Introduction to Algorithms
Lecture 13 Number Theoretic Algorithm

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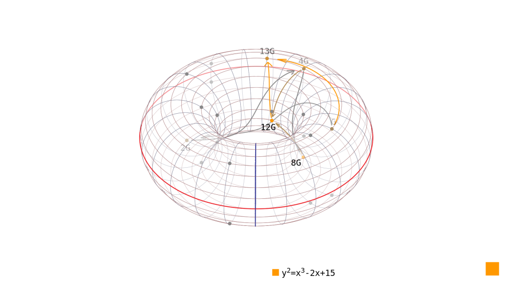
Outline

Introduction

Many algorithms uses number theory

1. Hash functions
2. Coding theory
3. Cryptography



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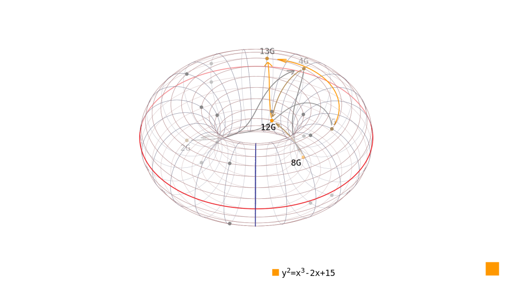

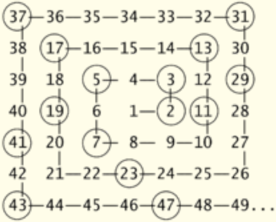

Primes are the backbone of Number theory

Most Fundamental Problem in Number Theory

How to find a large prime number p?

Set up

Our Focus

Given n say 10^3 or 10^4, find a prime number of n digits (in binary).

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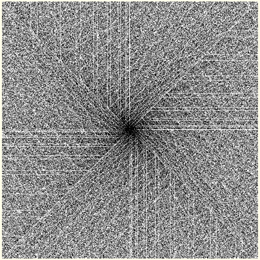
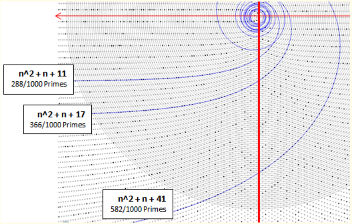
Given n say 10^3 or 10^4, find a prime number of n digits (in binary).

1. Basic idea: Sieve algorithm?
2. But we want a number between 2^{n-1} and $2^n - 1$!
3. Lots of interesting math: prime number THM, Euclid's ALGO, Fermat's THM, chinese remainder THM, primality testing





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2. How to test whether q is a prime or not efficiently?
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2. How to test whether q is a prime or not efficiently?
3. Many tools ...
4. To the best of my knowledge, fastest algorithm in time $\approx n^3$

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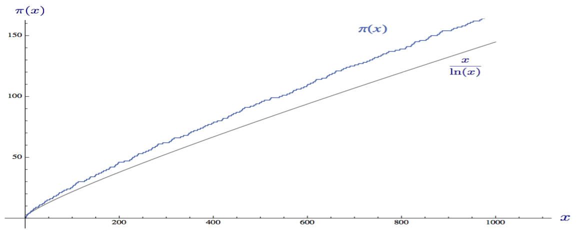

Prime Number Theorem — Most fundamental theorem of primes

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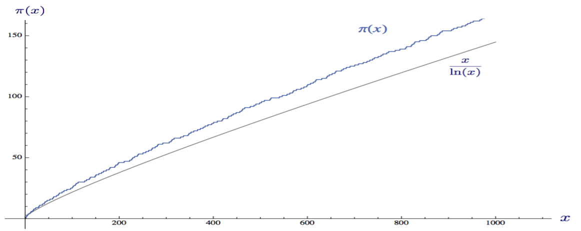

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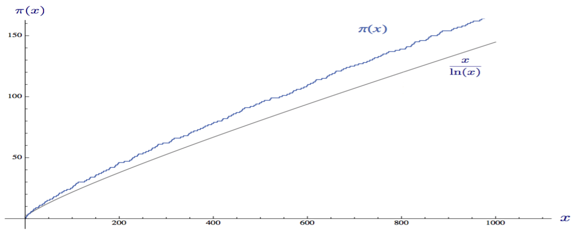

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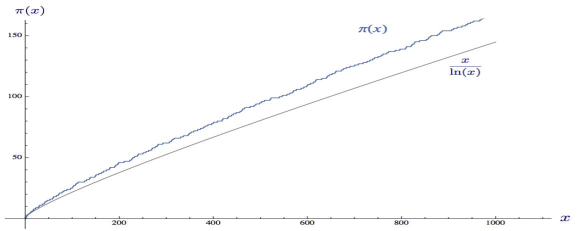

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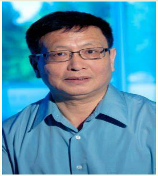

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3. Twin prime number THM by Yitang Zhang 2013

Legendre

Weak Proof by Chebyshev

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2. On the 1st hand, $\binom{2N}{N} = \Theta(2^{2N}/\sqrt{N})$
3. On the 2nd hand, let us consider the contribution of $p \leqslant N$ in $\binom{2N}{N}$
   — Question: Can we bound k s.t. at most k factors of p in $\binom{2N}{N}$?

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Basic Property

For any $a \in \{1, \ldots, p-1\}$, $\exists b$ such that $ab \equiv 1 \mod p$, called a^{-1}.

In fact, we could find a^{-1} in time $O(\log p) = O(n)$ ☺

Revisit Euclid's Algorithm

Recall that Euclid's algorithm computes $gcd(a, b)$

Algorithm Euclid's algorithm for GCD

```
function EUCLID(a, b)
    if b=0 then
        return a
    else
        return Euclid(b, a mod b)
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Recall that Euclid's algorithm computes $gcd(a, b)$

Algorithm Extended Euclid's algorithm

function EUCLID(a, b)
  if b=0 **then**
    return $(x = 1, y = 0)$
  else
    $(x_0, y_0) = \text{Euclid}(b, a \mod b)$
    Return $(y_0, x_0 - [a/b] \cdot y_0)$

Extension

For any a, b, it finds $x, y \in \mathbb{Z}$ such that $ax + by = gcd(a, b)$

Properties of Prime

Fermat's little Theorem

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7. Let us introduce Euler function $\phi(N)$

Property: Prime Factorization

Unique Factorization

For any integer N, there is an unique factorization of primes $N = p_1^{e_1} \cdot p_2^{e_2} \cdots p_r^{e_R}$.

1. Definition of Euler Function: $\phi(N) = (1 - 1/p_1) \cdots (1 - 1/p_r) \cdot N$ if N's distinct prime factors are $p_1, \ldots, p_r$

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2. Fact: Number of $a \in Z_N$ with $(a, N) = 1$ is $\phi(N)$
3. Extension of Fermat's little THM: $a^{\phi(N)} \equiv 1 \mod N$ for any $(a, N) = 1$.

Composite Numbers

To distinguish prime numbers from composites, what properties does a composite have?

Chinese Remainder THM

Given distinct primes $p_1, \ldots, p_r$, consider $(a_1, \ldots, a_r) \in Z_{p_1} \times \cdots \times Z_{p_r}$.
There is a 1-1 map between $a \in Z_{p_1 \cdot p_2 \cdots p_r}$ and $(a_1, \ldots, a_n)$

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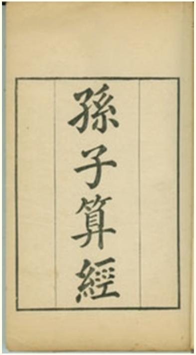

1. Most fundamental THM in group/ring theory and number theory
2. Both directions are useful in algorithm design and analysis

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Key Observation

1. If q is a prime, $x^2 \equiv 1$ has at most 2 roots. What are they?
2. If q is not a prime, $x^2 \equiv 1$ has at least 4 roots. Why?

New Idea

Taking Square Root

If we can generate a random root of $x^2 \equiv 1 \mod q$, it tells whether q is a prime or not — prime q has at most 2 roots and composite q has $\geqslant 4$ roots

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   - If $a^{\frac{q-1}{2}} \neq \pm 1$, claim q is composite
   - If $a^{\frac{q-1}{2}} = -1$, give up this a
   - Else if $a^{\frac{q-1}{2}} = 1$, keep trying $a^{\frac{q-1}{4}}, a^{\frac{q-1}{8}}, \ldots$

Formal Description

Pick many random $a \sim Z_q$ and call WITNESS(a) for each one — output "composite" if any call does so

---

Algorithm Miller-Rabin Tester

---

function WITNESS(a, q)
    Decompose $q - 1 = 2^t \cdot u$
    $x_0 \equiv a^u$
    for $i = 1, \ldots, t$ **do**
        $x_i \equiv x_{i-1}^2$ // x_{i-1} is a square root of x_i
        if $x_i \equiv 1$ and $x_{i-1} \neq \pm 1$ **then**
            Return Composite
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Running Time

$O(\log^3 q)$: $t = O(\log q)$ — but $x_i^2 \mod q$ takes $O(\log^2 q)$ time

Analysis

Correctness

If q is composite, at least half a make $\textsc{Witness}(a, q) = $ Composite.

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6. When $i = j + 1$, half elements will be caught

Summary

1. Primality tester faster than $O(n^3)$?
2. How to generate primes efficiently?
3. Many other problem: factoring, ...

Questions?