$$j_x(r_0,\ t_0) = \frac{k}{2\pi}\,\epsilon_{x\mu\nu}\,\partial^\mu A^\nu(r_0,\ t_0) = \\ \frac{k}{2\pi}\,\epsilon_{xty}\,\partial^t A^y(r_0,\ t_0) + \frac{k}{2\pi}\,\epsilon_{xyt}\,\partial^y A^t(r_0,\ t_0) = \frac{k}{2\pi}\left[-\partial^t A^y(r_0,\ t_0) + \partial^y \phi(r_0,\ t_0)\right] = \frac{k}{2\pi}\left[\partial_t A^y(r_0,\ t_0) + \partial_y \phi(r_0,\ t_0)\right] = -\frac{k}{2\pi} E_y \tag{4.12}$$

Q: What happens if we have a metal?

A: For metals, the charge degrees of freedom must also be considered. So we have both gauge fields and charge. The situation is much more complicated and in general, we cannot get a clear answer of σ_{xy}.

4.1.3. Electric charge and magnetic field in a quantum Hall system

The Cherm-Simon's gauge theory

$$L_{CS} = \frac{k}{4\pi}\int d\vec{r}\,\epsilon_{\mu\nu\lambda}\,A^\mu\,\partial^\nu A^\lambda \tag{4.13}$$

Charge density:

$$\rho(r_0,\ t_0) = \frac{\delta S_{\text{material}}}{\delta A_0(r_0,\ t_0)} = \frac{\delta}{\delta A_0(r_0,\ t_0)}\,\frac{k}{4\pi}\int d\vec{r}\,dt\,\epsilon_{\mu\nu\lambda}\,A^\mu(r,\ t)\,\partial^\nu A^\lambda(r,\ t) = \\ \frac{k}{4\pi}\int d\vec{r}\,dt\,\epsilon_{\mu\nu\lambda}\,\frac{\delta A^\mu(r,\ t)}{\delta A_0(r_0,\ t_0)}\,\partial^\nu A^\lambda(r,\ t) + \frac{k}{4\pi}\int d\vec{r}\,dt\,\epsilon_{\mu\nu\lambda}\,A^\mu(r,\ t)\left[\partial^\nu\,\frac{\delta A^\lambda(r,\ t)}{\delta A_0(r_0,\ t_0)}\right] \\ = \frac{k}{4\pi}\int d\vec{r}\,dt\ \epsilon_{\mu\nu\lambda}\,\delta_{\mu,0}\,\delta(r-r_0)\,\delta(t-t_0)\,\partial^\nu A^\lambda(r,\ t) + \frac{k}{4\pi}\int d\vec{r}\,dt\,\epsilon_{\mu\nu\lambda}\,A^\mu(r,\ t)\left[\partial^\nu\,\delta_{\lambda,0}\,\delta(r-r_0)\,\delta(t-t_0)\right] \\ = \frac{k}{4\pi}\,\epsilon_{0\nu\lambda}\,\partial^\nu A^\lambda(r_0,\ t_0) - \frac{k}{4\pi}\,\epsilon_{\mu\nu 0}\,\partial^\nu A^\mu(r,\ t) = \frac{k}{2\pi}\,\epsilon_{0\mu\nu}\,\partial^\mu A^\nu(r_0,\ t_0) = \frac{k}{2\pi}\,\nabla\times\vec{A} = \frac{k}{2\pi}\,B \tag{4.14}$$

Bottom line: charge density is propotional to magnetic field

$$\rho(r_0,\ t_0) = \frac{k}{2\pi}\,B \tag{4.15}$$

Integral on both sides:

$$\int\!\!\int_D dr\,\rho(r_0,\ t_0) = \frac{k}{2\pi}\int\!\!\int_D dr\,B \tag{4.16}$$

We find that electric charge is proportional to the magnetic flux

$$q = \frac{k}{2\pi}\,\Phi_B \tag{4.17}$$

Flux attachment:

- Electrons in an IQH system carries magnetic flux $\Phi_B = 2\pi/k$ (the Chern-Simon's gauge theory attaches a magnetic flux $\Phi_B = 2\pi/k$ to each of the electrons).
- Magnetic flux in an IQH system carries charge q=$k\,\frac{\Phi_B}{2\pi}$

Physical reason: Faraday's law+Hall effect

Let's add an magnetic flux in an IQH system, by increasing the magnetic field in some region D. $B(r,\ t_i) \to B(r,\ t_f)$. If $B(t_f) > B(t_i)$, we added some magnetic flux into the system

$$\Delta\Phi_B = \int\!\!\int_D dr\,B(r,\ t_f) - \int\!\!\int_D dr\,B(r,\ t_i) \tag{4.18}$$

Increasing magnetic flux will create some electric field (Faraday's law)

$$\oint E\cdot dl = -\frac{d\Phi_B}{dt} \tag{4.19}$$

If we have E, it will creates Hall current perpendicular to E(Hall effect)

$$j = \sigma_{xy}\, E \tag{4.20}$$

Total current flowing away from the region D:

$$I = \oint j \cdot d\, n = \sigma_{xy} \oint E \cdot d\, l = -\sigma_{xy} \frac{d\,\Phi_B}{d\,t} \tag{4.21}$$

Total charge accumulated due to this current flow:

$$\Delta q = q(t_f) - q(t_i) = -\int d\,t\, I(t) = \sigma_{xy} \int d\,t \frac{d\,\Phi_B}{d\,t} = \sigma_{xy}\left[\Phi_B(t_f) - \Phi_B(t_i)\right] = \sigma_{xy}\, \Delta\Phi_B \tag{4.22}$$

If we create an flux Φ_B in a IQH system, some charge a accumulated at this flux: $q = \sigma_{xy}\, \Phi_B$

Chern-Simon's gauge theory: $q = k\, \frac{\Phi_B}{2\pi}$, adding units back $q = k\, \frac{e}{\hbar}\, \frac{\Phi_B}{2\pi} = k\, \frac{e}{h}\, \Phi_B = \sigma_{xy}\, \Phi_B$

4.1.4. Why k is quantized? (a not-so-rigorous proof)

(For more a rigorous proof, see Quantum Field Theory of Many-body Systems, Xiao G, Oxford 2004).

Integral over the whole system:

$$\int\int_D d\, r\, \rho(r_0,\, t_0) = \frac{k}{2\pi} \int\int_D d\, r\, B \tag{4.23}$$

Total charge is proportional to the total magnetic flux

$$N_e = k \int\int_D d\, r\, \frac{B}{2\pi} = k\, N_m \tag{4.24}$$

N_e is always an integer. On a closed manifold, N_m is quantized to integer values (the monopole charge)

If we increase the magnetic field by a little bit such that N_m turns into $N_m + 1$, the charge of the system need to change from N_e to $N_e + k$. Because charge is always an integer, k must be an integer.

If we increase the charge by 1, $N_e \to N_e + 1$, for general k (if k is not ± 1), N_m will increase by a fractional value $1/k$ in order to satisfies the relation: $N_e = k\, N_m$. This means that if we have a IQHE with N_e electrons, by adding one electron, the system will no longer be in an IQHE effect. This is reasonable because quantum Hall effect is an insulator. For an insulator, if we add one more electron, it becomes a metal.

4.1.5. Chern-Simon's theory for an open manifold (with edges)

$$A_\mu \to A_\mu{}' = A_\mu + \partial_\mu \chi \tag{4.25}$$

$$S_{CS} \to S_{CS}{}' = S_{CS} + \frac{k}{4\pi} \int d\,\vec{r}\, d\,t\, \epsilon_{\mu\nu\lambda}\, \partial^\mu \left(\chi\, \partial^\nu A^\lambda\right) = S_{CS} + \text{surface term} \tag{4.26}$$

It is not gauge invariant near the boundar. This is because we have (metallic) chiral edge states near the boundary, which also violate the gauge symmetry (known as gauge anomaly), and the violation of the gauge theory there introduce an extra term to the action, which cancles the gauge dependent term here.

4.1.6. Gauge anomaly (NOT required)

Ref: Quantum theory of fields, Weinberg, VOL 2, Chapter 22, Anomalies

A stand approximation: ignore high energy states, because it cost too much energy to excite the system into such state. This approximation is NOT always valid and could be problematic.

Example (statistical physics): at 1K, what is the probability of finding a hydrogen atom in its ground state?

Energy levels of a hydrogen atom: $E = -\frac{13.6\,\text{eV}}{n^2}$

Boltzmann distribution $P \propto \exp(-E_n/T)$. Low energy state have a larger probably. For very high energy states, the probability is much smaller.

Ground state energy of a hydrogen atom: $E_1 = -13.6$ eV

First excited states: $E_2 = -3.4$ eV

Temperature $T = 1\ K \approx 10^{-4}$ eV.

$T << \text{E}_2 - E_1$, so let's ignore the high energy states beyond $n > 2$, only focusing on the lowest to states.

$$P_{n=1} = \frac{\exp(E_1 / T)}{\exp(E_1 / T) + \exp(E_2 / T)} \approx \frac{10^{59\,064}}{10^{59\,064} + 10^{14\,766}} = 1 \tag{4.27}$$

This answer is WRONG! One cannot ignore high energy states. Their probably is small, but their number is huge.

$$P_{n=1} = \frac{\exp(-E_1 / T)}{\sum \exp(-E_n / T)} \approx \frac{10^{59\,064}}{10^{59\,064} + 10^{14\,766} + \ldots 1 + 1 + 1 + 1 + \ldots} = \frac{10^{59\,064}}{\infty} = 0 \tag{4.28}$$

One cannot ignore the high energy states here. If one ignore the high energy states, the probably is 1. But if one include the high energy states, one get 0. In other words, the high energy states changed the probably by "-1", which is an "anomaly".

Sometimes, high energy states (ultraviolet) can change the low energy behavior (infrared) very strongly. Such a contribution from the high energy states are often called an "anomaly". Anomaly here means that one cannot see it in the low energy theory. So in the low energy theory, it seems to be something comes from nowhere. But in really, there is nothing abnormal here. Just one cannot ignore the high energy physics.

In gauge theory, there is such an effect, which was first noticed in a high-energy experiments: pion decay

pion decay: $\pi^0 \to 2\,\gamma$

Pions are bosons made by quark and anti-quark pairs.

Quarks are fermions, in 3+1D, we have left moving and right moving quarks. In the effective low energy (when we ignore the mass of fermions), the number of left and right moving quarks are separately conserved. This is one important conservation law in QED and QCD.

It is not an exact conservation law, because fermions have some mass. But it is a very good approximate conservation law.

This conservation law suppresses the pion decay rate very strongly. If we keep this (approximate) conservation law in mind, the decay rate is $1.9 \times 10^{13}\ s^{-1}$(theory).

In experiments, the decay rate is 1.19×10^{16}, which is 1000 time larger.

The problem is the conservation law. At low energy, (if we ignore all the high energy states), there is such an conservation law (particle numbers in the left and right moving sectors are conserved separately). But the high energy state cannot be ignored in this system and destroys this conservation law, which increases the decay rate of π^0 by 3 order of magnitudes, which is known as the chiral anomaly.

A baby version of the chiral anomaly,

Let's consider a 1D condensed matter system, electrons hopping on a lattice. The dispersion is a

Low energy effective theory

$$H = \sum_k v_F\, k\, L_k^\dagger\, L_k - \sum_k v_F\, k\, R_k^\dagger\, R_k \tag{4.29}$$

We have two species of fermions, L and R. Their dispersions are $\pm v_F\, k$ (left and right moving).

Particle numbers for left moving particles:

$$N_L = \sum_k L_k^\dagger\, L_k \tag{4.30}$$

Particle numbers for right moving particles:

$$N_R = \sum_k R_k^\dagger\, R_k \tag{4.31}$$

N_L and N_R are both conserved quantities

$$[N_L,\ H] = [N_R,\ H] = 0 \tag{4.32}$$

This is NOT true! Because if we cannot ignore high energy states here. If we taken into account the high energy states, we find that the left and right moving branches are connected at energy far away from the chemical potential (near the band bottom). So in reality, we only have one conservation law, which is the total fermion number $N_L + N_R$. $N_L - N_R$ is NOT conserved. We only have one conserved quantity. This is the reason why one get wrong answer in π decay rate, by assuming there are two conserved quantities. The two separate conservation laws for N_L and N_R are fake. It is because one ignored high energy states, which should not be ignored in this case.

In high energy, this problem is addressed in a slightly different way, using gauge theory. Because gauge field couples to the phase of fermions and phase is related to fermion numbers, it is reasonable to use gauge theory to describe the system. In high energy, they way people describe this effect is to couple the left moving particles with a gauge field, and right moving particles with another gauge field. For left moving particles, one effect is to couple the left moving particles with a gauge field, and right moving particles with another gauge field. For left moving particles, one find that the gauge theory is NOT gauge invariant (high energy states gives an extra term to the theory, which is not gauge invariant): $\epsilon_{\mu\nu\lambda} \frac{1}{4\pi} \chi \partial^\mu A^\nu$ Similarly, the right moving theory is NOT gauge invariant either, and the gauge anomaly takes the same form but with opposite sign $-\frac{1}{4\pi} \chi \nabla A^0$. To keep gauge invariance, one must have one left moving channel and one right moving channel and their gauge anomaly cancels each other. So at the end of the day, only the total number is conserved, and one cannot write down a theory which conserved the number for both left and right moving particles.

For IQHE, the edge state only have one branch (left modes only). So, the high energy theory tells us that we will have a gauge anomaly and we cannot keep the gauge theory. This gauge anomaly is precisely the extra piece we get in the CS gauge theory near the boundary. And they cancel each other, keep the whole system gauge invariant.