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Topological insulator part I: Phenomena

(Part II and Part III discusses how to understand a topological insluator based band-structure theory and gauge theory)

(Part IV discusses more complicated T-reversal invariance topological insulators)

One phenomena: bulk insulator: surface/edge is metallic.

Example: integer quantum Hall effect

2.1. Quantum Hall effect

Hall effect:

- Electrons moving in a 2D plane (2D electron gas or 2DEG): Apply an electric field E, we get some current j. Typically, $E \ // \ J$. Let's assume they are in the x direction E_x and j_x. Resistivity $\rho_{xx} = E_x / j_x$
- Apply a B field perpendicular to the 2D plane. Lorentz force leads to charge accumulation for the top and bottom edge. This gives a E field perpendicular to the current (E_y). Hall resistivity: $\rho_{xy} = E_y / j_x$
- Rotational symmetry: $\rho_{xx} = \rho_{yy}$ and $\rho_{xy} = \rho_{yx}$
- Classical mechanics: $\rho_{xy} = B$

$$e E_y = e v B \tag{2.1}$$

$$j_x = e v n \tag{2.2}$$

$$\rho_{xy} = E_y / j_x = \frac{v B}{e v n} = \frac{B}{e n} \tag{2.3}$$

Remark: very useful experimental technique. It determines the carrier density.

Integer Quantum Hall effect

ρ_{xy}: plateaus. $\sigma_{xy} = \frac{1}{\rho_{xy}}$ is quantized: $\nu e^2 / h$ (ν is an integer, known as the filling factor or filling fraction). In the same time, $\rho_{xx} = 0$ at these plateaus.

Between two plateaus, not universal (sample dependent).

σ_{xy} quantization. Very accurate. The second best way to measure the fine structure constant

The fine structure constant is one of the key fundamental constants. In CGS unit

$$\alpha = \frac{e^2}{\hbar c} \tag{2.4}$$

c is the speed of light, whose value is exactly known (no error). Quantum Hall effect gives e^2 / h, so we can get α.

The most accurate way to determine α comes from g-2

Magnetic moment of an electron:

$$\mu = g\,\mu_B\,S/\hbar \tag{2.5}$$

where μ_B is a Bohr magneton and S is the spin of an electron.

Dirac equation tells us that $g = 2$, but in reality, $g = 2.0023193043622$

This difference is known as "anomalous magnetic moment". This anomaly is due to the interaction between the gauge fields and the particles (Dirac equation only describes the motion of a fermoin without gauge fields), which renormalized the value of g slightly away from 2.

$$a = \frac{g-2}{2} \tag{2.6}$$

This renormalization of g can be computed theoretically in Quantum electrodynamics (QED) using loop expansions. According to QED, a is directly related to the fine structure constant.

$$a = \frac{\alpha}{2\pi} + \sharp 2\,\alpha^2 + \sharp 3\,\alpha^3 + \sharp 4\,\alpha^4 + \ldots \tag{2.7}$$

Currently, we know the first 4 coefficients. However, because α is very small $\sim\frac{1}{127}$, the contributions from the higher order terms are very small (in the order of 10^{-10}). If we ignore the higher order terms, we can get α from the value of a.

Remarks:

- For measuring α, g-2 gives a more accurate result than IQH. the error bar from g-2 is 0.37×10^{-9}. Using IQH, it is 24×10^{-9}
- On the other hand, in g-2, one need to relies on QED to get α. But IQH offers an measurement of α independent of QED.
- The fact that these two measurements agree with each other offers a direct check to the theory of QED.
- Finally, we condensed matter physicists have something that almost as accurate as particle physics.

Difference between theories and experiments in condensed matter physics, typically: a few percent to a few hundred percent (can be larger). Much larger than particle physics

- Strongly correlated (cannot trust perturbation theory, in direct contrast to QED where α<<1):
  - QED: the strength of interactions (energy scale of interaction/energy scale of single-particle kinetic energy)

$$\alpha = \frac{e^2}{\hbar\,c} \sim 1\big/127 \tag{2.8}$$

  - CMP: the strength of interactions (energy scale of interaction/energy scale of single-particle kinetic energy)

$$\alpha_{\text{CMP}} = \frac{e^2}{\hbar\,v_F} \tag{2.9}$$

  - Fermi velocity v_F is typically 1/100-1/1000 of c

$$\alpha_{\text{CMP}} = \frac{e^2}{\hbar\,v_F} \sim 1-10 \tag{2.10}$$

  - Perturbation theory cannot be used. one must need to compute all powers to get a reasonable answer, which is impossible
- Dirty background (impurities): extremely important for transport. σ is typically sample dependent (not universal). Depends on the density of impieties.

Why IQH is so accurate? (interactions and impurities do not matter). Please keep these questions and we will come back to them later, after we learn more about TIs.

Now , let's look at ρ_{xx}, which is 0 in a quantum Hall state.

- Q: Zero resistivity ρ_{xx}. A superconductor? A pefect metal?
- A: No. It is not a perfect conductor, not a superconductor. It is not even a conductor. It is an insulator.
- A clue: Conductivity is zero

Two formulars we are very familiar with:

$$j = \sigma\,E \tag{2.11}$$

$$E = \rho\,j \tag{2.12}$$

So:

$$\rho = 1/\sigma \tag{2.13}$$

This naive formula requires a very important assumption: j and E are in the same direction. If we apply E in the x direction, the current must also in the x direction and there cannot be any j_y, and vice versa.

This assumption is true sometimes, but not always

More generic formula:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{2.14}$$

If all off-diagonal terms are zero ($\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$) and all the diagonal terms are the same we got:

$$j = \sigma\, E \tag{2.15}$$

$$\text{So } \rho = \frac{1}{\sigma} \tag{2.16}$$

Key: conductivity is a tensor (a $d \times d$ matrix). Not a scalar.

For 2D

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{2.17}$$

- Q: How about resistivity?
- A: Of course, it is also a matrix

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} \tag{2.18}$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{2.19}$$

So

$$\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \tag{2.20}$$

The resitivity tensor is the (matrix) inverse of the conductivity tensor, and vice versa.

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}^{-1} = \frac{1}{\rho_{xx}\,\rho_{yy} - \rho_{xy}\,\rho_{yx}} \begin{pmatrix} \rho_{yy} & -\rho_{xy} \\ -\rho_{yx} & \rho_{xx} \end{pmatrix} \tag{2.21}$$

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Inverse[{{ρxx, ρxy}, {ρyx, ρyy}}]
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$$\left\{\left\{\frac{\rho yy}{-\rho xy\,\rho yx + \rho xx\,\rho yy}, -\frac{\rho xy}{-\rho xy\,\rho yx + \rho xx\,\rho yy}\right\}, \left\{-\frac{\rho yx}{-\rho xy\,\rho yx + \rho xx\,\rho yy}, \frac{\rho xx}{-\rho xy\,\rho yx + \rho xx\,\rho yy}\right\}\right\}$$

For a quantum Hall system, at each plateaus: $\rho_{xx} = \rho_{yy} = 0$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \frac{1}{-\rho_{xy}\,\rho_{yx}} \begin{pmatrix} 0 & -\rho_{xy} \\ -\rho_{yx} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/\rho_{yx} \\ 1/\rho_{xy} & 0 \end{pmatrix} \tag{2.22}$$

$$\sigma_{xx} = \sigma_{yy} = 0 \tag{2.23}$$

No conducivity. A insulator.

Between two IQH plateaus (ρ_{xx} has a peak),

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \frac{1}{\rho_{xx}\,\rho_{yy} - \rho_{xy}\,\rho_{yx}} \begin{pmatrix} \rho_{yy} & -\rho_{xy} \\ -\rho_{yx} & \rho_{xx} \end{pmatrix} \tag{2.24}$$

Finite σ_{xx}. A metal. (It is a metal. σ_{xx} depends on impurity densities, non-universal, just like other metals).

2.2. Why a IQH state is an insulator: Landau levels

a charge neutral particle in 2D:

$$-i\,\partial_t\,\psi(x,\,y) = \left[\frac{1}{2\,m}\,(-i\,\hbar\,\partial_x)^2 + \frac{1}{2\,m}\,\left(-i\,\hbar\,\partial_y\right)^2\right]\psi(x,\,y) \tag{2.25}$$

$$H = \frac{1}{2\,m}\,(-i\,\hbar\,\partial_x)^2 + \frac{1}{2\,m}\,\left(-i\,\hbar\,\partial_y\right)^2 \tag{2.26}$$

a particle with charge e in 2D. Just change the momentum operator $\vec{p}$ into $\vec{p} + e\,\vec{A}\big/c$, where $\vec{A}$ is the vector potential, and change $-i\,\partial_t$ into $-i\,\partial_t - e\,\Phi/c$, where Φ is the Electric potential. (minimal coupling).

$$(-i\,\partial_t - e\,\Phi/c)\,\psi(x,\,y) = \left[\frac{1}{2\,m}\left(-i\,\hbar\,\partial_x - \frac{e}{c}\,A_x\right)^2 + \frac{1}{2\,m}\left(-i\,\hbar\,\partial_y - \frac{e}{c}\,A_y\right)^2\right]\psi(x,\,y) \tag{2.27}$$

$$-i\,\partial_t\,\psi(x,\,y) = \left\{\left[\frac{1}{2\,m}\left(-i\,\hbar\,\partial_x - \frac{e}{c}\,A_x\right)^2 + \frac{1}{2\,m}\left(-i\,\hbar\,\partial_y - \frac{e}{c}\,A_y\right)^2\right] + e\,\Phi/c\right\}\psi(x,\,y) \tag{2.28}$$

$$H = \frac{1}{2\,m}\left(-i\,\hbar\,\partial_x - \frac{e}{c}\,A_x\right)^2 + \frac{1}{2\,m}\left(-i\,\hbar\,\partial_y - \frac{e}{c}\,A_y\right)^2 + e\,\Phi/c \tag{2.29}$$

For us, E=0, so we can set Φ=0.

The vector potential satisfies

$$\nabla \times A = \partial_x\,A_y - \partial_y\,A_x = B \tag{2.30}$$

A is NOT a physical observable. For a fixed B field, A is not unique. If A is the vector potential for B ($\nabla \times A = B$), $A' = A + \nabla\,\chi$ is also the vector potential for B ($\nabla \times A' = B$). We can choose A arbitrarily and they all give the same physical results. This is known as a gauge choice.
For us, B is a constant (a uniform magnetic field), so we can choose

$$A_x = 0 \text{ and } A_y = B\,x \text{ (Landau gauge)} \tag{2.31}$$

$$A_x = -\frac{B\,y}{2} \text{ and } A_y = \frac{B\,x}{2} \text{ (symmetric gauge)} \tag{2.32}$$

Or any other gauge.

If we choose the Landau gauge,

$$H = \left[\frac{\hbar^2}{2\,m}\,(-i\,\partial_x)^2 + \frac{1}{2\,m}\left(-i\,\hbar\,\partial_y - \frac{e}{c}\,B\,x\right)^2\right] \tag{2.33}$$

- Q: Eigenstates of H?

H is invariant under translations along the y axis $y \to y' = y + \text{constant}$

So, $\left[p_y,\ H\right] = 0$

We can find common eigenstates for p_y and H. So the eigen states for H must take this form

$$\psi = f(x)\,\exp\!\left(-i\,k_y\,y\right) \tag{2.34}$$

$$H\,\psi = \epsilon\,\psi \tag{2.35}$$

$$-\frac{\hbar^2}{2m} f''(x) + \frac{1}{2m}\left(\hbar k_y - \frac{e}{c} B x\right)^2 f(x) = \epsilon f(x) \tag{2.36}$$

$$-\frac{\hbar^2}{2m} f''(x) + \frac{e^2 B^2}{2 m c^2}\left(x - \frac{c \hbar}{e B} k_y\right)^2 f(x) = \epsilon f(x) \tag{2.37}$$

A harmonic oscillator:

$$-\frac{\hbar^2}{2m} \frac{d}{d x^2} \phi(x) + \frac{k}{2}(x - x_0)^2 \phi(x) = \epsilon \phi(x) \tag{2.38}$$

x_0 is the equilibrium position. k is the spring constant

Exactly the same.

$$x_0 = \frac{c \hbar}{e B} k_y \tag{2.39}$$

$$k = \frac{e^2 B^2}{m c^2} \tag{2.40}$$

$$\psi_{n, k_y} = \phi_n(x - x_0) \exp(-i k_y y) \tag{2.41}$$

$$\epsilon_{n,k_y} = \left(n + \frac{1}{2}\right)\hbar \omega = \left(n + \frac{1}{2}\right)\hbar \sqrt{\frac{k}{m}} = \left(n + \frac{1}{2}\right)\frac{e B \hbar}{c m} \tag{2.42}$$

- ϵ as a function of k_y
  - Looks like energy bands (n--- band index and k_y--- the momentum)
  - All the states with same n (but different k_y) have the same energy (flat bands)
- μ in the gap: insulator (QH plateaus). μ cross with one of the bands (Landau levels): metal (between two plateaus)
- ϵ proportional to B
  - As B increases, the energy for each band increases continuously.
  - For a fixed μ, as B increases, μ in the gap → cross with Landau level n → in the gap → cross with n-1→ in the gap …

2.3. Edge states with Quantization of the Hall conductivity

Electrons at the edge can move along the edge in one direction (the direction is determined by the B field).

This edge states gives quantized Hall conductivity.

Top edge:

$$N = \int \frac{d p\, d r}{2 \pi \hbar} = L \int \frac{d p}{2 \pi \hbar} \tag{2.43}$$

W is the width of the sample

$$n = \frac{N}{L} = \int \frac{d p}{2 \pi \hbar} \tag{2.44}$$

$$\delta n_{\text{top}} = \int_{k_F}^{k_{F\,\text{top}}} \frac{d p}{2 \pi \hbar} = \frac{k_{F\,\text{top}} - k_F}{2 \pi \hbar} = \frac{\mu_{\text{top}} - \mu}{2 \pi \hbar v_{F\,\text{top}}} \tag{2.45}$$

net current for top edge:

$$I_{\text{top}} = e v_{F\,\text{top}} \delta n_{\text{top}} = e v_{F\,\text{top}} \frac{\mu_{\text{top}} - \mu}{2 \pi \hbar v_{F\,\text{top}}} = e \frac{\mu_{\text{top}} - \mu}{2 \pi \hbar} = e \frac{e V/2}{2 \pi \hbar} = \frac{e^2}{2 \pi \hbar} \frac{V}{2} \tag{2.46}$$

Similarly,

$$I_{\text{bottom}} = \frac{e^2}{2\pi\hbar}\frac{V}{2} \tag{2.47}$$

Notice that the current for both edges flows in the same directions! top edge: more particles moving to the right. So the direction of total current is right. Bottom edge: less particles moving to the left, so the net current is also to the right.

$$I = I_{\text{top}} + I_{\text{bottom}} = \frac{e^2}{2\pi\hbar} V = \frac{e^2}{h} V \tag{2.48}$$

Hall Conductance

$$\frac{1}{R_H} = \frac{I}{V} = \frac{e^2}{h} \tag{2.49}$$

Hall Conductivity:

$$\sigma = j/E = \frac{I}{L_y} \Big/ \frac{U}{L_y} = I/U = \frac{1}{R_H} = \frac{e^2}{h} \tag{2.50}$$

If one has n copies of edge states, each edge states will give

$$I_i = \frac{e^2}{2\pi\hbar} V \tag{2.51}$$

Total currents for all edge states

$$I = n\, I_i = n\frac{e^2}{h} V \tag{2.52}$$

Conductivity:

$$\sigma = \frac{I}{V} = n\frac{e^2}{h} \tag{2.53}$$

- Q: impurity scattering?
- A: No backward scatterings. (one-way current, no reflections)

This is one of the reason, why σ_{xy} is quantized so precisely. Impurities will not change it.

2.4. Why some insulators have chiral edge states?

A piece of plastic is an insulator. IQH is also an insulator. Why IQH has chiral edge state but a piece of plastic doesn't?

Puzzle to solve: Why insulators are different from each other? How many different types of insulators do we have in this world?

What tool should we use? Where should we start to look for an answer?

Hint 0: This is a quantum problem: σ_{xy} depends on $\hbar$.

- So we should look at quantum mechanics.

Hint 1: Transport is the motion of charge. In quantum mechanics, charge is closely related with one quantity.

Hint 1.5: They are conjugate variables to each other, like r and p.

- Phase. invariant under $\phi \to \phi + \delta\phi$ implies the conservation of charge.

Hint 2: Phase can often lead to quantization

- Wavefunction is single valued. If we go around a circle, the change of the phase can only be 2π n where n is an integer.
- Example: the quantization of magnetic flux.

Q: Phase of a wavefunction: important or not?

A: The relative phase (the phase difference between two states) is important (interference), but the absolute value of a phase is not important

If $|\psi\rangle$ is an eigenstate, $e^{i\phi}|\psi\rangle$ is the wavefunction for the same state. All the observables remains the same.

$\langle\psi|A|\psi\rangle = \langle\psi|e^{-i\phi}|A|e^{i\phi}|\psi\rangle$

A^{+} answer:

The absolute value of a phase is not important, but the fact that phase is unimportant is VERY IMPORTANT.