8.3. Feynman diagram

8.3.1. multi-particle Green's functions

Single particle Green's function:

$$G(1, 1') = \left(\frac{1}{i}\right) \langle \mathcal{T}\psi(1)\, \psi^\dagger(1') \rangle \tag{8.72}$$

Two-particle Green's function:

$$G_2(1, 2; 1', 2') = \left(\frac{1}{i}\right)^2 \langle \mathcal{T}\psi(1)\, \psi(2)\, \psi^\dagger(2')\, \psi^\dagger(1') \rangle \tag{8.73}$$

three-particle Green's function:

$$G_3(1, 2, 3; 1', 2', 3') = \left(\frac{1}{i}\right)^3 \langle \mathcal{T}\psi(1)\, \psi(2)\, \psi(3)\, \psi^\dagger(3')\, \psi^\dagger(2')\, \psi^\dagger(1') \rangle \tag{8.74}$$

n-particle Green's function:

$$G_n(1, 2, \ldots, n; 1', 2', \ldots, n') = \left(\frac{1}{i}\right)^n \langle \mathcal{T}\psi(1)\, \psi(2)\, \ldots \psi(n)\, \psi^\dagger(n')\, \ldots \psi^\dagger(2')\, \psi^\dagger(1') \rangle \tag{8.75}$$

8.3.2. the equations of motion of the Green's functions

$$i\, \frac{\partial \psi(r_0, t)}{\partial t} = [\psi(r_0), H] = \int dr\, \frac{\nabla\, \delta(r - r_0)\, \nabla\, \psi(r)}{2\, m} +$$

$$\frac{1}{2} \int dr\, dr'\, \delta(r - r_0)\, V(\,|\,r - r'\,|)\, \psi^\dagger(r')\, \psi(r')\, \psi(r) + \frac{1}{2} \int dr\, dr'\, V(\,|\,r - r'\,|)\, \delta(r' - r_0)\, \psi^\dagger(r)\, \psi(r')\, \psi(r) \tag{8.76}$$

$$= -\frac{\nabla^2 \psi(r_0, t)}{2\, m} + \int dr\, V(\,|\,r_0 - r\,|)\, \psi^\dagger(r)\, \psi(r)\, \psi(r_0)$$

$$\left(i\, \partial_t + \frac{1}{2\, m}\, \nabla^2\right) G(1, 2) = \delta(r_1 - r_2)\, \delta(t_1 - r_2) + \frac{1}{i} \int dr\, V(\,|\,r_0 - r\,|)\, \langle \mathcal{T}\psi^\dagger(r_1, t_1)\, \psi(r_1, t_1)\, \psi(r_0, t_1)\, \psi^\dagger(r_2, t_2) \rangle \tag{8.77}$$

$$\left(i\, \partial_t + \frac{1}{2\, m}\, \nabla^2\right) G(1, 1') = \delta(1 - 1') + i \int dr\, V(\,|\,r_1 - r_2\,|) \left(\frac{1}{i}\right)^2 \langle \mathcal{T}\left[\psi^\dagger(r_2, t_1)\, \psi(r_2, t_1)\, \psi(r_1, t_1)\right] \psi^\dagger(r_{1'}, t_{1'}) \rangle =$$

$$\delta(1 - 1') + i \int dr\, V(\,|\,r_1 - r_2\,|) \left(\frac{1}{i}\right)^2 \langle \mathcal{T}\psi^\dagger(r_2, t_1 + \delta)\, \psi(r_2, t_1)\, \psi(r_1, t_1)\, \psi^\dagger(r_{1'}, t_{1'}) \rangle =$$

$$\delta(1 - 1') \pm i \int dr\, V(\,|\,r_1 - r_2\,|) \left(\frac{1}{i}\right)^2 \langle \mathcal{T}\psi(r_1, t_1)\, \psi(r_2, t_1)\, \psi^\dagger(r_2, t_1 + \delta)\, \psi^\dagger(r_{1'}, t_{1'}) \rangle = \tag{8.78}$$

$$\delta(1 - 1') \pm i \int dr\, V(\,|\,r_1 - r_2\,|)\, G_2(1, 2; 1', 2^+)\, |_{t_1 = t_2}$$

Here 2^+ means that the time argument for 2^+ is slightly larger than 2, to keep the operators in the right order ($t_2{}^+ = t_2 + \delta$).

$$\left(i\, \partial_t + \frac{1}{2\, m}\, \nabla^2\right) G(1, 1') = \delta(1 - 1') \pm i \int dr_2\, V(\,|\,r_1 - r_2\,|)\, G_2(1, 2; 1', 2^+)\, |_{t_1 = t_2} \tag{8.79}$$

This equation tells us that in order to get $G(1, 2)$, we need to know $G_2(1, 2; 1', 2')$. So we need to write down the EOM for G_2

$$\left(i\, \partial_t + \frac{1}{2\, m}\, \nabla^2\right) G_2(1, 2; 1', 2') = \delta(1 - 1')\, G(2, 2') \pm \delta(1 - 2')\, G(2, 1') \pm i \int dr_2\, V(\,|\,r_1 - r_2\,|)\, G_3(1, 2, 3; 1', 2', 3^+)\, |_{t_1 = t_3} \tag{8.80}$$

This equation tells us that in order to get G_2, we need to know G_3.

Repeat the same procedure, we find that if we want to know G_n, we need to know G_{n+1}.

$$\left(i \, \partial_t + \frac{1}{2 \, m} \nabla^2\right) G_n(1, 2, \ldots, n; 1', 2', \ldots, n') = \delta(1 - 1') \, G_{n-1}(2, \ldots, n; 2', \ldots, n') \pm \delta(1 - 2') \, G_{n-1}(2, \ldots, n; 1', 3', \ldots, n') +$$

$$\ldots \pm i \int d \, r_2 \; V(\mid r_1 - r_2 \mid) \, G_{n+1}(1, 2, \ldots, n + 1; 1', 2', \ldots, n + 1) \mid_{t_1 = t_{n+1}} \tag{8.81}$$

So we cannot get a close set of equations. In order words, there is no way to solve these equations.

8.3.3. How to solve this equation? A: The perturbation theory

$G(1, 1')$ depends on interaction strength V. Let's expand G as a power series of V

$$G(1, 1') = G^{(0)}(1, 1) + O(V) + O(V^2) + O(V^3) + \ldots \tag{8.82}$$

Notice that G_{n-1} is related to $V \times G_n$. Therefore, to get G up to the order of $O(V^n)$, we just need to keep G_2 to the order of $O(V^{n-1})$, and G_3 to $O(V^{n-2})$ ... G_{n+1} to $O(V^0)$ and set $G_{n+2} = 0$

8.3.4. the zeroth-order approximation (free-particle approximation, or say non-interacting approximation)

If we want to get G to the zeroth order, we need to set $G_2 = 0$

$$\left(i \, \partial_{t_1} + \frac{1}{2 \, m} \nabla_{r_1}{}^2\right) G^{(0)}(1, 1') = \delta(1 - 1') \tag{8.83}$$

One equation, one unknown. It can be solved easily.

$$G^{(0)}(k, i\omega_n) = \frac{1}{i\omega_n - \frac{k^2}{2 \, m}} \tag{8.84}$$

8.3.5. the first-order approximation (the Hartree-Fock approximation)

If we want to get G to the first order, we need to set $G_3 = 0$ and keep G_2 to $O(V^0)$

$$\left(i \, \partial_{t_1} + \frac{1}{2 \, m} \nabla_{r_1}{}^2\right) G(1, 1') = \delta(1 - 1') \pm i \int d \, r_2 \; V(\mid r_1 - r_2 \mid) \, G_2(1, 2; 1', 2^+) \mid_{t_1 = t_2} \tag{8.85}$$

$$\left(i \, \partial_{t_1} + \frac{1}{2 \, m} \nabla_{r_1}{}^2\right) G_2(1, 2; 1', 2') = \delta(1 - 1') \, G(2, 2') \pm \delta(1 - 2') \, G(2, 1') + O(V \, G_3) = \delta(1 - 1') \, G^{(0)}(2, 2') \pm \delta(1 - 2') \, G^{(0)}(2, 1') \tag{8.86}$$

two equations and two unknowns. The solution for the second equation is very simple:

$$G_2(1, 2; 1', 2') = G^{(0)}(1, 1') \, G^{(0)}(2, 2') \pm G^{(0)}(1, 2') \, G^{(0)}(2, 1') \tag{8.87}$$

Let's check it

$$\left(i \, \partial_{t_1} + \frac{1}{2 \, m} \nabla_{r_1}{}^2\right) G^{(0)}(1, 1') \, G^{(0)}(2, 2') = \delta(1 - 1') \, G^{(0)}(2, 2') \tag{8.88}$$

$$\left(i \, \partial_{t_1} + \frac{1}{2 \, m} \nabla_{r_1}{}^2\right) G^{(0)}(1, 2') \, G^{(0)}(2, 1') = \delta(1, 2') \, G^{(0)}(2, 1') \tag{8.89}$$

So we can go back to the first equation to get G.

$$\left(i \, \partial_{t_1} + \frac{1}{2 \, m} \nabla_{r_1}{}^2\right) G(1, 1') =$$

$$\delta(1 - 1') \pm i \int d \, r_2 \; V(\mid r_1 - r_2 \mid) \, G^{(0)}(1, 1') \, G^{(0)}(2, 2^+) \mid_{t_1 = t_2} + i \int d \, r_2 \; V(\mid r_1 - r_2 \mid) \, G^{(0)}(1, 2^+) \, G^{(0)}(2, 1') \mid_{t_1 = t_2} \tag{8.90}$$

$$\left(i \, \partial_{t_1} + \frac{1}{2 \, m} \nabla_{r_1}{}^2\right) G(1, 1') = \delta(1 - 1') + f(1, 1') \tag{8.91}$$

This equation can be separated into two equations:

$$\left(i\,\partial_{t_1} + \frac{1}{2\,m}\,\nabla_{r_1}{}^2\right) G^{(0)}(1,\,1') = \delta(1-1') \tag{8.92}$$

$$\left(i\,\partial_{t_1} + \frac{1}{2\,m}\,\nabla_{r_1}{}^2\right) G^{(1)}(1,\,1') = f(1,\,1') \tag{8.93}$$

and $G = G^{(0)} + G^{(1)}$.

The first equation have been solved before. It is just the zeroth-order equation (the free theory)

$$\left(i\,\partial_{t_1} + \frac{1}{2\,m}\,\nabla_{r_1}{}^2\right) G^{(0)}(1,\,1') = \delta(1-1') \tag{8.94}$$

For the second equation, the solution is straightforward using the Green's function technique

$$G^{(1)}(1,\,1') = \int d\,r_3 \int d\,t_3\, G^{(0)}(1,\,3)\, f(3,\,1') \tag{8.95}$$

Let's check this

$$\left(i\,\partial_{t_1} + \frac{1}{2\,m}\,\nabla_{r_1}{}^2\right) \int d\,r_3 \int d\,t_3\, G^{(0)}(1,\,3)\, f(3,\,1') = \int d\,r_3 \int d\,t_3\, \delta(1-3)\, f(3,\,1') = f(1,\,1') \tag{8.96}$$

So

$$G^{(1)}(1,\,1') = \int d\,r_3 \int d\,t_3\, G^{(0)}(1,\,3)\, f(3,\,1') = \pm i \int d\,r_3 \int d\,t_3 \int d\,r_2\, G^{(0)}(1,\,3)\, V(\,|\,r_3 - r_2\,|)\, G^{(0)}(3,\,1')\, G^{(0)}(2,\,2^+)\,|_{t_3=t_2}$$
$$+ i \int d\,r_3 \int d\,t_3 \int d\,r_2\, G^{(0)}(1,\,3)\, V(\,|\,r_3 - r_2\,|)\, G^{(0)}(3,\,2^+)\, G^{(0)}(2,\,1')\,|_{t_3=t_2} \tag{8.97}$$

For $G(1,\,1')$

$$G(1,\,1') = G^{(0)}(1,\,1') \pm i \int d\,r_3 \int d\,t_3 \int d\,r_2\, G^{(0)}(1,\,3)\, V(\,|\,r_3 - r_2\,|)\, \delta(t_3 - t_2)\, G^{(0)}(3,\,1')\, G^{(0)}(2,\,2^+) +$$
$$i \int d\,r_3 \int d\,t_3 \int d\,r_2\, G^{(0)}(1,\,3)\, V(\,|\,r_3 - r_2\,|)\, \delta(t_3 - t_2)\, G^{(0)}(3,\,2^+)\, G^{(0)}(2,\,1')\,|_{t_3=t_2} \tag{8.98}$$

The first term is the free propagator, the second therm is known as the Hartree term and the last term is the Fock term.

8.3.6. second order:

If we want to get G to the order of $O(V^2)$, we set $G_4 = 0$, keep G_3 to $O(V^0)$ and G_2 to $O(V^1)$

$$\left(i\,\partial_{t_1} + \frac{1}{2\,m}\,\nabla_{r_1}{}^2\right) G(1,\,1') = \delta(1-1') \pm i \int d\,r_2\, V(\,|\,r_1 - r_2\,|)\, G_2(1,\,2;\,1',\,2^+)\,|_{t_1=t_2} \tag{8.99}$$

$$\left(i\,\partial_t + \frac{1}{2\,m}\,\nabla^2\right) G_2(1,\,2;\,1',\,2') = \delta(1-1')\, G(2,\,2') + \delta(1-2')\, G(2,\,1') \pm i \int d\,r_2\, V(\,|\,r_1 - r_2\,|)\, G_3(1,\,2,\,3;\,1',\,2',\,3^+)\,|_{t_1=t_3} \tag{8.100}$$

$$\left(i\,\partial_{t_1} + \frac{1}{2\,m}\,\nabla_{r_1}{}^2\right) G_3(1,\,2,\,3;\,1',\,2',\,3') = \delta(1-1')\, G_2(2,\,3;\,2',\,3') \pm \delta(1-2')\, G_2(2,\,3;\,1',\,3') + \delta(1-3')\, G_2(2,\,3;\,1',\,2') \tag{8.101}$$

The last equation give us (keep only $O(V^0)$ terms)

$$G_3(1,\,2,\,3;\,1',\,2',\,3') = G^{(0)}(1,\,1')\, G^{(0)}(2,\,2')\, G^{(0)}(3,\,3') \pm G^{(0)}(1,\,1')\, G^{(0)}(2,\,3')\, G^{(0)}(2,\,3') \pm G^{(0)}(1,\,2')\, G^{(0)}(2,\,1')\, G^{(0)}(3,\,3') +$$
$$G^{(0)}(1,\,2')\, G^{(0)}(2,\,3')\, G^{(0)}(3,\,1') + G^{(0)}(1,\,3')\, G^{(0)}(2,\,1')\, G^{(0)}(3,\,2') \pm G^{(0)}(1,\,3')\, G^{(0)}(2,\,2')\, G^{(0)}(3,\,1') \tag{8.102}$$

So we can get G_2 and then G

- Too complicated!

- Hard to compute when n is large.

There is a simple way to directly get the finally answer we want, thanks the very smart technique designed by Feynman**.**

8.3.7. Feynman diagrams and Feynman rules

- Each integration coordinate is represented by a point;
- A propagator, G(1,1'), is represented by a solid line;
- A creation operator is represented by a solid line attached to the point with an arrow from the point;
- An annihilation operator is represented by a solid line attached to the point with an arrow to the point;
- The interaction $V(1-2)$ is represented by a dashed line connecting two points (r_1 and r_2) and each ending point has a creation operator and an annihilation operator.
- Connect the lines, and keep the direction of the arrows

Find all the diagrams obey the rules described above, each of them represent a term in the power series expansion of G.

8.3.8. k,ω-space

Assuming the system have momentum and energy conservation laws. In the k,ω-space, we use the same diagrams and in addition, we also:

- Assign momentum and frequency to each line and keep the momentum and energy conservation law at each ending point or crossing points.
- For each loop, there is a pair of unknown q and $i\,\Omega_n$ and we need to integrate/sum over them.
- For each fermionic loop, we get an extra factor of (-1), which comes the commutation relation.

$$G(k,\,i\omega_n) = G^{(0)}(k,\,i\omega_n) \pm i \int d q \sum_{i\,\Omega_n} G^{(0)}(k,\,i\omega_n)\,G^{(0)}(k,\,i\omega_n)\,V(0,\,0)\,G^{(0)}(k,\,i\,\Omega_n) +$$

$$i \int d q \sum_{i\,\Omega_n} G^{(0)}(k,\,i\omega_n)\,G^{(0)}(k,\,i\omega_n)\,G^{(0)}(k+q,\,i\omega_n + i\,\Omega_n)\,V(q,\,i\,\Omega_n) + ... \tag{8.103}$$

8.4. the Dyson's equation

8.4.1. The sum of a geometric series

How do we compute the sum of a geometric series:

$$X = a + a\,q + a\,q^2 + a\,q^3 + ... \tag{8.104}$$

First, we notice that

$$q\,X = a\,q + a\,q^2 + a\,q^3 + a\,q^4 + ... \tag{8.105}$$

then, we rewrite X as

$$X = a + a\,q + a\,q^2 + a\,q^3 + ... = a + q\,X \tag{8.106}$$

So,

$$X - q\,X = a \tag{8.107}$$

$$X = \frac{a}{1-q} \tag{8.108}$$

We can use the same trick to sum many Feynman diagrams.

8.4.2. example: Hartree-Fock

Rewrite Hartree and Fork term as:

$$G(k,\,i\omega_n) = G^{(0)}(k,\,i\omega_n) + G^{(0)}(k,\,i\omega_n)\,\Sigma_{\mathrm{HF}}(k,\,i\omega_n)\,G^{(0)}(k,\,i\omega_n) \tag{8.109}$$

where Σ is called the self-energy:

$$\Sigma_{\mathrm{HF}}(k,\,i\omega_n) = \pm i \int d q \sum_{i\,\Omega_n} V(0,\,0)\,G^{(0)}(k,\,i\,\Omega_n) + i \int d q \sum_{i\,\Omega_n} G^{(0)}(k+q,\,i\omega_n + i\,\Omega_n)\,V(q,\,i\,\Omega_n) \tag{8.110}$$

Now, we can include many higher order terms (we didn't include all the terms here, but we included a lot)

$$G(k, i\omega_n) = G^{(0)}(k, i\omega_n) + G^{(0)}(k, i\omega_n) \Sigma(k, i\omega_n) G^{(0)}(k, i\omega_n) + G^{(0)}(k, i\omega_n) \Sigma(k, i\omega_n) G^{(0)}(k, i\omega_n) \Sigma(k, i\omega_n) G^{(0)}(k, i\omega_n) +$$
$$G^{(0)}(k, i\omega_n) \Sigma(k, i\omega_n) G^{(0)}(k, i\omega_n) \Sigma(k, i\omega_n) G^{(0)}(k, i\omega_n) \Sigma(k, i\omega_n) G^{(0)}(k, i\omega_n) + \ldots \tag{8.111}$$

the sum of these diagram give us

$$G(k, i\omega_n) = \frac{G^{(0)}(k, i\omega_n)}{1 - \Sigma_{\mathrm{HF}}(k, i\omega_n) G^{(0)}(k, i\omega_n)} = \frac{1}{\frac{1}{G^{(0)}(k,i\omega_n)} - \Sigma_{\mathrm{HF}}(k, i\omega_n)} = \frac{1}{i\omega_n - \epsilon(k) - \Sigma_{\mathrm{HF}}(k, i\omega_n)} \tag{8.112}$$

We found that effectively, the interactions changes the energy of a particle from $\epsilon(k)$ into $\in (k) - \Sigma_{\mathrm{HF}}(k, i\omega_n)$. This is why it is called the self-energy correction.

8.4.3. higher order diagrams and One-particle irreducible diagrams

This trick can be easily generalized to higher order diagrams and works for any diagram (just make sure that double counting are avoided). By using more complicated self-energy,

$$\Sigma(k, i\omega_n) = \pm i \int d q \sum_{i\,\Omega_n} V(0, 0) G^{(0)}(k, i\,\Omega_n) + i \int d q \sum_{i\,\Omega_n} G^{(0)}(k + q, i\omega_n + i\,\Omega_n) V(q, i\,\Omega_n) + \ldots \tag{8.113}$$

we can get more accurate $G(k, i\omega_n)$.

The rule to avoid double counting is easy. One just need to include 1-particle irreducible diagrams in the self-energy Σ

One-particle irreducible diagrams means that if we cut one internal diagram, the diagram is still connected.

8.5. Screening and the Random Phase approximation

It turns out that the Hartree-Fock approximation doesn't work well for metals (with $1/r$ e-e interactions). In fact, the free fermion theory works much better. Notice that the Hartree-Fock is the first order approximation, while the free-fermion theory is the zeroth order approximation, so this means that by using a "better" (more accurate) theory, we are getting further way from the truth. Why?

It turns out that one needs to consider more diagrams than just the first order Hartree-Fock terms to get the right answer. In fact, after the "necessary" diagrams are all included, one find that the correct theory is very similar to a free-fermion theory. We only need to adjust three parameters in the free-fermion theory to get the correct low-energy theory for interacting electrons.

8.5.1. Coulomb interaction in the momentum

Fourier transform for the Coulomb interaction

$$V(q) = \int d r \, V(r) \, e^{-i\,q \cdot r} = ? \tag{8.114}$$

$$V(r) = \frac{e^2}{r} = e \, U(r) \tag{8.115}$$

where $U(r) = \frac{e}{r}$ is the potential induced by a point charge e.

$$\nabla^2 \, U(r) = -4 \pi e \, \delta(r) \tag{8.116}$$

In q-space

$$-q^2 \, U(q) = -4 \pi e \tag{8.117}$$

$$U(q) = 4 \pi \frac{e}{q^2} \tag{8.118}$$

$$V(q) = \frac{4 \pi e^2}{q^2} \tag{8.119}$$

$V(q)$ is a singular function at $q \to 0$.

8.5.2. Permittivity and the Dyson's equation for interaction $V(q, i\,\omega_n)$

Notice that the interaction is also a line in Feynman diagrams, similar to G. So we can also treat V as a propagator similar as G.

It is worthwhile to emphasize here that this is just another way to group and organize the diagrams. We are not adding any extra diagrams. Neither did we change any physics here.

To avoid confusion, let's call the original interaction $V^{(0)}(q, i\omega_n) = 4\pi e^2/q^2$.

$$V(q, i\omega_n) = V^{(0)}(q, i\omega_n) \pm V^{(0)}(q, i\omega_n) \Pi^{(0)}(q, i\omega_n) V^{(0)}(q, i\omega_n) + \ldots \tag{8.120}$$

Here, the leading order correction is $\Pi^{(0)}$ is called a bubble. This name comes from the fact that the Feynman diagram for $\Pi^{(0)}$ looks like a bubble

$$\Pi^{(0)}(q, i\omega_n) = \int dk \sum_{i\Omega_n} G^{(0)}(k, i\Omega_n) G^{(0)}(k + q, i\Omega_n + i\omega_n) \tag{8.121}$$

The Dyson's equation

$$V(q, i\omega_n) = \frac{1}{V^{(0)}(q, i\omega_n)^{-1} \mp \Pi(q, i\omega_n)} \tag{8.122}$$

Physical meaning of this Dyson's equation:

$$V(q, i\omega_n) = \frac{V^{(0)}(q, i\omega_n)}{1 \mp \Pi(q, i\omega_n) V^{(0)}(q, i\omega_n)} = \frac{V^{(0)}(q, i\omega_n)}{\epsilon(q, i\omega_n)} \tag{8.123}$$

Inside a material, the e-e interaction is different from the Coulomb's interaction in vacuum. This is described by the "permittivity" ϵ. The vacuum permittivity is ϵ_0, but in a material it is different. This Dyson's compute the permittivity.

In insulators, ϵ is a number in the limit $q \to 0$ and $\omega \to 0$. But in a metal,

$$\text{Lim}_{q\to 0, \omega\to 0} \epsilon(q, \omega) = \infty \tag{8.124}$$

So the permittivity changes the interaction dramatically.

8.5.3. Screening

Q: What is the E field in a perfect metal? (consider only the static E field)

A: $E = 0$

Ohm's law

$$j = \sigma E \tag{8.125}$$

For a perfect metal, $\sigma \to \infty$, so if we want to keep j finite, $E = 0$.

Another way to see this is to notice that electrons in a perfect metal can move freely. If we put a test charge inside a perfect metal, the electrons will redistribute and "screen" the charge perfectly.

So the Coulomb's law in a metal is:

$$V(q, \omega = 0) = 0 = \frac{V^{(0)}(q, \omega = 0)}{\epsilon(q, \omega = 0)} \tag{8.126}$$

So, $\epsilon(q, \omega = 0) = \infty$

In a real metal, the "screening" is not prefect, so $\epsilon(q, \omega = 0)$ is not always ∞, but $\epsilon (q = 0, \omega = 0) = \infty$. The potential $V(q, \omega = 0)$ is not always 0, but $V(r, \omega = 0)$ decays exponentially as a function of r (instead of $1/r$). In other words, in a metal the e-e interactions are short-range interactions (not $1/r$).

8.5.4. The random phase approximation

The self-energy

$$\Pi(q, i\omega_n) = \Pi^{(0)}(q, i\omega_n) + O(V^1) \tag{8.127}$$

If we only keep the leading order term $\Pi^{(0)}(q, i\omega_n)$, this is known as the RPA approximation

$$V^{\text{RPA}}(q, i\omega_n) = \frac{V^{(0)}(q, i\omega_n)}{\epsilon^{\text{RPA}}(q, i\omega_n)} \tag{8.128}$$

where

$$\epsilon^{RPA}(q, i\,\omega_n) = 1 + \Pi^{(0)}(q, i\,\omega_n)\,V^{(0)}(q, i\,\omega_n) \tag{8.129}$$

Then, we use V^{RPA} to substitute V and compute the Hartree-Fock self-energy Σ^{RPA}.

At $T = 0$,

$$\Pi^{(0)}(q, i\,\omega_n) = \text{constant} + O(q, i\,\omega_n) \tag{8.130}$$

And this constant is proportional to the density of state N(0).

For insulators, the DOS=0

$$\epsilon^{RPA}(q = 0, z = 0) = 1 \tag{8.131}$$

(Here, we are using a simplified model. We didn't consider the contributions from lattice sites etc. If those effect are included, $\epsilon(q = 0, \omega = 0)$ shall not be 1, but it is also a constant)

For metals, the DOS>0

$$\epsilon^{RPA}(q, z = 0) = 1 + \frac{\text{constant}}{q^2}\,4\,\pi\,e^2 \tag{8.132}$$

$$\text{Lim}_{q\to 0}\,\epsilon^{RPA}(q, z = 0) = \infty \tag{8.133}$$

$$V^{RPA}(q, i\,\omega_n = 0) = \frac{4\,\pi\,e^2}{q^2 + \text{constant}\,4\,\pi\,e^2} \tag{8.134}$$

Transfer back to the real space

$$V^{RPA}(r, i\,\omega_n = 0) = A\,\exp(-r/r_0) \tag{8.135}$$

The interaction decays exponentially as r increases.

For a Fermi liquid, the RPA shows pretty good agreement with experiments and is the minimum approximation to describe an interacting Fermi liquid.

$$G(k, i\omega_n) = \frac{1}{i\omega_n - \epsilon(k) - \Sigma^{RPA}(k, i\omega_n)} \tag{8.136}$$

$$G(k, z) = \frac{1}{z - \epsilon(k) - \Sigma^{RPA}(k, z)} \tag{8.137}$$

For a Fermi liquid at T=0 K, we focus on $k \sim k_F$ and z~0.

Near k_F, $\epsilon(k) = \mu + v(k - k_F) + O(k - k_F)^2$.

$$G^{(0)}(k_F, z) = \frac{1}{z - v(k - k_F) - \mu + O(k - k_F)^2} \tag{8.138}$$

Using this $G^{(0)}$, we can compute Σ^{RPA}

For $\Sigma^{RPA}(k, z)$, we expand it around $k \sim k_F$ and z~0.

$$G(k_F, z) = \frac{1}{z - v(k - k_F) - \mu - \Sigma^{RPA}(k_F, 0) - a(k - k_F) + b\,z + O(k - k_F)^2 + O(z)^2} \tag{8.139}$$

It is important to notice that b is real and positive (z^2 term has a complex coefficient).

$$G^{RPA}(k_F, z) = \frac{Z}{z - \tilde{v}(k - k_F) - \tilde{\mu}} + O(k - k_F)^2 + O(z)^2 \tag{8.140}$$

here $Z = 1/(1 + b) < 1$. $\tilde{v} = \frac{v+a}{1+b}$ and $\tilde{\mu} = \frac{\mu + \Sigma^{RPA}(k_F, 0)}{1+b}$.

Notice that G^{RPA} (k$_F$, z) looks just like the Green's function for a free fermion, with a renoramlized Fermi velocity and a renormalized chemical potential. This is why we can treat electrons as "free" particles in a Fermi liquid, although the interaction is pretty strong.

In addition, we have an extra factor Z, which is known as the Fermi residue

$$G^{\text{RPA}}(k_F, z) = Z \, G^{\text{Free}}(k_F, z) \tag{8.141}$$

This extra factor is $0 < Z < 1$. The physical meaning is that now at k_F the jump in occupation number $n(k_F - \delta) - n(k_F + \delta) = Z$ not 1.