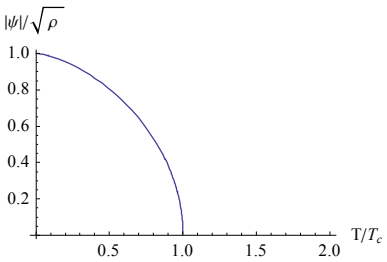

Fig. 4. $|\psi|$ as a function of T.

For $T > T_c$, F has only one minimum with ψ=0. So we only have one ground state. However at $T < T_c$, the ψ=0 state becomes unstable and there are infinite number of minimum of F on the ring of $|\psi| = \sqrt{\alpha_0(T_c - T)/2\beta}$. This means that the system have infinite number of degenreate ground state. Which ground state the system falls into is determined determined by chance and they all have the same opportunity.

Here the Hamiltonian has the U(1) phase symmetry. $b \to b\, e^{i\phi}$. However, the ground state doesn't have this U(1) symmetry. ($\langle b \rangle$ is not invariant under $b \to b\, e^{i\phi}$, because $\langle b \rangle \to \langle b \rangle\, e^{i\phi}$). So the low temperature superfluid phase breaks the U(1) phase symmetry.

6.3. Cooper Pair and the BCS theory

Superconductivity and superfluid are very similar (no resistivity vs no viscosity), which implies that the fundamental physics may be similar. But electrons are fermions, and we cannot have more than 1 electron on quantum state, so there is no way to have BEC for fermions (which requires many particles to stay on the ground state). How can fermions form a BEC state? Well, one fermion cannot condensate, but a pair of fermions is a boson and these pairs can condensate. These fermion pairs are known as Cooper pairs, and this theory of superconductivity is known as the BCS theory (for Bardeen, Cooper and Schrieffer).

Originally this idea sounds like crazy, because electrons the same charge so they have repulsive interactions. How can a bunch of particles

6.3.1. Electron phonon interactions

A rigorous treatment on electron-phonon interactions requires quantum field theory. Due to time-limitation here are just describe the physical picture without rigorous proof. Key conclusions:

- Electron-phonon interactions are crucial for conventional superconductors.
- Electron-phonon interactions induce an effective attractive interactions between electrons.

The first conclusion are based on the isotopic effect in conventional superconductors. If superconductor are purely due to electrons and lattices places no roles in it, the transition temperature Tc should be independent on the mass of the nucleons. However, experiments shows that the transition temperature T_c depends strongly on the mass of the nucleons, if we use different isotopes of a given element type of atom (e.g. Hg 203, Hg 202, Hg 200, Hg 199, Hg 198). This shows a hint that superconductor are related with the lattice motion (phonons).

The second conclusion are based on this physics picture

- An electron carries negative charge, which will attract nearby nucleons. So locally, the spacing between the nucleons will be reduced.
- Electron mass is much smaller than nucleon mass. So electrons moves much faster.
- When electron flies away, the lattice distortion will not recover immediately (because lattice moves slower). So locally, there is a higher density of positive charges here, which will attract other electrons.
- Combining all the things mentioned above, we found that an electron will attract other electrons, via lattice distortions. And this is phonon mediated attractions between electrons.

Theoretically, the attraction are induced when two electrons exchange a virtue phonon, which can shown rigorously in quantum field theory.

6.3.2. Model Hamiltonian

The key conclusion in the BCS theory is that: in a Fermi liquid, as long as there are some attractive interactions (no matter how weak the interactions are), the Fermi liquid state will become unstable below some temperature T_c.

Now, let's consider electrons with attractive interactions. The Hamiltonian is

The energy contains to parts, kinetic energy and the potential energy.

$$E = \sum_{k,\sigma} \epsilon(k)\, n_\sigma(k) + \frac{1}{2} \sum_{r,r'} \sum_{\sigma,\sigma'} V(r - r')\, n_\sigma(r)\, n_{\sigma'}(r') \tag{6.16}$$

If we translate this into the form of second quantization:

$$H = \sum_{k,\sigma} \epsilon(k)\, c_{k\sigma}{}^{\dagger}\, c_{k\sigma} + \frac{1}{2} \sum_{\sigma,\sigma'} \sum_{r,r'} V(r-r')\, c_{r\sigma}{}^{\dagger}\, c_{r\sigma}\, c_{r'\sigma'}{}^{\dagger}\, c_{r'\sigma'} \tag{6.17}$$

Transfer the interaction part into the k space.

$$H = \sum_{k,\sigma} \epsilon(k)\, c_{k,\sigma}{}^{\dagger}\, c_{k,\sigma} + \frac{1}{2} \sum_{q,k,k'} \sum_{\sigma,\sigma'} V_q\, c_{k+q/2,\sigma}{}^{\dagger}\, c_{k-q/2,\sigma}\, c_{k'-q/2,\sigma'}{}^{\dagger}\, c_{k'+q/2,\sigma'} \tag{6.18}$$

Let's simplify further the interactions:

$$H = \sum_{k,\sigma} \epsilon(k)\, c_{k,\sigma}{}^{\dagger}\, c_{k,\sigma} + \sum_{k,k'} V_{k,k'}\, c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger}\, c_{k'\downarrow}\, c_{-k'\uparrow} \tag{6.19}$$

This interaction is part of the electron interactions. Here we ignored all other type of interactions. This turns out to be a good approximation, because this interaction is the key which induces the pairing while all other interactions have very little effects.

For simplicity, we assume that $V_{k,k'} = V$.

$$H = \sum_{k,\sigma} \epsilon(k)\, c_{k,\sigma}{}^{\dagger}\, c_{k,\sigma} + V \sum_{k,k'} c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger}\, c_{k'\downarrow}\, c_{-k'\uparrow} \tag{6.20}$$

6.3.3. the Mean field approximation:

In BEC, the order parameter is $\langle b^{+} \rangle$. Here, pairs of electrons forms a BEC state, so the order parameter is $\sum_k \langle c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \rangle$. In other words, in the condensed phase, $\langle c^{\dagger}\, c^{\dagger} \rangle \neq 0$. We can write:

$$c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} = \langle c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \rangle + \left(c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - \langle c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \rangle \right) \tag{6.21}$$

$$c_{k'\downarrow}\, c_{-k'\uparrow} = \langle c_{k'\downarrow}\, c_{-k'\uparrow} \rangle + \left(c_{k'\downarrow}\, c_{-k'\uparrow} - \langle c_{k'\downarrow}\, c_{-k'\uparrow} \rangle \right) \tag{6.22}$$

The physical meaning for these two formulas are: the operator $c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger}$ has average value $\langle c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \rangle$. On top of the average value, the operator fluctuates around this average value with fluctuations being $c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - \langle c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \rangle$. If we believe that the fluctuations of the operator is small, the last term in the formula above is small. Here we can define $X_{k'} = \langle c_{k'\downarrow}\, c_{-k'\uparrow} \rangle$, as a result, $X_k{}^{*} = \langle c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \rangle$. So we have

$$c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} = X_k{}^{*} + \left(c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - X_k{}^{*} \right) \tag{6.23}$$

$$c_{k'\downarrow}\, c_{-k'\uparrow} = X_{k'} + \left(c_{k'\downarrow}\, c_{-k'\uparrow} - X_{k'} \right) \tag{6.24}$$

We rewrite the interactions in terms of the average values $X_{k'}$ and $X_k{}^{*}$ and fluctuations, $\left(c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - X_k{}^{*} \right)$ and $\left(c_{k'\downarrow}\, c_{-k'\uparrow} - X_{k'} \right)$

$$\begin{aligned} &\frac{V}{2} \sum_{k} c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger}\, c_{k'\downarrow}\, c_{-k'\uparrow} = V \sum_{k,k'} \left[X_k{}^{*} + \left(c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - X_k{}^{*} \right) \right] \left[X_{k'} + \left(c_{k'\downarrow}\, c_{-k'\uparrow} - X_{k'} \right) \right] = \\ &\quad V \sum_{k,k'} \left[X_k{}^{*}\, X_{k'} + X_k{}^{*} \left(c_{k'\downarrow}\, c_{-k'\uparrow} - X_{k'} \right) + \left(c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - X_k{}^{*} \right) \right) X_{k'} + \left(c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - X_k{}^{*} \right) \left(c_{k'\downarrow}\, c_{-k'\uparrow} - X_{k'} \right). \end{aligned} \tag{6.25}$$

The last term is fluctuations times fluctuations. If fluctuations is small, the last term would be the smallest part.

Mean-field approximation: ignore the fluctuations*fluctuations part, so the Hamiltonian becomes:

$$H_{\mathrm{MF}} = \sum_{k} \left[\epsilon(k)\, c_{k,\uparrow}{}^{\dagger}\, c_{k,\uparrow} + \epsilon(k)\, c_{k,\downarrow}{}^{\dagger}\, c_{k,\downarrow} \right] + V \sum_{k} X_k{}^{*} \sum_{k'} c_{k'\downarrow}\, c_{-k'\uparrow} + V \sum_{k} X_k \sum_{k} c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} - V \sum_{kk'} X_k{}^{*}\, X_{k'} \tag{6.26}$$

Within the mean-field approximation, the mean-field Hamiltonian only have quadratic terms in c and $c^{\dagger}$ (very similar to an free Fermi system).

Define $\Delta^{*} = V \sum_k X_k{}^{*}$ and $\Delta = V \sum_k X_k$,

$$H_{\mathrm{MF}} = \sum_{k} \left[\epsilon(k)\, c_{k,\uparrow}{}^{\dagger}\, c_{k,\uparrow} + \epsilon(k)\, c_{k,\downarrow}{}^{\dagger}\, c_{k,\downarrow} + \Delta^{*}\, c_{k\downarrow}\, c_{-k\uparrow} + \Delta \sum_{k} c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \right] - \frac{\Delta^{*}\, \Delta}{V} \tag{6.27}$$

If we work with fixed chemical potential, μ, it make s more sense to consider $H - \mu\, N$, instead of H

$$H_{\mathrm{MF}} - \mu\, N = \sum_{k} \left\{ [\epsilon(k) - \mu]\, c_{k,\uparrow}{}^{\dagger}\, c_{k,\uparrow} + [\epsilon(k) - \mu]\, c_{k,\downarrow}{}^{\dagger}\, c_{k,\downarrow} + \Delta^{*}\, c_{k\downarrow}\, c_{-k\uparrow} + \Delta\, c_{k\uparrow}{}^{\dagger}\, c_{-k\downarrow}{}^{\dagger} \right\} - \frac{\Delta^{*}\, \Delta}{V} \tag{6.28}$$

We can write it in terms of a two-by-two matrix

$$H_{\text{MF}} - \mu N = \sum_k \begin{pmatrix} c_{k,\uparrow}^\dagger & c_{-k,\downarrow} \end{pmatrix} \begin{pmatrix} \epsilon(k) - \mu & \Delta \\ \Delta^* & \mu - \epsilon(k) \end{pmatrix} \begin{pmatrix} c_{k,\uparrow} \\ c_{-k,\downarrow}^\dagger \end{pmatrix} - \frac{\Delta^* \Delta}{V} \tag{6.29}$$

This is very similar to the Hamiltonian of a 2-band systems.

6.3.4. Bogoliubov transformation

Similar as in a 2-band systems, we should diagonalize the matrix using a unitary transformation, and here, this transformation is known as the Bogoliubov transformation. Define:

$$\gamma_{1k} = u\, c_{k,\uparrow} + v\, c_{-k,\downarrow}^\dagger \tag{6.30}$$

$$\gamma_{2k} = -v^*\, c_{k,\uparrow} + u^*\, c_{-k,\downarrow}^\dagger \tag{6.31}$$

where u and v are complex numbers with $|u|^2 + |v|^2 = 1$. It is easy to check that the conjugate operators are

$$\gamma_{1k}^\dagger = u^*\, c_{k,\uparrow}^\dagger + v^*\, c_{-k,\downarrow} \tag{6.32}$$

$$\gamma_{2k}^\dagger = -v\, c_{k,\uparrow}^\dagger + u\, c_{-k,\downarrow} \tag{6.33}$$

The inverse transformation is:

$$c_{k,\uparrow} = u^*\, \gamma_{1k} - v\, \gamma_{2k} \tag{6.34}$$

$$c_{-k,\downarrow}^\dagger = v^*\, \gamma_{1k} + u\, \gamma_{2k} \tag{6.35}$$

and

$$c_{k,\uparrow}^\dagger = u\, \gamma_{1k}^\dagger - v^*\, \gamma_{2k}^\dagger \tag{6.36}$$

$$c_{-k,\downarrow} = v\, \gamma_{1k}^\dagger + u^*\, \gamma_{2k}^\dagger \tag{6.37}$$

It is also easy to check that the new operators γ_{1k}, γ_{2k}, $\gamma_{1k}^\dagger$ and $\gamma_{2k}^\dagger$ satisfies the anti-commutation relations (so they are fermions). Here, I check on anti-commutation relation

$$\begin{aligned} &\left\{\gamma_{1k}^\dagger,\, \gamma_{1k}\right\} = \\ &\quad \left\{u\, c_{k,\uparrow}^\dagger + v\, c_{-k,\downarrow},\, u\, c_{k,\uparrow} + v\, c_{-k,\downarrow}^\dagger\right\} = u^2 \left\{c_{k,\uparrow}^\dagger,\, c_{k,\uparrow}\right\} + v^2 \left\{c_{-k,\downarrow},\, c_{-k,\downarrow}^\dagger\right\} + u\, v \left\{c_{k,\uparrow}^\dagger,\, c_{-k,\downarrow}^\dagger\right\} + u\, v \left\{c_{-k,\downarrow},\, c_{-k,\downarrow}\right\} = u^2 + v^2 = 1. \end{aligned} \tag{6.38}$$

Therefore, these operators creates and annihilates fermions, and these fermions are known as Bogoliubov quasi-particles.

$$H_{\text{MF}} - \mu N = \sum_k \begin{pmatrix} \gamma_{1k}^\dagger & \gamma_{2k}^\dagger \end{pmatrix} \cdot \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \cdot \begin{pmatrix} \epsilon(k) - \mu & \Delta \\ \Delta^* & \mu - \epsilon(k) \end{pmatrix} \cdot \begin{pmatrix} u^* & -v \\ v^* & u \end{pmatrix} \begin{pmatrix} \gamma_{1k} \\ \gamma_{2k} \end{pmatrix} - \frac{\Delta^* \Delta}{V} \tag{6.39}$$

$$\begin{aligned} &\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \cdot \begin{pmatrix} \epsilon(k) - \mu & \Delta \\ \Delta^* & \mu - \epsilon(k) \end{pmatrix} \cdot \begin{pmatrix} u^* & -v \\ v^* & u \end{pmatrix} = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \cdot \begin{pmatrix} (\epsilon(k) - \mu)\, u^* + \Delta\, v^* & -(\epsilon(k) - \mu)\, v + \Delta\, u \\ \Delta^*\, u^* + (\mu - \epsilon(k))\, v^* & -\Delta^*\, v + (\mu - \epsilon(k))\, u \end{pmatrix} = \\ &\quad \begin{pmatrix} (\epsilon(k) - \mu)\, u^* u + \Delta\, v^* u + \Delta^*\, u^* v + (\mu - \epsilon(k))\, v^* v & -2\,(\epsilon(k) - \mu)\, v\, u + \Delta\, u^2 - \Delta^*\, v^2 \\ -2\,(\epsilon(k) - \mu)\, u^*\, v^* - \Delta\, v^{*2} + \Delta^*\, u^{*2} & (\epsilon(k) - \mu)\, v\, v^* - \Delta\, u\, v^* - \Delta^*\, u^*\, v + (\mu - \epsilon(k))\, u^*\, u \end{pmatrix} \end{aligned} \tag{6.40}$$

Since we want to diagonalize the matrix, $-2\,(\epsilon(k) - \mu)\, v\, u + \Delta\, u^2 - \Delta^*\, v^2 = 0$

$$2\,(\epsilon(k) - \mu)\, v\, u = \Delta\, u^2 - \Delta^*\, v^2 \tag{6.41}$$

Define:

$$u = e^{i\,\phi_u} \cos\chi \quad \text{and} \quad v = e^{i\,\phi_v} \sin\chi \quad \text{and} \quad \Delta = |\Delta|\, e^{i\,\phi} \tag{6.42}$$

$$2\,(\epsilon(k) - \mu) \cos\chi \sin\chi\, e^{i\,\phi_u + i\,\phi_v} = e^{2\,i\,\phi_u + i\,\phi}\, \Delta\, \cos^2\chi - e^{2\,i\,\phi_v - i\,\phi}\, \Delta^* \sin^2\chi \tag{6.43}$$

So we have

$$\phi_u + \phi_v = 2\,\phi_u + \phi = 2\,\phi_v - \phi \tag{6.44}$$

and

$$2\,(\epsilon(k)-\mu)\cos\chi\sin\chi = |\Delta|\left(\cos^2\chi-\sin^2\chi\right) \tag{6.45}$$

For the phases, we get $\phi_u-\phi_v=-\phi$, while $\phi_u+\phi_v$ can take any value. Without loss of generality, we can choose $\phi_u=-\phi_v=-\phi/2$. For χ we have,

$$(\epsilon(k)-\mu)\sin 2\chi = |\Delta|\cos 2\chi \tag{6.46}$$

So $\tan 2\chi=\frac{|\Delta|}{\epsilon(k)-\mu}$.

In conclusion, if we choose $u=e^{-i\phi/2}\cos\chi$ and $v=e^{i\phi/2}\sin\chi$ with $\chi=\frac{1}{2}\arctan\frac{|\Delta|}{\epsilon(k)-\mu}$, the two-by-two matrix turns into a diagonal matrix:

$$\begin{pmatrix}(\epsilon(k)-\mu)\cos 2\chi+\Delta\sin 2\chi & 0\\ 0 & -(\epsilon(k)-\mu)\cos 2\chi-\Delta\sin 2\chi\end{pmatrix}=\begin{pmatrix}\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2} & 0\\ 0 & -\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}\end{pmatrix} \tag{6.47}$$

$$H_{\rm MF}-\mu N=\sum_k\begin{pmatrix}\gamma_{1k}^\dagger & \gamma_{2k}^\dagger\end{pmatrix}\cdot\begin{pmatrix}E_k & 0\\ 0 & -E_k\end{pmatrix}\begin{pmatrix}\gamma_{1k}\\ \gamma_{2k}\end{pmatrix}-\frac{\Delta^*\Delta}{V} \tag{6.48}$$

with $E_k=\sqrt{(\epsilon_k-\mu)^2+(|\Delta|)^2}$.

For the top band, we have energy $E_k\geq|\Delta|$. for the bottom band, the dispersion relation $-E_k\leq-|\Delta|$. So we have energy gap 2|Δ|, if |Δ|>0.

A superconductor is an insulator of Bogoliubov quasi-particles!

6.3.5. Δ=?

The value of Δ is determined by self-consistent condition.

$$\begin{aligned}\Delta &= V\sum_k\langle c_{-k\downarrow}c_{k\uparrow}\rangle=V\sum_k\left\langle\left(v\,\gamma_{1k}^\dagger+u^*\gamma_{2k}^\dagger\right)\left(u^*\gamma_{1k}-v\,\gamma_{2k}\right)\right\rangle=\\ &V\sum_k\left[u^*v\left\langle\gamma_{1k}^\dagger\gamma_{1k}\right\rangle-u^*v\left\langle\gamma_{2k}^\dagger\gamma_{2k}\right\rangle\right]=V\sum_k\left[u^*v\,\frac{1}{\exp\left[\frac{E_k}{k_BT}\right]+1}-u^*v\,\frac{1}{\exp\left[\frac{-E_k}{k_BT}\right]+1}\right]\end{aligned} \tag{6.49}$$

$$|\Delta|e^{i\phi}=Ve^{i\phi}\sum_k\frac{\sin 2\chi}{2}\left[\frac{1}{\exp\left[\frac{E_k}{k_BT}\right]+1}-\frac{1}{\exp\left[\frac{-E_k}{k_BT}\right]+1}\right] \tag{6.50}$$

$$\begin{aligned}|\Delta| &= V\sum_k\frac{\sin 2\chi}{2}\left[\frac{\exp\left[-\frac{E_k}{2k_BT}\right]}{\exp\left[\frac{E_k}{2k_BT}\right]+\exp\left[-\frac{E_k}{2k_BT}\right]}-\frac{\exp\left[\frac{E_k}{2k_BT}\right]}{\exp\left[-\frac{E_k}{2k_BT}\right]+\exp\left[\frac{E_k}{2k_BT}\right]}\right]=\\ &V\sum_k\frac{\sin 2\chi}{2}\,\frac{\exp\left[-\frac{E_k}{2k_BT}\right]-\exp\left[\frac{E_k}{2k_BT}\right]}{\exp\left[\frac{E_k}{2k_BT}\right]+\exp\left[-\frac{E_k}{2k_BT}\right]}=-V\sum_k\frac{|\Delta|}{2\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}}\tanh\left(\frac{E_k}{2k_BT}\right)\end{aligned} \tag{6.51}$$

$$|\Delta|=-V\sum_k\frac{|\Delta|}{2\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}}\tanh\left(\frac{E_k}{2k_BT}\right) \tag{6.52}$$

|Δ|=0 is always a solution for this equation. If |Δ|≠0,

$$1=-\frac{V}{2}\sum_k\frac{1}{\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}}\tanh\left(\frac{\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}}{2k_BT}\right) \tag{6.53}$$

For repulsive interactions, this equation has no solution, because the l.h.s.>0 while the r.h.s.<0. However, for V<0, we have

$$1 = \frac{|V|}{2}\sum_k \frac{1}{\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}}\tanh\left(\frac{\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}}{2k_BT}\right) \tag{6.54}$$

The equations may have some solution. In a Fermi liquid, typically only fermions near the Fermi surface will contribute and near the Fermi surface, the sum of momentum k can be written as an integer of energy.

$$1 = \frac{|V|}{2}\int N(0)\,d\epsilon\,\frac{1}{\sqrt{(\epsilon-\mu)^2+|\Delta|^2}}\tanh\left(\frac{\sqrt{(\epsilon-\mu)^2+|\Delta|^2}}{2k_BT}\right) \tag{6.55}$$

where N(0) is known as the density of the states, which can be measure directly in STM. The range of the interaction is determined by the Debye frequency, which is the typical energy scales of the phonons (remember that the attractive interaction comes from phonons).

$$1 = \frac{|V|}{2}N(0)\int_{\mu-\epsilon_D}^{\mu+\epsilon_D} d\epsilon\,\frac{\tanh\left(\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}\,\Big/\,2k_BT\right)}{\sqrt{(\epsilon(k)-\mu)^2+|\Delta|^2}} \tag{6.56}$$

If we define $\xi=\epsilon-\mu$,

$$1 = \frac{|V|}{2}N(0)\int_{-\epsilon_D}^{+\epsilon_D} d\xi\,\frac{\tanh\left(\sqrt{\xi^2+|\Delta|^2}\,\Big/\,2k_BT\right)}{\sqrt{\xi^2+|\Delta|^2}} = |V|\,N(0)\int_0^{+\epsilon_D} d\xi\,\frac{\tanh\left(\sqrt{\xi^2+|\Delta|^2}\,\Big/\,2k_BT\right)}{\sqrt{\xi^2+|\Delta|^2}} \tag{6.57}$$

For $T > T_c$, there is only one solution for the self-consistency equation, $\Delta=0$.

For $T < T_c$, we have one solution $\Delta=0$, which corresponds to the unstable solution for the G-L free energy. In addition, we also have a solution with $|\Delta|=\Delta(T)$, where $\Delta(T)$ is a function of T.

Because $|\Delta|=0$ at T_c, we know that

$$1 = |V|\,N(0)\int_0^{+\epsilon_D} d\xi\,\frac{\tanh(\xi/2k_BT_C)}{\xi} = |V|\,N(0)\int_0^{\epsilon_D/2k_BT_C} dx\,\frac{\tanh(x)}{x} \tag{6.58}$$

In the weakly coupling limit [$|V|<<1/N(0)$], $k_BT_c << \epsilon_D$, so the upper limit of the integral is very large $\epsilon_D/2k_BT_C >> 1$. For this limit,

$$1 = |V|\,N(0)\int_0^{\epsilon_D/2k_BT_C} dx\,\frac{\tanh(x)}{x} \approx N(0)\,|V|\,\ln\left(1.13\,\frac{\epsilon_D}{k_BT_C}\right) \tag{6.59}$$

So,

$$T_c = 1.13\,\frac{\epsilon_D}{k_B}\exp\left[-\frac{1}{N(0)\,|V|}\right] \tag{6.60}$$

The critical temperature is proportional to the Debye frequency. In addition, it depends on the density of states N(0) and interaction strengths |V|. As the interaction becomes weaker and weaker, T_c goes down to zero.

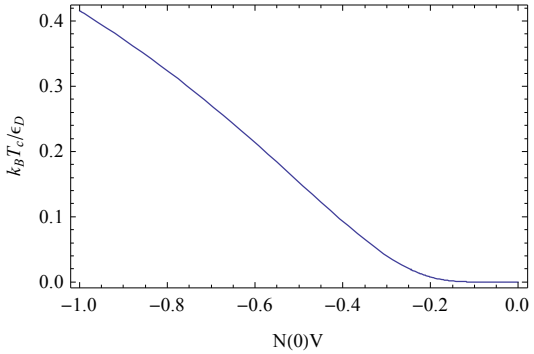

Fig. 5. T_c as a function of the interaciton strength.

Order parameter at zero temperature can be found as:

$$1 = \left| V \right| N(0) \int_0^{+\epsilon_D} d\xi \frac{1}{\sqrt{\xi^2 + |\Delta|^2}} = \left| V \right| N(0) \ln\left[\frac{\epsilon_D + \sqrt{|\Delta_0|^2 + \epsilon_D{}^2}}{|\Delta_0|} \right] \tag{6.61}$$

In the weak coupling limit [$V\, N(0) << 1$], we have $|\Delta_0| << \epsilon_D$

$$1 = \left| V \right| N(0) \ln\left[\frac{\epsilon_D + \sqrt{|\Delta_0|^2 + \epsilon_D{}^2}}{|\Delta_0|} \right] \approx \left| V \right| N(0) \ln \frac{2\,\epsilon_D}{|\Delta_0|} \tag{6.62}$$

$$\Delta_0 = 2\,\epsilon_D \exp\left(-\frac{1}{N(0)\,|V|} \right) \tag{6.63}$$

In particular, the ratio between Δ_0 and T_c is a universal constant:

$$\frac{\Delta_0}{k_B\, T} = \frac{2}{1.13} = 1.764 \tag{6.64}$$

This is true for all weakly-correlated superconductors.