

## 6

# Superconductivity

## 6.1. Phenomena

For some metal, if one reduce the temperature, the resistivity suddenly drops to zero at some critical temperature  $T_C$ .

- Zero resistivity
- A second order phase transition at  $T_c$  (discontinuity in the specific heat)
- Meissner effect ( $B=0$  in a superconductor).
- Magnetic field Suppresses  $T_C$ . If  $B$  is larger than some critical value  $B_C$ , the superconducting phase disappears.

## 6.2. Superfluid and Bose–Einstein condensate

### Phenomena:

In liquid helium-4 (bosons), the viscosity (friction) vanishes below certain temperature (2.17 K).

Experiment 1: Connecting two containers with a tube and put some He4 liquid into both containers. Due to the viscosity and frictions, if we want to push the liquid He from one container to the other, we need to apply a pressure and the pressure becomes bigger if we reduce diameter of the tube. However, below the critical temperature, the fluid can flow from one side to the other without applying a pressure. This fluid with zero viscosity are known as a superfluid, which has no friction/viscosity.

Experiment 2: Put a torsion pendulum into liquid He, at high temperature, because of the viscosity, the amplitude of the rotation decreases with time. At low temperature (below  $T_c$ ), the friction and viscosity remain none zero.

Puzzle: if we measure the viscosity using experiment #1, we find it is zero in the superfluid phase. However, experiment #2 tells us that the viscosity is none-zero. Contradiction?

### Two fluid picture:

Below  $T_c$ , the He4 liquid is a mixture of two types of fluids: some fraction of the He atoms forms a superfluid while other He4 atoms form a normal fluid. The density for the superfluid part is called  $\rho_S$  and the density for the normal component is called  $\rho_N$ . The density of the He liquid is the sum of them  $\rho = \rho_S + \rho_N$ . The superfluid part has no viscosity but the normal fluid has viscosity. Above  $T_c$ ,  $\rho_S = 0$  and  $\rho = \rho_N$ . Below  $T_c$ ,  $\rho_S$  increases from 0 as temperature is reduced.

In experiment #1, when we push the fluids, the superfluid component can flow between the two containers without any viscosity, so we always get some flow even if the pressure is very small (so we get viscosity=0). The normal fluid will not flow when the pressure is weak and don't contribute to the flow. In other words, we not see the normal fluid in this experiment (this experiment is blind to the normal component and can only see the superfluid part).

In experiment #2, both the normal and superfluid components touch the pendulum. The normal fluid part induces damping. The superfluid has no friction, but the pendulum cannot see it. The pendulum only sees the normal component, which gives it damping. So we find nonzero viscosity.

### Bose-Einstein condensate

For simplicity, let's ignore interactions and consider a free boson gas (in helium-4 the interactions are fairly strong and it is not a good approximation to ignore interactions. However, as far as superfluidity is concerned, we can get the same phenomenon even if we ignore the interaction). The occupation number for the quantum state with momentum  $p$  is:

$$n(p) = \frac{1}{\exp[(\epsilon_p - \mu)/k_B T] - 1} \quad (6.1)$$

Here  $\epsilon_p = p^2/2m$ . Because the occupation number is nonnegative  $n(p) \geq 0$ ,  $\mu < \epsilon_p$  for any  $p$ . For  $\epsilon_p = p^2/2m$ , this means that  $\mu \leq 0$ .

Total number of particles:

$$N = \sum_k \frac{1}{\exp[(\epsilon_k - \mu)/k_B T] - 1} = \frac{V}{(2\pi\hbar)^d} \int d^d p \frac{1}{\exp[(\epsilon_p - \mu)/k_B T] - 1} = \frac{V}{(2\pi\hbar)^3} 4\pi \int p^2 dp \frac{1}{\exp[(\epsilon_p - \mu)/k_B T] - 1} \quad (6.2)$$

The density of particles

$$\rho = \frac{N}{V} = \frac{1}{2\pi^2\hbar^3} \int p^2 dp \frac{1}{\exp[(\frac{p^2}{2m} - \mu)/k_B T] - 1} = \frac{(2mk_B T)^{3/2}}{2\pi^2\hbar^3} \int x^2 dx \frac{1}{\exp(x^2 - \mu/k_B T) - 1} \quad (6.3)$$

where  $x = \sqrt{p/2mk_B T}$ .

The density  $\rho$  is a function of  $\mu$  and  $T$ :  $\rho = \rho(\mu, T)$ . If we plot  $n$  as a function of  $\mu$  and  $T$ , we get the following figures. The l.h.s. shows a 3D plot of  $\rho(\mu, T)$ . The r.h.s. is the same function shown in a contour plot.

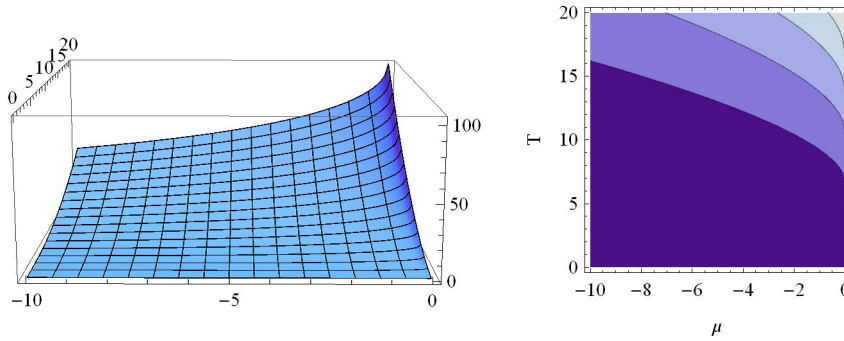


Fig. 1. Density as a function of  $\mu$  and  $T$  (in arbitrary units)

From the contour plot, we can see that if we fix the density, for each temperature, we can find the corresponding chemical potential. However, the constant  $n$  curve stops at certain critical temperature  $T_c$ , at which the chemical potential reaches zero (remember that  $\mu$  cannot be positive). This means that below  $T_c$ , we cannot find a value of  $\mu$  to keep the density fixed. The value of  $T_c$  is determined by the density. At  $T_c$ ,  $\mu=0$  so we have:

$$\rho = \frac{(2mk_B T_c)^{3/2}}{2\pi^2\hbar^3} \int_0^\infty x^2 dx \frac{1}{\exp(x^2) - 1} = \frac{(2mk_B T_c)^{3/2}}{2\pi^2\hbar^3} \frac{1}{4} \sqrt{\pi} \zeta_{\frac{3}{2}} = \left(m k_B T_c / 2\pi\hbar^2\right)^{3/2} \zeta_{\frac{3}{2}} \quad (6.4)$$

Here  $\zeta_{3/2} = 2.61238$ , with  $\zeta_s = \sum_{n=1}^\infty n^{-s}$

$$T_c = \frac{2\pi\hbar^2}{m k_B} (\rho / \zeta_{3/2})^{2/3} \quad (6.5)$$

Q: What happens below  $T_c$ ?

A: Bose-Einstein condensate

It turns out that the equation we have above is not entirely correct, because we used the integral  $\int d^d p$ . For a quantum system (with finite size), we know that all quantum states are discretized, so we should use sum instead of integral. When the system size becomes larger and larger, the separation between different states becomes smaller and smaller, so that we can turn the sum into an integral. However, in order to turn a sum into an integral, we need to assume that the function is smooth enough and has no “spikes”, but this condition is violated at and below  $T_c$ .

At  $T_c$ , we have  $\mu=0$ , so the occupation number is:

$$n(p) = \frac{1}{\exp\left[\frac{p^2}{2mk_B T}\right] - 1} \quad (6.6)$$

This function has a singularity at  $p = 0$ .  $n(p = 0) = 1/0 = \infty$ . This means that there are large number of particles in the state  $p = 0$  (the ground state). However, when we compute the total number of particles, we use integral to substitute the sum.

$$N = \sum_k \frac{1}{\exp[(\epsilon_k - \mu)/T] - 1} = \frac{V}{(2\pi\hbar)^3} 4\pi \int p^2 dp \frac{1}{\exp\left[\left(\frac{p^2}{2m} - \mu\right)/T\right] - 1} \quad (6.7)$$

In this integral, the particles on the ground state  $p = 0$  are not taken into consideration, because there is a  $p^2$  factor in the integral. At  $p = 0$ , this prefactor  $p^2$  is zero. Therefore, using an integral is a bad approximation here, because it ignores the contribution from the ground state, which is the most important state below  $T_c$  and contains a very large number of particle (As will be shown below, above  $T_c$  it is safe to use the integral formula).

So here, we should add the ground state contribution back into the total particle number

$$N = N_0 + \frac{4\pi V}{(2\pi\hbar)^3} \int p^2 dp \frac{1}{\exp\left[\left(\frac{p^2}{2m} - \mu\right)/T\right] - 1} \quad (6.8)$$

where  $N_0$  is the particle number in the ground state, and the integral give us the particle number on all excited states. Similarly, for density, we have

$$\rho = \rho_0 + \frac{4\pi}{(2\pi\hbar)^3} \int p^2 dp \frac{1}{\exp\left[\left(\frac{p^2}{2m} - \mu\right)/T\right] - 1} \quad (6.9)$$

where  $\rho_0$  is the density of the particles on the ground states and the integral gives the density for all particles on the excited states. Above  $T_c$ ,

$$\rho_0 = \frac{n(p=0)}{V} = \frac{1}{V} \frac{1}{\exp(-\mu/T) - 1} \quad (6.10)$$

In the thermodynamic limit ( $V \rightarrow \infty$ ),  $\rho_0 = 0$ . So we get our old integral formula back:

$$\rho = \frac{4\pi}{(2\pi\hbar)^3} \int p^2 dp \frac{1}{\exp\left[\left(\frac{p^2}{2m} - \mu\right)/T\right] - 1} \quad (6.11)$$

However, below  $T_c$ ,  $\rho_0$  cannot be ignored because:

$$\rho_0 = \frac{n(p=0, \mu=0)}{V} = \frac{\infty}{\infty} = ? \quad (6.12)$$

How can we determine  $\rho_0$ ? We know  $\rho = \rho_0 + \rho_{\text{excited}}$ , so

$$\rho_0 = \rho - \rho_{\text{excited}} = \rho - \frac{4\pi}{(2\pi\hbar)^3} \int p^2 dp \frac{1}{\exp\left(\frac{p^2}{2m}/T\right) - 1} = \rho - (mk_B T / 2\pi\hbar^2)^{3/2} \zeta_{\frac{3}{2}} = \left(\frac{mk_B}{2\pi\hbar^2}\right)^{3/2} \zeta_{\frac{3}{2}}(T_c^{3/2} - T^{3/2}) \quad (6.13)$$

We can plot  $\rho_0$  as a function of  $T$ .

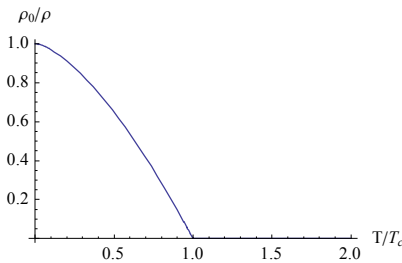


Fig. 2.  $\rho_0$  as a function of  $T$ .

### Bose-Einstein condensate and two fluid picture

Here we separate particles into two groups: particles in the ground state and particles in the excited states, and they are treated separately. This is closely related with the two-fluid picture in superfluid liquid He.  $\rho_0$  is the superfluid part and  $\rho_{\text{excited}}$  is the normal part.

- For free bosons, BEC is a third order transition (the third order derivative of the free energy shows some discontinuity). For interacting bosons, the transition turns into a second order one.
- For free bosons, the normal component vanishes at  $T = 0$ . For interacting particles, there will always be some normal component left even at  $T = 0$ .

### ***U(1) Phase symmetry breaking***

$\rho_0 \neq 0$  is the necessary condition for superfluidity but not sufficient. We know that any phase transition need to be associated with a symmetry breaking (unless it is a topological transition). So which symmetry is broken in the superfluid transition?

To answer this question, let's consider a theory for the "normal" component. Here, we treat the particles in the ground state as the "background" (or say as our "vacuum"). Because we are not considering particle in the ground state, we can use the integral formulism safely (quantum field theory). However, there is a price one needs to pay for using this intergral formulas. Because we ignored the ground states, the particle conservation law is violated. so  $\langle b(r) \rangle \neq 0$ .

For an operator  $\hat{O}$ , its expectation value is defined as the following

- At  $T = 0$ ,  $\langle \hat{O} \rangle$  is defined as  $\langle G | \hat{O} | G \rangle$ , where  $|G\rangle$  is the ground state.
- At  $T > 0$ ,  $\langle \hat{O} \rangle$  is defined as  $\text{tr}(\hat{O} \exp(\beta \hat{H})) = \sum_n \langle \psi_n | \exp\left(\frac{\hat{H}}{k_B T}\right) \hat{O} | \psi_n \rangle$  where  $\{\psi_n\}$  is a complete basis of the Hilbert space,  $\hat{H}$  is the Hamiltonian operator. If  $|\psi_n\rangle$  are the eigenstates of  $H$  with eigenenergy  $E_n$ ,  $\langle \hat{O} \rangle$  becomes  $\sum_n \exp(-E_n / k_B T) \langle \psi_n | \hat{O} | \psi_n \rangle$ , which recovers the Canonical ensemble we learned in classical statistical mechanics.

For operator  $b(r)$ , its expectation in the normal phase must be zero, due to the particle conservation law. However, in the superfluid phase, because we ignored the ground states, which has a large number of particles,  $\langle b \rangle \neq 0$  and  $\langle b \rangle \propto \sqrt{\rho_0}$ .

How do we know that  $\langle b \rangle \propto \sqrt{\rho_0}$ ? The simplest way to show it is via dimension analysis.  $\int d^d r \langle b^\dagger(r) b(r) \rangle = N$  is the total number of particle.  $N$  is a dimensionless quantity, so  $\langle b^\dagger(r) b(r) \rangle$  must have the dimension of  $r^{-d}$  (density). So  $\langle b \rangle$  must have the dimension of the square root density. So,  $\langle b \rangle \propto \sqrt{\rho_0}$ . Therefore, we know that  $\langle b \rangle = 0$  above  $T_c$  and  $\langle b \rangle \neq 0$  below  $T_c$ . So we can use  $\langle b \rangle$  as our order parameter. Define  $\psi(r) = \langle b(r) \rangle$  to be our order parameter, we can write down an phenomenological theory (the Ginsburg-Landau free energy):

$$F(\psi, \psi^*) = \int \gamma(T) \nabla \psi^* \nabla \psi + \alpha(T) \psi^* \psi + \beta(T) (\psi^* \psi)^2 + \dots \quad (6.14)$$

For stability reasons, we assume  $\gamma > 0$  and  $\beta > 0$  and  $\alpha = \alpha_0(T - T_c)$ . similar to what we learned from our thermaldynamics class, we need to minimize the free energy. Because  $\gamma > 0$ , we'd better have  $\nabla \psi = 0$  (uniform  $\psi$ ).

$$F(\psi, \psi^*) = \int \alpha_0(T - T_c) (|\psi|^2) + \beta(T) (|\psi|^2)^2 \quad (6.15)$$

For  $T > T_c$ , the minimum of  $F$  is reached at  $\psi=0$ . For  $T < T_c$ , the minimum of  $F$  has  $|\psi| = \sqrt{\alpha_0(T_c - T)/2\beta}$ . However, the phase of  $\psi$  can take any value. The phase is choosen "randomly". This can be seen by plotting  $F$  as a function of the real and imaginary parts of  $\psi$ , we get the famous Mexican hat free energy:

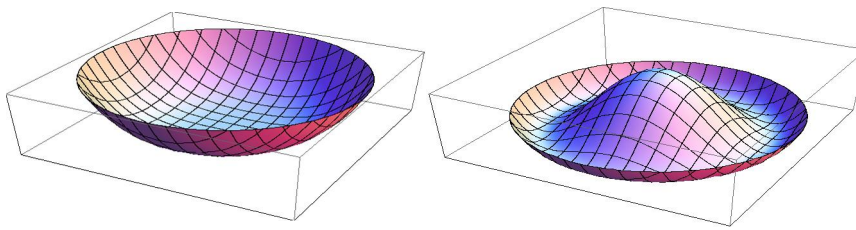


Fig. 3.  $F$  as a function of  $\text{Re}(\psi)$  and  $\text{Im}(\psi)$

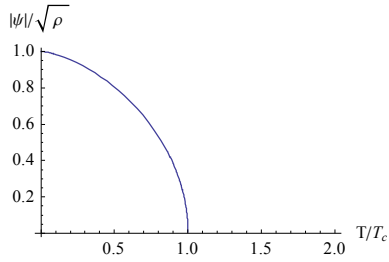


Fig. 4.  $|\psi|$  as a function of  $T$ .

For  $T > T_c$ ,  $F$  has only one minimum with  $\psi=0$ . So we only have one ground state. However at  $T < T_c$ , the  $\psi=0$  state becomes unstable and there are infinite number of minimum of  $F$  on the ring of  $|\psi| = \sqrt{\alpha_0(T_c - T)/2\beta}$ . This means that the system have infinite number of degenerate ground state. Which ground state the system falls into is determined by chance and they all have the same opportunity.

Here the Hamiltonian has the U(1) phase symmetry.  $b \rightarrow b e^{i\phi}$ . However, the ground state doesn't have this U(1) symmetry. ( $\langle b \rangle$  is not invariant under  $b \rightarrow b e^{i\phi}$ , because  $\langle b \rangle \rightarrow \langle b \rangle e^{i\phi}$ ). So the low temperature superfluid phase breaks the U(1) phase symmetry.