$$1 = \left|V\right| N(0) \int_0^{+\epsilon_D} d\xi \frac{1}{\sqrt{\xi^2 + |\Delta|^2}} = \left|V\right| N(0) \ln\left[\frac{\epsilon_D + \sqrt{|\Delta_0|^2 + \epsilon_D{}^2}}{|\Delta_0|}\right] \tag{6.61}$$

In the weak coupling limit [$V\,N(0) << 1$], we have $|\Delta_0| << \epsilon_D$

$$1 = \left|V\right| N(0) \ln\left[\frac{\epsilon_D + \sqrt{|\Delta_0|^2 + \epsilon_D{}^2}}{|\Delta_0|}\right] \approx \left|V\right| N(0) \ln \frac{2\,\epsilon_D}{|\Delta_0|} \tag{6.62}$$

$$\Delta_0 = 2\,\epsilon_D \exp\left(-\frac{1}{N(0)\,|V|}\right) \tag{6.63}$$

In particular, the ratio between $\Delta_0$ and $T_c$ is a universal constant:

$$\frac{\Delta_0}{k_B\,T} = \frac{2}{1.13} = 1.764 \tag{6.64}$$

This is true for all weakly-correlated superconductors.

## 6.4. Effective guage theory

### 6.4.1. Gauge symmetry

In a superconductor, the complex order parameter $\psi(r) = \langle c(r)\,c(r)\rangle$ is nonzero. By minimizing the GL free energy, we fixed the value of $|\psi(r)|$, but the phase of $\psi(r)$ is not determined.

Q: What will happen when we perform a gauge transformation?

The gauge field:

$$A_\mu(r) \to A_\mu(r) + \partial_\mu \Lambda(r) \tag{6.65}$$

The matter field:

$$c(r) \to c(r) \exp(i\,e\,\Lambda(r)/\hbar) \tag{6.66}$$

The order parameter:

$$\psi(r) = \langle c(r)\,c(r)\rangle \to \langle c(r) \exp(i\,e\,\Lambda(r)/\hbar)\, c(r) \exp(i\,e\,\Lambda(r)/\hbar)\rangle = \psi(r) \exp(2\,i\,e\,\Lambda(r)/\hbar) \tag{6.67}$$

The charge “2e” is because a cooper pair contains 2 electrons.

Derivative of $\psi(r)$

$$\partial_\mu \psi(r) \to \partial_\mu [\psi(r) \exp(2\,i\,e\,\Lambda(r)/\hbar)] = \partial_\mu \psi(r) + 2\,i\,e/\hbar\ \partial_\mu \Lambda(r)\,\psi(r) \tag{6.68}$$

Therefore, it is easy to check that $\left[\partial_\mu - 2\,i\,e\ A_\mu(r)/\hbar\right] \psi(r)$ is gauge invariant.

$$\partial_\mu \psi(r) - 2\,i\,e\ A_\mu(r)\,\psi(r)/\hbar \to \partial_\mu \psi(r) - 2\,i\,e\ A_\mu(r)\,\psi(r)/\hbar \tag{6.69}$$

This implies that if we write down the Lagrangian, L, any spatial derivative of the order parameter ($\partial_\mu \psi(r)$) must be combined with a term $-2\,i\,e\,A_\mu(r)/\hbar$. In fact, in order to keep the gauge symmetry, we need to replace any $\partial_\mu \psi$ by $\left[\partial_\mu - 2\,i\,e\ A_\mu(r)/\hbar\right] \psi(r)$. This is just the minimal coupling for charged particles we discussed before (replacing the momentum $p_\mu = -i\,\hbar\,\partial_\mu$ by $p_\mu + q\,A = -i\,\hbar\,\partial_\mu + q\,A_\mu$)

If we only focus on the phase of the order parameter, $\psi(\mathrm{r}) = |\psi| \exp[i\,\phi(\mathrm{r})]$

$$\begin{aligned}\left[\partial_\mu - 2\,i\,e\ A_\mu(r)/\hbar\right] \psi(r) &= |\psi| \left[\partial_\mu - 2\,i\,e\ A_\mu(r)/\hbar\right] \exp[i\,\phi(r)] = |\psi| \exp[i\,\phi(r)] \left[i\,\partial_\mu \phi(r) - 2\,i\,e\ A_\mu(r)/\hbar\right] = \\ &\quad i\,|\psi| \exp[i\,\phi(r)] \left[\partial_\mu \phi(r) - 2\,e\ A_\mu(r)/\hbar\right] \propto \partial_\mu \phi(r) - 2\,e\ A_\mu(r)/\hbar\end{aligned} \tag{6.70}$$

### 6.4.2. the Lagrangian.

If we write down the Lagrangian for a superconductor, we know that it must contain gauge fields and the phase of the order parameter. The electrons should not appear if we only care about low enegy physics smaller than the superconducting gap (remember that supercondcutors are electrons should not appear if we only care about low enegy physics smaller than the superconducting gap (remember that supercondcutors are insualtors for Bogoliubov quasi-particles, so the fermions cannot move similar to an insulator. Similar to an insulator, we don't need to consider the motion of electrons). The amplitude of the order parameter is not important either, because their value is fixed by the GL free energy. So have can write down the theory as:

$$L = \frac{1}{2}\int d\vec{r}\left(\epsilon_0 E^2 - \frac{B^2}{\mu_0}\right) + L_S\left[\partial_\mu \phi(r) - 2\,e\;A_\mu(r)\big/\hbar\right] \tag{6.71}$$

It is important to notice that the superconductor part $L_S$ cannot depends on $\phi$ itself, because $\phi$ is not gauge invariant. However, it can be a function of $\partial_\mu \phi(r) - 2\,e\;A_\mu(r)\big/\hbar$, which is gauge invariant.

for simplicity, lets use the theorist units: 2e=ℏ=1.

$$L = \frac{1}{2}\int d\vec{r}\left(\epsilon_0 E^2 - \frac{B^2}{\mu_0}\right) + L_S\left[\partial_\mu \phi(r) - A_\mu(r)\right] \tag{6.72}$$

### 6.4.3.charge and currents

Currents and charge

$$J_i = \frac{\delta \mathrm{L}_S}{\delta \mathrm{A}_i} = -\frac{\delta \mathrm{L}_S}{\delta\,\partial_i \phi} \tag{6.73}$$

$$\rho = \frac{\delta \mathrm{L}_S}{\delta \mathrm{A}_0} = -\frac{\delta \mathrm{L}_S}{\delta\,\partial_0 \phi} \tag{6.74}$$

### 6.4.4.Hamiltonian and $\partial_\mu \phi = A_\mu$

$$H = \frac{1}{2}\int d\vec{r}\left(E^2 + B^2\right) + H_S\left[\partial_\mu \phi(r) - A_\mu(r)\right] \tag{6.75}$$

If we do a Taylor expansion,

$$H = \frac{1}{2}\int d\vec{r}\left(E^2 + B^2\right) + \frac{\kappa_0}{2}\int d\vec{r}\,[\partial_0 \phi(r) - A_0(r)]^2 + \sum_i \frac{\kappa_i}{2}\int d\vec{r}\,[\partial_i \phi(r) - A_i(r)]^2 + \text{higher order terms} \tag{6.76}$$

$\kappa_0$ and $\kappa_i$ are positive due to stability reasons.

To minimize the energy, $\partial_\mu \phi = A_\mu$. This equation is the key to understand superconductivity. It means that in a superconductor, the gauge field and the phase of the order parameters are directly connected to each other. This relation is a unique property of superconductors.

A guage field which is a total derivative is known as a "pure gauge'. It means that if the system is singly-connected, there is no E=B=0. In a singly connected space, we can define a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ with $\Lambda = -\phi$. After this gauge transformation, $A'_\mu = 0$. so E=B=0.

### 6.4.5. Meissner effect and magnetic levitation

Because $\partial_\mu \phi = A_\mu$, the magnetic field

$$B = \nabla \times A = \nabla \times \nabla \phi = 0 \tag{6.77}$$

There is no magnetic field inside a superconductor. This is the Meissner effect.

### 6.4.6. Critical field $B_c$

The Meissner effect expels the B field, which costs energy $B^2/2$ (times the size of the system). If this energy cost is larger than the energy gain to form Cooper pairs $\Delta$, the system will not want to form a superconducting state. This defines a critical field $B_C = \sqrt{2\,\Delta}$ . If B field is above this value, the system turns into a normal metal (paramagnetic)

### 6.4.7. quantization of a magnetic flux.

If we have a superconducting ring, we can integrate the gauge field around the ring, which is just the magnetic flux.

$$\oint A \cdot d\,r = \int\!\!\int B \cdot d\,S \tag{6.78}$$

We know that $\partial_\mu \phi = A_\mu$, so

$$\oint A \cdot d r = \oint \partial_\mu \phi \cdot d r = 2 n \pi \tag{6.79}$$

Here we used the fact that the order parameter is single-valued. Therefore, if we go around the ring, the phase of the order parameter can only change by 2 n $\pi$. If we put the unit back,

$$\oint A \cdot d r = \oint \partial_\mu \phi \cdot d r = 2 n \pi \frac{\hbar}{2 e} = \frac{n \pi \hbar}{e} \tag{6.80}$$

Notice that the denominator here is 2e instead of e. This means that the fundamental particles in a superconductor carries electronic charge 2e, which is a cooper pair. The quantization of the magnetic flux (to $2\pi\hbar/2e$ instead of $2\pi\ \hbar/e$) is a direct experimental evidence for the formation of Cooper pairs.

This is also one of the best techniques to measure $\hbar$

### 6.4.8. Zero resistivity

We know

$$\rho = \frac{\delta \mathrm{L}_S}{\delta \mathrm{A}_0} = -\frac{\delta \mathrm{L}_S}{\delta\, \partial_0 \phi} \tag{6.81}$$

We also know that the canonical momentum $\pi$ is defined as

$$\pi = \frac{\delta \mathrm{L}_S}{\delta\, \partial_0 \phi} \tag{6.82}$$

So, $\rho = -\pi$. Therefore, the Hamiltonian should be considered a function of -$\rho$ and $\phi$, instead of $\partial_0 \phi$ and $\phi$.

The equation of motion is:

$$\partial_0 \phi = \frac{\delta \mathrm{H}_S}{\delta \pi} = -\frac{\delta \mathrm{H}_S}{\delta \rho} \tag{6.83}$$

On the other hand, we know that

$$\frac{\delta \mathrm{H}_S}{\delta \rho(r)} = V(r) \tag{6.84}$$

where V is the voltage. This is because $E = \int \rho(r)\, V(r)\, d r$

Therefore,

$$\partial_0 \phi(r) = -V(r) \tag{6.85}$$

For a static system, the phase of the order parameter should be time-independent, $\partial_0 \phi = 0$. So we have $V(r)$=0.

This means that if we have a constant current in the system (the system is static), $V = 0$. No voltage but finite current. This is superconductivity.

### 6.4.9. The Josephson junction.

Josephson junction has two superconductors separated by a thing layer of insulator. The Lagrange of a Josephson junction $L_J$ depends on the phase difference between the two superconductors, which is also gauge invariant

$$L = L_{E \text{ and } B} + L_{\mathrm{S1}} + L_{\mathrm{S2}} + L_{\text{junction}} \tag{6.86}$$

$$L_{\text{junction}} = \mathcal{A}\, F(\Delta\phi) \tag{6.87}$$

where $\mathcal{A}$ is the area of the junction. $F(\Delta\phi)$ is a function of the phase difference between the two superconductors ($\Delta\phi = \phi_L - \phi_R$).

It is easy to notice that F($\Delta\phi$) must be a periodic function with periodicity 2 n $\pi$ $\hbar$/2 e=n $\pi$ $\hbar$/e. Again the denominator 2 e is due to the fact that Cooper pairs have charge 2e.

$$F(\Delta\phi) = F(\Delta\phi + n \pi \hbar / e) \tag{6.88}$$

In the presence of a gauge field $A$, the gauge invariance $\Delta\phi$ is

$$\Delta\phi = \int d r \cdot (\nabla \phi - A) \tag{6.89}$$

Therefore the current cross the junction is

$$J = \frac{\delta \text{L}_{\text{junction}}}{\delta \text{A}} = \mathcal{A}\, F'(\Delta\phi) \frac{\delta \Delta\phi}{\delta \text{A}} = \mathcal{A}\, F'(\Delta\phi) \frac{\delta \text{A}}{\delta \text{A}} = \mathcal{A}\, F'(\Delta\phi) \tag{6.90}$$

Now we apply a voltage $V$ across the junction. Because we know that

$$\partial_0 \phi(r) = -V(r), \tag{6.91}$$

it is easy to notice that

$$\Delta\phi = -V\, t + \text{constant} \tag{6.92}$$

Therefore:

$$J = \mathcal{A}\, F'(-V\, t + \text{constant}) \tag{6.93}$$

By applying a fixed V, we found that the current is changing with time. Because F is a periodic function, F' is also a periodic function. So J is aperiodic function of t, and the periodicity is $\pi\, \hbar/\text{e}\, \Delta\text{V}$