Introduction to Algorithms
Lecture 10 All Pairs Shortest Paths

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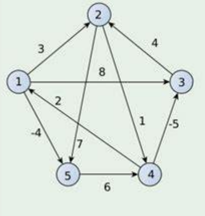
Outline

All Pairs Shortest Paths (APSP)

Problem Description

Given a weighted direct-graph G, compute the shortest paths between all pairs $s \in V$ and $t \in V$.



	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
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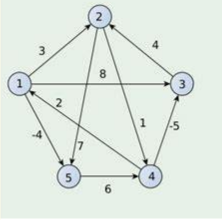

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1. Fundamental problem in CS, e.g., compute the table of fastest transportation between major cities.
2. Use Adjacency matrix to store $\binom{n}{2}$ shortest paths
3. Notation: $w(i, j)$ denotes (negative) weight of edge (i, j)
   $\delta(i, j)$ denotes shortest distance between i and j
   $d(i, j)$ denotes ALG's output

Overview

Known: Single Source Shortest Paths

1. Non-negative weights: Dijkstra's algorithm in time $O(m \log n)$
   $\Rightarrow$ APSP for non-negative weights in time $O(nm \log n)$

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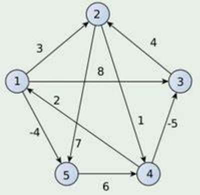

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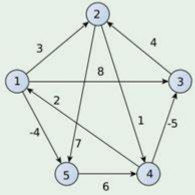

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1. Dense graph: Floyd-Warshall Algorithm in time $O(n^3)$ — better than running Bellman-Ford$\times n$ times
2. Sparse graph: Johnson's algorithm in time $O(nm \log n)$ — generalizes Dijkstra's ALGO to negative weights

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Basic Idea

Consider dynamic programming instead of the greedy method (b.c. of negative weights)

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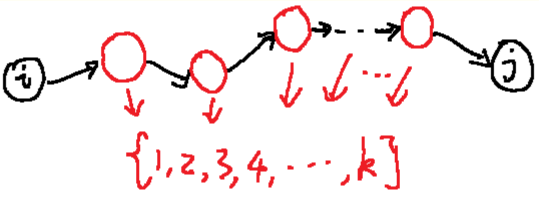
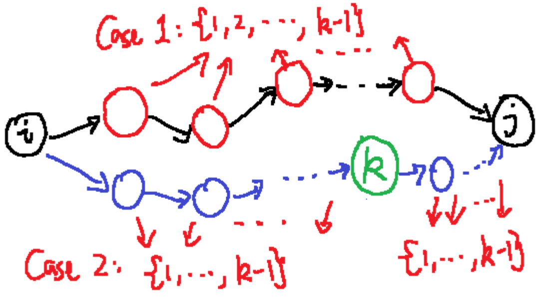
2. 2nd idea: Define $d^k(i,j)$ as the shortest distance from i to j over all paths where all intermediate vertices are in $\{1,2,\ldots,k\}$



— called Floyd-Warshall Algorithm

Dynamic Programming



Recursive Solution

$$d^k(i,j) = \begin{cases} w(i,j) & \text{if } k = 0; \\ \min\Big\{ \underbrace{d^{k-1}(i,j)}_{Case\ 1}, \underbrace{d^{k-1}(i,k) + d^{k-1}(k,j)}_{Case\ 2} \Big\} & \text{if } k > 0; \end{cases}$$

Algorithm Description

procedure FLOYD-WARSHALL(G)
  Initialize $d(i,j) = w(i,j)$ for all i and j
  for $k = 1, \ldots, n$ **do**
    for $i = 1, \ldots, n$ **do**
      for $j = 1, \ldots, n$ **do**
        $d(i,j) = \min\{d(i,j), d(i,k) + d(k,j)\}$

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2. Record decisions $\Pi(i,j)$ to find the path
3. Elegant and easy to implement
4. Next: While it works well for dense graphs, how about sparse graphs?

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Motivation

Recall Single-Source Shortest Paths

1. Dijkstra's algorithm in time $O(m \log n)$ for non-negative weights
2. Bellman-Ford algorithm in time $O(mn)$ for negative weights

Goal: Design an APSP algorithm with 2 properties
(1) Time $O(n \cdot m \log n)$ (2) Allow negative weights.

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1. Faster than Floyd-Warshall algorithm for sparse graphs
2. New idea: Reweight all edges by potential functions s.t.
   (a) All new weights are non-negative
   (b) New weights "preserve" the length of each path

Main Idea of preservation

1. Define potential function $h : V \to \mathbb{R}$ on every vertex
2. Reweight each edge $w(i, j)$ as $\hat{w}(i, j) = w(i, j) + h(i) - h(j)$.

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2. Reweight each edge $w(i,j)$ as $\hat{w}(i,j) = w(i,j) + h(i) - h(j)$.
3. Length of path $i_1 \to i_2 \to \ldots \to i_\ell$ becomes

$$\sum_{j=1}^{<\ell} \hat{w}(i_j, i_{j+1}) = \sum_{j=1}^{<\ell} \left[w(i_j, i_{j+1}) + h(i_j) - h(i_{j+1}) \right]$$
$$= h(i_1) - h(i_\ell) + \sum_{j=1}^{<\ell} w(i_j, i_{j+1})$$

— is preserved

4. Next: How to find a potential function h?

Choose h

Goal

$\hat{w}(i,j) = w(i,j) + h(i) - h(j) \geqslant 0$ for all i, j

1. Recall that for single-source shortest paths: $d(s,j) \leqslant d(s,i) + w(i,j) \Rightarrow w(i,j) + d(s,i) - d(s,j) \geqslant 0$ for all (i,j) — how to choose s?

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5. Finally apply Dijkstra's algorithm n times on the non-negative weighted graph

Description

JOHNSON(G, w)

compute G', where $G'.V = G.V \cup \{s\}$,
  $G'.E = G.E \cup \{(s, v) : v \in G.V\}$, and
  $w(s, v) = 0$ for all $v \in G.V$
if BELLMAN-FORD$(G', w, s) ==$ FALSE
  print "the input graph contains a negative-weight cycle"
else for each vertex $v \in G'.V$
    set $h(v)$ to the value of $\delta(s, v)$
      computed by the Bellman-Ford algorithm
  for each edge $(u, v) \in G'.E$
    $\widehat{w}(u, v) = w(u, v) + h(u) - h(v)$
  let $D = (d_{uv})$ be a new $n \times n$ matrix
  for each vertex $u \in G.V$
    run DIJKSTRA$(G, \widehat{w}, u)$ to compute $\widehat{\delta}(u, v)$ for all $v \in G.V$
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Running time: $O(nm \log n)$.

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Matrix Multiplication?

Back to the idea: Define $d^{\ell}(i,j)$ as the shortest distance from i to j over all paths of $\leqslant \ell$ edges

$$\text{Example: } d^2(i,j) = \min_k \left\{ d^1(i,k) + d^1(k,j) \right\}$$

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Let $A^{(\ell)}(i,j)$ denote the matrix with entries $d^{\ell}(i,j)$. If we can define some algebraic operation $(\odot, \oplus)$:

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3. Then $A^{(2)} = A^{(1)} \odot A^{(1)}$ and $A^{(n)} = A^{(n-1)} \odot A^{(1)} = (A^{(1)})^{\odot n}$
4. If we have a faster matrix multiplication algorithm for these algebraic operation ... ☺

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An easier problem: Transitive Closure

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3. 1st try: Define $\odot$ as $\wedge$ and $\oplus$ as $\vee$ (See Chapter 25.1 and 25.2 in CLRS) — ☹ but they are not standard operation in $\mathbf{F}_2$

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6. Determine bit $k-1$: Check $A^{2^{k-1}+2^{k-2}-1}(i,j)$, and so on...
7. Reorganize it s.t. the binary search is applied for all $\delta(i,j)$ — Easier to determine the last bit by consider A^2 reversely
8. Challenge: find those paths and weighted graphs!

On the All-Pairs-Shortest-Path Problem

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Abstract

The following algorithm solves the distance version of the all-pairs-shortest-path problem for undirected, unweighted n-vertex graphs in time $O(M(n) \log n)$, where $M(n)$ denotes the time necessary to multiply two $n \times n$ matrices of small integers (which is currently known to be $o(n^{2.376})$):

Input: $n \times n$ 0-1 matrix A, the adjacency matrix of undirected, connected graph G

Output: $n \times n$ integer matrix D, with d_{ij} the length of a shortest path joining vertices i and j in G

function APD(A : $n \times n$ 0-1 matrix) : $n \times n$ integer matrix
  let $Z = A \cdot A$
  let B be an $n \times n$ 0-1 matrix, where $b_{ij} = 1$ iff $i \neq j$ **and** ($a_{ij} = 1$ **or** $z_{ij} > 0$)
  if $b_{ij} = 1$ for all $i \neq j$ **then return** $n \times n$ matrix $D = 2B - A$
  let T = APD(B)
  let $X = T \cdot A$
  return $n \times n$ matrix D, where $d_{ij} = \begin{cases} 2t_{ij} & \text{if } x_{ij} \geq t_{ij} \cdot degree(j) \\ 2t_{ij} - 1 & \text{if } x_{ij} < t_{ij} \cdot degree(j) \end{cases}$

We also address the problem of actually finding a shortest path between each pair of vertices and present a randomized algorithm that matches APD() in its simplicity and in its expected running time.

Questions?