MATH6161
24-25-Deterministic OR Methods for Data Scientists

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Problem Introduction:
A bride and groom recently got engaged and are currently planning their wedding, to which they have invited 29 guests. After the ceremony, all the guests will sit down for a banquet. However, the bride and groom are having difficulty deciding where everyone should sit. The venue has 5 tables available and each table seats 6 people. They would like their guests to sit with as many familiar people as possible and, to this aim, they have organised their guests into groups as shown in Table 1. Ideally, all groups would remain intact, but the bride and groom understand that this might not be possible and that some groups might need to be split between tables. However, they have requested that groups should not be split between more than 2 tables. They have asked you to find the optimal seating positions such as to minimise the number of split groups.

Group	Members
1	Rachel, Marah, Sam
2	Andrew, Daisy, Ben, Abby
3	Briony, Henry
4	Emma, Edward, Arun, Will, Lilly
5	Steve, Lucy, Lewis, Harry, Emily, Katie
6	Lisa, Jessica, Rhiannon, Jake
7	Sophie, Claire, Nick, Terry, Amy

Table 1: Guest Groups

Solution Introduction

The optimal solution for the minimum number of split groups can be obtained by defining a binary variable Z_{ij} indicating whether the i-th group is assigned to table j, and the sum of Z_j from $j = 1$ to $j = 5$ represents the total number of split groups for the i-th group, and $\sum_{i=1}^{7}(\sum_{j=1}^{5} Z_{ij} - 1)$ gives the total number of extra split groups across all groups. Constriants are obtained from different question details.

Solutions and explanations

1. Formulate a mathematical program to model the seat assignment problem.

Define the variables used in the as follows which are also used in later questions:

- i - Index representing the i-th group.
- j - Index representing the j-th table.
- Z_{ij} - A binary variable indicating whether the i-th group is assigned to table j:

$$Z_{ij} = \begin{cases} 1 & \text{if group } i \text{ is assigned to table } j, \\ 0 & \text{otherwise.} \end{cases}$$

- G_i - The total number of people in the i-th group.
- y_{ij} - An integer variable representing the number of people from the i-th group assigned to table j.

Ans: 6 constraints can be designed for the basic question:
(1) There are at most 6 people at one table.
(2) The total number of arranged people must exactly equal the total number of people in the group (everyone must be assigned).
(3) The number of people from the j-th group arranged at table i must not exceed the total number of people in group j, provided the i-th group is assigned to the table.
(4) Each group can be split into at most 2 subgroups.
(5) Symmetric solutions from tables are ignored (people are arranged in order from table 1 to table 5).
(6) All variables must be integers.

Objective: Minimize

$$\sum_{i=1}^{7} \left(\sum_{j=1}^{5} Z_{ij} - 1 \right)$$

Constraints:

$$(1) \quad \sum_{i=1}^{7} y_{ij} \leq 6, \quad \forall j = 1, 2, \ldots, 5,$$

$$(2) \quad \sum_{j=1}^{5} y_{ij} = |G_i|, \quad \forall i = 1, 2, \ldots, 7,$$

$$(3) \quad y_{ij} \leq |G_i| Z_{ij}, \quad \forall i = 1, 2, \ldots, 7, \ \forall j = 1, 2, \ldots, 5,$$

$$(4) \quad \sum_{j=1}^{5} Z_{ij} \leq 2, \quad \forall i = 1, 2, \ldots, 7,$$

$$(5) \quad \sum_{i=1}^{7} y_{i1} \geq \sum_{i=1}^{7} y_{i2} \geq \cdots \geq \sum_{i=1}^{7} y_{i5},$$

$$(6) \quad y_{ij}, Z_{ij} \geq 0 \text{ and integer}, \quad \forall i, j.$$

Here constraint (5) is to decrease the number of sysmetric answers from the tables.

2. Solve the model in Xpress and present your solution (use default options for the solver).

Ans:
The optimal minimum number of split groups is 1. That means we only need to split one group into two. But There are many arrangement options, and here we provide possible two solutions:

(a) Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 3 with 4 people
Group 3 is assigned to Table 3 with 2 people
Group 4 is assigned to Table 5 with 5 people
Group 5 is assigned to Table 4 with 6 people
Group 6 is assigned to Table 1 with 3 people
Group 6 is assigned to Table 2 with 1 people
Group 7 is assigned to Table 2 with 5 people

(b) Optimal Objective Value: 1
Group 1 is assigned to Table 3 with 3 people
Group 2 is assigned to Table 1 with 1 people
Group 2 is assigned to Table 3 with 3 people
Group 3 is assigned to Table 2 with 2 people
Group 4 is assigned to Table 5 with 5 people
Group 5 is assigned to Table 4 with 6 people
Group 6 is assigned to Table 2 with 4 people
Group 7 is assigned to Table 1 with 5 people

More solutions are provided in codes.

3. The bride and groom are concerned that the current model might cause exactly one person to be separated from the rest of their group. Find a way to update your model such that groups cannot be split into single individuals. Find and present the new solution.

Ans:
Here we can add one constraint (Constraint 7):
The number of people in i-th group assigned to table j must be greater than 2 given the i-th group is assigned to the table.
Objective: Minimize

$$\sum_{i=1}^{7}\left(\sum_{j=1}^{5} Z_{ij} - 1\right)$$

Constraints:

$$\begin{aligned}
&(1) \quad \sum_{i=1}^{7} y_{ij} \leq 6, \quad \forall j = 1, 2, \ldots, 5, \\
&(2) \quad \sum_{j=1}^{5} y_{ij} = |G_i|, \quad \forall i = 1, 2, \ldots, 7, \\
&(3) \quad y_{ij} \leq |G_i| Z_{ij}, \quad \forall i = 1, 2, \ldots, 7, \ \forall j = 1, 2, \ldots, 5, \\
&(4) \quad \sum_{j=1}^{5} Z_{ij} \leq 2, \quad \forall i = 1, 2, \ldots, 7. \\
&(5) \quad \sum_{i=1}^{7} y_{i1} \geq \sum_{i=1}^{7} y_{i2} \geq \cdots \geq \sum_{i=1}^{7} y_{i5} \\
&(6) \quad y_{ij}, Z_{ij} \geq 0 \text{ and integer}, \quad \forall i, j \\
\textbf{Added: } &(7) \quad y_{ij} \geq 2Z_{ij}, \quad \forall i = 1, 2, \ldots, 7, \ \forall j = 1, 2, \ldots, 5.
\end{aligned}$$

The constraint (7) guarantees that if $Z_{ij} = 1$, $y_{ij} > 1$, otherwise $y_{ij} \geq 0$.
Here we provide two possible arrangements. More arrangements can be found in codes.

(a) Solution 1
Objective Value: 1
Group 1 is assigned to Table 4 with 3 people
Group 2 is assigned to Table 1 with 4 people
Group 3 is assigned to Table 3 with 2 people
Group 4 is assigned to Table 1 with 2 people
Group 4 is assigned to Table 4 with 3 people
Group 5 is assigned to Table 2 with 6 people
Group 6 is assigned to Table 3 with 4 people
Group 7 is assigned to Table 5 with 5 people

(b) Solution 2
Optimal Objective Value: 1

Group 1 is assigned to Table 3 with 3 people
Group 2 is assigned to Table 2 with 4 people
Group 3 is assigned to Table 1 with 2 people
Group 4 is assigned to Table 2 with 2 people
Group 4 is assigned to Table 3 with 3 people
Group 5 is assigned to Table 4 with 6 people
Group 6 is assigned to Table 1 with 4 people
Group 7 is assigned to Table 5 with 5 people

4. The day before the wedding, one of the tables breaks. There aren't any spares, and instead the venue replaces it with 2 tables, one of which seats 4 and the other seats 2 (for a total of 6 tables). Update your model to accommodate this change. Find and present the new solution.

Ans:
Here we assume table 5 is broken and is replaced by table 6 and 7 which have 4 and 2 seats separately.
So we can update Constraint 1 with two more constraints (Constraint 2 and Constraint 3) and require the corresponding number of seats in each table.

Objective: Minimize

$$\sum_{i=1}^{7}\left(\sum_{j=1}^{5} Z_{ij} - 1\right)$$

Constraints:

$$\textbf{Updated:}\ (1) \quad \sum_{i=1}^{7} y_{ij} \leq 6, \quad \forall j = 1, 2, \ldots, 4,$$

$$\textbf{Updated:}\ (2) \quad \sum_{i=1}^{7} y_{ij} \leq 4, \quad \forall j = 5$$

$$\textbf{Updated:}\ (3) \quad \sum_{i=1}^{7} y_{ij} \leq 2, \quad \forall j = 6$$

$$(4) \quad \sum_{j=1}^{5} y_{ij} = |G_i|, \quad \forall i = 1, 2, \ldots, 7,$$

$$(5) \quad y_{ij} \leq |G_i| Z_{ij}, \quad \forall i = 1, 2, \ldots, 7,\ \forall j = 1, 2, \ldots, 5,$$

$$(6) \quad \sum_{j=1}^{5} Z_{ij} \leq 2, \quad \forall i = 1, 2, \ldots, 7.$$

$$(7) \quad \sum_{i=1}^{7} y_{i1} \geq \sum_{i=1}^{7} y_{i2} \geq \cdots \geq \sum_{i=1}^{7} y_{i5}$$

$$(8) \quad y_{ij}, Z_{ij} \geq 0 \text{ and integer}, \quad \forall i, j$$

$$(9) \quad y_{ij} \geq 2Z_{ij}, \quad \forall i = 1, 2, \ldots, 7, \ \forall j = 1, 2, \ldots, 5.$$

Table 5 has 4 seats and Table 6 has 2 seats.
Here we provide two possible arrangements, and more arrangements can be found in codes.

(a) Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 6 with 2 people
Group 4 is assigned to Table 1 with 3 people
Group 4 is assigned to Table 3 with 2 people
Group 5 is assigned to Table 2 with 6 people
Group 6 is assigned to Table 3 with 4 people
Group 7 is assigned to Table 4 with 5 people

(b) Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 3 with 2 people
Group 4 is assigned to Table 1 with 3 people
Group 4 is assigned to Table 6 with 2 people
Group 5 is assigned to Table 2 with 6 people
Group 6 is assigned to Table 3 with 4 people
Group 7 is assigned to Table 4 with 5 people

5. The bride and groom understand that some groups might need to be split in order to accommodate all of their guests. However, when deciding which groups to split, they state that some groups should not be split with high priority over others. As a motivating example, think about the case when, e.g., group 1 contains the bridesmaids and should not be split unless absolutely necessary. The same might hold for group 2, which contains close family members, and ideally should not be split, but this is not as important as it is for group 1. Demonstrate how you can modify your model to accommodate the groups having varying priorities. In particular, supported by experimental results, choose an assignment of the parameters in the new model for

which the group(s) that was split in Q4 is (are) now not split.

Ans:
Here we can use the total number of splits in the i-th group $\sum_{j=1}^{j=7} Z_{ij}$ and require the order of group.

For example: $\sum_{j=1}^{j=7} Z_{4j} \leq \sum_{j=1}^{j=7} Z_{3j}$ means that the split number of group 4 should be smaller than the number of group 3 (group 3 has a higher priority to split).

Here, we assume that the splitted group 4 in the Question 4 has the lowest split priority (SP):

$$SP(4) \leq SP(1) \leq SP(2) \leq SP(3) \leq SP(5) \leq SP(6) \leq SP(7) \tag{1}$$

Then we can update our Constraints as:
Constraints:

$$
\begin{aligned}
&(1) \quad \sum_{i=1}^{7} y_{ij} \leq 6, \quad \forall j = 1, 2, \ldots, 4, \\
&(2) \quad \sum_{i=1}^{7} y_{ij} \leq 4, \quad \forall j = 5 \\
&(3) \quad \sum_{i=1}^{7} y_{ij} \leq 2, \quad \forall j = 6 \\
&(4) \quad \sum_{j=1}^{5} y_{ij} = |G_i|, \quad \forall i = 1, 2, \ldots, 7, \\
&(5) \quad y_{ij} \leq |G_i| Z_{ij}, \quad \forall i = 1, 2, \ldots, 7,\ \forall j = 1, 2, \ldots, 5, \\
&(6) \quad \sum_{j=1}^{5} Z_{ij} \leq 2, \quad \forall i = 1, 2, \ldots, 7. \\
&(7) \quad \sum_{i=1}^{7} y_{i1} \geq \sum_{i=1}^{7} y_{i2} \geq \cdots \geq \sum_{i=1}^{7} y_{i5} \\
&(8) \quad y_{ij}, Z_{ij} \geq 0 \text{ and integer}, \quad \forall i, j \\
&(9) \quad y_{ij} \geq 2 Z_{ij}, \quad \forall i = 1, 2, \ldots, 7,\ \forall j = 1, 2, \ldots, 5, \\
\textbf{Added:}&(10) \quad \sum_{i=1}^{5} Z_{i4} \leq \sum_{i=1}^{5} Z_{i1} \leq \cdots \leq \sum_{i=1}^{5} Z_{i7}
\end{aligned}
$$

In this case, Group 4 has the lowest priority to split, and Group 7 has the highest priority to split.

(a) Solution 1
Optimal Objective Value: 1
Group 1 is assigned to Table 3 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 2 with 2 people
Group 4 is assigned to Table 4 with 5 people
Group 5 is assigned to Table 1 with 6 people
Group 6 is assigned to Table 2 with 4 people
Group 7 is assigned to Table 3 with 3 people
Group 7 is assigned to Table 6 with 2 people

(b) Solution 2
Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 6 with 2 people
Group 4 is assigned to Table 4 with 5 people
Group 5 is assigned to Table 3 with 6 people
Group 6 is assigned to Table 2 with 4 people
Group 7 is assigned to Table 1 with 3 people
Group 7 is assigned to Table 2 with 2 people

Since Group 7 has the highest priority for splitting, the first two solutions illustrate different ways to split Group 7.

Appendices

Codes

Here we show the code below for the Question 5, and more codes can be found in the repository: https://github.com/phy-guanzh/Optimal_Seats_Arrangment/tree/main

```
model SeatingArrangement
    uses "mmxprs"

    declarations
        num_groups = 7
        num_tables  = 6
        group_sizes: array(1..7) of integer
```

```
        split_priority: array(1..7) of integer
        Z: array(1..7, 1..6) of mpvar
        y: array(1..7, 1..6) of mpvar
    end-declarations


    group_sizes :: [3, 4, 2, 5, 6, 4, 5]
    split_priority :: [4, 1, 2, 3, 5, 6, 7]

    forall(i in 1..num_groups, j in 1..num_tables) do
        Z(i,j) is_binary
        y(i,j) is_integer
    end-do


    Objective := sum(i in 1..num_groups) (sum(j in 1..num_tables)
        Z(i,j) - 1)

    forall(j in 1..num_tables-1) do
        sum(i in 1..num_groups) y(i,j) <= 6
    end-do

    forall(j in num_tables-1..num_tables-1) do
        sum(i in 1..num_groups) y(i,j) <= 4
    end-do

    forall(j in num_tables..num_tables) do
        sum(i in 1..num_groups) y(i,j) <= 2
    end-do

    forall(i in 1..num_groups) do
        sum(j in 1..num_tables) y(i,j) = group_sizes(i)
    end-do


    forall(i in 1..num_groups, j in 1..num_tables) do
        y(i,j) <= group_sizes(i) * Z(i,j)
    end-do


    forall(i in 1..num_groups) do
        sum(j in 1..num_tables) Z(i,j) <= 2
    end-do

    forall(j in 1..num_tables-2) do
```

```
        sum(i in 1..num_groups) y(i, j) >= sum(i in 1..num_groups) y(i, j+1)
    end-do



    forall(i in 1..num_groups, j in 1..num_tables) do
        2*Z(i,j) <= y(i,j)
    end-do

    forall(i in 1..num_groups-1) do
        sum(j in 1..num_tables) Z(split_priority(i), j) <= sum(j in 1..num_tables) Z(split_priority(i+1), j)
    end-do


    #These settings below are used to require the number of solutions we get, more details can be checked in:
    #https://www.fico.com/fico-xpress-optimization/docs/latest/evalguide2/dhtml/eg2sec3.html
    #and examples\getting_started\Mosel\folioenumsol.mos

    setparam("XPRS_ENUMDUPLPOL",3)

    setparam("XPRS_MIPDUALREDUCTIONS", 0)

    setparam("XPRS_enummaxsol", 10)

    minimize(XPRS_ENUM,Objective)
    forall(k in 1..getparam("XPRS_enumsols")) do
        selectsol(k)
        writeln("Solution", k)
        writeln("Optimal_Objective_Value:␣", getobjval)
        forall(i in 1..num_groups) do
            forall(j in 1..num_tables) do
                if getsol(Z(i,j)) > 0.5 then
                    writeln("Group", i, "is_assigned_to_Table", j, "with", getsol(y(i,j)), "people")
                end-if
            end-do
        end-do
    end-do

end-model
```

Listing 1: Xpress Optimization Code

Printout

```
Solution 1
Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 3 with 4 people
Group 3 is assigned to Table 3 with 2 people
Group 4 is assigned to Table 5 with 5 people
Group 5 is assigned to Table 4 with 6 people
Group 6 is assigned to Table 1 with 3 people
Group 6 is assigned to Table 2 with 1 people
Group 7 is assigned to Table 2 with 5 people
Solution 2
Optimal Objective Value: 1
Group 1 is assigned to Table 3 with 3 people
Group 2 is assigned to Table 1 with 1 people
Group 2 is assigned to Table 3 with 3 people
Group 3 is assigned to Table 2 with 2 people
Group 4 is assigned to Table 5 with 5 people
Group 5 is assigned to Table 4 with 6 people
Group 6 is assigned to Table 2 with 4 people
Group 7 is assigned to Table 1 with 5 people
```

Figure 1: The printout for Question2

```
Solution 1
Optimal Objective Value: 1
Group 1 is assigned to Table 4 with 3 people
Group 2 is assigned to Table 1 with 4 people
Group 3 is assigned to Table 3 with 2 people
Group 4 is assigned to Table 1 with 2 people
Group 4 is assigned to Table 4 with 3 people
Group 5 is assigned to Table 2 with 6 people
Group 6 is assigned to Table 3 with 4 people
Group 7 is assigned to Table 5 with 5 people
Solution 2
Optimal Objective Value: 1
Group 1 is assigned to Table 3 with 3 people
Group 2 is assigned to Table 2 with 4 people
Group 3 is assigned to Table 1 with 2 people
Group 4 is assigned to Table 2 with 2 people
Group 4 is assigned to Table 3 with 3 people
Group 5 is assigned to Table 4 with 6 people
Group 6 is assigned to Table 1 with 4 people
Group 7 is assigned to Table 5 with 5 people
```

Figure 2: The printout for Question3

```
Solution 1
Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 6 with 2 people
Group 4 is assigned to Table 1 with 3 people
Group 4 is assigned to Table 3 with 2 people
Group 5 is assigned to Table 2 with 6 people
Group 6 is assigned to Table 3 with 4 people
Group 7 is assigned to Table 4 with 5 people
Solution 2
Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 3 with 2 people
Group 4 is assigned to Table 1 with 3 people
Group 4 is assigned to Table 6 with 2 people
Group 5 is assigned to Table 2 with 6 people
Group 6 is assigned to Table 3 with 4 people
Group 7 is assigned to Table 4 with 5 people
```

Figure 3: The printout for Question4

```
Solution 1
Optimal Objective Value: 1
Group 1 is assigned to Table 3 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 2 with 2 people
Group 4 is assigned to Table 4 with 5 people
Group 5 is assigned to Table 1 with 6 people
Group 6 is assigned to Table 2 with 4 people
Group 7 is assigned to Table 3 with 3 people
Group 7 is assigned to Table 6 with 2 people
Solution 2
Optimal Objective Value: 1
Group 1 is assigned to Table 1 with 3 people
Group 2 is assigned to Table 5 with 4 people
Group 3 is assigned to Table 6 with 2 people
Group 4 is assigned to Table 4 with 5 people
Group 5 is assigned to Table 3 with 6 people
Group 6 is assigned to Table 2 with 4 people
Group 7 is assigned to Table 1 with 3 people
Group 7 is assigned to Table 2 with 2 people
```

Figure 4: The printout for Question5