Introduction to Algorithms: Lecture 4

Xue Chen
xuechen1989@ustc.edu.cn
2025 spring in

HW & Experiment

1. HW 2 & Experiment 1 are out
2. Office hours of Week 4 and Week 5 are in classroom 3A103

Outline

Data Structure

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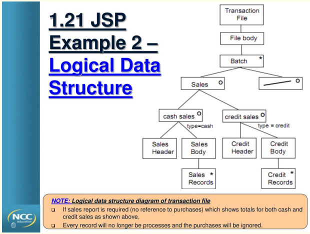

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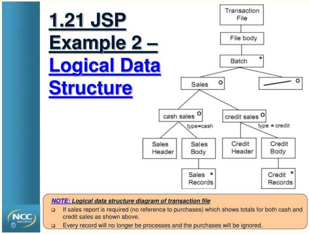

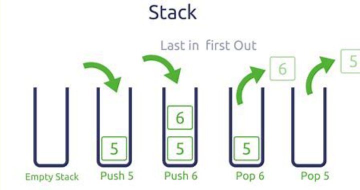

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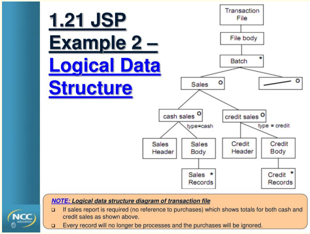
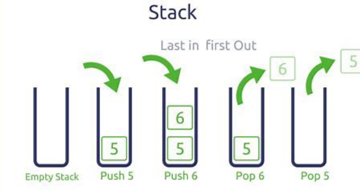
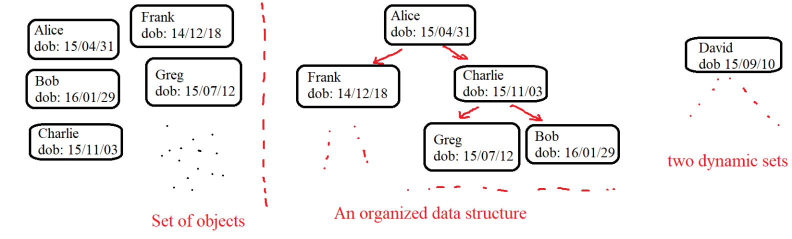




2. Different types: Linear (queues, stacks, linked lists, . . .), Non-Linear (trees,graphs,. . .)
3. Other properties: static vs dynamic, homogenous, . . .

Overview

This course considers data structures as a way to represent finite dynamic sets (of various elements).



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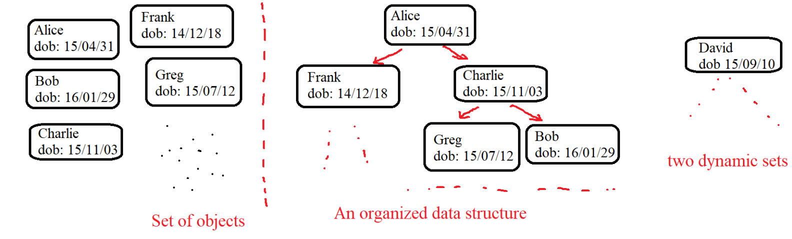

The goal is

1. Maintain it efficiently
2. Answer queries like (1) who is the oldest kid? (2) How many kids born in August 2015? ...

Overview (II)

Each element has a unique ID/pointer (like “Alice”) and a key value $k \in \mathbb{Z}$ with a total order (like “dob: 15/04/31”).

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5. $\textsc{Successor}(S, x)$: Given x, return the next element after x in S
6. $\textsc{Union}(S, T)$: Unites the two dynamic sets S and T
7. $\textsc{Count}(S, k_1, k_2)$: Given a total order and an interval $[k_1, k_2]$, return the number of elements in S with a key value k in $[k_1, k_2]$.

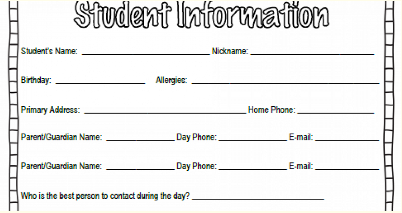
Outline

Introduction

Hash maintains a dynamic set S of keys for the dictionary problem

Example

Maintain all students information — enroll, graduate, search by id and name, ...


Student Information

Student's Name: ____________ Nickname: ____________

Birthday: ____________ Allergies: ____________

Primary Address: ____________ Home Phone: ____________

Parent/Guardian Name: ____________ Day Phone: ____________ E-mail: ____________

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Who is the best person to contact during the day? ____________

Description

$x.key$ denotes the unique key of each element x — for convenience, only consider one key in $\mathbb{Z}$.

1. SEARCH(k): Find x in S with $x.key = k$
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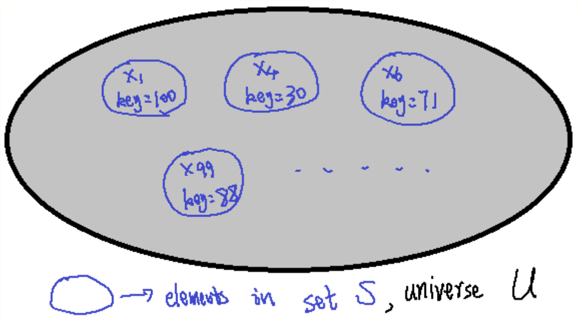

Notations

1. U: the domain of keys
2. S: the dynamic set in U
3. m: the max elements in S

Think $|U| = 10^9$ and $m = 10^3$ or 10^6

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(2) If we only have $O(m)$ storage-space, what shall we do?

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1. Solution 1: Maintain $T : U \rightarrow$ node s.t. $T[x.key] = x$ for any x
2. Solution 2: Maintain an array or a chain.

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Solution

1. Prepare a hash function $f : U \to [m]$
   like $f_{a,b}(x) = (a \cdot x.key + b) \mod m$
2. Maintain a truth table $T : [m] \to$ node

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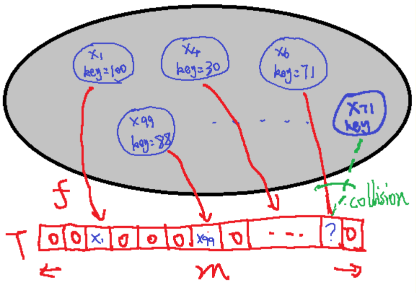

Assume no collision, how to support SEARCH, INSERT, DELETE?

Handle Collisions (I)

Collisions are unavoidable unless $|S| \leqslant \sqrt{m}$ — birthday paradox.

Theorem

For a perfectly random hash h,

$$Pr_h\left[h(x_1), \ldots, h(x_k) \text{ have no collision}\right] \leqslant e^{-\binom{k}{2}/m}.$$

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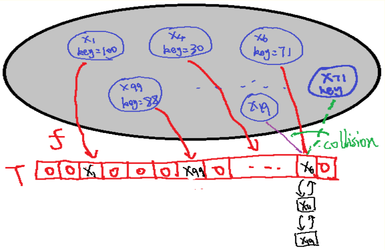


For each node x, introduce *x.left* and *x.right* to maintain the chain

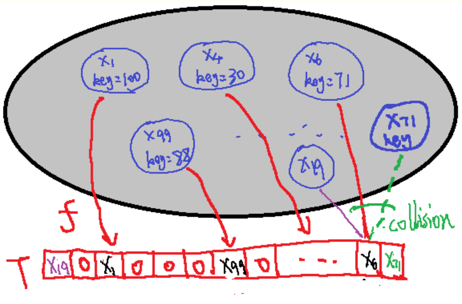
Solution 1

Maintain a chain for each position in T

Collisions 2

Solution 2

Put x into the next empty box in T — open-addressing method



Solution 3: Power of 2-choice

A surprising powerful idea

Prepare multiple hash functions, say h_1 and h_2, and put x into $h_1(x)$ or $h_2(x)$

1. Multiple choices hash
2. Always-Go-Left Hash
3. Cuckoo Hash: When $n < m/2$, at most one ball in every bin.

Summary

As a warm-up, Hash is the most fundamental data structure in CS

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5. More: memory-hierarchy hash, ...

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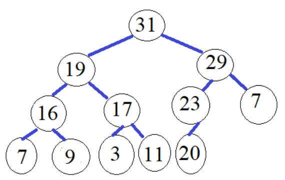

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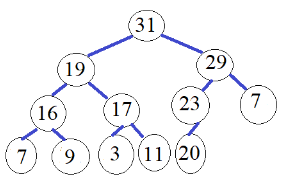

Main Property

1. For any node v (not root), parent[v].key $\geqslant$ v.key
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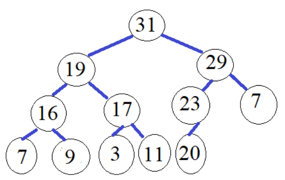

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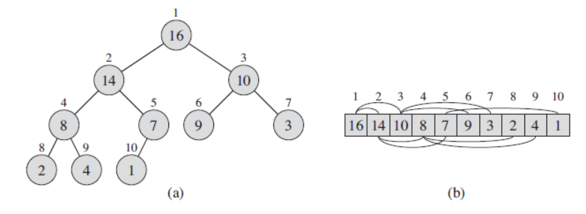
Remarks: (1) Only consider one dynamic set and $n :=$ its size. (2) Neglect corner cases in these slides.

Details

Two ways to view the heap because it is nearly complete.



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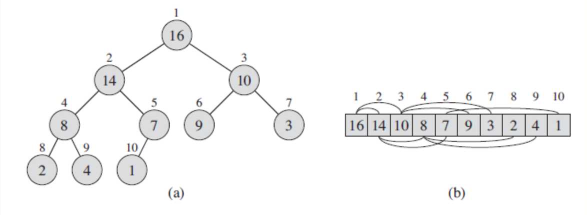

Implement it as an array

For each node with label i,

$$\begin{aligned} Key(i):&\quad A[i] \\ Parent(i):&\quad \lfloor i/2 \rfloor \\ Left(i):&\quad 2i \\ Right(i):&\quad 2i+1. \end{aligned}$$

Remark: $Left(i) = \emptyset$ if $2i > n$ and vice versa for $Right(i)$.

Maintaining the heap

Operations

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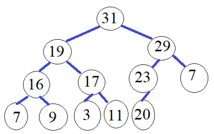
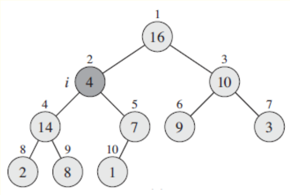


Key property: the height h is $\leqslant \lceil \log_2 n \rceil + 1$.

MAX-HEAPIFY

Task: After decreasing $A[i]$, maintain A as a heap.



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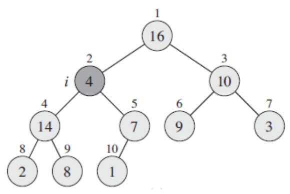


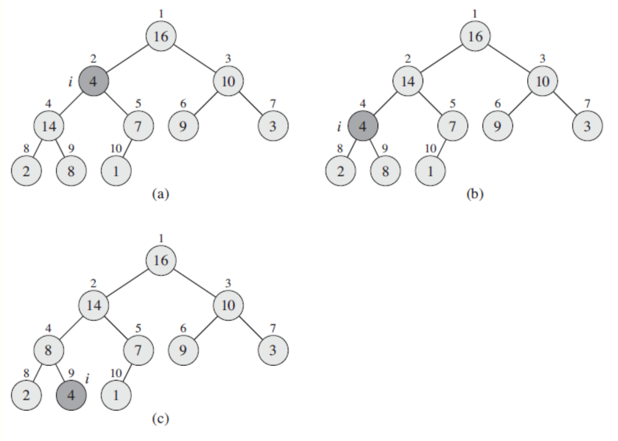
Example: Adjust $A[2]$ to maintain the properties of a heap

Recall two main properties: (1) $A[i] \geqslant A[2i]$ & $A[2i+1]$ (2) a nearly complete binary tree

---

procedure MAX-HEAPIFY(A, i)
  while $A[i] \leqslant A[2i]$ or $A[i] \leqslant A[2i+1]$ **do**
    $largest = \arg\max\{A[2i], A[2i+1]\}$
    Exchange $A[i]$ with $A[largest]$
    $i = largest$

Example run of MAX-HEAPIFY



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procedure BUILD-MAX-HEAP(A)
  for $i = [n/2], \ldots, 1$ **do**
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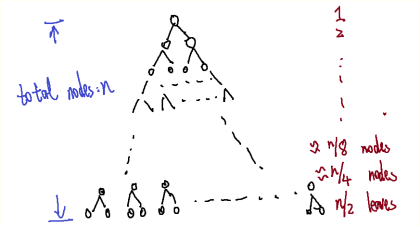
Remarks

1. The loop i has to go from $[n/2]$ to 1. Otherwise it violates 1st property.
2. Question: Running time?

Running time of BUILD-MAX-HEAP

While the total height is $\log_2 n$, only 1 node (root) will have $\log_2 n$ exchanges possibly; only 2 nodes will have $\log_2 n - 1$ exchanges possibly, ...



$$1 \cdot \log_2 n + 2 \cdot (\log_2 n - 1) + \ldots + n/4 \cdot 1 = O(n).$$

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procedure HEAP-EXTRACT-MAX(A)

$\quad \mathsf{max} = A[1]$

$\quad A[1] = A[n]$

$\quad n = n - 1$

$\quad$ MAX-HEAPIFY(A, 1)

$\quad$ Return max

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Question

Use BUILD-MAX-HEAP and HEAP-EXTRACT-MAX to design a sort algorithm?

HEAP-INCREASE-KEY

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procedure HEAP-INCREASE-KEY(A, i, key)
    $A[i] = key$
    while $i > 1$ and $A[parent(i)] < A[i]$ **do**
        Exchange $A[i]$ with $A[parent(i)]$
        $i = Parent(i)$

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Question

How to delete an arbitrary element?

MAX-HEAP-INSERT

Task: Insert a new element with value *key*

---

procedure HEAP-INCREASE-KEY(A, *key*)

  $n = n + 1$

  HEAP-INCREASE-KEY(A, n, *key*)

Summary of Heap

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2. An almost complete binary tree
3. Easy to implement and works well in practice
4. Support many operations in $O(\log_2 n)$-time
5. While heap does not support SEARCH(KEY), one could combine it with Hash
6. But neither of them could find the kth largest in $o(n)$ time.

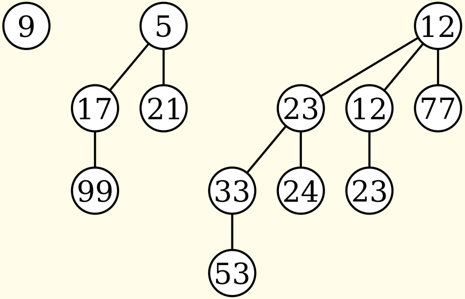
Extensions

Support UNION in $o(n)$ time

Binomial heap and Fibonacci Heap

Basic idea: Allow each node to have more children.

While they are faster and more powerful, complicated to implement.



Questions?