Introduction to Algorithms
Lecture 11 Minimum Spanning Trees

Xue Chen
xuechen1989@ustc.edu.cn
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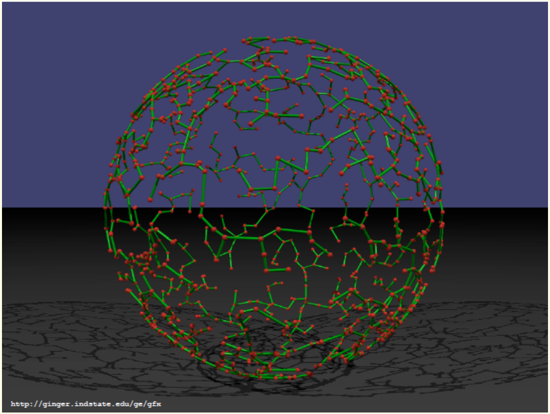
Outline

1. Introduction
2. Basic Properties
3. Prim's Algorithm
4. Kruskal's Algorithm
5. Data Structures for Disjoint Sets

Overview

Spanning trees are fundamental objects in graph theory

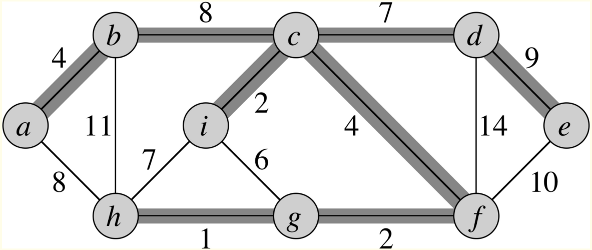


1. Connectivity: DFS/BFS trees
2. Single Source Shortest Paths Tree
3. Today: Minimum Spanning Trees
4. More: trees for cut, ...

Problem Description

Minimum Spanning Tree (MST)

Give a undirected & weighted graph G, define the weight of a tree T to be the sum of all weights on its edges. The goal is to find a Tree with minimum weight.



1. Network Design: Traffic, eletrical, ...
2. Algorithm Design: traveling salesman prob, Steiner tree, ...

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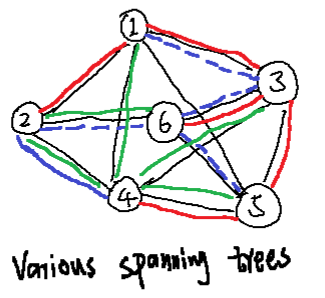
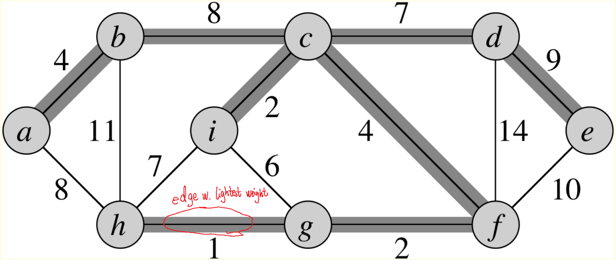
Basic Properties of Trees

1. Spanning tree is a minimal way to connect a graph
2. Remove a cycle edge can not disconnect a graph
3. A tree on n nodes has $n-1$ edges
4. Any connected graph $G=(V,E)$ with $|E|=|V|-1$ is a tree
5. A connected graph is a tree iff $\exists$ a unique path between any two nodes



Cut Property

1. True of False: If $\exists$ a unique edge e with the lightest weight, e in any MST.



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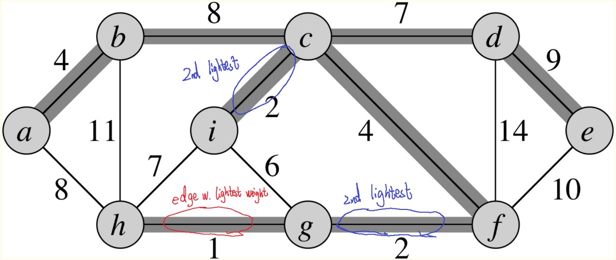
2. Question: So (h, g) is always in MST, how about 2nd lightest edges (i, c) and (g, f)?



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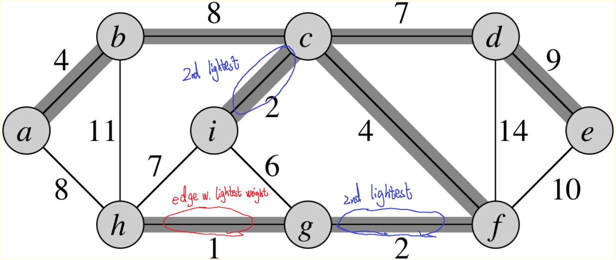

Greedy method in Theorem 23.1 of CLRS

Let A be a subset of edges that is included in some MST. Pick any cut $(S, \overline{S})$ such that S and $\overline{S}$ are NOT connected by edges in A.
Then for any lightest edge e on $S \times \overline{S}$, $\exists$ a MST that contains e.

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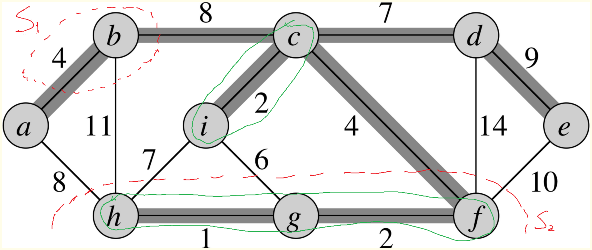

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Two Algorithms from opposite directions

Define $\text{cut}(S, \overline{S}) := \{(u, v) : u \in S, v \notin S\}$

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2. Kruskal's Algorithm: Add an edge e and check $A \cup \{e\}$ has a cycle or not.

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Based on the cut property, consider a greedy method to grow a tree S:

1. Start from $S = \{\text{root}\}$ and add a neighbor with the lightest crossing edge to S every time

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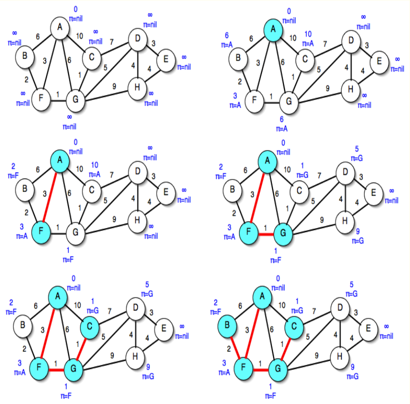

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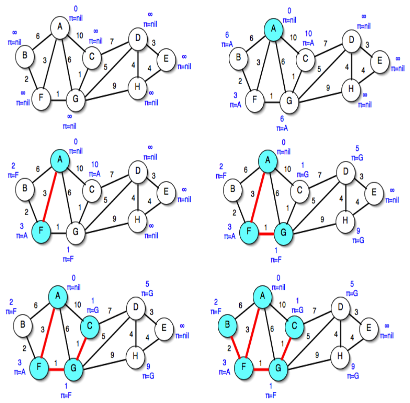
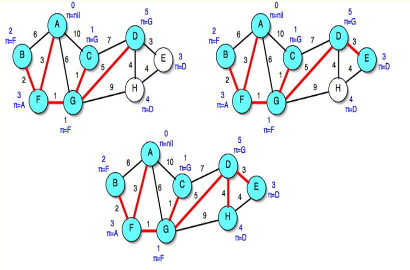
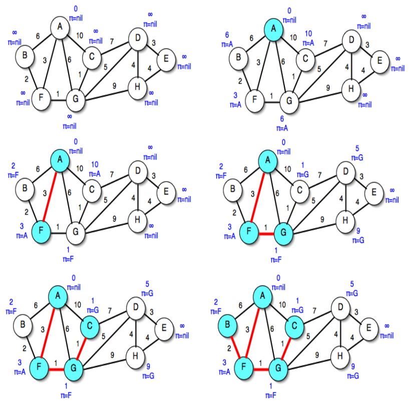


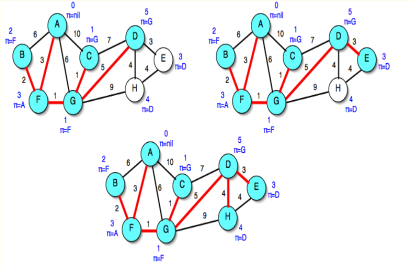


Overview

Based on the cut property, consider a greedy method to grow a tree S:

1. Start from $S = \{\text{root}\}$ and add a neighbor with the lightest crossing edge to S every time
2. Use a heap to maintain the lightest cross edge to v for every $v \notin S$





1. $u.key := \min_{v \in S} w(u, v)$ for $u \notin S$
2. $u.\pi$ denotes its argmin (as its father in MST)
3. Q maintains $\overline{S}$ according to $u.key$

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
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Time Complexity

$O(m \log n)$: Each vertex will be extracted once; so every edge will make at most one change in the heap

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An alternate greedy approach:

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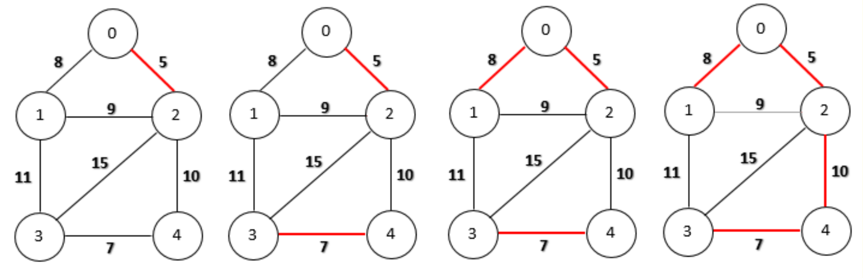

Description

MST-KRUSKAL(G, w)

```
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

Example: Pseudo-code from CLRS

Time Complexity

SORT(m edges) $+ n \times$ UNION $+ m \times$ FIND-SET

Next: Disjoint sets supports UNION and FIND-SET in almost constant time: RADIX-SORT + Disjoint sets is faster than $O(m \log n)$

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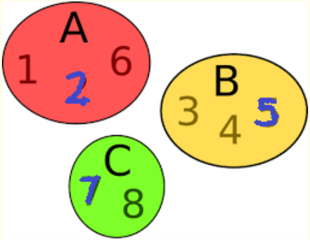
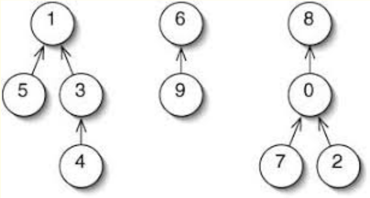


Applications

Maintenance of connected components in graphs and maps, equivalence relation, least-common ancestors problem, ...

Implementation



Maintain those sets as trees

1. Root is the unique representation
2. Question:

   How to implement MAKE-SET and FIND-SET?

   How about UNION operation?

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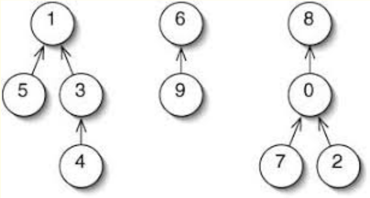

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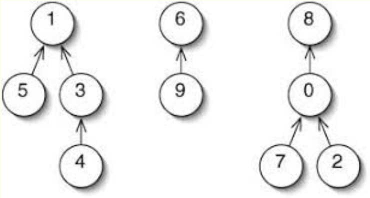

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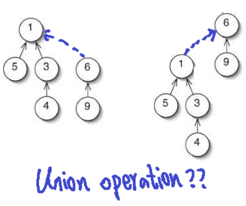
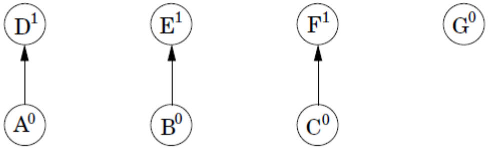


Rank of elements

To maintain heights, define a rank for each element:

After $\texttt{makeset}(A), \texttt{makeset}(B), \ldots, \texttt{makeset}(G)$:

After $\texttt{union}(A, D), \texttt{union}(B, E), \texttt{union}(C, F)$:



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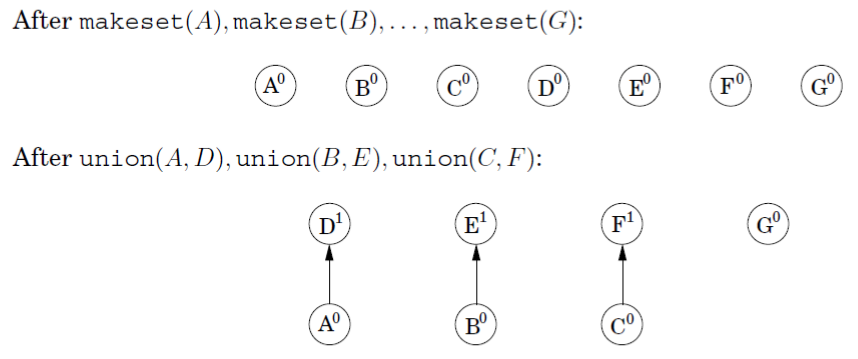
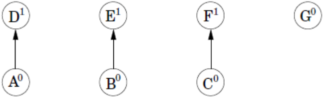


UNION(x, y)

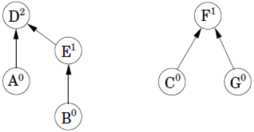
1. Let r_x and r_y be their representatives
2. If $rank(r_x) > rank(r_y)$: $\pi(r_y) = r_x$
3. Else: $\pi(r_x) = r_y$ and Update $rank(r_y)$ to $\max\{rank(r_y), rank(r_x) + 1\}$

After makeset(A), makeset(B), $\ldots$, makeset(G):

After union(A, D), union(B, E), union(C, F):



After union(C, G), union(E, A):



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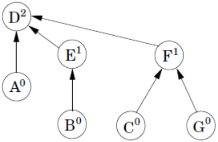

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A Small trick improves time significantly

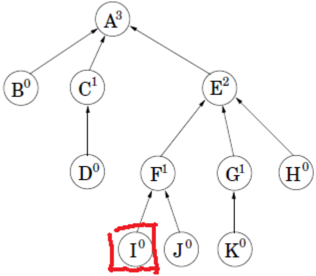
Path compression: In FIND-SET, set $\pi(x)$ to be the root for all x on the path

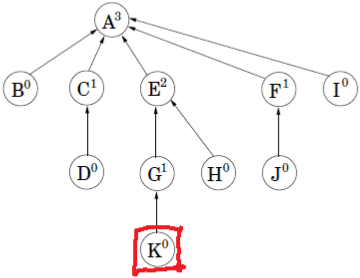
Formal Description

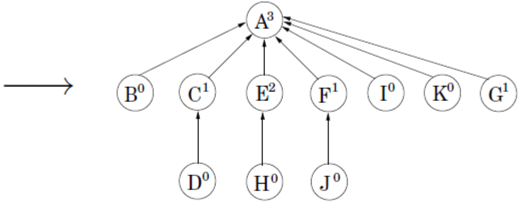
function FIND-SET(x)
  if $\pi(x) \neq x$ **then** //x is not a root
    $\pi(x) = \text{FIND-SET}(\pi(x))$ // Path Compression Trick
  return $\pi(x)$

procedure MAKE-SET(x)
  $\pi(x) = x$ and $rank(x) = 0$

procedure UNION-SET(x, y)
  $r_x = \text{FIND-SET}(x)$ and $r_y = \text{FIND-SET}(y)$
  if $rank(r_x) > rank(r_y)$ **then**
    $\pi(r_y) = r_x$
  else
    $\pi(r_x) = r_y$ and $rank(r_y) = \max\{rank(r_y), rank(r_x) + 1\}$







Find(I) followed by Find(k)

Amortized Analysis

Let $\log^* n$ be number of log operations that bring n down to 1, e.g., $\log^* 1000 = 4$ and $\log^* 2^{65536} = 5$

THM: Running time of Disjoint Sets

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1. Consider those ranks— path compression does not affect their ranks
2. Divide all ranks into $\log^* n + O(1)$ intervals:

$\longleftarrow$ $\log^* n$ *intervals* $\longrightarrow$

$\{1\}, \{2\}, \{3,4\}, \{5,6,\cdots,16\}, \{17,\cdots,2^{16}=65536\}, \{65537,\cdots,\}, \cdots$

$\leq \frac{n}{2}$ nodes, $\leq \frac{n}{4}$, $\cdots$, $\leq \frac{n}{32} + \cdots + \frac{n}{2^{16}}$ nodes, $\cdots$

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3. Amortized analysis with accounting method: For each x whose rank $\in [k+1, \ldots, 2^k]$, assign a budget 2^k — total budget $= n \cdot \log^* n$
4. Next: $m \cdot \log^* n + \textit{budgets}$ bounds time of $m \times$ FIND-SET

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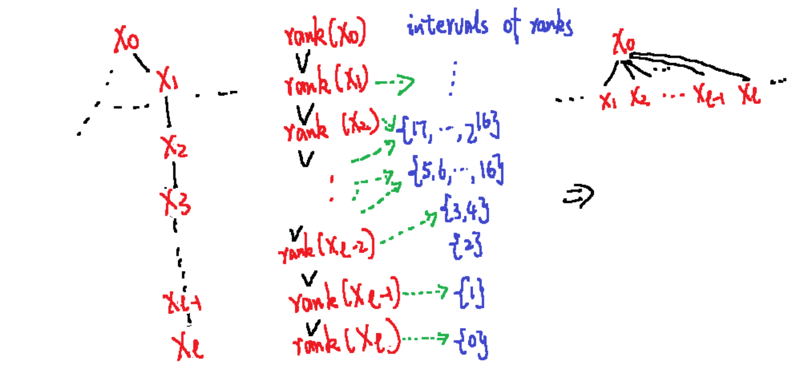

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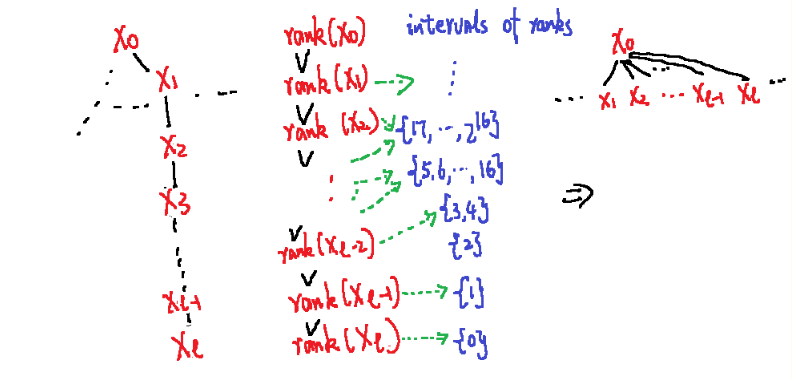

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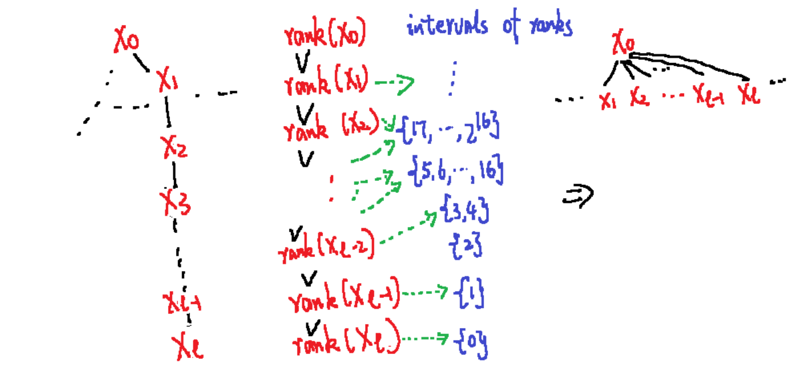

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4. OBS: Once $\left[rank(x_i), rank(\pi(x_i))\right]$ crossed an interval, it keeps crossing those intervals in the future — this part is at most $\log^* n$
5. O.w. pay the cost by the budget of x_i — If $rank(x_i) \in [k+1, \ldots, 2^k]$, after x_i paid $\leqslant 2^k$ times, $rank(\pi(x_i))$ falls into above case and x_i stops paying

Summary

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2. Prim's ALG runs in $O(m \log n)$ via heaps (faster by Fibonacci-heap)
3. Kruskal's ALG runs in $O(m \log^* n)$ after sorting

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2. Prim's ALG runs in $O(m \log n)$ via heaps (faster by Fibonacci-heap)
3. Kruskal's ALG runs in $O(m \log^* n)$ after sorting
4. For disjoint sets, a small change makes a big difference: from $O(\log n)$ to 5!

Questions?