7

Frustrated magnets and 1D quantum system

7.1. magnetization and frustrated magnets (in any dimensions)

Consider a lattice of spins with spin interactions. The Heisenberg model:

$$H = J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j = J \sum_{<ij>} \left(S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z\right) \tag{7.1}$$

If J<0, the spins want to point in the same direction as their neighbors (ferromagnetic). If J>0, neighboring spins want to point in the opposite directions to minimize the energy (anti-ferromagnetic).

In real materials (which may be anisotropic), the couplings in different directions may be different, which can be described by the XYZ model:

$$H = \sum_{<ij>} \left(J_x S_i^x S_j^x + J_y S_i^y S_j^y + J_z S_i^z S_j^z\right) \tag{7.2}$$

If $J_x = J_y$, we get the so called XXZ model

$$H = \sum_{<ij>} \left(J_x S_i^x S_j^x + J_x S_i^y S_j^y + J_z S_i^z S_j^z\right) \tag{7.3}$$

If $J_x = J_y$ and $J_z = 0$, we get the so called XY model

$$H = J \sum_{<ij>} \left(S_i^x S_j^x + S_i^y S_j^y\right) \tag{7.4}$$

If $J_x = J_y = 0$ and $J_z \neq 0$, we get the so called Ising model

$$H = J \sum_{<ij>} S_i^z S_j^z \tag{7.5}$$

7.1.1. Frustration

For ferromagnetic couplings, the ground state is simple: all spin pointing in the same directions (plus quantum fluctuations). For anti-ferromagnetic couplings, it depends on geometry. For example, on a square lattice, the spins can find a way to make all spins happy (all neighboring spins stay in opposite directions). But on a triangle, there is no way to make all the spin happy. So the spins are “frustrated”, and they don’t know in which direction they should be pointing. This type of systems are known as frustrated magnets.

The reason that frustrated magnets are interesting is because their ground states are not obvious. Therefore, some of these system may have very strange ground state, with exotic properties, such as spin liquids.

As T goes down, in most systems, some form of ordering will be formed (liquid turns in to solids, paramagnetism turns into ferromagnetism, etc.). Motion of particles, spins etc. are frozen at low T. However, if the quantum fluctuation is too strong, the system cannot order even at T=0 (liquid down to zero T). In spin systems, such a liquid is known as a spin liquid.

Typically, in theoretical studies, one maps spins into a strongly correlated Fermi gas (or boson gas), which is known as the slave fermion (or slave boson) approach. For the slave fermion approach, the fermion may form a insulator or superconductor (gaped spin liquid), semi-metal like graphene (gapless spin liquid, algebra spin liquid), or a metal (Bose metal and the Fermi surfaces for this metal is known as the spinon Fermi surface).

The mapping from spin to fermions/bosons can be rigorously constructed in any dimensions. However, one needs to pay a prize; these fermions/-

bosons have strong interactions, and thus the problem cannot be solved in general. However, in 1D, there is a very straightforward mapping to fermions and the fermions may not have any interactions (for the XY model). In 1D, bosons and fermions are not fundamentally different from each other.

7.2. Spin-1/2 and hard-core bosons (in any dimensions)

This section describes a baby version of the slave boson approach. Hard-core bosons are bosons hopping on a lattice, but we assume that there can be at most one boson per site. This physics can be realized if one introduces strong (short-range) repulsive interactions between bosons, so the boson will not want to stay together.

Consider one single lattice site, the size (dimension) of the Hilbert space of hard core bosons is 2: occupied or empty. For spin-1/2, the Hilbert space has the same size: spin up and spin down. So it is possible to make a connection between spins and hard-core bosons.

We can identify the spin up state for spin-1/2 with the occupied state for a hard core boson. Spin down state with an empty state.

Define hard core boson creation and annihilation operators b_i and $b_i{}^\dagger$.

$$b^\dagger \left|0\right\rangle = \left|1\right\rangle \qquad b\left|0\right\rangle = 0 \tag{7.6}$$

$$b^\dagger \left|1\right\rangle = 0 \qquad b\left|1\right\rangle = \left|0\right\rangle \tag{7.7}$$

Define spin rising and lowering operator $S_i{}^+ = S_x + i\, S_y$ and $S_i{}^- = S_x - i\, S_y$.

$$S^+ \left|\text{down}\right\rangle = \left|\text{up}\right\rangle \qquad S_i{}^- \left|\text{down}\right\rangle = 0 \tag{7.8}$$

$$S^+ \left|\text{up}\right\rangle = 0 \qquad S_i{}^- \left|\text{up}\right\rangle = \left|\text{down}\right\rangle \tag{7.9}$$

So, we find that $b^\dagger$ is S^+ and b is just S^-.

The density operator: $n = b^\dagger\, b$. It is easy to check that $n\left|1\right\rangle = 1\times\left|1\right\rangle$ and $n\left|0\right\rangle = 0\times\left|0\right\rangle$

For spin-1/2, the density operator turns into $n = S^+\, S^- = \left(S_x + i\, S_y\right)\left(S_x - i\, S_y\right) = S_z + 1/2$.

$$\begin{aligned} n &= S^+\, S^- = \left(S_x + i\, S_y\right)\left(S_x - i\, S_y\right) = \\ &S_x{}^2 + S_y{}^2 - i\left(S_x\, S_y - S_y\, S_x\right) = \left(\left|\vec{S}\right|\right)^2 - S_z{}^2 - i\left[S_x, S_y\right] = s(s+1) - S_z{}^2 + S_z = \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{4} + S_z = S_z + 1/2 \end{aligned} \tag{7.10}$$

It is easy to check that $(S_z + 1/2)\left|\text{up}\right\rangle = 1\times\left|\text{up}\right\rangle$ and $(S_z + 1/2)\left|\text{down}\right\rangle = 0\times\left|\text{down}\right\rangle$

One can further verify the commutation relation.

$$[b, b] = \left[b^\dagger, b^\dagger\right] = 0 \tag{7.11}$$

$$\left[b, b^\dagger\right]\left|0\right\rangle = b\, b^\dagger\left|0\right\rangle - b^\dagger\, b\left|0\right\rangle = \left|0\right\rangle \tag{7.12}$$

$$\left[b, b^\dagger\right]\left|1\right\rangle = b\, b^\dagger\left|0\right\rangle - b^\dagger\, b\left|0\right\rangle = -\left|0\right\rangle \tag{7.13}$$

So,

$$\left[b, b^\dagger\right] = (1 - 2\, n) \tag{7.14}$$

For a lattice with different sites,

$$\left[b_i, b_j{}^\dagger\right] = (1 - 2\, n)\, \delta_{\text{ij}} \tag{7.15}$$

For spins, we have the same commutation relations.

$$\left[S_i{}^-, S_j{}^-\right] = \left[S_i{}^+, S_j{}^+\right] = 0 \tag{7.16}$$

$$\left[S_i{}^+, S_j{}^-\right] = \left[S_x + i\, S_y,\; S_x - i\, S_y\right] = -i\left[S_x,\; S_y\right] + i\left[S_y, S_x\right] = 2\, i\left[S_y, S_x\right] = 2\, S_z = (2\, n - 1) = \left[b_j{}^\dagger, b_i\right] \tag{7.17}$$

7.2.1. XXZ model

$$H = \sum_{<ij>} J_x \left(S_i^x S_j^x + S_i^y S_j^y \right) + J_z S_i^z S_j^z = \\ \sum_{<ij>} J_x \left[S^+ S^- + i (S_x S_y - S_y S_x) \right] + J_z S_i^z S_j^z = \sum_{<ij>} J_x \left[(S^+ S^- - S_z) + J_z S_i^z S_j^z \right. \tag{7.18}$$

$$S_i^+ S_j^- + S_j^+ S_i^- = (S_i^x + i S_i^y) \left(S_j^x - i S_j^y \right) + \left(S_j^x + i S_j^y \right) (S_i^x - i S_i^y) = \\ 2 S_i^x S_j^x + 2 S_i^y S_j^y + i S_i^y S_j^x - i S_i^x S_j^y + i S_j^y S_i^x - i S_j^x S_i^y = 2 S_i^x S_j^x + 2 S_i^y S_j^y \tag{7.19}$$

$$H = \frac{J_x}{2} \sum_{<ij>} \left(S_i^+ S_j^- + S_j^+ S_i^- \right) + J_z \sum_{<ij>} (n_i - 1/2) \left(n_j - 1/2 \right) = \\ \frac{J_x}{2} \sum_{<ij>} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + J_z \sum_{<ij>} n_i n_j - J_z \sum_{\langle ij \rangle} (n_i - 1/4) = \frac{J_x}{2} \sum_{<ij>} \left(b_i^+ b_j^- + b_j^+ b_i^- \right) + J_z \sum_{<ij>} n_i n_j - \text{constant} \tag{7.20}$$

$$H = t \sum_{<ij>} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + V \sum_{<ij>} n_i n_j \tag{7.21}$$

with $t = J_x/2$ and $V = J_z$.

It is worthwhile to emphasize that these bosons are hard core bosons, which are bosons with a huge on-site repulsion. Using ordinary bosons, this model becomes

$$H = t \sum_{<ij>} \left(a_i^\dagger a_j + a_j^\dagger a_i \right) + V \sum_{<ij>} n_i n_j + U \sum_i n_i (n_i - 1) \tag{7.22}$$

with $U \to \infty$. So this is a boson model with infinite interactions, which cannot be solved.

7.3. Spin-1/2 and fermions (1D)

This section is a baby version of the slave fermion approach. The problem of bosons is that we need infinite interactions to keep avoid many bosons occupying the same state. To make sure that the size of the Hilbert space matches, we need to ensure that there are at most 1 boson per site. If we use fermions, the problem of multi-occupation will not appear at all, due to the Pauli exclusive principle. So fermion seems to be our friends here. (Is it true?)

7.3.1. a single site

Consider one single lattice site, the size (dimension) of the Hilbert space of spinless fermion is 2 occupied or empty. For spin-1/2, the Hilbert space has the same size: spin up and spin down. So it seems possible to make a connection between spins and hard-core bosons.

We can identify the spin up state for spin-1/2 with the occupied state for a hard core boson. Spin down state with an empty state.

Define hard core boson creation and annihilation operators c_i and $c_i^\dagger$.

$$c^\dagger \left| 0 \right\rangle = \left| 1 \right\rangle \qquad c \left| 0 \right\rangle = 0 \tag{7.23}$$

$$c^\dagger \left| 1 \right\rangle = 0 \qquad c \left| 1 \right\rangle = \left| 0 \right\rangle \tag{7.24}$$

Define spin rising and lowering operator $S_i^+ = S_x + i S_y$ and $S_i^- = S_x - i S_y$.

$$S^+ \left| \text{down} \right\rangle = \left| \text{up} \right\rangle \qquad S_i^- \left| \text{down} \right\rangle = 0 \tag{7.25}$$

$$S^+ \left| \text{up} \right\rangle = 0 \qquad S_i^- \left| \text{up} \right\rangle = \left| \text{down} \right\rangle \tag{7.26}$$

So, we find that $c^\dagger$ is S^+ and c is just S^-.

The density operator: $n = c^\dagger c$. It is easy to check that $n \left| 1 \right\rangle = 1 \times \left| 1 \right\rangle$ and $n \left| 0 \right\rangle = 0 \times \left| 0 \right\rangle$

For spin-1/2, the density operator turns into $n = S^+ S^- = \left(S_x + i S_y \right) \left(S_x - i S_y \right) = S_z + 1/2$.

$$n = S^+ S^- = \left(S_x + i S_y \right) \left(S_x - i S_y \right) = \\ S_x^2 + S_y^2 - i \left(S_x S_y - S_y S_x \right) = \left(\left| \vec{S} \right| \right)^2 - S_z^2 - i \left[S_x, S_y \right] = s(s+1) - S_z^2 + S_z = \frac{1}{2} \left(\frac{1}{2} + 1 \right) - \frac{1}{4} + S_z = S_z + 1/2 \tag{7.27}$$

It is easy to check that $(S_z + 1/2)\,|\,\text{up}\rangle = 1 \times |\,\text{up}\rangle$ and $(S_z + 1/2)\,|\,\text{down}\rangle = 0 \times |\,\text{down}\rangle$

One can further verify the anti-commutation relation.

$$\{c, c\} = \left\{c^\dagger, c^\dagger\right\} = 0 \tag{7.28}$$

$$\left\{c, c^\dagger\right\} = 1 \tag{7.29}$$

$$\{S^-, S^-\} = \{S^+, S^+\} = 0 \tag{7.30}$$

$$\begin{aligned}\{S^+, S^-\} &= \left\{S_x + i\,S_y,\ S_x - i\,S_y\right\} = \\ &2\left(S_x^2 + S_y^2\right) - i\ S_x\,S_y + i\,S_y\,S_x - i\ S_y\,S_x + i\,S_x\,S_y = 2\left(S_x^2 + S_y^2\right) = 2\left[s(s+1) - S_z^2\right] = 2\left[\frac{1}{2}\left(\frac{1}{2} + 1\right) - \frac{1}{4}\right] = 1\end{aligned} \tag{7.31}$$

7.3.2. two sites

$$\left\{c_i, c_j\right\} = \left\{c_i^\dagger, c_j^\dagger\right\} = 0 \tag{7.32}$$

$$\left\{S_i^-, S_j^-\right\} = \left\{S_i^+, S_j^+\right\} = 0 \tag{7.33}$$

$$\left\{c_1, c_2^\dagger\right\} = 0 \tag{7.34}$$

$$\{S_1^-, S_2^+\} \neq 0 \quad \text{but } [S_1^-, S_2^+] = 0 \tag{7.35}$$

One cannot naively identify S^+, S^- with $c^\dagger$ and c.

The correct mapping is

$$S_1^+ = c_1^\dagger \tag{7.36}$$

$$S_2^+ = c_2^\dagger(-1)^{n_1} \tag{7.37}$$

This extra factor $(-1)^{n_1}$ will generate the correct anti-commutation relation.

7.3.3. Many-sites

Consider many sites i=1,2,3…N

$$S_i^+ = c_i^\dagger(-1)^{\sum_{j=1}^{j=i-1} n_j} \tag{7.38}$$

This is known as the Jordan–Wigner transformation.

7.3.4. XXZ model

Mapping to fermions, we get

$$H = t\sum_{<\text{ij}>}\left(c_i^\dagger\,c_j + c_j^\dagger\,c_i\right) + V\sum_{<\text{ij}>} n_i\,n_j \tag{7.39}$$

with $t = J_x/2$ and $V = J_z$.

For XY model with $J_z = 0$, this is just a free fermion problem.

Notice that 1D is very special. In higher dimensions, the JW transformation will generate long range interactions. But in 1D there is no such problem.