Introduction to Algorithms: Lecture 2b

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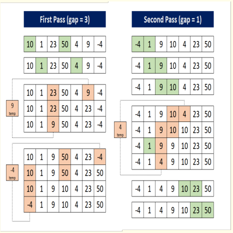
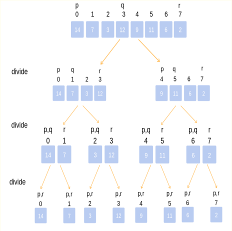
Outline

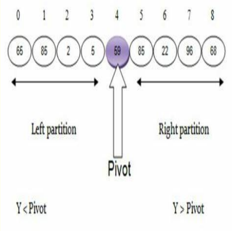
Motivation



(a) SHELLSORT



(b) MERGESORT



(c) QUICKSORT

Questions

Several algorithms sort n elements in $O(n \log n)$ time
— Faster Algorithms? Is $O(n)$ possible?

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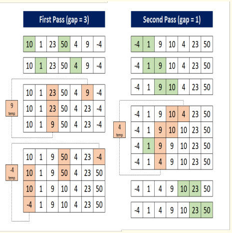

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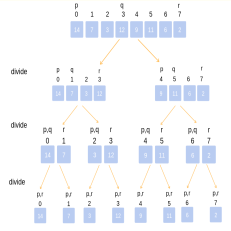

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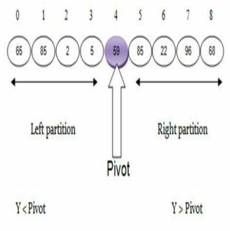

(f) QUICKSORT

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Several algorithms sort n elements in $O(n \log n)$ time
— Faster Algorithms? Is $O(n)$ possible?

Remark: $\Omega(n)$ time to read all elements

Overview

Part 1: $\Omega(n \log n)$ lower bound for comparison sorts

All previous sorting algorithm can sort strings, real numbers, and any objects — too flexible to get $O(n)$ time.

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 Demonstration of string sorting using Bubble sort in C++
Strings in sorted order are :
 String 1 is Asia
 String 2 is Educba
 String 3 is India
 String 4 is Institute
 String 5 is Python
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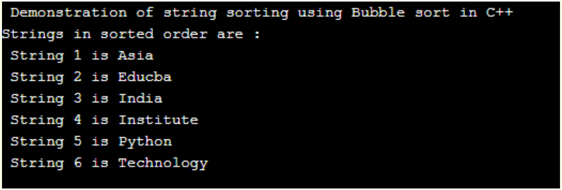
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Part 2: $O(n)$ sorting

A linear-time algorithm for sorting bounded integers.

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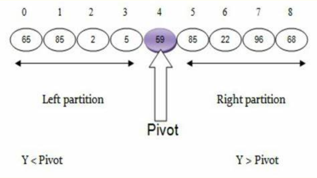
If pivot $x = A[i]$, $A[j]$ is in front of $A[i]$ for any $A[j] < x$.



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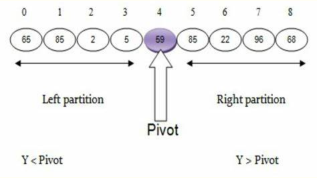

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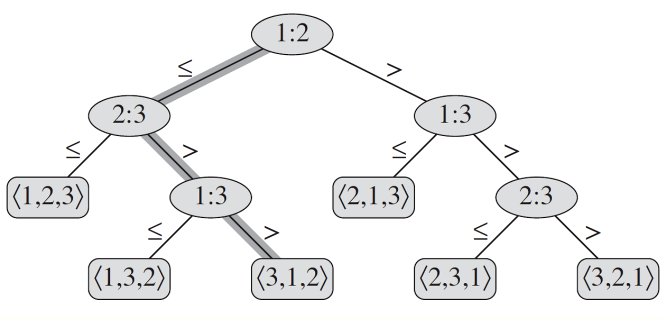
For convenience, consider A as a permutation of $[n] := \{1, 2, \ldots, n\}$.

Comparisons

Any deterministic sort (excluding QUICKSORT) $\Rightarrow$ a decision tree:

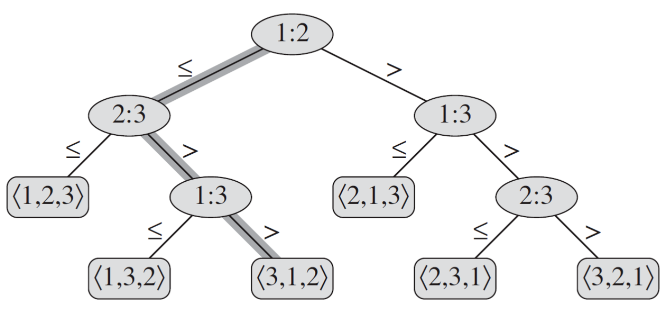
1. Starting from the root, each node (i, j) corresponds to a comparison



Comparisons

Any deterministic sort (excluding QUICKSORT) $\Rightarrow$ a decision tree:

1. Starting from the root, each node (i, j) corresponds to a comparison
2. each edge has two labels "$<$" and "$>$"
3. each leaf corresponds to a termination with a correct order



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Observation: Worst running time $\geqslant$ length (longest path from the root).

THM 8.1 in CLRS

Any comparison sort algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

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1. Consider the decision tree corresponding to the sort algorithm.
2. The decision has $n!$ permutations on its leaves
   — so its depth is $\geqslant \log_2(n!)$.

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Extensions: 1) The average-case running time is $\Omega(n \log n)$.
2) The running time of any randomized comparison sort is $\Omega(n \log n)$ with high prob.

Outline

Counting Sort

Suppose all elements are integers in $[0, 1, \ldots, k]$ for small $k = O(n)$.

Basic Idea

Instead of comparing them, one could count how many elements $= 0, = 1, \ldots, = k$ separately.

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Basic Idea

Instead of comparing them, one could count how many elements $= 0, = 1, \ldots, = k$ separately.

$\Rightarrow$ Positions of elements with value ℓ are between $(\#\text{ elements} < \ell) + 1$ and $(\#\text{ elements} < \ell + 1)$.

Implementation: After counting $(\#\text{ elements} = 0), \ldots, (\#\text{ elements} = k)$, sum them up.

Description

```
COUNTING-SORT(A, B, k)
let C[0..k] be a new array
for i = 0 to k
    C[i] = 0
for j = 1 to A.length
    C[A[j]] = C[A[j]] + 1
// C[i] now contains the number of elements equal to i.
for i = 1 to k
    C[i] = C[i] + C[i - 1]
// C[i] now contains the number of elements less than or equal to i.
for j = A.length downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```

on Page 195 of CLRS

Analysis

The correctness follows from the main properties of C.

Running time

Its running time is $O(n + k)$.

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Next Question

What if k is huge say $k = n^{O(1)}$ or $k = 2^{64}$?

Radix Sort

If k is huge, consider the binary representation of all numbers.
Example:

$$
\begin{aligned}
A[1] &= 683 &&= (1010101011)_2 \\
A[2] &= 121 &&= (0001111001)_2 \\
&\vdots \\
A[n-1] &= 794 &&= (1100011010)_2 \\
A[n] &= 835 &&= (1101000011)_2
\end{aligned}
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1st idea: Sort according to 1st bit, then 2nd bit, 3rd bit, ...

$$\begin{aligned} A[2] &= 121 && = ({\color{red}0}001111001)_2 \\ &\vdots && \\ A[i-1] &= 301 && = ({\color{red}0}100101101)_2 \\ A[i] &= 648 && = ({\color{red}1}010001000)_2 \\ &\vdots && \\ A[1] &= 683 && = ({\color{red}1}010101011)_2 \end{aligned}$$

Key Idea

Then sort the two groups separately.

Group 0

$$\begin{aligned} A[2] &= 121 &= (0001111001)_2 \\ &\vdots \\ A[i-1] &= 301 &= (0100101101)_2 \end{aligned}$$

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However, this needs to store $\log_2 k$ levels (and many groups).

Question

Can we find a simpler solution?

An elegant solution: Radix-sort

Recall stable: numbers with the same value appear in the output are in the same order as their order in the input

---

procedure RADIX-SORT(d)
  for $i = 1, \ldots, d$ **do**
    use a stable sort (e.g., COUNTINGSORT) to sort array A on digit i

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Example: $1010, 0101, 1111, 0000, 0001, 0100, 1110, 0011$

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$1010, 0000, 0100, 1110, 0101, 1111, 0001, 0011$

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1. Consider the highest digit h where $A[i]$ and $A[j]$ are different. After the iteration $i = h$, $A[i]$ is in front of $A[j]$.
2. Then $A[i]$ is always in front of $A[j]$ because of the stable sort.

Running time

$O(d(n+k))$ if there are d digits where each digit takes up to k values.

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Setting parameters

Let $b = \log_2\big(\max_i A[i]\big)$, we could sort r bits (like one digit with $k = 2^r$) s.t. the time becomes $O\big(\frac{b}{r} \cdot (n + 2^r)\big)$.

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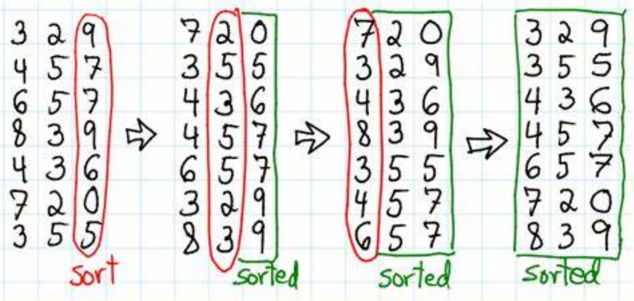
1. Choose $r = \log n + O(1)$ s.t. the running time becomes $O(\frac{b}{\log n} \cdot n)$.
2. When $b = O(\log n)$ — all numbers are in $\mathrm{poly}(n)$, linear time! ☺

Extensions

It works for strings, dates, and objects with several keywords.

Conclusion



1. COUNTSORT and RADIXSORT are fast and easy to implement.
2. Some restrictions.

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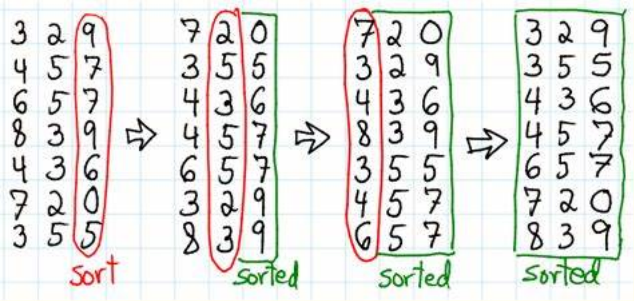

1. CountSort and RadixSort are fast and easy to implement.
2. Some restrictions.
3. Two keys in RadixSort are (1) a delicate property called stable; (2) adjusting parameters.

Summary of sorting algorithms

Type	Time	Method	
SHELLSORT	$O(n \log^2 n)$	INSERTIONSORT	(1) $O(1)$-extra space; (2) easy to implement
MERGESORT	$O(n \log n)$	Divide & Conquer	(1) $O(n)$-extra space; (2) big constant in O
QUICKSORT	$O(n \log n)$	Divide & Conquer	(1) Most widely used; (2) Randomized
RADIXSORT	$O(n)$	COUNTINGSORT	(1) For integers $\leqslant n^{O(1)}$; (2) big constant in O

Table of sorting algorithms

More: (1) A lower bound $\Omega(n \log n)$ for comparison sort.
(2) Many algorithms could be applied to sort strings and other objects.

Questions?