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Problem set #1

1.1. A one-band model on a square lattice

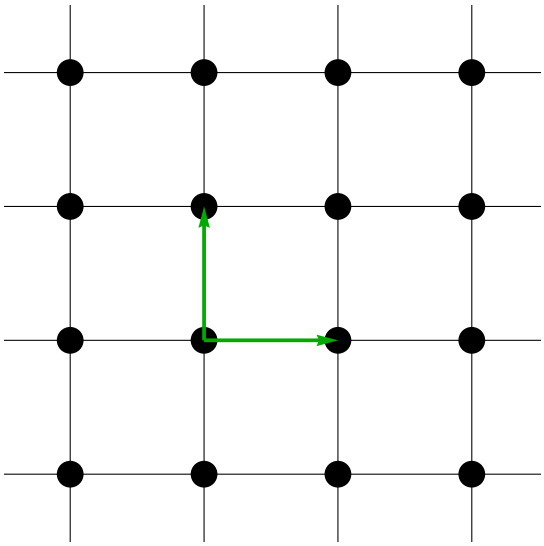


Fig. 1

Consider a square lattice with only nearest-neighbor hoppings (as shown in the figure above):

$$H = -t \sum_{\langle i,j \rangle} a_i^\dagger a_j \tag{1.1}$$

where ⟨i,j⟩ stands for nearest neighbors (solid bonds in the figure above). Here we assume that the hopping strength t is real.

In this model, all lattice sites are equivalent and thus each unit cell contains only one site. As a result, the band structure for this model contains only one band.

1.1.1. k-space

This model contains two bonds per unit cell (one along the x-direction and the other along y). So we can rewrite the Hamiltonian as:

$$H = -t \sum_i a_{\vec{r}_i+\vec{e}_x}^\dagger a_{\vec{r}_i} - t \sum_i a_{\vec{r}_i+\vec{e}_y}^\dagger a_{\vec{r}_i} - t \sum_i a_{\vec{r}_i}^\dagger a_{\vec{r}_i+\vec{e}_x} - t \sum_i a_{\vec{r}_i}^\dagger a_{\vec{r}_i+\vec{e}_y} \tag{1.2}$$

Here $\vec{e}_x = (a, 0)$ and $\vec{e}_y = (0, a)$ are the lattice vectors (the two green vectors shown in the figure above) with a the length of the nearest-neighbor bonds. The first term in the Hamiltonian describes hoppings along the +x direction. The second one shows hoppings along +y. The last two terms are the Hermitian conjugate of the first two, and they are hoppings along -x and -y directions. The sum $\sum_i$ is over all unit cells.

Rewrite the Hamiltonian in k-space as $H = \sum_k \epsilon_1\left(\vec{k}\right) a_{\vec{k}}^\dagger a_{\vec{k}}$ using the Fourier transform:

$$a_k = \frac{1}{\sqrt{N}} \sum_i a_i e^{i \vec{k}\cdot\vec{r_i}} \qquad \text{and} \qquad a_i = \frac{1}{\sqrt{N}} \sum_k a_k e^{-i \vec{k}\cdot\vec{r_i}} \tag{1.3}$$

Find the dispersion relation $\epsilon_1\left(\vec{k}\right)$ and show that $\epsilon_1(k_x, k_y)$ is a periodic function

$$\epsilon_1(k_x + 2n\pi/a, k_y + 2m\pi/a) = \epsilon_1(k_x, k_y) \tag{1.4}$$

for any integers m and n.

1.1.2. Complex hoppings

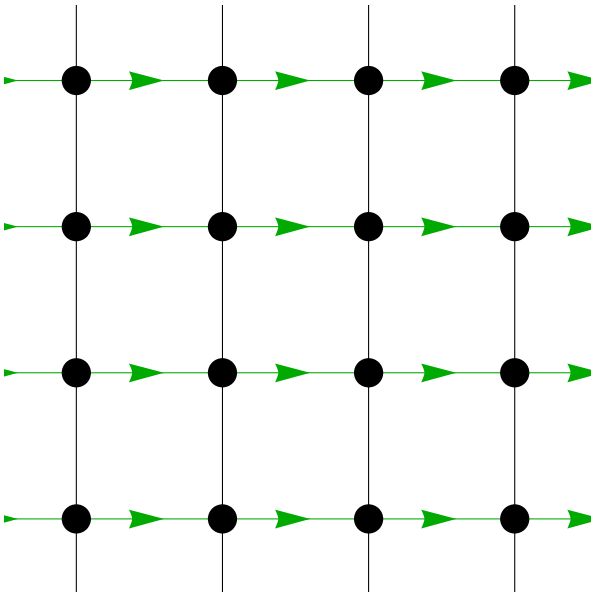


Fig. 2

Let's consider the same lattice, but now we allow the hopping strength to be complex. Here, we assume that the hoppings along the y-axis have real coefficient t. For hoppings along the x-axis, the hopping strength is $t\, e^{i\phi}$ if the hopping is toward the +x direction (along the arrows shown in Fig. 2). Due to the Hermitian condition $H^\dagger = H$, the hopping toward the -x direction has coefficient $t\, e^{-i\phi}$. The Hamiltonian now reads:

$$H = -t\, e^{i\phi} \sum_i a_{\vec{r_i}+\vec{e_x}}^\dagger a_{\vec{r_i}} - t \sum_i a_{\vec{r_i}+\vec{e_y}}^\dagger a_{\vec{r_i}} - t\, e^{-i\phi} \sum_i a_{\vec{r_i}}^\dagger a_{\vec{r_i}+\vec{e_x}} - t \sum_i a_{\vec{r_i}}^\dagger a_{\vec{r_i}+\vec{e_y}} \tag{1.5}$$

Rewrite the Hamiltonian in k-space as $H = \sum_k \epsilon_2\left(\vec{k}\right) a_{\vec{k}}^\dagger a_{\vec{k}}$ using Fourier transformations. Find the dispersion relation $\epsilon_2\left(\vec{k}\right)$. Show that $\epsilon(k_x, k_y)$ is also a periodic function

$$\epsilon_1(k_x + 2n\pi/a, k_y + 2m\pi/a) = \epsilon_1(k_x, k_y) \tag{1.6}$$

for any integers m and n.

1.1.3. Compare the two models

Compare the band structure find in problem 1.1.1 and 1.1.2, i.e.. $\epsilon_1(k_x, k_y)$ and $\epsilon_2(k_x, k_y)$. Show that these two dispersions are actually the same function if one shifts the momentum by some constant:

$$\epsilon_1(k_x, k_y) = \epsilon_2(k_x + \alpha, k_y + \beta) \tag{1.7}$$

Find α and β.

1.1.4. Why the two models have the same band structure (up to a shift of $\vec{k}$)?

As mentioned in the lecture, the phase of the hopping strength for each individual bond doesn't have any physical meaning. The only thing matters

here is the flux through each plaquette (sum over all phases around a plaquette). Check the flux through each plaquette for both these two models and show that they are the same.

If two models have the same flux pattern, they are actually the same model, although the phase on each bond may be different. In other words, the two model are connected by a gauge transformation. Let's consider the Hamiltonian of problem 1.1.2

$$H = -t\, e^{i\phi} \sum_i a^\dagger_{\vec{r}_i+\vec{e}_x} a_{\vec{r}_i} - t \sum_i a^\dagger_{\vec{r}_i+\vec{e}_y} a_{\vec{r}_i} - t\, e^{-i\phi} \sum_i a^\dagger_{\vec{r}_i} a_{\vec{r}_i+\vec{e}_x} - t \sum_i a^\dagger_{\vec{r}_i} a_{\vec{r}_i+\vec{e}_y} \tag{1.8}$$

Perform the following gauge transformation:

$$a_{\vec{r}_i} \to \tilde{a}_{\vec{r}_i} = \exp\left(-i\ \phi\ \frac{x_i}{a}\right) \tilde{a}_{\vec{r}_i} \tag{1.9}$$

$$a^\dagger_{\vec{r}_i} \to \tilde{a}^\dagger_{\vec{r}_i} = \exp\left(i\ \phi\ \frac{x_i}{a}\right) \tilde{a}^\dagger_{\vec{r}_i} \tag{1.10}$$

Here x_i is the x component of $\vec{r}_i$. Now, rewrite the Hamiltonian using $\tilde{a}_{\vec{r}_i}$ and $\tilde{a}^\dagger_{\vec{r}_i}$ as our annihilation and creations operators. Please show that in terms of $\tilde{a}_{\vec{r}_i}$ and $\tilde{a}^\dagger_{\vec{r}_i}$, the Hamiltonian turns into

$$H = -t \sum_i \tilde{a}^\dagger_{\vec{r}_i+\vec{e}_x} \tilde{a}_{\vec{r}_i} - t \sum_i \tilde{a}^\dagger_{\vec{r}_i+\vec{e}_y} \tilde{a}_{\vec{r}_i} - t \sum_i \tilde{a}^\dagger_{\vec{r}_i} \tilde{a}_{\vec{r}_i+\vec{e}_x} - t \sum_i \tilde{a}^\dagger_{\vec{r}_i} \tilde{a}_{\vec{r}_i+\vec{e}_y} \tag{1.11}$$

which is exactly the Hamiltonian we studied in problem 1.1.1. So indeed the two models are identical.

1.2. A two-band model on a square lattice with Dirac points

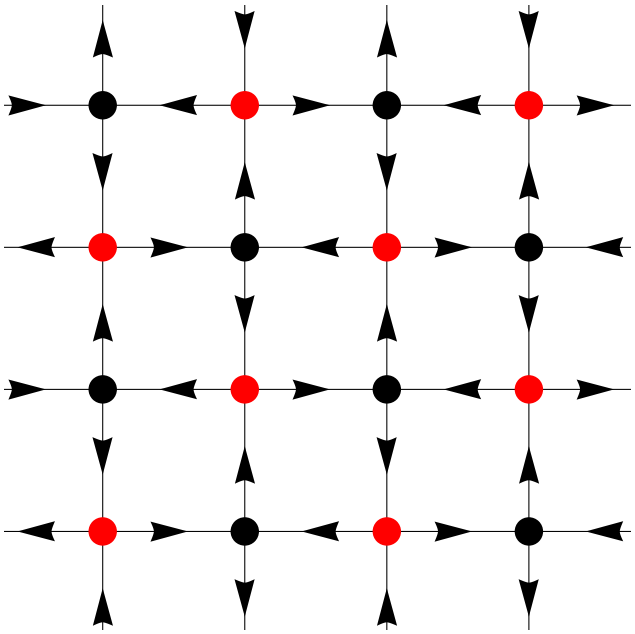


Fig. 3

Let's consider the same lattice. Now we choose the phase of hoppings according to the arrows shown in Fig. 3.

$$H = -t \sum_{\langle i,j \rangle} e^{i\phi_{ij}} a_i^\dagger b_j + h.c. \tag{1.12}$$

Here, we assume that electrons can hop only between nearest neighbors and we also assume all the nearest-neighbor hopping strength to have the same absolute value (t, which is real and positive). For the complex phase ϕ_{ij}, if the hopping is along the direction of the arrow, $\phi_{ij} = \pi/4$, so that the hopping strength is $t\, e^{i\pi/4}$. In the opposite direction (against the arrow), the hopping strength is $t\, e^{-i\pi/4}$. For this model, there are two different types of sites (red and black). For the red sites, the arrows at this site point away from the site along x and towards the site along y. For the black sites, the arrows point towards the site along x and away from the site along y. To indicate that we have two different types of site, we use a_i and $a_i^\dagger$ to represent fermion creation and annihilation operators for the red sites and using b_i and $b_i^\dagger$ for the black site. Because we have two types of

sites, we know that the model will give us two bands.

1.2.1. magnetic flux and time-reversal symmetry

Consider a square formed by four nearest-neighbor bonds, which is the smallest plaquette on this lattice. If we goes around the square (in the counter-clock direction), the Aharonov– Bohm phase picked up by the electron is:

$$\phi_1 + \phi_2 + \phi_3 + \phi_3 = 4 \times \pi/4 = \pm\pi \tag{1.13}$$

For this lattice, half of the squares have $+\pi$ and the other half have $-\pi$. As we learned in the lecture, a $\pm\pi$-flux doesn't break the time-reversal symmetry, because $\mathcal{T} e^{i\pi} = e^{-i\pi} = e^{i\pi}$. Therefore, this model is time-reversally invariant, similar to graphene (although we do have complex hoppings here, they don't break the time-reversal symmetry)

1.2.2. find the Band structure

This model contains four bonds per unit cell (the four bonds around one black site).

$$\begin{aligned} H = &-t\, e^{-i\pi/4} \sum_i a_{\vec{r}_i}^\dagger b_{\vec{r}_i + \vec{e}_x} - t\, e^{-i\pi/4} \sum_i a_{\vec{r}_i}^\dagger b_{\vec{r}_i - \vec{e}_x} - t\, e^{i\pi/4} \sum_i a_{\vec{r}_i}^\dagger b_{\vec{r}_i + \vec{e}_y} - t\, e^{i\pi/4} \sum_i a_{\vec{r}_i}^\dagger b_{\vec{r}_i - \vec{e}_y} - \\ &t\, e^{i\pi/4} \sum_i b_{\vec{r}_i}^\dagger a_{\vec{r}_i + \vec{e}_x} - t\, e^{i\pi/4} \sum_i b_{\vec{r}_i}^\dagger a_{\vec{r}_i - \vec{e}_x} - t\, e^{-i\pi/4} \sum_i b_{\vec{r}_i}^\dagger a_{\vec{r}_i + \vec{e}_y} - t\, e^{-i\pi/4} \sum_i b_{\vec{r}_i}^\dagger a_{\vec{r}_i - \vec{e}_y} \end{aligned} \tag{1.14}$$

Here, $\vec{e}_x = (a, 0)$ and $\vec{e}_y = (0, a)$ with a being the length of the nearest-neighbor bonds. The first term in the Hamiltonian describes hoppings from b to a along the +x direction. The second, third and fourth ones are along -x, +y and -y directions respectively. The last four terms are the Hermitian conjugate of the first four, which describes hoppings from a sites to b sites. The sum $\sum_i$ here sums over all unit cells.

Rewrite the Hamiltonian in the k-space as

$$H = \sum_k \left(a_k^\dagger,\ b_k^\dagger\right) \begin{pmatrix} \mathcal{H}^{(0)}{}_{a\,a}(k) & \mathcal{H}^{(0)}{}_{a\,b}(k) \\ \mathcal{H}^{(0)}{}_{b\,a}(k) & \mathcal{H}^{(0)}{}_{b\,b}(k) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{1.15}$$

Find the two-by-two matrix, which is the kernel of the Hamiltonian

$$\mathcal{H}^{(0)}(k) = \begin{pmatrix} \mathcal{H}^{(0)}{}_{a\,a}(k) & \mathcal{H}^{(0)}{}_{a\,b}(k) \\ \mathcal{H}^{(0)}{}_{b\,a}(k) & \mathcal{H}^{(0)}{}_{b\,b}(k) \end{pmatrix} \tag{1.16}$$

Find the eigenvalues of $\mathcal{H}^{(0)}(k)$ as a function of k_x and k_y, $\epsilon_\pm(k_x, k_y)$

1.2.3. Dirac points

Show that the two bands touch each other at $\vec{k} = \left(\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ and $\vec{k} = \left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right)$,

$$\epsilon_+\left(\frac{\pi}{2a}, \frac{\pi}{2a}\right) = \epsilon_-\left(\frac{\pi}{2a}, \frac{\pi}{2a}\right) \tag{1.17}$$

$$\epsilon_+\left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right) = \epsilon_-\left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right) \tag{1.18}$$

Expand the dispersion $\epsilon_\pm(k_x, k_y)$ near the point $\vec{k} = \left(\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ as a power series:

$$\epsilon_+\left(\frac{\pi}{2a} + q_x, \frac{\pi}{2a} + q_y\right) = v_F \sqrt{q_x^2 + q_y^2} + O(q^2) \tag{1.19}$$

$$\epsilon_-\left(\frac{\pi}{2a} + q_x, \frac{\pi}{2a} + q_y\right) = -v_F \sqrt{q_x^2 + q_y^2} + O(q^2) \tag{1.20}$$

Find the Fermi velocities v_F. Here we find two band crossing points and near the band crossing points, the energy is a linear function of the momentum $q = \sqrt{q_x^2 + q_y^2}$. In other words, we find two Dirac points in this model, similar to graphene.

1.2.4. the Brillouin zone

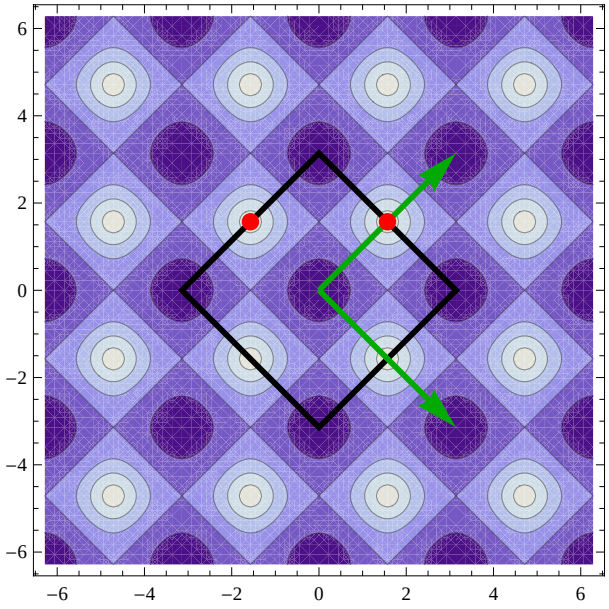


Fig. 4 Contour plot for $\epsilon_-(k)$ as a function of k_x and k_y. Here I set a = 1. Darker color means smaller ϵ_-. The two green arrows are the vectors Q_1 and Q_2 defined below. The black square marks the first Brillouin zone and the two red dots marks the two Dirac points.

Show that $\epsilon_+\left(\vec{k}\right)$ is a periodic function (the same is true for ϵ_-)

$$\epsilon_+\left(\vec{k} + m\,\vec{Q_1} + n\,\vec{Q_2}\right) = \epsilon_+\left(\vec{k}\right) \tag{1.21}$$

for any integers m and n. Here $\vec{Q_1} = \left(\frac{\pi}{a}, \frac{\pi}{a}\right)$ and $\vec{Q_2} = \left(\frac{\pi}{a}, -\frac{\pi}{a}\right)$. As a result, the Brillouin zone is the square with four corner points located at $(\pi/a, 0)$, $(0, \pi/a)$, $(-\pi/a, 0)$ and $(0, -\pi/a)$. The Brillouin zone and the two vectors $\vec{Q_1}$ and $\vec{Q_2}$ are shown on the figure above, as well as the Dirac points.

1.3. two-band model on a square lattice with a nontrivial Chern number

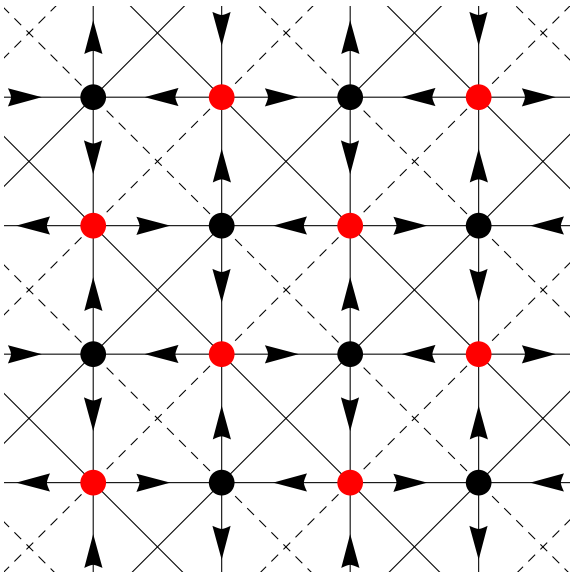


Fig. 5

Let's add next-nearest-neighbor hoppings to the model we considered above in Problem 1.2. Here we keep the nearest-neighbor hoppings the same as in Problem 1.2.

$$H == H_{\text{NN}} + H_{\text{NNN}} \tag{1.22}$$

Here H_{NN} describes the nearest-neighbor hoppings and it is the same as the Eq. (1.12).

$$H_{\text{NN}} = -t \sum_{\langle i,\, j\rangle} e^{i\,\phi_{ij}}\, a_i^{\dagger}\, b_j + h.c. \tag{1.23}$$

H_{NNN} are the next-nearest-neighbor hoppings which are marked by the diagonal lines in Fig. 5. Here we assume that all the next-nearest-neighbor hoppings are real. Along the solid diagonal lines, the hopping strength is t'. Along dashed lines, the strength is -t'. (They have opposite signs)

$$H_{\text{NNN}} = -\sum_{\langle\langle i,\, j\rangle\rangle} t_{\text{ij}}'\, a_i^{\dagger}\, a_j - \sum_{\langle\langle i,\, j\rangle\rangle} t_{\text{ij}}'\, b_i^{\dagger}\, b_j \tag{1.24}$$

1.3.1. magnetic flux and time-reversal symmetry

Consider a triangle formed by two nearest-neighbor bonds and one next-nearest-neighbor bond, which is the smallest plaquette on this lattice. If we goes around the square (in the counter-clock direction), the Aharonov– Bohm phase picked up by the electron is:

$$\phi_1 + \phi_2 + \phi_3 \tag{1.25}$$

The phase for nearest-neighbor hoppings are $\pm\pi/4$ ($\phi_1 = \phi_2 = \pm\pi/4$). For next-nearest-neighbor bonds, the phase is 0 if $t_{\text{ij}}' > 0$ and π if $t_{\text{ij}}' < 0$. Therefore the total phase is

$$\phi_1 + \phi_2 + \phi_3 = \pm\frac{\pi}{2} \qquad \text{or} \qquad \phi_1 + \phi_2 + \phi_3 = \pm\frac{\pi}{2} + \pi \tag{1.26}$$

These fluxes are NOT integer times π. As we learned in the lecture, they break the time-reversal symmetry. So this model is NOT time-reversally invariant.

1.3.2. Band structure

H_{NN} has been taken care of in the previous problem. Now we convert H_{NNN} into k-space. For red-red hoppings, there are two next-nearest-neighbor bonds per unit cell, one along the 45-degree line and the other along 135-degree line. In addition, there are two other NNN bonds for black-black hoppings. As a result the Hamiltonian is:

$$H_{\text{NNN}} = t' \sum_i a_{\vec{r}_i+\vec{v}_1}^{\dagger}\, a_{\vec{r}_i} - t' \sum_i a_{\vec{r}_i+\vec{v}_2}^{\dagger}\, a_{\vec{r}_i} - t' \sum_i b_{\vec{r}_i+\vec{v}_1}^{\dagger}\, b_{\vec{r}_i} + t' \sum_i b_{\vec{r}_i+\vec{v}_2}^{\dagger}\, b_{\vec{r}_i} + h.c. \tag{1.27}$$

Here, $\vec{v}_1 = (a,\, a)$ and $\vec{v}_2 = (-a,\, a)$ with a being the length of the nearest-neighbor bonds. The sum $\sum_i$ here sums over all unit cells.

Rewrite H_{NNN} in the k-space as

$$H_{\text{NNN}} = \sum_k \left(a_k^{\dagger},\ b_k^{\dagger}\right) \begin{pmatrix} \mathcal{H}^{(1)}{}_{a\,a}(k) & \mathcal{H}^{(1)}{}_{a\,b}(k) \\ \mathcal{H}^{(1)}{}_{b\,a}(k) & \mathcal{H}^{(1)}{}_{b\,b}(k) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{1.28}$$

Find the two-by-two matrix, which is the kernel of the Hamiltonian for H_{NNN}

$$\mathcal{H}^{(1)}(k) = \begin{pmatrix} \mathcal{H}^{(1)}{}_{a\,a}(k) & \mathcal{H}^{(1)}{}_{a\,b}(k) \\ \mathcal{H}^{(1)}{}_{b\,a}(k) & \mathcal{H}^{(1)}{}_{b\,b}(k) \end{pmatrix} \tag{1.29}$$

As a result, the full Hamiltonian $H = H_{\text{NN}} + H_{\text{NNN}}$ take the following form in k-space

$$H = \sum_k \left(a_k^{\dagger},\ b_k^{\dagger}\right) \begin{pmatrix} \mathcal{H}_{a\,a}(k) & \mathcal{H}_{a\,b}(k) \\ \mathcal{H}_{b\,a}(k) & \mathcal{H}_{b\,b}(k) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{1.30}$$

where $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$.

Find the matrix $\mathcal{H}\left(\vec{k}\right)$ and find its eigenvalues $\epsilon_{\pm}\left(\vec{k}\right)$.

Show that gaps open up at the two Dirac points by computing the gap:

$$\Delta = \epsilon_{+}\left(\frac{\pi}{2\,a},\ \frac{\pi}{2\,a}\right) - \epsilon_{-}\left(\frac{\pi}{2\,a},\ \frac{\pi}{2\,a}\right) \tag{1.31}$$

$$\Delta = \epsilon_+\left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right) - \epsilon_-\left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right) \tag{1.32}$$

Find Δ and show Δ>0 as long as t' is nonzero.

1.3.3. Eigenvectors

To find the eigenvectors of $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$, we first rewrite the two×two matrix $\mathcal{H}$ in terms of identity and Pauli matrices.

$$\mathcal{H}\left(\vec{k}\right) = \mathcal{H}_0\left(\vec{k}\right) I + \mathcal{H}_x\left(\vec{k}\right)\sigma_x + \mathcal{H}_y\left(\vec{k}\right)\sigma_y + \mathcal{H}_z\left(\vec{k}\right)\sigma_z \tag{1.33}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{and} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{and} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{and} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.34}$$

Find the four coefficients $\mathcal{H}_0$, $\mathcal{H}_x$, $\mathcal{H}_y$ and $\mathcal{H}_z$ and show that

$$u^{(I)}{}_-(k) = \frac{1}{\mathcal{N}^{(I)}}\begin{pmatrix} \mathcal{H}_z(k) - \left|\vec{\mathcal{H}}(k)\right| \\ \mathcal{H}_x(k) + i\,\mathcal{H}_y(k) \end{pmatrix} \tag{1.35}$$

is one eigenvector for the lower band.

In addition, please show that

$$u^{(\mathrm{II})}{}_-(k) = \frac{1}{\mathcal{N}^{(\mathrm{II})}}\begin{pmatrix} -\mathcal{H}_x(k) + i\,\mathcal{H}_y(k) \\ \mathcal{H}_z + \left|\vec{\mathcal{H}}(k)\right| \end{pmatrix} \tag{1.36}$$

is also a eigenvector for the lower band.

1.3.4. Can one use one wavefunction to cover the whole Brillouin zone?

Compute $u^{(I)}{}_-(k)$ and $u^{(\mathrm{II})}{}_-(k)$ at $k = \left(\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ and show that one of them is singular.

Compute $u^{(I)}{}_-(k)$ and $u^{(\mathrm{II})}{}_-(k)$ at $k = \left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ and show that one of them is singular.

Can one use a single wavefunction $u^{(I)}$ or $u^{(\mathrm{II})}$ to cover the whole Brillouin zone?

1.3.5. Compute the Chern number

Draw a small circle around the point $k = \left(\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ to separate the Brillouin zone into two regions: inside and outside. Inside the circle, use wavefunction $u^{(I)}{}_-(k)$ and outside use $u^{(\mathrm{II})}{}_-(k)$. Now there is no singular point. As we have proved in class, at the boundary of these two regions (the circle), the two wavefunctions are connected by a gauge transformation

$$e^{i\,\chi(k)} = \frac{\left|\mathcal{H}_x(k) + i\,\mathcal{H}_y(k)\right|}{\mathcal{H}_x(k) + i\,\mathcal{H}_y(k)} \tag{1.37}$$

Expand k near $\left(\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ as

$$\vec{k} = \left(\frac{\pi}{2a} + q_x, \frac{\pi}{2a} + q_y\right) \tag{1.38}$$

At small q, show that

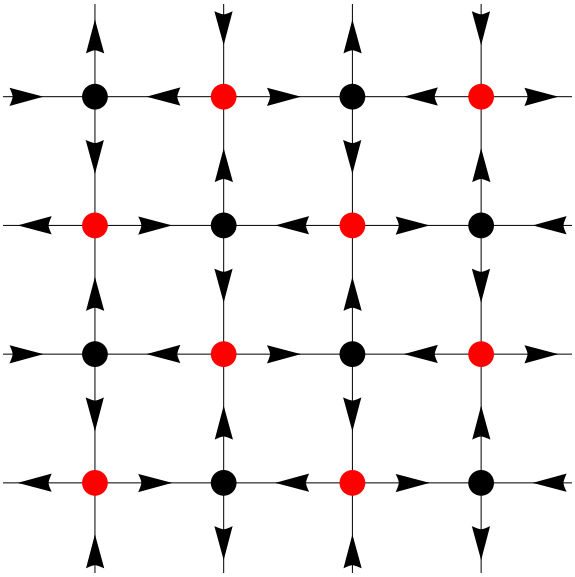
$$e^{i\,\chi(k)} = \frac{\left|\mathcal{H}_x(k) + i\,\mathcal{H}_y(k)\right|}{\mathcal{H}_x(k) + i\,\mathcal{H}_y(k)} \approx \frac{\left|q_x + i\,q_y\right|}{(-1)^{3/4}\left(q_x + i\,q_y\right)} = e^{-i\,\theta - i\,3\pi/4} \tag{1.39}$$

where θ is the polar angle. As a result, the Chern number is

$$C = \frac{1}{2\pi}[\chi(\theta = 0) - \chi(\theta = 2\pi)] = 1 \tag{1.40}$$

So this is a topological insulator, if we fill the lowest band.

1.4. (optional) two-band model on a square lattice: topologically trivial insulator



Let's consider the same model as in Problem 1.2 (with only nearest-neighbor hoppings). Now we add different potential energy for a sites and b sites

$$H = -t \sum_{\langle i,j \rangle} e^{i\phi_{ij}} a_i^\dagger b_j + h.c. + V \sum_i \left(a_i^\dagger a_i - b_i^\dagger b_i\right) \tag{1.41}$$

Find the kernel of the Hamiltonian $\mathcal{H}$ for this model, which is a two-by-two matrix. Find its eigenvalues and show that gaps are opened at the two Dirac points.

Expand the kernel $\mathcal{H}$ in terms of identity and Pauli matrices.

$$\mathcal{H}\left(\vec{k}\right) = \mathcal{H}_0\left(\vec{k}\right) I + \mathcal{H}_x\left(\vec{k}\right)\sigma_x + \mathcal{H}_y\left(\vec{k}\right)\sigma_y + \mathcal{H}_z\left(\vec{k}\right)\sigma_z \tag{1.42}$$

Compute $u^{(I)}\_(k)$ and $u^{(\mathrm{II})}\_(k)$ at $k = \left(\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ and show that one of the is singular. Compute $u^{(I)}\_(k)$ and $u^{(\mathrm{II})}\_(k)$ at $k = \left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ and show that one of the is singular.

Can one use a single wavefunction $u^{(I)}$ or $u^{(\mathrm{II})}$ to cover the whole momentum space?