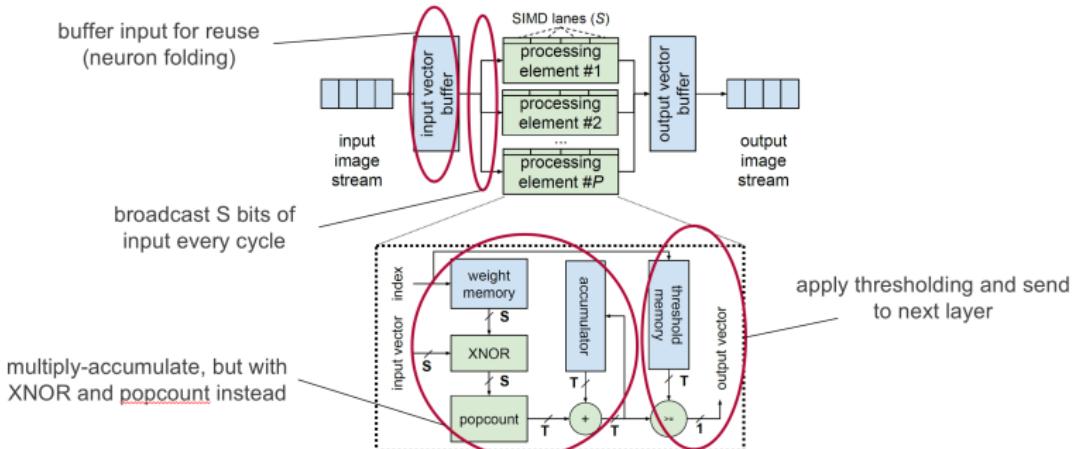
Heterogeneous Streaming Architecture



- One hardware layer per BNN layer, parameters built into bitstream
 - Both inter- and intra-layer parallelism
- Heterogeneous: Avoid “one-size-fits-all” penalties
 - Allocate compute resources according to FPS and network requirements
- Streaming: Maximize throughput, minimize latency
 - Overlapping computation and communication, batch size = 1

Matrix-Vector Threshold Unit (MVTU)

- Core computational element of FINN, tiled matrix-vector multiply
- Computes a (P) row $\times (S)$ column chunk of matrix every cycle, per-layer configurable tile size

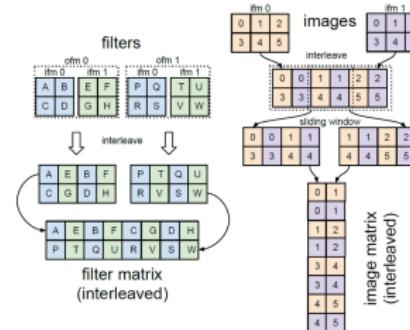
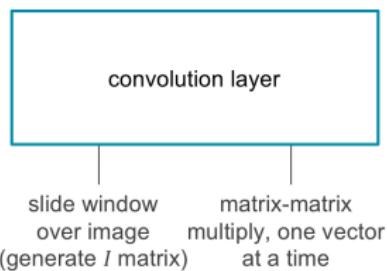


Convolutional Layers

► Lower convolutions to matrix-matrix multiplication, $W \cdot I$

- W : filter matrix (generated offline)
- I : image matrix (generated on-the-fly)

► Two components:



Performance

FINN

Prior Work

	Accuracy	FPS	Power (chip)	Power (wall)	kFPS / Watt (chip)	kFPS / Watt (wall)	Precision
MNIST, SFC-max	95.8%	12.3 M	7.3 W	21.2 W	1693	583	1
MNIST, LFC-max	98.4%	1.5 M	8.8 W	22.6 W	177	269	1
CIFAR-10, CNV-max	80.1%	21.9 k	3.6 W	11.7 W	6	2	1
SVHN, CNV-max	94.9%	21.9 k	3.6 W	11.7 W	6	2	1
MNIST, Alemdar et al.	97.8%	255.1 k	0.3 W	-	806	-	2
CIFAR-10, TrueNorth	83.4%	1.2 k	0.2 W	-	6	-	1
SVHN, TrueNorth	96.7%	2.5 k	0.3 W	-	10	-	1

Max accuracy loss: ~3%

10 – 100x better performance

CIFAR-10/SVHN energy efficiency comparable to TrueNorth ASIC

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Symmetric Quantisation (SYQ) [Far+18]

- To compute quantised weights from FP weights

$$\mathbf{Q}_I = \text{sign}(\mathbf{W}_I) \odot \mathbf{M}_I \quad (6)$$

with,

$$M_{I,i,j} = \begin{cases} 1 & \text{if } |W_{I,i,j}| \geq \eta_I \\ 0 & \text{if } -\eta_I < W_{I,i,j} < \eta_I \end{cases} \quad (7)$$

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise} \end{cases} \quad (8)$$

where \mathbf{M} represents a masking matrix, η is the quantization threshold hyperparameter (0 for binarised)

Symmetric Quantisation (SYQ) [Far+18]

- Make approximation $W_I \approx \alpha_I Q_I$, $Q_I \in \mathcal{C}$
- \mathcal{C} is the codebook, $\mathcal{C} \in \{C_1, C_2, \dots\}$ e.g. $\mathcal{C} = \{-1, +1\}$ for binary, $\mathcal{C} = \{-1, 0, +1\}$ for ternary
- A diagonal matrix α_I is defined by the vector $\alpha_I = [\alpha_I^1, \dots, \alpha_I^m]$:

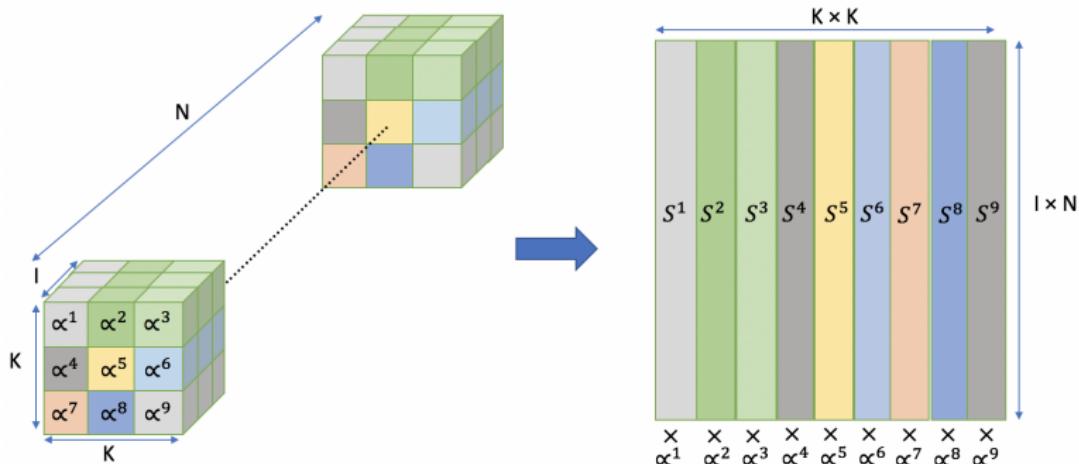
$$\alpha = \text{diag}(\alpha) := \begin{bmatrix} \alpha^1 & 0 & .. & 0 & 0 \\ 0 & \alpha^2 & .. & : & 0 \\ : & : & .. & \alpha^{m-1} & : \\ 0 & 0 & .. & 0 & \alpha^m \end{bmatrix}$$

- Train by solving

$$\alpha_I^* = \operatorname{argmin}_{\alpha} E(\alpha, \mathbf{Q}) \quad \text{s.t.} \quad \alpha \geq 0, \quad Q_{I,i,j} \in \mathcal{C}$$

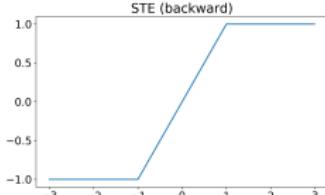
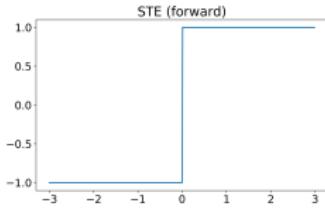
Subgroups

- Finer-grained quantisation improves weight approximation
- Pixel-wise shown, layer-wise has similar accuracy



Dealing with Non-differentiable Functions

- Recall (6) $\mathbf{Q}_I = \text{sign}(\mathbf{W}_I) \odot \mathbf{M}_I$
- This step function has a derivative which is zero everywhere: *vanishing gradients* problem
- Address via a straight through estimator (STE)
- Consider $q = \text{sign}(r)$ and $g_r \approx \frac{\partial C}{\partial q}$ then $\frac{\partial C}{\partial r} \approx g_q \mathbf{1}_{|r| \leq 1}$



SYQ Forward

- Quantise the inference step

Initialize: Set subgrouping granularity for S_I^i and set $\alpha_{I_0}^i$.

Inputs: Minibatch of inputs & targets (I, Y), Error function $E(Y, \hat{Y})$, current weights W_t and learning rate, γ_t

Outputs: Updated W_{t+1} , α_{t+1} and γ_{t+1}

SYQ Forward:

for $I=2$ to $L-1$ **do**

$Q_I = sign(W_I) \odot M_I$ **with** η

for i th subgroup in I th layer **do**

 Apply α_I^i to S_I^i

end for

end for

$\hat{Y} = \text{SYQForward}(I, Y, Q_I, \alpha_I)$

SYQ Backward

- Do the backprop step with FP32 weights
- Compute \mathbf{W} and α

SYQ Backward:

$\frac{\partial \hat{E}}{\partial \mathbf{Q}_I} = \text{WeightBackward}(\mathbf{Q}_I, \alpha_I, \frac{\partial \hat{E}}{\partial \hat{Y}})$

$\frac{\partial \hat{E}}{\partial \alpha_I} = \text{ScalarBackward}(\frac{\partial \hat{E}}{\partial \mathbf{Q}_I}, \alpha_I, \frac{\partial \hat{E}}{\partial \hat{Y}})$

$\mathbf{W}_{t+1} = \text{UpdateWeights}(\mathbf{W}_t, \frac{\partial \hat{E}}{\partial \mathbf{Q}_I}, \gamma)$

$\alpha_{t+1} = \text{UpdateScalars}(\alpha_t, \frac{\partial \hat{E}}{\partial \alpha_I}, \gamma)$

$\gamma_{t+1} = \text{UpdateLearningRate}(\gamma_t, t)$

Results for 8-bit activations

Model		Bin	Tern	FP32	Reference
AlexNet	Top-1	56.6	58.1	56.6	57.1
	Top-5	79.4	80.8	80.2	80.2
VGG	Top-1	66.2	68.7	69.4	-
	Top-5	87.0	88.5	89.1	-
ResNet-18	Top-1	62.9	67.7	69.1	69.6
	Top-5	84.6	87.8	89.0	89.2
ResNet-34	Top-1	67.0	70.8	71.3	73.3
	Top-5	87.6	89.8	89.1	91.3
ResNet-50	Top-1	70.6	72.3	76.0	76.0
	Top-5	89.6	90.9	93.0	93.0

- Our ResNet and AlexNet reference results are obtained from <https://github.com/facebook/fb.resnet.torch> and <https://github.com/BVLC/caffe>

Alexnet Comparison

Model	Weights	Act.	Top-1	Top-5
DoReFa-Net [Zho+16]	1	2	49.8	-
QNN [Hub+16]	1	2	51.0	73.7
HWGQ [Cai+17]	1	2	52.7	76.3
SYQ	1	2	55.2	78.4
DoReFa-Net [Zho+16]	1	4	53.0	-
SYQ	1	4	56.2	79.4
BWN [Ras+16]	1	32	56.8	79.4
SYQ	1	8	56.6	79.4
SYQ	2	2	55.7	79.1
FGQ [Mel+17]	2	8	49.04	-
TTQ [Zhu+16]	2	32	57.5	79.7
SYQ	2	8	58.1	80.8

Outline

1 Neural Networks

2 Arithmetic

3 Case Studies

Integer Quantisation

FINN: A Binarised Neural Network

SYQ: Low Precision DNN Training

4 Summary

5 Tutorial

Summary

- Reducing precision
 - Significantly reduce computational costs in DNNs
 - Data may now fit entirely on chip, avoiding external memory accesses
 - Computations greatly simplified
 - Key dimension for optimisation in CPU/GPU/FPGA implementations
- Convolutional layer can be computed as a MM
- Still an active research topic

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Installation

- To load:

```
docker load -i nova-quant-docker.tar
```

- To run:

```
docker run -it -p8888:8888 \
pytorch-jupyter-docker_jupytertorch:latest bash
```

- Then:

```
jupyter notebook --notebook-dir=/data --ip='*' \
--allow-root --port=8888 --no-browser
```

- On your browser:

```
http://127.0.0.1:8888/?token=1b14e...
```

Questions

- 1 Read the QPyTorch Functionality Overview at

https://github.com/Tiiiger/QPyTorch/blob/master/examples/tutorial/Functionality_Overview.ipynb

- 2 Follow the instructions in the following jupyter notebooks to complete the tutorial.

- sround.ipynb an implementation of stochastic rounding (slide 21).
- mnist.ipynb training of a low-precision MNIST inference network.

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