算法基础 2025 春 Homework 1

任课老师：陈雪 due: March 6, 13:59

作业要求：说明思路与符号，清晰简洁的伪代码，必要的时间复杂度分析和必要的正确性分析。可以直接调用基本的数据库和已讨论过的算法/程序（如排序、找中位数、二分查找等）。

问题 1 (20 分). 回答以下问题，并给出必要的计算过程。

(a) 考虑将 N 个球分别随机投入 N 个盒子中（即每个球以 $1/N$ 的概率被独立地投入到其中一个盒子中）。求第一个盒子中恰好有 3 个球的概率。请求出准确的概率然后尽量化简。

(b) 在 (a) 中，期望有多少个盒子中恰好有 3 个球？

(c) 将序列 $1, 2, \ldots, n$ 等概率随机重新排列为 $i_1, i_2, \ldots, i_n$（即对于每种排列，最后的结果恰是此排列的概率均为 $1/n!$），求满足 $i_k = k\,(k \in [n])$ 的个数 k 的期望值。

(d) 考虑一个 ℓ 面的骰子。假设其每次投掷都是独立的，且每面向上的概率都为 $1/\ell$，求第一次看到第一面向上的期望投掷次数。

问题 2 (30 分). 回顾 weighted majority 算法：

Algorithm 1 5.2

```
procedure WEIGHTED MAJORITY(n, T)
    c_i^(0) = 1 for all i ∈ [n]
    for t = 1 to T do
        Prediction p^(t) ←Majority(∑_{i:q_i^(t)=0} c_i^(t-1), ∑_{i:q_i^(t)=1} c_i^(t-1))
        Update c_i^(t) ← c_i^(t-1)/2 if expert i is wrong on event t
    end for
end procedure
```

Let $s^{(t)}$:=the number of mistakes by our algorithm in the first t events and $s_i^{(t)}$:=the number of mistakes by expert i in the first t events. We have shown

$$s^{(t)} \leq 2.41(s_i^{(t)} + \log n) \text{ for any } i \in [n] \text{ at any moment } t. \tag{1}$$

a) Change the update rule: reward expert i if her prediction is correct $c_i^{(t)} \leftarrow c_i^{(t-1)} \cdot 2$; still keep the penality $c_i^{(t)} \leftarrow c_i^{(t-1)}/2$ if she is wrong.

   Prove the new guarantee of (1) and argue that reward does not make a big difference.

b) Change the update rule to guarantee $s^{(t)} \leq (2+\epsilon)s_i^{(t)} + O_\epsilon(\log n)$.

c) Design an instance such that WeightedMajority has $s^{(T)} \geq \log_2 n$ but the best expert has $s_i^{(T)} = 0$. This demonstrates the $\log n$ term in (1) is necessary — in fact, this term is necessary for ANY algorithm.

d) Design another instance such that $\min_i s_i^{(T)} \geq 1\% \cdot T$ and $s^{(T)} \geq 2 \cdot \min_i s_i^{(T)}$ — assume that when $\sum_{i:q_i^{(t)}=0} c_i^{(t-1)} = \sum_{i:q_i^{(t)}=1} c_i^{(t-1)}$, WeightedMajority always choose 0.

This matches the upper bound $2+\epsilon$ in part (b).

问题 3 (20 分)**.** 判断以下命题的正误。若正确，请给出证明；若错误，请给出反例。

(a) $n \cdot 2^{(\log n)^{0.9}} = n^{1+o(1)}$

(b) $n^{100} = 2^{\Omega(\log n)}$

(c) $n \cdot \log n = O(n \cdot 2^{\sqrt{\log n}})$

(d) $\binom{n}{k} = O(n^k)$ for any $k \in [n]$ — $\binom{n}{k}$ 表示从 $[n] = \{1, \dots, n\}$ 中选取 k 个元素的组合数

问题 4 (30 分)**.** 阅读以下伪代码，对于每一份代码，解释其功能，并提供代码运行时间的严格分析。

Algorithm 2

Require: An array a of length n

```
procedure COUNT(a, n)
    res ← 0
    for i ← 1 to n do
        for j ← 1 to i − 1 do
            for k ← i + 1 to n do
                if a[i] < a[j] ∧ a[i] < a[k] then
                    res ← res + 1
                end if
            end for
        end for
    end for
    return res
end procedure
```

Algorithm 3

Require: Two positive integers a, b

```
procedure G(a, b)
    if b = 0 then
        return a
    else
        return G(b, a mod b)
    end if
end procedure
```

Algorithm 4

Require: Integers n , m

```
procedure COUNT(n, m)
    if n = 1 then
        return 1
    end if
    total ← 0
    for x ← 0 to m do
        total ← total + COUNT(n − 1, m − x)
    end for
    return total
end procedure
```