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Topological insulator part II: Berry Phase and Topological index

3.1. Last chapter

Topological insulator: an insulator in the bulk and a metal near the boundary (surface or edge)

Quantum Hall insulator: $\sigma_{xx} = 0$

- 2D metal (weak lattice or no lattice potential), $\epsilon = k^2 / 2m$. As long as $\mu > 0$, metal (assuming there is no impurities and no interactions)
- B field: Landau levels: $\epsilon_{n,k} = (n + \frac{1}{2}) \hbar \omega$ (very similar to energy bands). If μ in the gap, it is an insulator. If μ coincides with a Landau level, the system is a metal.
- These insulating states are topological insulators (bulk insulator with chiral edge states).

Chiral edge states gives quantized Hall conductivity $\sigma_{xy} = n e^2 / h$

- Q: Why is Hall conductivity nonzero? A: chiral edge states.
 - For a chiral edge state, the top and lower edges have opposite velocities (say electrons near the top edge goes to the right and the lower edge to the left).
 - If we apply $+V/2$ for top edge and $-V/2$ for bottom edge. We will have a higher electron density at the top edge and a lower density at the lower edge (because electrons like positive voltage). So we have more electrons moving to the right (top edge has more electrons) and less electrons moving to the left (low edge has less electrons). So there is a net current in the x direction.
 - Another way to see this: if we pass a current (from left to right), we need to put more charge near the top edge, and less charge near the bottom edge. Charge imbalance induces a E field in the y direction.
- Q: Why Hall conductivity is quantized? A: chiral edge states. One edge contributes to e^2 / h . If we have n edge states: $n e^2 / h$
- Q: Why are impurities unimportant? A: chiral edge states. Electrons cannot be reflected by an impurity.

3.2. Some insulators have chiral edge state but others don't. Why?

A piece of plastic is an insulator. a IQH state is also an insulator. Why does the IQH system have chiral edge states but a piece of plastic doesn't?

Q: Why insulators are different from each other? How many different types of insulators do we have in this world?

What tool should we use? Where should we start to look for an answer?

Hint 1: σ_{xy} depends on \hbar . So this phenomenon must be due to some quantum physics.

Hint 2: Transport is the motion of charge. In quantum mechanics, charge is closely related with one quantity. What is that?

- Hint 2.5: It is the conjugate variable of charge (like p is the conjugate variable of r)
- Phase: invariant under $\phi \rightarrow \phi + \delta\phi$ implies the conservation of charge.

Hint 3: Phase can often lead to quantization (e.g. quantization of $L_z = n \hbar$)

- Wavefunction is single valued. If we go around a circle, the change of the phase can only be $2\pi n$ where n is an integer.

Hint 4: Phase of a wavefunction: quantum physics. (This is good, because we know that the quantum Hall effect is a quantum phenomena).

So: we will focus on “phase”.

- Q: Phase of a wavefunction: important or not?
- A: The relative phase (the phase difference between two states) is important (which leads to interference), but the absolute value of a phase is not important

If $|\psi\rangle$ is an eigenstate, $|\psi'\rangle = e^{i\phi} |\psi\rangle$ is the wavefunction for the same state. All the observables remains the same.

$$\langle\psi|A|\psi\rangle = \langle\psi|e^{-i\phi}|A|e^{i\phi}|\psi\rangle = \langle\psi'|A|\psi'\rangle \quad (3.1)$$

Relative phase is important

$$|\Psi\rangle = \frac{1}{\sqrt{2}} |\psi_1\rangle + \frac{1}{\sqrt{2}} |\psi_2\rangle \quad (3.2)$$

$$\langle\Psi|P|\Psi\rangle = \frac{\langle\psi_1|P|\psi_1\rangle + \langle\psi_2|P|\psi_2\rangle}{2} + \frac{\langle\psi_1|P|\psi_2\rangle + \langle\psi_2|P|\psi_1\rangle}{2} \quad (3.3)$$

The physical meaning of the first term: system has half of the probably on state $|\psi_1\rangle$ and half on $|\psi_2\rangle$. So the momentum of $|\Psi\rangle$ is the average of $|\psi_1\rangle$ and $|\psi_2\rangle$, just like in classical systems.

The physical meaning of the second term: interference between the two quantum state. (relies on the relative phase between state 1 and 2)

$$|\Psi'\rangle = \frac{1}{\sqrt{2}} |\psi_1\rangle + \frac{e^{i\phi}}{\sqrt{2}} |\psi_2\rangle \quad (3.4)$$

$$\begin{aligned} \langle\Psi'|P|\Psi'\rangle &= \frac{\langle\psi_1|P|\psi_1\rangle + \langle\psi_2|P|\psi_2\rangle}{2} + \frac{e^{i\phi}\langle\psi_1|P|\psi_2\rangle + e^{-i\phi}\langle\psi_2|P|\psi_1\rangle}{2} = \\ &= \frac{\langle\psi_1|P|\psi_1\rangle + \langle\psi_2|P|\psi_2\rangle}{2} + \cos\phi \frac{\langle\psi_1|P|\psi_2\rangle + \langle\psi_2|P|\psi_1\rangle}{2} \end{aligned} \quad (3.5)$$

Example: flux quantization

- an A^+ answer: The absolute value of a phase is not important, but the fact that phase is unimportant is VERY IMPORTANT.

3.3. Example: Central charge and quantum group

References:

- Weinberg, The quantum theory of fields, Vol. 1, Chapter 2.2 and 2.7.
- Polchinski, String theory, Vol. 1
- Di Francesco, Mathieu, Senechal, Conformal Field theory

Remarks on central charge:

- Not directly related to topological insulators
- Same physics principle: phase of a quantum state is unimportant, which is very important
- Useful when we discuss 1+1D quantum physics in later chapters
- It is always good to know the connections between different pieces of knowledge.

3.3.1. Commutation relations:

Commutation relations are crucial for quantum systems.

There are two types of commutation relations:

- Type 1: Obvious e.g. $[L_x, L_y] = i L_z$
- Type 2: Not so obvious: central charge

3.3.2. Example of the first type: $[L_x, L_y] = i L_z$

- Q: Why angular momentum operators don't commute with each other.
- A: Rotating a system along the x axis first then along the y axis is different from rotating a system along the y axis first then along the x axis

Angular momentum is relation with space rotation. (generators of rotations)

$$R_x(\theta) = \exp(-i \theta L_x) \quad (3.6)$$

$$R_y(\theta) = \exp(-i \theta L_y) \quad (3.7)$$

$$R_z(\theta) = \exp(-i \theta L_z) \quad (3.8)$$

$$L_x = i \partial_\theta R_x(\theta) |_{\theta \rightarrow 0} \quad (3.9)$$

$$L_y = i \partial_\theta R_y(\theta) |_{\theta \rightarrow 0} \quad (3.10)$$

$$L_z = i \partial_\theta R_z(\theta) |_{\theta \rightarrow 0} \quad (3.11)$$

All rotations form a "Group" (SO(3)), a set with product.

- Product: the product of two elements X and Y is another element of this set $Z = X Y$. (Closure)
 - $X Y$ and $Y X$ don't need to be the same.
 - For rotations, for any two rotations R_1 and R_2 , their product $R_2 R_1$ is defined as we first do the rotation R_2 and then R_1 , which is another rotation $R_3 = R_2 R_1$. For example $R_x(\theta_1) R_x(\theta_2) = R_x(\theta_1 + \theta_2)$
- Associativity: $A (B C) = (A B) C$
- Identity element I . For any element X , $I X = X I = X$.
 - For rotations, I is rotation by zero degree. $I = R(0)$.
- Inverse element: For any element X , there is another element X^{-1} which satisfies $X X^{-1} = X^{-1} X = I$
 - For rotations, the inverse of $R_x(\theta)$ is $R_x(-\theta)$, rotating along the same axis but we flip the sign of the angle.

For rotations, the order of the product matters.

$$R_x(\theta_1) R_y(\theta_2) \neq R_y(\theta_2) R_x(\theta_1) \quad (3.12)$$

So

$$[R_x(\theta_1), R_y(\theta_2)] \neq 0 \quad (3.13)$$

Rotations don't commute with each other. This tells us that their generators cannot commute with each other.

$$[L_x, L_y] \neq 0 \quad (3.14)$$

If $[L_x, L_y] = 0$, we will have $[\exp(-i \theta_1 L_x), \exp(-i \theta_2 L_y)] = 0$, which is obviously incorrect.

Using the commutation relation between rotations, we can find that:

$$[L_x, L_y] = i L_z \quad (3.15)$$

A geometry problem. No quantum physics involved.

3.3.3. Example of the second type: Why is mass NOT an operator? Galilean transformation and space translation: commute or not?

Space translations: $T(a)$ changes $x \rightarrow x' = x + a$. The generator is the momentum operator p :

$$T(a) = \exp(-i p a) \quad (3.16)$$

$$p = i \partial_a T(a) |_{a \rightarrow 0} \quad (3.17)$$

Galilean transformation: $G(v)$ changes $x \rightarrow x' = x + v t$. The generator is called K .

$$G(v) = \exp(-i K v) \quad (3.18)$$

$$K = i \partial_v G(v) |_{v \rightarrow 0} \quad (3.19)$$

- Q: $[K, p] = ?$
- A: Classical physics and quantum physics have different answer.

Classical physics: $[T, G] = 0$ so $[K, p] = 0$

$$G_v : x \rightarrow x' = x + v t \quad (3.20)$$

$$T_a : x \rightarrow x' = x + a \quad (3.21)$$

$$G_v \psi(x, t) = \psi(x + v t, t) \quad (3.22)$$

$$T_a G_v \psi(x, t) = T_a \psi(x + v t, t) = \psi(x + v t + a, t) \quad (3.23)$$

$$T_a \psi(x, t) = \psi(x + a, t) \quad (3.24)$$

$$G_v T_a \psi(x, t) = G_v \psi(x + a, t) = \psi(x + v t + a, t) \quad (3.25)$$

$$T_a G_v \psi(x, t) = G_v T_a \psi(x, t) \quad (3.26)$$

$$T_a G_v = G_v T_a \quad (3.27)$$

$$[T_a, G_v] = 0 \quad (3.28)$$

$$\text{So} \quad (3.29)$$

$$\partial_a \partial_v [T_a, G_v] = 0 \quad (3.30)$$

$$[\partial_a T_a, \partial_v G_v] = 0 \quad (3.31)$$

$$[i \partial_a T_a, i \partial_v G_v] = 0 \quad (3.32)$$

$$[K, p] = \lim_{a \rightarrow 0} \lim_{v \rightarrow 0} [i \partial_a T_a, i \partial_v G_v] = 0 \quad (3.33)$$

Quantum physics:

Galilean transformation change velocity from u to $u+v$. So it change momentum p to $p+m v$ where m is the mass

$$G(v) p G^{-1}(v) = p + m v \quad (3.34)$$

$$\exp(-i K v) p \exp(i K v) = p + m v \quad (3.35)$$

At small v :

$$\exp(-i K v) = 1 - i v K + O(v^2) \quad (3.36)$$

$$\exp(i K v) = 1 + i v K + O(v^2) \quad (3.37)$$

$$(1 - i v K) p (1 + i v K) = p + m v \quad (3.38)$$

$$p - i v K p + i v p K = p + m v \quad (3.39)$$

$$-i K p + i p K = m \quad (3.40)$$

$$K p - p K = i m \quad (3.41)$$

$$[K, p] = i m \quad (3.42)$$

Q: Why do we have two different answers?

A: Phase is unimportant in QM, which is very important.

$$T_a G_v x = x + v t + a \quad (3.43)$$

$$G_v T_a x = x + a + v t \quad (3.44)$$

$$G_v T_a x = T_a G_v x \quad (3.45)$$

This equation tells us that $T_a G_v$ and $G_v T_a$ are the same transformation. For classical physics, it simply means $T_a G_v = G_v T_a$.

In quantum mechanics, $T_a G_v$ and $G_v T_a$ are the same transformation means that $T_a G_v \psi(x, t)$ and $G_v T_a \psi(x, t)$ must give the same quantum state. However, they don't need to have the same phase, because phase itself is unimportant. This means that

$$T_a G_v \psi(x, t) = e^{i\phi(v, a)} G_v T_a \psi(x, t) \quad (3.46)$$

$$T_a G_v = e^{i\phi(v, a)} G_v T_a \quad (3.47)$$

- Key: in quantum mechanics “the same” means equal up to a phase difference.

$$[T, G] = T_a G_v - G_v T_a = (e^{i\phi(v, a)} - 1) G_v T_a \neq 0, \text{ if the phase is nonzero } \phi(v, a) \neq 0. \quad (3.48)$$

The operations also form a group, but their products are different from classical physics. This is known as a quantum group (or projective representation).

For generators, this extra phase means that we need to add some extra piece to their commutation relations, which is known as central charge.

For K and p , this implies that $[K, p] = iC$, where C is known as a central charge. It is just the mass.

- The central charge needs to satisfy some self-consistency condition: $X(YZ) = (XY)Z$.
- For many groups, this condition implies $C=0$ (no extra piece. Same as classical physics.) Example: rotational groups $SO(3)$, Lorentz group ...
- The central charge must commute with all other symmetry operators due to the self-consistency condition, i.e. it is a number.

Because mass is a central charge, mass must commute with all other operators. In other words, mass is a number in nonrelativistic QM.

Bottom line: phase is unimportant implies that we can have a central charge in the commutation relations. This is a quantum phenomena with no parallel example in classical physics.

3.3.4. A more important example of central charge: string theory and 1D quantum physics

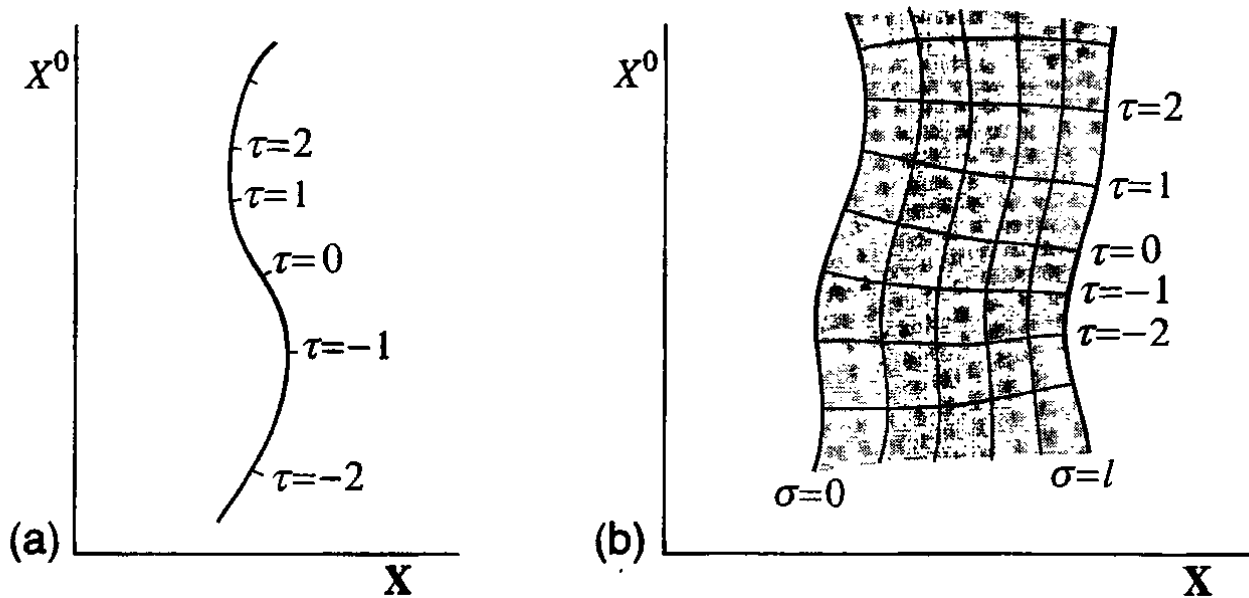


Figure from Polchinski, String theory, Vol. 1. (a) World-line of a particle and (b) world-sheet of a string.

For a particle moving in D -dimensional space-time.

- Q: What is the trace of this particle?
- A: 1D curve: world-line.

$$X^\mu = X^\mu(\tau) \quad (3.49)$$

with $\mu=t, x, y, z$ and τ is a parameter along the world-line.

For a string moving in D dimensional space-time.

- Q: What is the trace of this string?

- A: 2D surface: world-sheet.

$$X^\mu = X^\mu(\sigma, \tau) \quad (3.50)$$

σ and τ are the coordinate on the 2D world sheet, which lives in a D dimensional space.

Quantum 1D system in condensed matter. Same description, same physics and same technique (conformal field theory). For example, density of electrons in a 1D Fermi gas:

$$\rho(x, t) \quad (3.51)$$

String theory and critical 1D quantum systems (including 1D free fermions or bosons) have a very special symmetry: 2D conformal symmetry.

Conformal symmetry:

for a 2D system, we can define the complex coordinate $z = x + i t$ (or $(z = \sigma + i \tau)$ for the string theory)

A conformal transformation: a mapping $z \rightarrow f(z)$ for any an analytic function $f(z)$ (functions which can be expand into a Taylor series, including most of the functions we usually use: $e^{\alpha z}$, $\sin k z$, z^n , $\log z \dots$).

- Conformal transformation preserves the angles between two vectors (general definition in any dimensions).
- In higher dimensions, $d > 2$, there are only $\frac{(d+2)(d+1)}{2}$ of them.
- In $d=2$, there are infinite number of them. (infinite analytic functions)
- Translations are conformal transformations: $f(z) = z + x_0 + i t_0$
- Rotations are also conformal transformations: $f(z) = z e^{i \phi}$
- Dilation is also a conformal transformation: $f(z) = a z$

Conformal symmetry: systems invariant under all conformal transformations.

- Example: string theory or a 1D critical system
- Noether theorem: A symmetry implies one conservation law. Infinite conformal transformation, and the system is invariant under every one of them. This means that there are infinite numbers of conservation law.
- This is why string theory is easy to handle.

Generators of conformal transformations: l_n with n an integer:

$$l_n = -z^{n+1} \partial_z \quad (3.52)$$

- $n = -1$ gives the generators of translations $l_{-1} = -\partial_z = \frac{-\partial_x + i \partial_y}{2}$. Real part is just $-i p_x / 2$ and imaginary part gives $-i p_y / 2$
- $n = 0$ gives the generators of rotations and dilations $l_0 = -z \partial_z = (x + i y) \frac{-\partial_x + i \partial_y}{2} = -(x \partial_x + y \partial_y) + i (x \partial_y - y \partial_x)$. Imaginary part is proportional to $L_z = r \times p = -i (x \partial_y - y \partial_x)$.

Classical physics

$$[l_m, l_n] = (m - n) l_{m+n} \quad (3.53)$$

Quantum physics: (Virasoro algebra)

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m,-n} \quad (3.54)$$

c is the central charge.

Similarities and differences between studying string theory and learning to play guitar?

- Similarities: both studies the motions of strings (the equations of motion are pretty much the same)
- Differences: the formal studies quantum strings (with central charge), while the latter handle classical strings (no central charge)

Central charge in the string theory: why D=26?

For X^μ , each of them has $c=1$ and there are D of them, where D is the dimension of the space. In addition, one need ghost fields for gauge symmetry, which also have central charge.

Total central charge $C=D-26=0$

D=26. This is why we should study bosonic string theory in 26 dimentions (supersymmetry has extra fermions, so the $C=0$ condition is different and one works in D=10).

Central charge in condensed matter systems: (one example)

In condensed matter physics, 1D Fermi gas and Bose gas has the same Central charge $C=1$. They are totally equivalent to each other. In 1D, one can turn bosons into fermions or fermions into bosons and vice versa, which is known as “bosonization”.

For $C < 1$, we know pretty much everything about the critical behavior (scaling laws) by knowing the value of C .

For $C \geq 1$, a little bit more complicated. Some additional symmetry could be helpful (affine Lie algebras)

3.4. Example II: Phase of Bloch waves

3.4.1. Phase of a Bloch wavefunction

A particle moving in a periodic potential:

$$\left[-\frac{1}{2m} \nabla^2 + V(\vec{r}) \right] \psi(\vec{r}) = \epsilon \psi(\vec{r}) \quad (3.55)$$

with

$$V(\vec{r}) = V(\vec{r} + \vec{a}_i) \quad (3.56)$$

Eigen-wavefunctions: Bloch waves

$$\psi_{n,k}(r) = u_{n,k}(r) \exp(i k r) \quad (3.57)$$

■ Q: Phase of $u_{n,k}(r)$?

■ A: Doesn't matter. Phase itself is unimportant.

Bloch waves are invariant under a global phase shift in the momentum space. (Same as other quantum states)

If $\psi_{n,k}(r) = u_{n,k}(r) \exp(i k r)$ is a Bloch wave, then $\psi'_{n,k}(r) = e^{i \phi} u_{n,k}(r) \exp(i k r)$ is the same eigen state

In addition, Bloch waves are invariant under a local phase shift in the momentum space. (new and important)

If $\psi_{n,k}(r) = u_{n,k}(r) \exp(i k r)$ is a Bloch wave, then $\psi'_{n,k}(r) = e^{i \phi(k)} u_{n,k}(r) \exp(i k r)$ is the same eigen state

$$\left[-\frac{1}{2m} \nabla^2 + V(\vec{r}) \right] \psi_{n,k}(r) = \epsilon \psi_{n,k}(r) \quad (3.58)$$

$$\left[-\frac{1}{2m} \nabla^2 + V(\vec{r}) \right] \psi'_{n,k}(r) = \left[-\frac{1}{2m} \nabla^2 + V(\vec{r}) \right] e^{i \phi(k)} \psi_{n,k}(r) = \quad (3.59)$$

$$e^{i \phi(k)} \left[-\frac{1}{2m} \nabla^2 + V(\vec{r}) \right] \psi_{n,k}(r) = e^{i \phi(k)} \epsilon \psi_{n,k}(r) = \epsilon e^{i \phi(k)} \psi_{n,k}(r) = \epsilon \psi'_{n,k}(r)$$

The system is invariant under a local phase shift in the momentum space.

Requirement to have such a local symmetry.

1 Momentum conservation (no impurities)

2. No electron-electron interactions

So, a particle with momentum k always has momentum k . Bloch waves with different momentum doesn't talk to each other. So their phase can change independently.

3.5. Local phase symmetry

3.5.1. How to describe this local phase degrees of freedom?

Q: Do we know something similar?

A: Yes, we know a similar local phase degrees of freedom. Not in the k -space, but in the real space, which is the gauge symmetry.