4

Gauge theory of insulators

4.1. Gauge theory (E&M) for Chern insulators

In an crystal, we have nucleons, electrons, and they couple together through gauge fields (E and B). If we don't consider the spin degree of freedom (we don't consider magnetic ordering and magnetization here), the only degrees of freedom are the motion of charge and the gauge field. In an insulator, charged particles cannot move, so the only useful degrees of freedom are the gauge fields. So let's consider the gauge field in an insulator?

4.1.1. Hamiltonian of *E* and *B* field

E and B field carries energy. In vacuum, the energy (Hamiltonian) is

$$H = \frac{1}{2} \int d\vec{r} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \tag{4.1}$$

In an insulator (2D) with time-reversal symmetry, the energy of gauge fields must be a function of E and B. For weak fields (small E and B) we can do a power series expansion:

$$H = \frac{1}{2} \int d\vec{r} \left(\epsilon_x E_x{}^2 + \epsilon_y E_y{}^2 + 2 \epsilon_{xy} E_x E_y + \frac{1}{\mu_x} B_x{}^2 + \frac{1}{\mu_y} B_y{}^2 + B_z{}^2 \right) + \text{higher order terms} \tag{4.2}$$

We cannot have any other quadratic terms, like E·B, because it violates the time-reversal symmetry (B is odd under time − reversal but E is even). Higher order terms are in principle allowed and they lead to non-linear E&M response. However, these nonlinear effect are weak at small E and B, so we can ignore them.

The bottom line: the energy is still $E^2 + B^2$. Only the coefficient got renoramlized. So an insulator is NOT fundamentally different from vacuum (so we have nothing fancy here).

Q: what will happen if we break the T-reversal symmetry?

A: Additional terms are allowed by symmetry. e.g. E·B. This term means that we will induce magnetization by applying E field (and induce electric dipoles by applying B field). But this is a "small" effect, which we will not consider here. In addition to $E \cdot B$, the system can get another term which is the Chern-Simon's gauge theory

4.1.2. the Chern-Simon's gauge theory in 2+1D

For simplicity, we use the theorists' unit: $\epsilon = \mu =$ everything $= 1$.

The Cherm-Simon's theory is easier to handle using Lagrangian, instead of the Hamiltonian. The Lagrangian is

$$L_{\text{CS}} = \frac{k}{4\pi} \int d\vec{r}\, \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda \tag{4.3}$$

k is the coupling constant, μ, ν, λ run over t, x and y. (A^x, A^y) is the vector potential and A^t is the electric potential ϕ. $\epsilon_{\text{txy}} = \epsilon_{\text{xyt}} = \epsilon_{\text{ytx}} = 1$ and $\epsilon_{\text{tyx}} = \epsilon_{\text{yxt}} = \epsilon_{\text{xty}} = -1$ and $\epsilon = 0$ for any other subindcies.

The action is

$$S_{\text{CS}} = \int dt\, L = \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda \tag{4.4}$$

Gauge invariance:

Let's consider a gauge transformation:

$$A_\mu \to A_\mu{}' = A_\mu + \partial_\mu \chi \tag{4.5}$$

$$\begin{aligned} S_{\text{CS}} \to S_{\text{CS}}{}' &= \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} (A^\mu + \partial^\mu \chi)\left[\partial^\nu \left(A^\lambda + \partial^\lambda \chi\right)\right] = \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} (A^\mu + \partial^\mu \chi)\, \partial^\nu A^\lambda = \\ & \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda + \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} \partial^\mu \chi\, \partial^\nu A^\lambda = S_{\text{CS}} + \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} \partial^\mu \left(\chi\, \partial^\nu A^\lambda\right) = S_{\text{CS}} + \text{surface term} \end{aligned} \tag{4.6}$$

If we don't worry about the surface term (assuming the system has no boundary), the action is gauge invariant. So this is a valid gauge theory. The surface term will be consider later. In fact, the surface term of the Chern-Simon's gauge theory violates the gauge symmetry. But luckily, the edge metallic state (chiral state) that we have at the edge also violates the gauge symmetry through parity anomaly. And these two effects cancels each other, so the whole system is gauge invariant.

For the whole insulator:

$$S = S_{\text{CS}} + \frac{1}{2} \int d\vec{r} \left(\epsilon_x E_x{}^2 + \epsilon_y E_y{}^2 + 2\, \epsilon_{\text{xy}} E_x E_y + \frac{1}{\mu_x} B_x{}^2 + \frac{1}{\mu_y} B_y{}^2 + B_z{}^2 \right) + \text{higher order terms} \tag{4.7}$$

If the field A varies very slowly in space and time, ∂A is small. The CS term contains only one derivative, but E^2 and B^2 term contain two derivatives ($E$and B both take the form of ∂A). Two derivatives means it is much smaller if A changes very slowly. So here, the CS term gives the dominate contribution.

We can separate the action into two parts

$$S = S_{\text{vacuum}} + S_{\text{material}} \tag{4.8}$$

Here, S_{material} is the contribution from the material (the insulator)

$$S_{\text{material}} = S - S_{\text{vacuum}} \tag{4.9}$$

Electric Current:

$$\begin{aligned} j_i(r_0, t_0) &= \frac{\delta S_{\text{material}}}{\delta A_i(r_0, t_0)} = \frac{\delta}{\delta A_i(r_0, t_0)} \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} A^\mu(r, t)\, \partial^\nu A^\lambda(r, t) = \\ & \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} \frac{\delta A^\mu(r, t)}{\delta A_i(r_0, t_0)} \partial^\nu A^\lambda(r, t) + \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} A^\mu(r, t) \left[\partial^\nu \frac{\delta A^\lambda(r, t)}{\delta A_i(r_0, t_0)}\right] \\ &= \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda}\, \delta_{\mu,i}\, \delta(r - r_0)\, \delta(t - t_0)\, \partial^\nu A^\lambda(r, t) + \frac{k}{4\pi} \int d\vec{r}\, dt\, \epsilon_{\mu\nu\lambda} A^\mu(r, t) \left[\partial^\nu \delta_{\lambda,i}\, \delta(r - r_0)\, \delta(t - t_0)\right] \\ &= \frac{k}{4\pi} \epsilon_{i\nu\lambda}\, \partial^\nu A^\lambda(r_0, t_0) - \frac{k}{4\pi} \epsilon_{\mu\nu i}\, \partial^\nu A^\mu(r, t) = \frac{k}{2\pi} \epsilon_{i\mu\nu}\, \partial^\mu A^\nu(r_0, t_0) = -\frac{k}{2\pi} \vec{e}_i \times \vec{E}_j \end{aligned} \tag{4.10}$$

The Hall conductivity is (we set e and $\hbar$ to 1 in the calculations above and here we add them back):

$$\sigma_{\text{xy}} = \frac{k}{2\pi} \frac{e^2}{\hbar} = k \frac{e^2}{h} \tag{4.11}$$

So σ_{xy} is quantized.

Q: Why $\frac{k}{2\pi} \epsilon_{i\mu\nu}\, \partial^\mu A^\nu(r_0, t_0) = -\frac{k}{2\pi} \vec{e}_i \times \vec{E}_j$?

A: let's do $i = x$ as an example:

$$j_x(r_0,\ t_0) = \frac{k}{2\pi} \epsilon_{x\mu\nu} \partial^\mu A^\nu(r_0,\ t_0) =$$
$$\frac{k}{2\pi} \epsilon_{xty} \partial^t A^y(r_0,\ t_0) + \frac{k}{2\pi} \epsilon_{xyt} \partial^y A^t(r_0,\ t_0) = \frac{k}{2\pi}\left[-\partial^t A^y(r_0,\ t_0) + \partial^y \phi(r_0,\ t_0)\right] = \frac{k}{2\pi}\left[\partial_t A^y(r_0,\ t_0) + \partial_y \phi(r_0,\ t_0)\right] = -\frac{k}{2\pi} E_y \tag{4.12}$$

Q: What happens if we have a metal?

A: For metals, the charge degrees of freedom must also be considered. So we have both gauge fields and charge. The situation is much more complicated and in general, we cannot get a clear answer of σ_{xy}.