9

Quantum phase transition and quantum criticality

9.1. Basic Ideas

Classical phase transition: phase transition at finite temperature

Quantum phase transition: phase transition at $T = 0$.

9.1.1. Quantum phase transition and level crossing

Consider a quantum Hamiltonian:

$$H = (1 - x) H_1 + x H_2 \tag{9.1}$$

where x varies between 0 and 1. At $x = 0$, $H = H_1$. So the ground state at $x = 0$ is the ground state of H_1 ($| G_1\rangle$). At $x = 1$ $H = H_2$, so the ground state of the system is the ground state of H_2 ($| G_2\rangle$).

Q: At x increases from 0 to 1, how does the ground state wavefunction of H evolves from $| G_1\rangle$ to $| G_2\rangle$?

A: there are two possibilities. (1) $| G_1\rangle$ evolves adiabatically to $| G_2\rangle$ or (2) there is a level crossing point at some critical $x = x_C$.

The second case is known as a quantum phase transition.

Q: Why the level crossing? (Why not avoided level crossing)

A: If the symmetry properties of $| G_1\rangle$ and $| G_2\rangle$ are fundamentally different,

9.1.2. Example: transverse field Ising model

$$H = -J\, g \sum_i \sigma_i^x - J \sum_{\langle i,j\rangle} \sigma_i^z \sigma_j^z \tag{9.2}$$

The first term $-J\, g \sum_i \sigma_i^x$ is our H_1 and the second term $-J \sum_{\langle i,j\rangle} \sigma_i^z \sigma_j^z$ is our H_2.

Two special limits are easy to study:

At $g = +\infty$, the first term dominates so $H = H_1$. Now, the ground state is: all spins point to the $+x$ direction, which is our $| G_1\rangle$. At very large g (but not infinite g), most of the spin will be pointing in the $+x$ direction.

At $g = 0$, the first term vanishes so $H = H_2$. Now, there two degenerate ground states: all spins point to the $+z$ or -z direction, which we call $| G_2^+\rangle$ and $| G_2^-\rangle$. At $0 < g << 1$, not all the spins will be pointing into the same direction, but we will still have two ground states, one has most of the spin pointing to $+z$ the other have most spin pointing to $-z$.

Symmetry properties

The Hamiltonian preserves the symmetry: $z \to -z$. For the ground state $| G_1\rangle$, the symmetry is preserved. But for $| G_2^\pm\rangle$, this symmetry is broken spontaneously. If we change z→-z, $| G_2^+\rangle \leftrightarrow | G_2^-\rangle$.

G_1 has one ground state which preserves the $z \to -z$ symmetry. $| G_2^\pm\rangle$ have two ground states, which break the $z \to -z$ symmetry.

So, it is impossible for G_1 to evolve adiabatically to G_2 as g decreases from $+\infty$ to 0. So there must be (at least) one phase transition between $g = 0$ and $g = +\infty$.

9.1.3. Quantum critical point and quantum criticality

For some quantum phase transitions, the transition point shows scaling behavior. This type of quantum phase transitions are known as "second order" quantum phase transitions and the transition points for these transitions are known as quantum critical points.

Correlation functions:

$$\langle \sigma_i^z \sigma_j^z \rangle \propto \exp(|i - j| a/\xi) \tag{9.3}$$

At long distance $|i - j| >> 1$, this correlation function is typically an exponential function. Here ξ is known as the correlation length and a is the lattice spacing. Near a quantum critical point,

$$\xi \propto |g - g_c|^{-\nu} \tag{9.4}$$

these type of power-law behavior is known as scaling law.

The energy gap near a quantum critical point also obey the scaling law

$$\Delta \propto |g - g_c|^{z\,\nu} = \xi^{-z} \tag{9.5}$$

Notice that $\hbar/\Delta$ defines a time scale t

$$t = \hbar/\Delta \propto |g - g_c|^{z\,\nu} = \xi^z \tag{9.6}$$

So

$$t \propto \xi^z \tag{9.7}$$

Time counts as z space dimension. This z is known as dynamic critical exponent. In fact, one can usually prove that for a system in d space dimensions, a quantum critical point with dynamic critical exponent z can be mapped to a classical phase transition in $d + z$ space dimensions with the same symmetry breaking pattern.

For example, the transverse Ising model has $z = 1$. Therefore, a d-dimensional transverse Ising model is Equivalent to a d+1 dimensional classical Ising model.

9.2. Quantum Ising model in 1D

$$H = -J\,g \sum_i \sigma_i^x - J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \tag{9.8}$$

First let's rotate the system around y-axis by 90 degrees (changing coordinates, which doesn't effect any physics). $x \to -z$ and $z \to x$

$$H = J\,g \sum_i \sigma_i^z - J \sum_i \sigma_i^x \sigma_{i+1}^x \tag{9.9}$$

9.2.1. Jordan-Wigner transformation

$$\sigma_i^+ = (-1)^{\sum_{j=1}^{j=i-1} n_j} c_i^\dagger = \Pi_{j=1}^{j=i-1} (-1)^{n_j} c_i^\dagger = \Pi_{j=1}^{j=i-1} (1 - 2\,n_j)\, c_i^\dagger \tag{9.10}$$

$$\sigma_i^- = (-1)^{\sum_{j=1}^{j=i-1} n_j} c_i\, \Pi_{j=1}^{j=i-1} (-1)^{n_j} c_i = \Pi_{j=1}^{j=i-1} (1 - 2\,n_j)\, c_i \tag{9.11}$$

$$\sigma_i^z = 2\,S^z = 2\,(n_i - 1/2) = 2\,n_i - 1 \tag{9.12}$$

$$\sigma_i^x = \sigma_i^+ + \sigma_i^- = \Pi_{j=1}^{j=i-1} (1 - 2\,n_j) \left(c_i^\dagger + c_i\right) \tag{9.13}$$

$$\begin{aligned} H = J\,g \sum_i \sigma_i^z - J \sum_i \sigma_i^x \sigma_{i+1}^x &= J\,g \sum_i (2\,n_i - 1) - J \sum_i \Pi_{j=1}^{j=i-1} (1 - 2\,n_j) \left(c_i^\dagger + c_i\right) \Pi_{j=1}^{j=i} (1 - 2\,n_j) \left(c_{i+1}^\dagger + c_{i+1}\right) = \\ J\,g \sum_i \left(2\,c_i^\dagger c_i - 1\right) &- J \sum_i \Pi_{j=1}^{j=i-1} (1 - 2\,n_j) \left(c_i^\dagger + c_i\right) \Pi_{j'=1}^{j'=i-1} (1 - 2\,n_{j'}) (1 - 2\,n_i) \left(c_{i+1}^\dagger + c_{i+1}\right) = \\ J\,g \sum_i \left(2\,c_i^\dagger c_i - 1\right) &- J \sum_i \Pi_{j=1}^{j=i-1} \Pi_{j'=1}^{j'=i-1} (1 - 2\,n_j) (1 - 2\,n_{j'}) \left(c_i^\dagger + c_i\right) (1 - 2\,n_i) \left(c_{i+1}^\dagger + c_{i+1}\right) = \\ J\,g \sum_i \left(2\,c_i^\dagger c_i - 1\right) &- J \sum_i \Pi_{j=1}^{j=i-1} (1 - 2\,n_j)^2 \left(c_i^\dagger - c_i\right) \left(c_{i+1}^\dagger + c_{i+1}\right) \end{aligned} \tag{9.14}$$

The last step we used the fact that $c_i^\dagger (1 - 2\,n_i) = c_i^\dagger$ and $c_i (1 - 2\,n_i) = -c_i$

$$c_i^\dagger(1-2\,n_i)\,\big|0\big\rangle = c_i^\dagger\,\big|0\big\rangle = \,|1\big\rangle \tag{9.15}$$

$$c_i^\dagger(1-2\,n_i)\,\big|1\big\rangle = -c_i^\dagger\,\big|1\big\rangle = 0 \tag{9.16}$$

$$c_i(1-2\,n_i)\,|\,0\rangle = c_i\,|\,0\rangle = 0 \tag{9.17}$$

$$c_i(1-2\,n_i)\,|\,1\rangle = -c_i\,|\,1\rangle = \,|\,0\rangle \tag{9.18}$$

Compare with

$$c_i^\dagger\,\big|0\big\rangle = \,|1\big\rangle \tag{9.19}$$

$$c_i^\dagger\,\big|1\big\rangle = 0 \tag{9.20}$$

$$c_i\,|\,0\rangle = 0 \tag{9.21}$$

$$-c_i\,|\,1\rangle = \,|\,0\rangle \tag{9.22}$$

We find that $c_i^\dagger(1-2\,n_i) = c_i^\dagger$ and $c_i(1-2\,n_i) = -c_i$

$$H = J\,g\sum_i\left(2\,c_i^\dagger\,c_i-1\right) - J\sum_i \Pi_{j=1}{}^{j=i-1}\left(1-2\,n_j\right)^2\left(c_i^\dagger-c_i\right)\left(c_{i+1}{}^\dagger+c_{i+1}\right) \tag{9.23}$$

Because $\left(1-2\,n_j\right)^2 = 1$,

$$\begin{aligned} H &= J\,g\sum_i\left(2\,c_i^\dagger\,c_i-1\right) - J\sum_i\left(c_i^\dagger-c_i\right)\left(c_{i+1}{}^\dagger+c_{i+1}\right) = \\ &-J\sum_i\left(c_i^\dagger\,c_{i+1}+c_{i+1}{}^\dagger\,c_i\right) - J\sum_i\left(c_i^\dagger\,c_{i+1}{}^\dagger+c_{i+1}\,c_i\right) + 2\,J\,g\sum_i c_i^\dagger\,c_i - J\,g\,N \end{aligned} \tag{9.24}$$

The last term is a constant number, which can be ignored.

The first term describes the nearest-neighbor hoppings. The second term is a pairing term (cooper pairs in a superconductor). The third term is chemical potential $\mu = -2\,J\,g$.

Go to the momentum space

$$c_k = \frac{1}{\sqrt{N}}\sum_j c_j\,e^{-i\,k\,r_j} \tag{9.25}$$

$$\begin{aligned} H &= -J\sum_k\left(c_k^\dagger\,c_k\,e^{i\,k\,a}+c_k^\dagger\,c_k\,e^{-i\,k\,a}\right) - J\sum_i\left(c_{-k}{}^\dagger\,c_k^\dagger\,e^{-i\,k\,a}+c_{-k}\,c_k\,e^{-i\,k\,a}\right) + 2\,J\,g\sum_k c_k^\dagger\,c_k = \\ &J\sum_k\left[2\,(g-\cos k\,a)\,c_k^\dagger\,c_k + i\sin k\,a\left(c_{-k}{}^\dagger\,c_k^\dagger + c_{-k}\,c_k\right)\right] = J\sum_k\left(\,c_k^\dagger \quad c_{-k}{}^\dagger\,\right)\begin{pmatrix} g-\cos k\,a & i\sin k\,a \\ -i\sin k\,a & -g+\cos k\,a\end{pmatrix}\begin{pmatrix} c_k \\ c_{-k}\end{pmatrix} \end{aligned} \tag{9.26}$$

$$\text{Eigenvalues of the } 2\times 2 \text{ matrix gives the energy spectrum of } a \text{ quasi} - \text{particle.} \tag{9.27}$$

$$E_\pm = \pm J\sqrt{(g-\cos k\,a)^2+(\sin k\,a)^2} = \pm J\sqrt{1+g^2-2\,g\cos k\,a} \tag{9.28}$$

The gap between the two bands

$$\Delta(k) = E_+ - E_- = 2\,J\sqrt{1+g^2-2\,g\cos k\,a} \tag{9.29}$$

Minimum of Δ(k) is reached at k=0.

$$\Delta(k) \geq \Delta = 2\,J\sqrt{1+g^2-2\,g} = 2\,J\,\,|\,1-g\,| \tag{9.30}$$

Δ is the gap between the ground state and exacted states. As one can see that Δ=0 at g=1. So $g = 1$ is our quantum phase transition point. Near this quantum phase transition point, Δ shows critical scaling $\Delta \propto |\,g-g_c\,|$. So we have $z\,\nu = 1$.

Similarly, one can prove that $\xi \propto 1/\,|\,1-g\,|$ so ν=1. So we have $z = 1$ and ν=1.

9.2.2. degenerate ground states at g<1.

For g<1, we know that there should be two degenerate ground states. How can we see it in the fermion model?

As shown in the homework, this 1D superconductor is a topological superconductor for $-1 < g < 1$. And it is a topologically trivial superconductor for $|g| > 1$. The topological superconductor has two Majorana fermions at the two ends of the 1D chain. In addition, we also proved that if we have two Majorana fermions, the ground state is 2-fold degenerate. This is the two-fold degeneracy for the symmetry broken phase.