$$A_{+k} = -i \langle u_+ | \partial_k | u_+ \rangle = 0 \tag{3.410}$$

$$A_{+\phi} = -\frac{i}{k} \langle u_+ | \partial_\phi | u_+ \rangle = -\frac{i}{k} \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\phi} & 1 \end{pmatrix} \partial_\phi \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-i\phi} \\ 1 \end{pmatrix} = -\frac{i}{k} \frac{1}{2} \begin{pmatrix} e^{i\phi} & 1 \end{pmatrix} \begin{pmatrix} -i\, e^{-i\phi} \\ 1 \end{pmatrix} = -\frac{1}{2k} \tag{3.411}$$

Let's choose a small circle around the Dirac point, the flux through this circle is:

$$\chi = \int\int \mathcal{F}\, d\mathbf{k} = \oint \mathcal{A} \cdot d\mathbf{k} = \int_0^{2\pi} \mathcal{A}_\phi\, k\, d\phi = -\int_0^{2\pi} \frac{1}{2k} k\, d\phi = -\frac{1}{2} \int_0^{2\pi} d\phi = -\pi \tag{3.412}$$

3.16.4. Why Dirac point is so stable?

This is because the Berry fluxes are very stable. If one has $\mathcal{TI}$ symmetry, the Berry curvature must be

$$\mathcal{F}(k) = \sum_i n_i\, \pi\, \delta(k - k_i) \tag{3.413}$$

Now, if we continuously tune some control parameter in our system, $\mathcal{F}$(k) must change continuously. However, if we have $\mathcal{TI}$ symmetry, the form of $\mathcal{F}$ cannot change, the only thing that can change continuously is the location of the delta functions.

In other words, without break the $\mathcal{TI}$ symmetry, one can only move the Dirac points around in the k-space. They cannot disappear. (However, two Dirac points can annihilate each other).

3.17. Edge states (numerical calculations)

3.17.1. Why energy is a function of k_x and k_y

Consider one electron moving in an infinite 2D lattice (for simplicity, we consider a square lattice here). Because the lattice is invariant under lattice translations ($x \to x + a$ and $y \to y + a$), lattice momentum is a good quantum number (a conserved quantity). Momentum conservation means that

$$[k_x,\, H] = [k_y,\, H] = 0 \tag{3.414}$$

In addition, we know $[k_x,\, k_y] = 0$. With three operators which commute with each other, quantum mechanics tells us that we can find common eigenstates for all these three operators, and we can use these eigenstates as the basis for the Hilbert space. This set of basis looks like:

$$|\psi_{\epsilon_n, k_x, k_y}\rangle \tag{3.415}$$

This state is the eigenstates of H, k_x and k_y with eigenvalues ϵ_n, k_x and k_y respectively. Here, one need a sub-index (n) to distinguish different states with the same k_x and k_y, which is the band index. So we have the band structure:

$$\epsilon_n(k_x,\, k_y) \tag{3.416}$$

3.17.2. What if we have an finite system?

To study edge states, we must have an edge. So we cannot use an infinite 2D plan. Let's consider the simplest case here, a infinitely long stripe. We assumes that the system is infinitely long along x, but it has a finite width along y. Now, the translational symmetry along x is still preserved. But there is no translational symmetry along y (due to the existence of the edge). So k_x is still a good quantum number but k_y is NOT.

$$[k_x,\, H] = 0 \qquad \text{but} \qquad [k_y,\, H] \neq 0 \tag{3.417}$$

So we cannot define common eigenstates for H, k_x and k_y, but we can still find eigenstates for H and k_x, because they commute with each other.

$$|\psi_{\epsilon_m, k_x}\rangle \tag{3.418}$$

Here, again, we need an extra index distinguish different states with the same k_x. Here I emphasize that m is NOT just the band index n. It includes information for both n and k_y.

So now, energy is a bunch of function of k_x

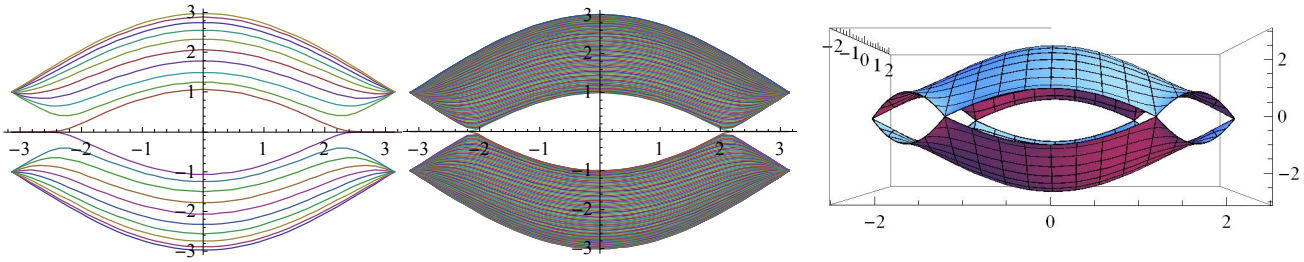
$$\epsilon_m(k_x) \tag{3.419}$$

where m = 1, 2, 3 …N_S. We have N_S functions of k_x. We can draw these N_x functions on the $k_x - \epsilon$ plane, which gives us N_S curves, which is our 1D dispersion.

3.17.3. Examples

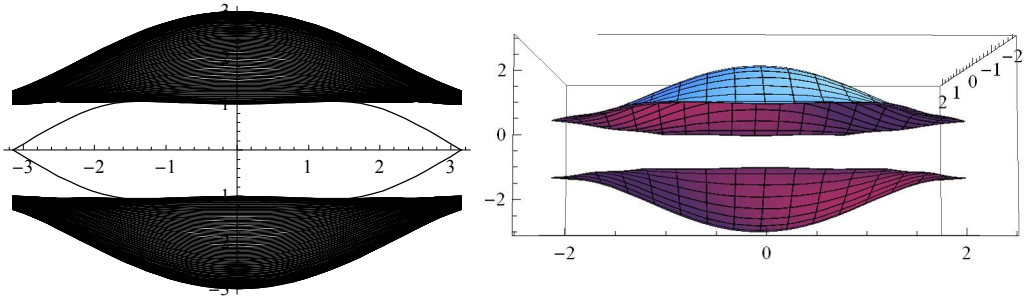
We consider the model of Haldane for a system with infinite size along x, by finite width along y. The Hamiltonian is invariant under translations along the x axis $\left(x \to x + \sqrt{3}\ a\right)$, so the x-component of the momentum is a conserved quantity $[k_x,\ H] = 0$. So we can find common eigenstates of k_x and H. The system has two edges: one on the top, one at the bottom. We can consider this lattice as a bunch of 1D (horizontal) lines coupled together, so the width are described by the number of chains one have in this infinite stripe (N). For $\epsilon_m(k_x)$, we have m=1,2,…2 N. This factor of 2 comes from the fact that we have two sites in each unit cell for a honeycomb lattice.

First, lets consider the case without NNN hopping (graphene, with two Dirac cones). Here, for the first panel, we show $\epsilon_m(k_x)$ for a system with width N=10 (so there are $N_S = 2\ N = 20$) lines. The second panel is the same calculation for a system with width N=100 ($N_S = 2\ N = 200$). The last panel shows the bulk band structure (side view).



The 2D figure of $\epsilon_m(k_x)$ is in some sense the 3D band structure $\epsilon_n\left(k_x,\ k_y\right)$ projected onto a 2D plane. We can see here an upper band and a lower band and 2 Dirac cones. The system is gapless.

Now, let's consider the topological case, with complex NNN hoppings. The first panel shows the energy dispersion for an infinite stripe and the second one shows the bulk band structure for an infinite 2D system.



Here, I choose the parameter $\phi=\pi/2$, $t_{\mathrm{NN}} = 1$ and $t_{\mathrm{NNN}} = 0.2$. As one can see, for the infinite stripe, there are two bands, whose energy spectrum coincide with the bulk band structure for infinite systems. But in addition, there are two extra lines shown inside the gap, which are the metallic edge states. By looking at the wavefunctions (see the interactive figures on the course website), one find that one of the in gap state is localized near the top edge, while the other is localized near the bottom edge.

If we consider the momentum region $(0,2\pi)$, instead of $(-\pi,\pi)$, we find that one of the two edge states has positive slop and the other one has negative slop (left moving). Because the slop is the velocity of electrons ($v = \partial\epsilon/\partial k$), one of the edge states has positive velocity (right moving) while the other one is right moving (negative velocity).