Introduction to Algorithms
Lecture 12 MAX FLOW

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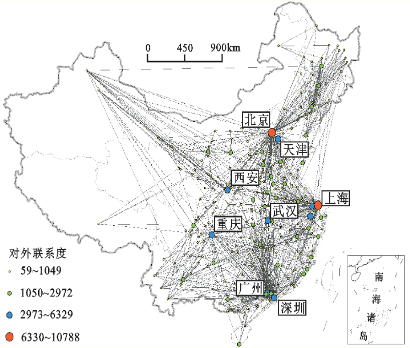
2025 spring in

Outline

Introduction

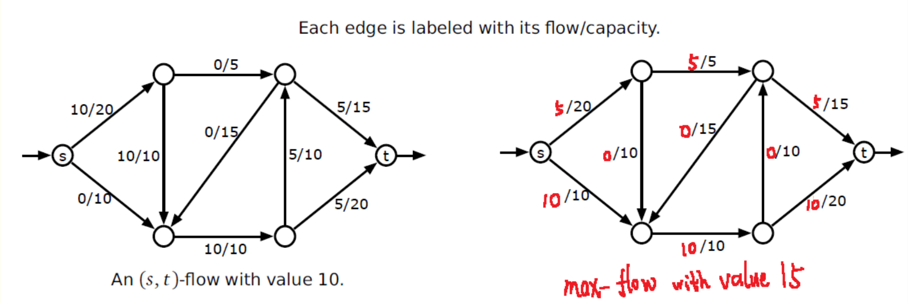
Flow networks are fundamental problems in many areas

1. Resource allocation and scheduling
2. Network routing
3. Traffic control



Overview

Basic problem: Given each edge's limit, schedule the max amount of flows from s to t



Lots of interesting algorithms:

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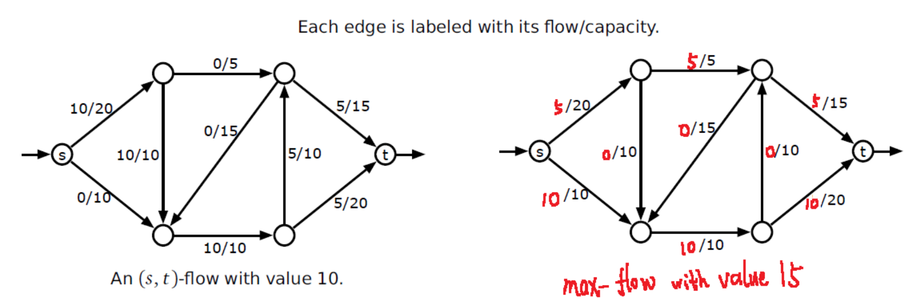
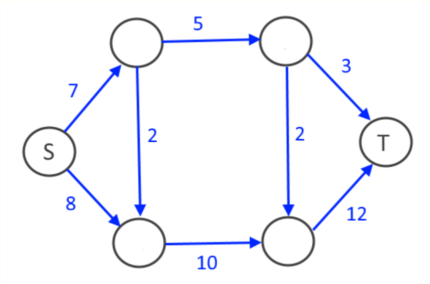


Lots of interesting algorithms:

1. How to compute a flow? How to prove the flow is max?
2. Duality: Max-Flow Min-Cut Theorem
3. Min-cut has lots of applications in data science ...

Formal Definition

A flow network $G = (V, E)$ is a directed graph where each edge (u, v) has a capacity $c(u, v) \geqslant 0$.



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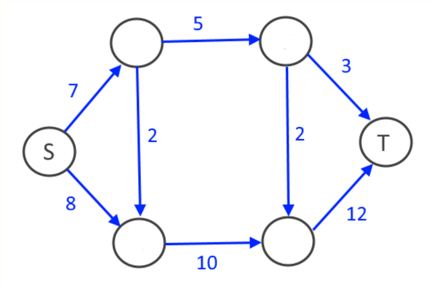

A flow $f : V \times V \to \mathbb{R}$ (with directions) satisfies:

1. Capacity constraint: $0 \leqslant f(u, v) \leqslant c(u, v)$
2. Flow conservation: For each $u \in V \setminus \{s, t\}$,
   flow-in $\sum_v f(v, u) =$ flow-out $\sum_v f(u, v)$

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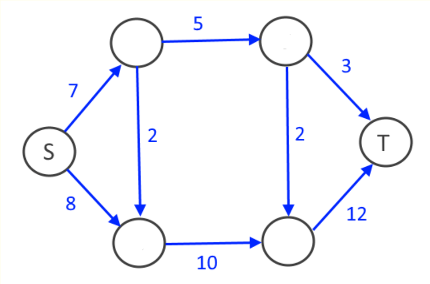

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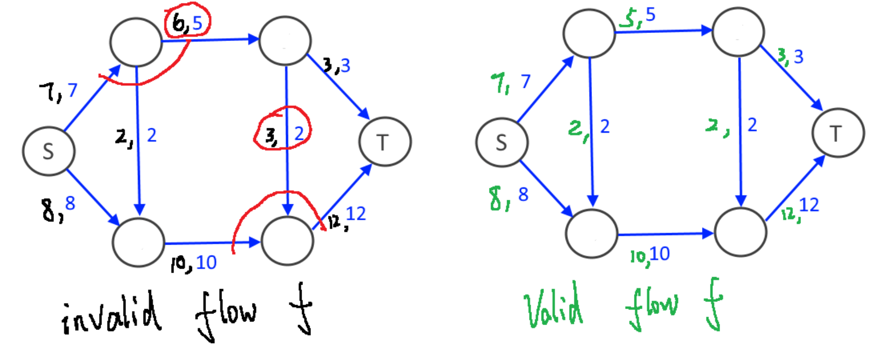
Moreover, define the value of f as $|f| := \sum_v f(s, v) = \sum_v f(v, t)$.

Examples


S
T
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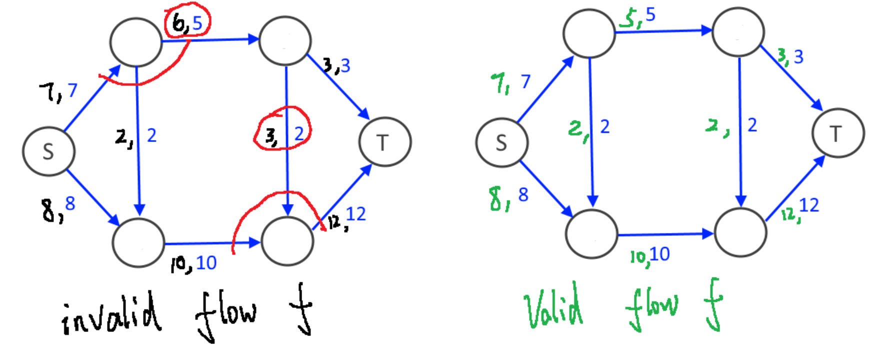

Overview:

1. Ford-Fulkerson Method: Residue graph and augmenting path
2. Correctness: Max-Flow Min-Cut Theorem
3. Greedy Methods control time

Outline

A general paradigm

procedure FORD-FULKERSON(G)
    $f : V \times V \to R$
    while $\exists$ augmenting paths in residual network G_f **do**
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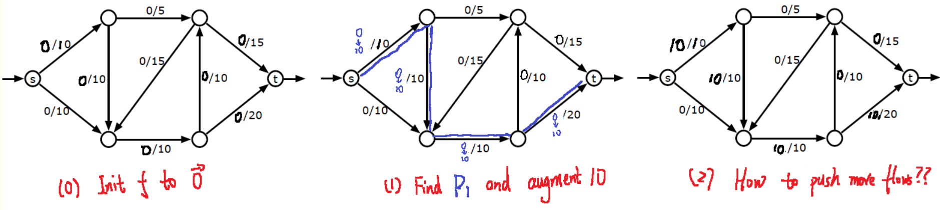

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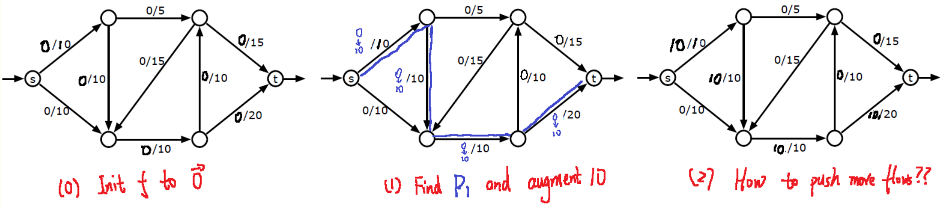

1. However, the naive implementation does not work
2. Define augmenting path and residual graph formally

Capacities in Residual Graph G_f

Roughly, G_f consists of all edges with a capacitie$>$flow.

Two Types of Edges in G_f: For an original $(u, v) \in E$,

1. its residual capacity $c_f(u, v) = c(u, v) - f(u, v)$
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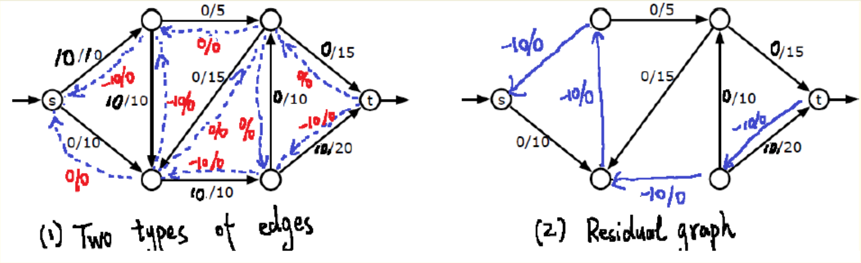
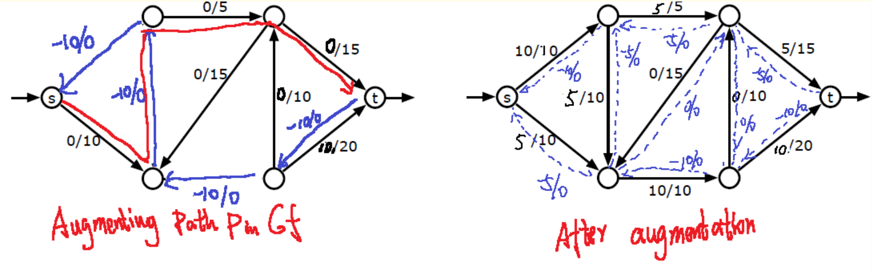


My implementation:

1. Set reverset capacity $c(v, u) = 0$ for any $(u, v) \in E$ but guarantee $f(v, u) = -f(u, v)$ for all edges at any moment
2. Then $c_f(u, v) = c(u, v) - f(u, v)$ for any u and v despite the direction

Augmenting Paths

An augmenting path P is a simple path $s \rightsquigarrow t$ in residual graph G_f



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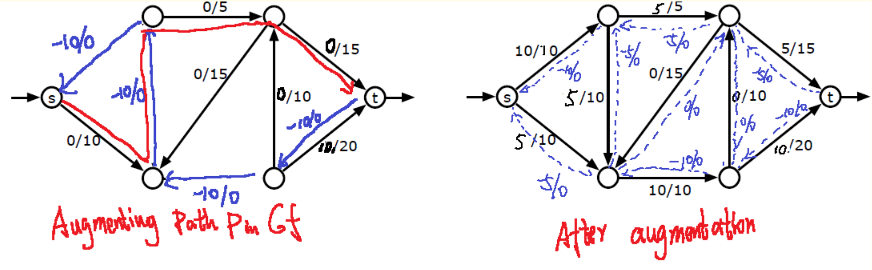

More about Augmenting a path

1. Define P's residual capacity as $c_f(P) = \min\{c_f(u, v) = c(u, v) - f(u, v) : (u, v) \in P\}$, i.e., max residual along P

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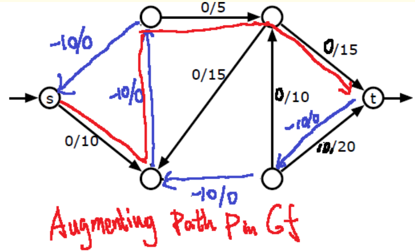

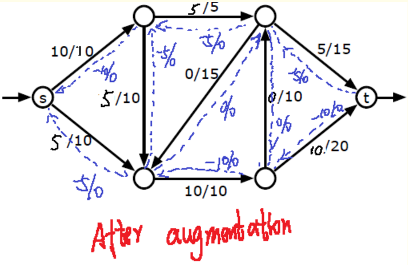

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1. Define P's residual capacity as $c_f(P) = \min\{c_f(u,v) = c(u,v) - f(u,v) : (u,v) \in P\}$, i.e., max residual along P
2. Then augment a flow of amount $c_f(P)$ along P
3. Increase $|f|$ by $c_f(P)$

Analysis

procedure BASIC-FORD-FULKERSON(G)
  $f : V \times V \to R$
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Next

1. Correctness: Why does it find a maximum flow?
2. Running time: Is it polynomial in $n \cdot m$?

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Definition

1. General Cut: Given a graph $G = (V, E)$, a cut is a partition $(S, \overline{S})$ where $S \subsetneq V$ and $S \neq \emptyset$
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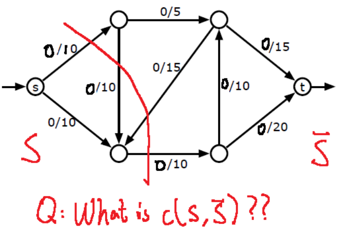

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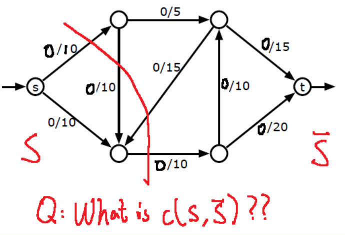
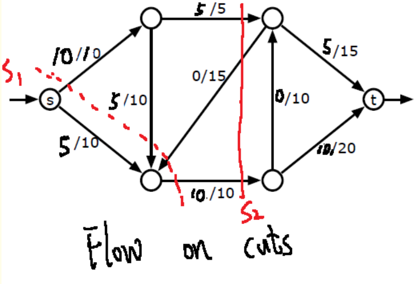


1. Graph cut is a fundamental object in CS: data mining, social networks, ...
2. Today, only consider S with minimum capacity — called min-cut

Basic Property



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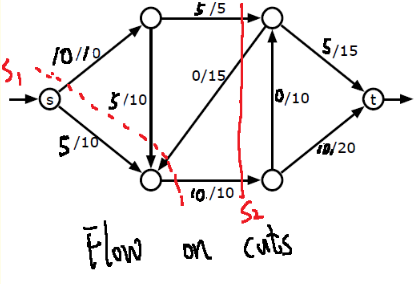

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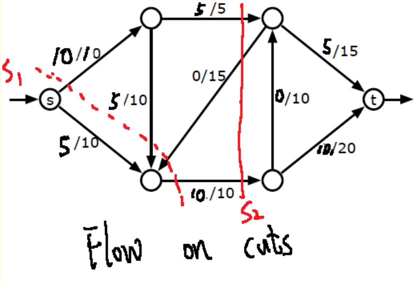

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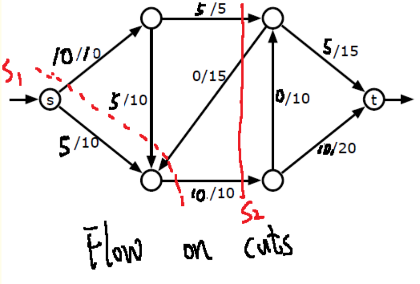

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2. Lemma 26.5 in CLRS: For any flow f and cut $(S, \overline{S})$, $f(S, \overline{S}) = |f|$
3. Corollary 26.6 in CLRS: $|f| \leqslant c(S, \overline{S})$ for any f and S

Main Theorem

Max-Flow Min-Cut Theorem (THM 26.6 in CLRS)

If f is a flow, the following 3 conditions are equivalent

1. f is a maximum flow
2. No augmenting path in G_f
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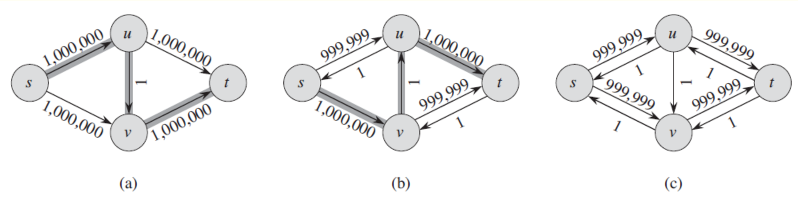
Time Complexity

One more issue

Basic version of Ford-Fulkerson Algorithm does not guarantee a polynomial-time.



Example: Figure 26.7 from CLRS

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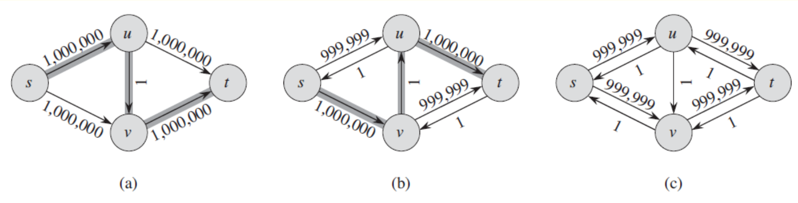

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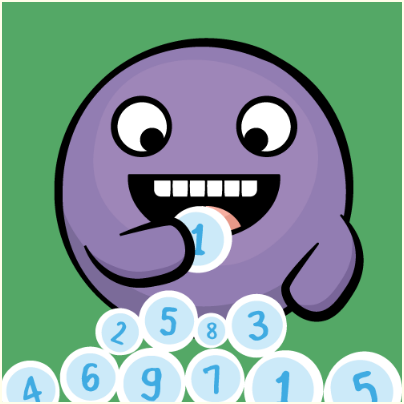
Question

How to improve running time?

Two greedy approaches



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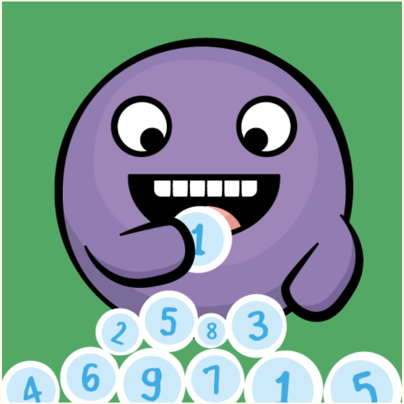

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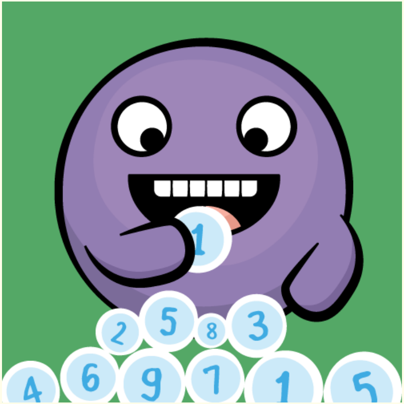

1. Finding the shortest augmenting path (i.e. with fewest edges) leads to at most nm paths, called Edmonds-Karp
2. Finding the fattest augmenting path (i.e. with largest residual) leads to at most $m \cdot \log |f^*|$ paths

Greedy I

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procedure EDMONDS-KARP(G)

  $f : V \times V \to R$

  while $\exists$ augmenting paths in the residual network G_f **do**

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    Augment/Push flow f on P

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For any v, the shortest-path distance $\delta_f(s, v)$ in the residual network G_f increases monotonically with each flow.

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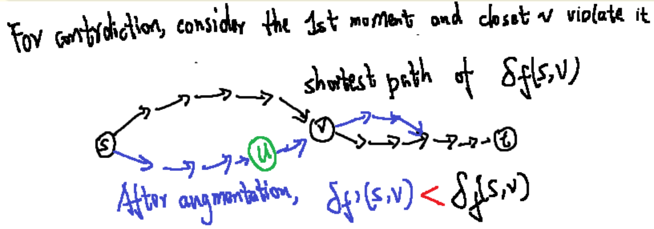

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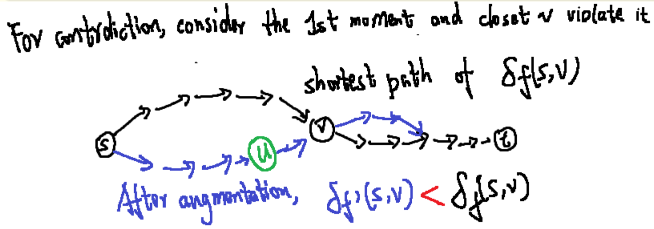

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3. Key claim: $(u, v) \notin G_f$
4. Only way that $(u, v) \notin G_f$ but $(u, v) \in G_{f'}$ is that the shortest augmenting path in G_f has the reverse (v, u) — leads to a contradiction

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Theorem 26.8 in CLRS

The Edmonds-Karp ALGO augments at most $O(nm)$ paths.

Analysis of Greedy II

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THM: Running time of Greedy II

The above greedy approaches finds at most $m \cdot \ln |f^*|$ paths where f^* is the amount of max-flow.

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3. Question: What are the subsets here?

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2. Consider each unit flow in f^* as an element, reformulate the problem as set cover
3. Question: What are the subsets here?
4. Recall that the greedy method in set cover needs $OPT \cdot \log |U|$ where OPT is the best solution and $|U|$ is # elements

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2. Residue graphs and augmenting paths: reverse edges!

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2. Residue graphs and augmenting paths: reverse edges!
3. Two greedy algorithms implement Ford-Fulkerson method
4. Many extensions: Max-Matching, Min-cost Max-Flow, ...

Questions?