Introduction to Algorithms: Lecture 5

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2025 spring in

Homework & Experiments

1. All office hours in April are in 3A103
2. Experiment 2 is due on Thursday
3. HW 3 will be due on next Wednesday

Outline

Overview

Next, discuss two powerful techniques in algorithm-design:

1. Dynamic programming
2. Greedy method

Similar to divide & conquer, they optimize the running time.

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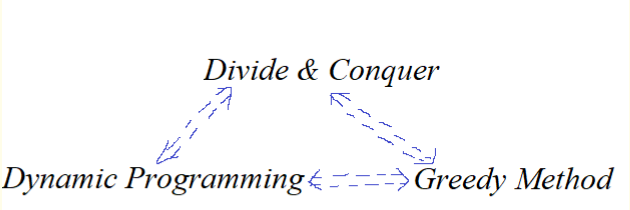

While these three methods have similar ideas, their implementations, analyses, and proofs are quite different.

Introduction

Dynamic Programming (DP): Divide & Conquer ++

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Recall 3 steps in divide & conquer

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2. Solve subproblems recursively
3. Combine solutions of subproblems

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1. Divide problem into subproblems
2. Solve subproblems recursively
3. Combine solutions of subproblems

Usually, we apply dynamic programming for optimization problems.

1. Similar to divide & conquer: Reduce to subproblems.
2. Differences: (1) division step in DP is more complicated; (2) DP handles subproblems by Memoization

Outline

Background

Machine learning and data science always need to compute a sequence of matrix multiplications.

Example: linear regression

Given $A \in \mathbb{R}^{n \times d}$ and $y \in \mathbb{R}^n$, $\min_\beta \|A\beta - y\|_2$ has $\beta^* = (A^\top A)^{-1} A^\top y$.

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Problem

Given dimensions $p_0, \ldots, p_n$ and matrices $A_1, \ldots, A_n$ with $A_i \in \mathbb{R}^{p_{i-1} \times p_i}$, fully parenthesize product $A_1 A_2 \cdots A_n$ to minimize the number of scalar multiplications.

Intuition

Problem

Given $p_0, \ldots, p_n \in \mathbb{Z}^+$ and matrices $A_1, \ldots, A_n$ with $A_i \in \mathbb{R}^{p_{i-1} \times p_i}$, fully parenthesize product $A_1 A_2 \cdots A_n$ to minimize the number of scalar multiplications.

There are $2^{\Omega(n)}$ methods to add parenthesizes on $A_1 A_2 \cdots A_n$:

$$(A_1 A_2) \cdot (A_3 A_4), ((A_1 A_2) A_3) \cdot A_4, (A_1 (A_2 A_3)) \cdot A_4, A_1 \cdot ((A_2 A_3) A_4), \ldots$$

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Basic idea

1. The last product $\cdot$ splits them into two independent subproblems.
2. Global optimal solution $\Leftrightarrow$ optimal on the two subproblems.

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Basic idea

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Question: How to split them? — Let us try all choices of splits.

Memoization

---

function $\text{ChainOrder}(s, t)$
    if $s = t$ **then**
        Return 0
    else if $s + 1 = t$ **then**
        Return $p_{s-1} \times p_s \times p_t$
    else
        $ans = +\infty$
        for $i \in [s, t)$ **do** //Enumerate the split
            $ans = \min\left\{ans, \text{ChainOrder}(s, i) + \text{ChainOrder}(i + 1, t) + p_{s-1} \times p_i \times p_t\right\}$
        Return ans

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While it does find the answer, $2^{\Omega(n)}$ calls of ChainOrder. However, there are only $\binom{n+1}{2}$ pairs (s, t).

Memoization

Solution

Once calculated $\textsc{ChainOrder}(s,t)$, Let $f[s,t]$ record the answer $\textsc{ChainOrder}(s,t)$ for future calls.

---

procedure $\textsc{ChainM}(s,t)$
// preprocess $s = t$ and $s + 1 = t$
    if $f[s,t] = +\infty$ **then**
        for $i \in [s,t)$ **do**
            $f[s,t] \;=\; \min\left\{f[s,t], \textsc{ChainM}(s,i) + \textsc{ChainM}(i+1,t) + \right.$
$\left. + p_{s-1} \times p_i \times p_t\right\}$
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  Return $f[s,t]$

Since we start by calling $\textsc{ChainM}(1,n)$, this is called top-down with memoization.

Bottom-Up Method

(s, i) and $(i+1, t)$ in loop are strictly smaller than (s, t) — we can compute $f[s, t]$ from smaller intervals to large ones, called bottom-up method.

procedure MAIN
  Pre-process $f[s, t]$ where $s = t$ or $s + 1 = t$
  for $\ell = 2, \ldots, n-1$ **do** // $\ell = t - s$
    for $s = 1, \ldots, n - \ell$ **do**
      $t = s + \ell$, $f[s, t] = +\infty$
      for $i \in [s, t)$ **do**
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    Return $f[1, n]$

We enumerate (ℓ, s) instead of (s, t) to guarantee $f[s, i]$ and $f[i+1, t]$ have been calculated.

Discussion

Analysis

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2. Running time $O(n^3)$ — 3 loops.

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procedure $\textsc{Parens}(s, t)$
  if $s = t$ **then**
    Print(A_i)
  else
    Print("(")
    Parens($s, d[s,t]$)
    Parens($d[s,t]+1, t$)
    Print(")")

---

1. Record $\arg\min_i$ as $d[s,t]$ in $\textsc{Main}$

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1. Record $\arg\min_i$ as $d[s, t]$ in $\textsc{Main}$
2. Print the construction from top-bottom.

Summary

While dynamic programming utilizes answers from subproblems, two differences between divide & conquer:

1. DP enumerates all splits.
2. DP memorize answers of all subproblems

Usually DP analyzes the problem from top to bottom but implement it from bottom to top (faster).

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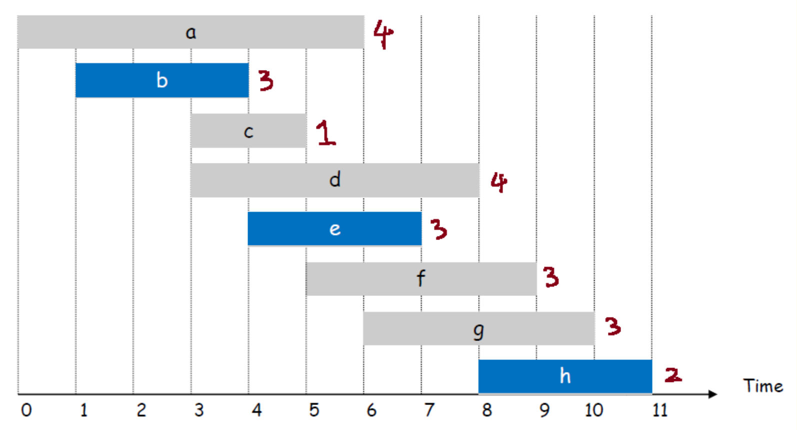
More examples

(1) Different problems consider various splits.
(2) Improve the running time of DP.

Outline

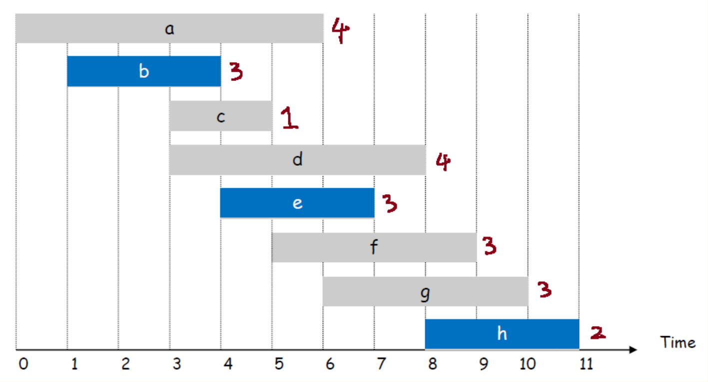
Interval Scheduling

1. n jobs in $[0, T]$
2. Job j starts from $h(j)$ and ends at $e(j)$ with weight w_j
3. Two jobs are compatible if they don't overlap.
4. Goal: find max-weight subset with compatible jobs.



Basic Idea

Reduce it to subproblems: max-weight subsets in $[0, H]$ and $[H, T]$



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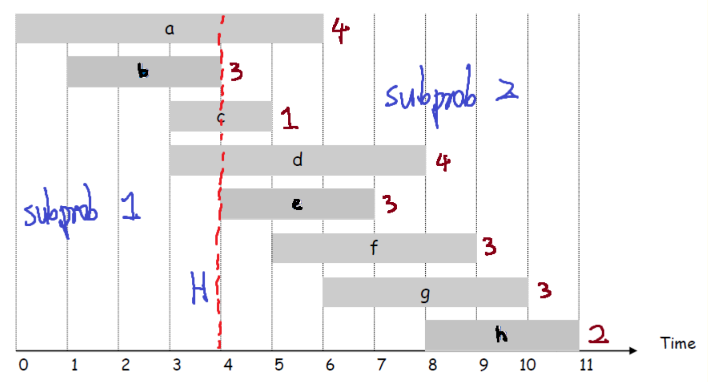

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The optimal solution is optimal again on the two subproblems!

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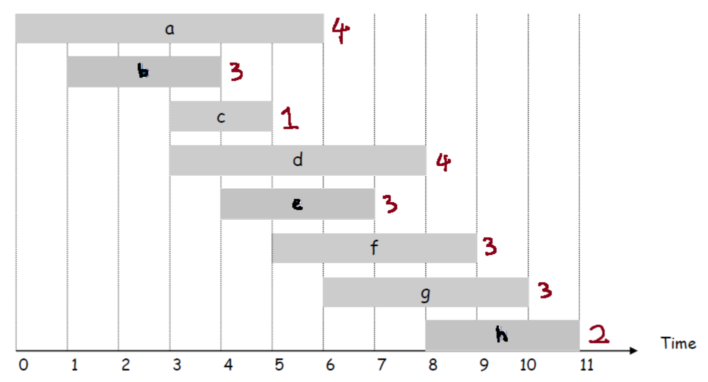

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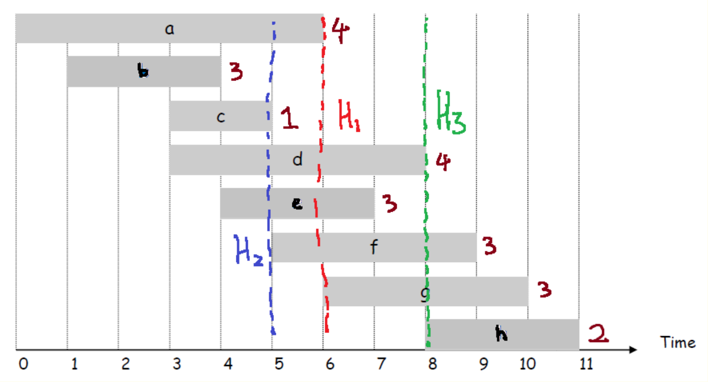

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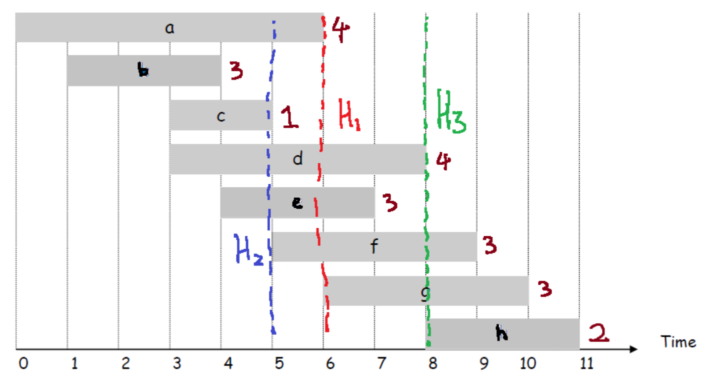
We do not know the optimal solutions yet — enumerate H

Subproblem

Let $OPT(s,t)$ denote the max-weight of compatible subsets from time s to t.



Enumerate the split H of optimal solutions:

$$OPT(s,t) = \max_{H \in (s,t)} \left\{ OPT(s,H) + OPT(H,t) \right\}.$$

Bottom-Up method

---

procedure MAIN
  for $\ell = 1, \ldots, T$ **do**
    for $s = 0, \ldots, T - s$ **do**
      $t = s + \ell, opt[s, t] = 0$
      for $j = 1, \ldots, n$ **do**   // solution with one interval in $[s, t]$
        if $\big[h[j], e[j]\big] \subseteq [s, t]$ **then**
          $opt[s, t] = \max\{opt[s, t], w[j]\}$
      for $H = s + 1, \ldots, t - 1$ **do** // solutions with $\geqslant 2$ intervals
        $opt[s, t] = \max\{opt[s, t], opt[s, H] + opt[H, t]\}$
  Return $opt[0, T]$

While it works, it is really slow — $O(T^3)$.

Question

How to improve it?

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procedure MAIN(n, h, e)

Sort n intervals s.t. $e(1) \leqslant e(2) \leqslant \ldots \leqslant e(n)$

Compute the last compatible interval $p(1), \ldots, p(n)$ for each j, i.e., $p(j) = \max_{i:e(i)\leqslant h(j)}\{i\}$

$g[0] = 0$

for $j = 1, \ldots, n$ **do**

$$g(j) = \max\left\{ g[j-1], w(j) + g[p(j)] \right\}$$

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   2. Compute $p\big(rank(1)\big), \ldots, p\big(rank(n)\big)$ by moving an index on $e(1) \leqslant e(2) \leqslant \cdots \leqslant e(n)$

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1. $O(T^3)$ time DP at first
2. Realize $f[s, t]$ could be reduce to $g(j)$ where $j = 1, \ldots, n$
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We can find solution again by memorizing all decisions.

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   $\Leftarrow$ Subproblems are independent s.t. any optimal solutions of them could be combined to a global optimal solution

Different from divide & conquer: The last two proofs could be tricky.

Summary (II)

To design DP,

1. Figure out the space of subproblems, i.e., define $OPT[s, t]$
2. Follow a bottom-up order
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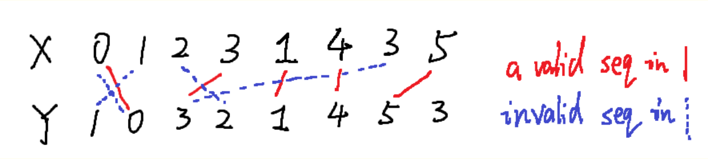
Roughly, running time is $O\Big(\text{size(space)} \times \#\text{choices}\Big)$

— Optimizing the running time is highly non-trivial!

Outline

Problem Intro

Given two sequences $X = (x_1, \ldots, x_m)$ and $Y = (Y_1, \ldots, Y_n)$, find the longest common sequence.



Formally, $Z = (z_1, \ldots, z_k)$ is a common sequence of X if $\exists$ strictly increasing seq $i_1 < \ldots < i_k$ of indices of X s.t. $z_j = X_{i_j}$ for $j \in [k]$.

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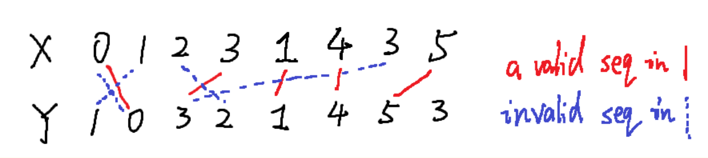

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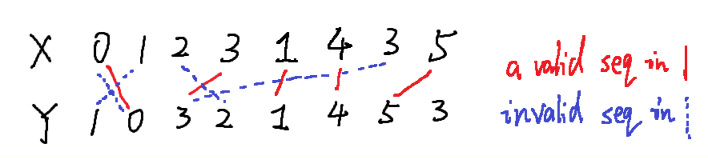
Goal

Find the longest sequence Z that is common in both X and Y.

Basic Idea

Each red line denotes one choice.



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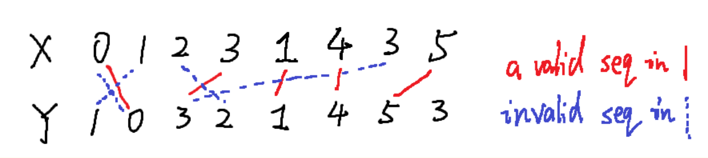

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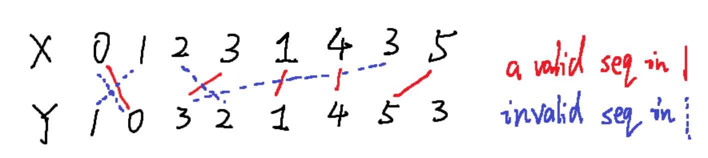

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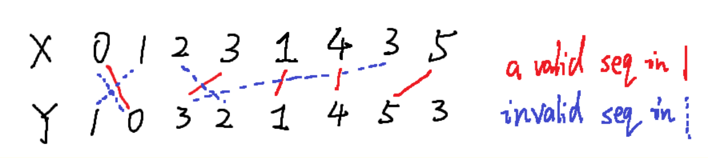

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2. The optimal solution must be optimal on subproblems $(x_1, \ldots, x_{i-1})$ and $(y_1, \ldots, y_{j-1})$
3. Moreover any optimal solution on the subproblem is good.

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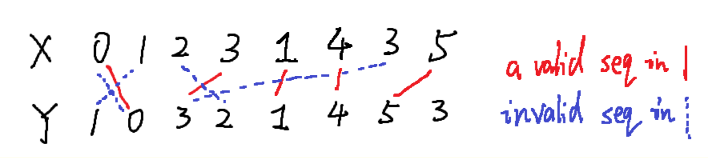

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2. The optimal solution must be optimal on subproblems $(x_1, \ldots, x_{i-1})$ and $(y_1, \ldots, y_{j-1})$
3. Moreover any optimal solution on the subproblem is good.

So the space of subproblems is $[1, \ldots, i] \times [1 \ldots, j]$ for all $i \in [n]$ and $j \in [m]$.

Formal Description

Let $c[k, \ell]$ be the length of the longest common sequence between $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_\ell)$.

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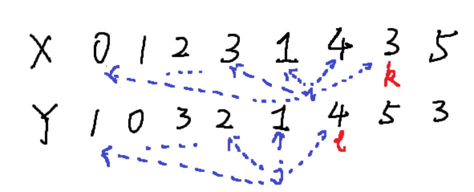

1. To compute it, try enumerating the position of the last line $x_i \leftrightarrow y_j$:

$$c[k, \ell] = \max_{i \leqslant k, j \leqslant \ell : x_i = y_j} \{c[i-1, j-1] + 1\}$$

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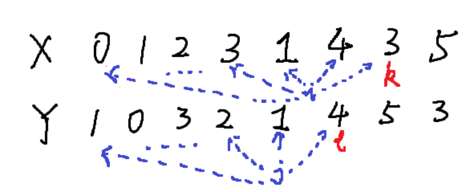

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2. While it is correct, running time is relatively slow $O(m^2 n^2)$.
3. Can we do better?

Improvement

Key OBS: Only need to check $x_k = y_j$ or not.

Theorem (15.1 in CLRS)

Let $Z = (z_1, \ldots, z_k)$ be any LCS of $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$.

1. $x_m = y_n$: $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. $x_m \neq y_n$: If $z_k \neq x_m$, Z must be an LCS of X_{m-1} and Y_n.
3. $x_m \neq y_n$: If $z_k \neq y_n$, Z must be an LCS of X_m and Y_{n-1}.

Faster DP

While we do not know Z to apply the THM, it still tells there are only 3 cases:

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ and } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max\left\{ c[i-1, j], c[i, j-1] \right\} & o.w. \end{cases}$$

The running time becomes $O(nm)$ ☺

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Constructing LCS

Remember where does $c[i,j]$ come from (called the decision) and trace back to output the sequence

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Usually it is highly non-trivial to improve the running time of DP.

1. Running time = Space of subproblems $\times$ enumeration
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Next

All previous DP algorithms are on lines/sequences. There are DP on trees (graphs) and matrices.

Outline

Problem Intro

Build a binary search tree to minimize the search time.

1. There are n keys $k_1, \ldots, k_n$ and $n+1$ dummy keys $d_0, \ldots, d_n$ s.t. $d_0 < k_1 < d_1 < k_2 < \cdots < k_n < d_n$.

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2. Each k_i appears with prob. p_i; each d_i appears with prob. q_i
3. $\sum_i p_i + \sum_i q_i = 1$
4. Design a BST T s.t. (1) Internal nodes are keys; (2) Leaves are dummy keys; (3) Minimize the expected cost of a search in T:

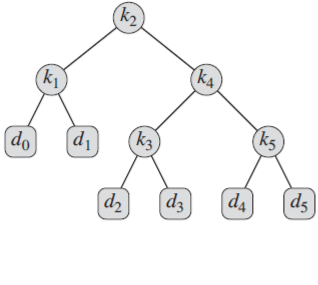
$$\mathbb{E}[\text{search cost in } T] := \mathop{\mathbb{E}}_{key \sim \{d_0, \ldots, d_n, k_1, \ldots, k_n\}} [\text{depth}(key) \text{ in } T].$$

Example

Figure 15.9 Two binary search trees for a set of $n = 5$ keys with the following probabilities:

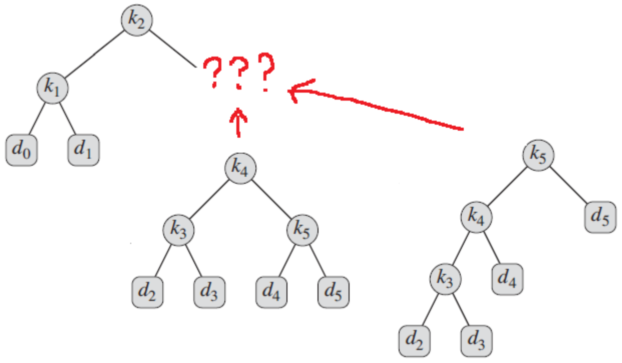
i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Two binary search trees with expected cost 2.8 and 2.75 separately:



(a) (b)

Step 1: Reduce to subproblems



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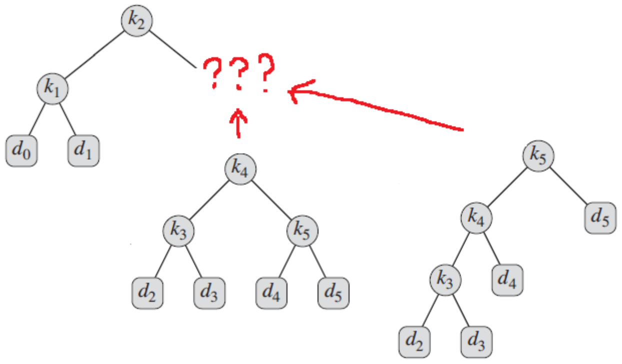

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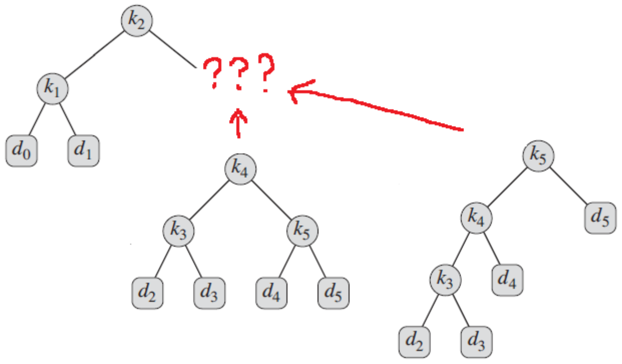

1. Say the optimal BST has root k_2 — consider its left/right subtrees.
2. Various possible subtrees with roots k_3, k_4, k_5 separately.
3. In the optimal BST, this subtree must be optimal among k_3, k_4, k_5 and $d_2, \ldots, d_5$.
4. Vice versa — any optimal subtree could be plugged into the optimal BST to give one optimal construction

Step: Recursive Formula

The subtree could contain any section of $k_1, \dots, k_n$.

1. Consider $e[i, j]$ as the min-expected-cost BST with keys $k_i, \dots, k_j$ and $d_{i-1}, \dots, d_j$.

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2. If $j = i - 1$, BST only has d_{i-1}: $e[i,j] = q_{i-1}$
3. Otherwise, enumerate the root r:

$$e[i,j] = \min_{r \in [i,j]} \{e[i, r-1] + e[r+1, j] + \text{ one more depth on them}\}$$

Note the last term $:= \underbrace{p_i + \cdots + p_j + q_{i-1} + \cdots + q_j}_{\text{define the sum as } w[i,j]}$.

OPTIMAL-BST(p, q, n)

```
let e[1..n+1, 0..n], w[1..n+1, 0..n],
        and root[1..n, 1..n] be new tables
for i = 1 to n + 1
    e[i, i-1] = q_{i-1}
    w[i, i-1] = q_{i-1}
for l = 1 to n
    for i = 1 to n - l + 1
        j = i + l - 1
        e[i, j] = ∞
        w[i, j] = w[i, j-1] + p_j + q_j
        for r = i to j
            t = e[i, r-1] + e[r+1, j] + w[i, j]
            if t < e[i, j]
                e[i, j] = t
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return e and root
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1. $O(n^3)$ because of the 3 loops

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Running Time

1. $O(n^3)$ because of the 3 loops
2. Improve it to $O(n^2)$: $root[i, j-1] \leqslant root[i, j] \leqslant root[i+1, j]$ by Knuth

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1. DP extends the methods of divide & conquer to optimization problems.
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2. Key structure: Optimal solutions of subproblems $\Leftrightarrow$ optimal solution
3. Running time: Space of subproblems $\times$ enumerate
4. Optimizing running time could be challenging.

Questions?