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# Instantons from Blow-up

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Abstract: The Nekrasov partition function for 4d $\mathcal{N}=2$ or 5d $\mathcal{N}=1$ gauge theory on the blow up of a point $\hat{\mathbb{C}}^2$ can be written in terms of the partition function on the flat space $\mathbb{C}^2$. At the same time, the partition function on the blow up is identical to the partition function on a flat space for sufficiently large class of examples. This relation enables us to compute the instanton partition functions for 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ gauge theories for arbitrary gauge theory with large class of matter representations without knowing explicit construction of the instanton moduli space. Remarkably, the instanton partition function is completely determined by the perturbative part. We obtain the partition function for the previously unknown theories: exceptional gauge groups $EFG$ with fundamental/spinor hypermultiplets and more. We also compute the case with $SU(6)$ with rank-3 antisymmetric tensor and compare with the topological vertex computation using the recently found 5-brane web configuration.

## Contents



## 1 Introduction

The Seiberg-Witten prepotential provides a complete description for the low energy dynamics of 4d $\mathcal{N}=2$ or 5d $\mathcal{N}=1$ gauge theory in its Coulomb branch [1, 2]. It is a function of the vacuum expectation value (VEV) of the scalar in the vector multiplet that parameterizes the Coulomb branch moduli space. Quantum correction to the prepotential is known to be one-loop exact, while there also exist non-perturbative corrections coming from Yang-Mills instantons.

An efficient way to compute the fully quantum corrected prepotential $\mathcal{F}$ is to study the Nekrasov partition function $\mathcal{Z}$ on $\Omega$-deformed $\mathbb{C}^2$ or $\mathbb{C}^2\times S^1$. It can be written as the product of the classical, one-loop, and instanton contributions,

$$\mathcal{Z}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q}) = \mathcal{Z}_{\text{class}}(\vec{a},\epsilon_1,\epsilon_2,\mathfrak{q})\ \mathcal{Z}_{\text{1-loop}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2)\ \mathcal{Z}_{\text{inst}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q}), \tag{1.1}$$

where the instanton piece is the fugacity sum over all multi-instanton contributions:

$$\mathcal{Z}_{\text{inst}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q}) = 1+\sum_{k=1}^{\infty}\mathfrak{q}^k\mathcal{Z}_k(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2). \tag{1.2}$$

Once the Nekrasov partition function is known, one can extract the prepotential via taking $\epsilon_i \to 0$ limit as $\mathcal{F} = \lim_{\epsilon_i \to 0} \epsilon_1 \epsilon_2 \log \mathcal{Z}$ [3–6].

The instanton part of the partition function in the Ω-background can be computed once we know appropriate instanton moduli space. For the classical gauge group, the ADHM construction of the moduli space provides a direct way compute the instanton partition function. The ADHM construction can be understood as the quantum mechanics described by the Dp-D(p+4) system. The Higgs branch moduli space of the Dp system gives the desired moduli space. Matter fields can be also introduced by considering the world-volume theory on the D0-branes of the D0-D4-D8 system. By using the localization on the 1d system on the D0-branes or its dimensional reduction [7, 8], the partition function has been obtained for variety of cases: classical gauge groups [9–13], exceptional gauge groups [14–17]. 3d N=4 Coulomb branch realization of the instanton moduli space: [18] The precise choice of the Contour of the ADHM integral has been derived in [19–21] following the Jeffrey-Kirwan residue formula in 2d elliptic genus [22, 23].

But there is no ADHM type construction for the exceptional gauge groups or generic type of matter fields. String-theoretic picture implies that they require strong-coupling dynamics or non-Lagrangian field theories to realize instanton moduli space of exceptional group as a vacuum moduli space. There has been a few results regarding the exceptional instantons.

In this paper, we generalize the approach of Nakajima-Yoshioka (NY) [5, 24, 25] for the pure YM theory. In [16], the NY blow-up formula were used to compute the instanton partition function for exceptional gauge group and tested against the superconformal index of 4d SCFT where the Higgs branch is given by the instanton moduli space [26]. We propose a general blow-up formula for general gauge theory with arbitrary representations, under the condition that the matter representation is not too large.[1] This enables us to compute the instanton partition functions for numerous gauge theories that have not been known before, without relying on to the explicit construction of the moduli space.

The basic idea is as follows: Let us consider a one-point blow up $\hat{\mathbb{C}}^2$ of the flat space $\mathbb{C}^2$. The *full* partition function on $\hat{\mathbb{C}}^2$ can be written in terms of the products of the *full* partition function of $\mathbb{C}^2$. But at the same time, the partition function on the blowup is identical to that of the flat space since we can smoothly blow-down $\hat{\mathbb{C}}^2$ to $\mathbb{C}^2$ as long as the matter representation is not 'too large'. This provides us a functional relation for the partition function, which turns out to be sufficient to determine the instanton partition function itself. Remarkably, this relation is completely determined by the perturbative part of the partition function. Therefore we arrive at a surprising conclusion: The *perturbative* physics determine the *non-perturbative* physics!

For example, we find the following universal expression for the 1-instanton partition

[1] This has to do with the 1-loop beta function coefficient as we will discuss in detail.

function with arbitrary gauge group and matters:

$$\boxed{Z_1 = \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma\in\Delta_l} \frac{e^{(h^\vee-1)a_\gamma/2}\prod_{\gamma\cdot w=1}(1-e^{a_w+m^{\mathrm{phy}}})}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-e_2})\prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})}} \tag{1.3}$$

where ...

## 2 Instanton Counting from Blow-up

The essential idea of using the blow-up of $\mathbb{C}^2$ for instanton counting is that the gauge theory partition function for a 4d $\mathcal{N}=2$ (or 5d $\mathcal{N}=1$) theory on the blow-up of a point $\hat{\mathbb{C}}^2$ (or $S^1\times\hat{\mathbb{C}}^2$) can be written in two different ways. This will allow us to write a recursion relation for the instanton partiton function that can be solved rather easily [5, 16, 24, 25].

### 2.1 Blowup equation

**Localization on the blow-up $\hat{\mathbb{C}}^2$** One of the expressions for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ comes from the Coulomb branch localization, which results that $\hat{\mathcal{Z}}$ can be obtained by patching together the flat-space partition function $\mathcal{Z}$ [27].

The blow-up $\hat{\mathbb{C}}^2$ is constructed from $\mathbb{C}^2$ by replacing the origin with a compact 2-cycle. In particular, the geometry is identical to the total space of the line bundle of degree $(-1)$ over $\mathbb{P}^1$. One can parametrize $\mathcal{O}(-1)\to\mathbb{P}^1$ using the homogeneous coordinates $(z_0,z_1,z_2)$, satisfying the projective condition $(z_0,z_1,z_2)\sim(\lambda^{-1}z_0,\lambda^1z_1,\lambda^1z_2)$ for any $\lambda\in\mathbb{C}^*$, where the two-cycle $\mathbb{P}^1\subset\hat{\mathbb{C}}^2$ corresponds to the locus $z_0=0$. We are interested in the $U(1)^2$ equivariant partition function, with the $U(1)^2$ action $V$ rotating the complex coordinates $(z_0,z_1,z_2)$ as follows:

$$(z_0,z_1,z_2)\mapsto(z_0,e^{\epsilon_1}z_1,e^{\epsilon_2}z_2). \tag{2.1}$$

Instantons are located at two fixed points of the $U(1)^2$ action, i.e., the north/south poles of the $\mathbb{P}^1$, whose coordinates are $(z_0,z_1,z_2)=(0,1,0)$ and $(0,0,1)$. Around these fixed points, the local weights under the $U(1)^2$ action $V$ are:

$$(z_0z_1,\ z_2/z_1)\mapsto(e^{\epsilon_1}z_0z_1,\ e^{\epsilon_2-\epsilon_1}z_2/z_1) \qquad \text{(near the north pole)} \tag{2.2}$$

$$(z_0z_2,\ z_1/z_2)\mapsto(e^{\epsilon_2}z_0z_2,\ e^{\epsilon_1-\epsilon_2}z_1/z_2) \qquad \text{(near the south pole)} \tag{2.3}$$

The full partition function $\hat{\mathcal{Z}}$ on $\hat{\mathbb{C}}^2$, which includes both the perturbative and instanton contributions, can be obtained by performing the localization on the Coulomb branch. On the Coulomb branch, the gauge group is generically broken to $U(1)^r$ where $r$ is the rank of the gauge group. The $U(1)^r$ equivariant parameters $\vec{a}$ naturally appear in the partition function. One needs to sum over all distinct field configurations with zero-sized instantons located at the north and south poles. All the inequivalent configurations are labeled by the $r$-dimensional vector $\vec{k}$ of the first Chern numbers, corresponding to different flux configurations on the two-cycle $\mathbb{P}^1$. When the gauge group has $U(1)$ factor, we can turn on the external flux that can be supported on the $\mathbb{P}^1$. We assume there is no such a factor in the gauge group. Summing up, $\hat{\mathcal{Z}}$ can be expressed in terms of the partition function $\mathcal{Z}$ on $\mathbb{C}^2$ as [27–31]

$$\hat{\mathcal{Z}}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m}) = \sum_{\vec{k}\in\Lambda} \mathcal{Z}^{(N)}(\vec{k})\mathcal{Z}^{(S)}(\vec{k}) \ , \tag{2.4}$$

where the flux sum is taken over the co-root lattice $\Lambda$ of the gauge algebra. Each factor represents the partition function localized at the $U(1)^2$ fixed points (north/south-poles of the $\mathbb{P}^1 \subset \hat{\mathbb{C}}^2$ given as

$$\begin{aligned} \mathcal{Z}^{(N)}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1,q,\vec{m}+\frac{1}{2}\epsilon_1) \ , \\ \mathcal{Z}^{(S)}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2,q,\vec{m}+\frac{1}{2}\epsilon_2) \ . \end{aligned} \tag{2.5}$$

In addition to the Coulomb branch parametres, the partition function depends on the Omega deformation parameters $\epsilon_1, \epsilon_2$ and also mass parameters $\vec{m}$. The parameter $q$ is the instanton parameter. For a 4d theory, it is given as $q = e^{2\pi i\tau} = \Lambda^{b_0}$ where $\tau$ is the complexified gauge coupling and $\Lambda$ being the dynamical scale of the gauge theory. The exponent $b_0$ is the 1-loop beta function coefficient. For a 5d theory, it is also given by a exponentiated gauge coupling as $q = e^{-\frac{1}{g^2}} \equiv e^{-m_0}$. Notice that the Coulomb parameter $\vec{a}$ gets an appropriate shift at each fixed point $p$, induced by the non-trivial magnetic flux $\vec{k}$ on the blown-up $\mathbb{P}^1$, with the proportionality constant $H|_p$. The values of the moment map $H$ for the $U(1)^2$ action $V$, *i.e.*, $dH = \iota_V\omega$, at the north and south poles are given as

$$H|_{\mathrm{NP}} = \epsilon_1 \text{ and } H|_{\mathrm{SP}} = \epsilon_2. \tag{2.6}$$

The mass parameters also get shifted since the hypermultiplet mass is twisted by $SU(2)_R$, which makes the combination $m + \frac{\epsilon_1+\epsilon_2}{2}$ invariant at the fixed points.[2]

**Partition function on $\hat{\mathbb{C}}^2$ vs $\mathbb{C}^2$** An alternative fact for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ is that $\hat{\mathcal{Z}}$ is actually identical to the flat-space partition function $\mathcal{Z}$ [5, 24, 25]. The blow-up $\hat{\mathbb{C}}^2$ is identical to $\mathbb{C}^2$ except for the origin, which is replaced by the blown-up sphere $\mathbb{P}^1$. Since the Nekrasov partition function gets contributions only from the small instantons localized at the fixed points of the $U(1)^2$ equivariant action $V$, the size of the divisor should not affect the partition function as we smoothly shrink it. So we expect that $\hat{\mathcal{Z}} = \mathcal{Z}$. This implies the following relation: [5, 16, 24, 25, 32, 33]

$$\mathcal{Z} = \hat{\mathcal{Z}} = \sum_{\vec{k}\in\Lambda} \mathcal{Z}^{(N)}(\vec{k})\mathcal{Z}^{(S)}(\vec{k}) \tag{2.7}$$

This blow-up identity can be thought of as a special case of more generalized to orbifold partition functions [31, 34, 35]. For example, the Nekrasov partition function on the orbifold

[2]One can instead use the shifted mass to simplify the formula involving mass. We use unshifted mass to match with the existing formulae in the literature.

$\mathbb{C}^2/\mathbb{Z}_2$ can be computed in two different ways, one is via formula analogous to (2.4) by combining the contributions from two fixed points of the blown-up geometry $\mathcal{O}(-2) \to \mathbb{P}^1$. The other way is to compute the partition function at the orbifold point using the ADHM construction for the orbifolds. The Nekrasov partition function still remains the same as we blow up or down the singular point.[3] The only difference in our case is that we blow-up or down a non-singular point instead of a singular point.

**Correlation functions in 4d** In fact, (2.7) is not enough to fix the partition function completely, since there are 3 unknown functions and only one relation. It turns out the necessary additional relations can be found from the insertion of non-trivial $\mathcal{Q}$-closed operators [5, 25] associated to the two-cycle on the blow-up.

In the 4d Donaldson-twisted theory, the $\mathcal{Q}$-invariant observable $\mathcal{O}_2$ associated to a two-cycle can be constructed by applying the topological descent procedure twice to the Casimir invariant $\mathcal{O}_0 = \mathrm{Tr}(\Phi^2)$ as [36]

$$\begin{aligned} 0 = \{\mathcal{Q}, \mathcal{O}_0\},\ d\mathcal{O}_0 = \{\mathcal{Q}, \mathcal{O}_1\},\ d\mathcal{O}_1 = \{\mathcal{Q}, \mathcal{O}_2\}, \\ d\mathcal{O}_2 = \{\mathcal{Q}, \mathcal{O}_3\},\ d\mathcal{O}_3 = \{\mathcal{Q}, \mathcal{O}_4\},\ d\mathcal{O}_4 = 0\ . \end{aligned} \tag{2.8}$$

In our case, we consider a $U(1)^2$-equivariant version of the topological descent procedure, that is to choose $\mathcal{Q} = Q_{\text{top}} + H$ to obtain the operator associated to the two-cycle. In terms of the component fields, it can be written as [37]

$$\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_2 = \int \left\{ \omega \wedge \mathrm{Tr}\Big( \Phi F + \frac{1}{2}\psi \wedge \psi \Big) + H\, \mathrm{Tr}\Big( F \wedge F \Big) \right\} \ . \tag{2.9}$$

Here $\omega$ and $H$ are the Kähler two-form on the $\mathbb{P}^1$ and the moment map $\iota_V \omega = dH$, respectively. The first part of (2.9) without $H$ is the non-equivariant version of the topological operator associated to two-cycle. It is convenient to study the generating function $\langle e^{t\,\mathcal{O}_{\mathbb{P}^1}} \rangle$ of the correlators $\langle \mathcal{O}_{\mathbb{P}^1} \dots \mathcal{O}_{\mathbb{P}^1} \rangle$. This cause a shift of instanton parameter by $q \to q\exp(tH)$ at the fixed points of the blow-up $\hat{\mathbb{C}}^2$ [5, 24, 25]. The expectation value of the generating function can be written as

$$\hat{\mathcal{Z}}^t \equiv \langle e^{t\mathcal{O}_{\mathbb{P}^1}} \rangle = \sum_{\vec{k} \in \Lambda} \mathcal{Z}^{(N),t}(\vec{k}) \cdot \mathcal{Z}^{(S),t}(\vec{k})\ , \tag{2.10}$$

where

$$\begin{aligned} \mathcal{Z}^{(N),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, q\exp(t\epsilon_1), \vec{m} + \tfrac{1}{2}\epsilon_1)\ , \\ \mathcal{Z}^{(S),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, q\exp(t\epsilon_2), \vec{m} + \tfrac{1}{2}\epsilon_2)\ . \end{aligned} \tag{2.11}$$

Now, as we shrink the two-cycle $\mathbb{P}^1$ to recover the flat $\mathbb{C}^2$, the effect of inserting $(\mathcal{O}_{\mathbb{P}^1})^d$ turns out to give a vanishing contribution for small $d$ due to the selection rule. We recall

[3] This simple picture does not necessarily hold when there are too many hypermultiplets, due to some subtle scheme dependence related to the wall-crossing [35].

that the instanton breaks the $U(1)_R$ symmetry to the discrete subgroup $\mathbb{Z}_{2b_0}$ with $b_0 = 2h^\vee - \sum_i I_2(\mathbf{R}_i)$ where the sum is over all hypermultiplets and $h^\vee$ is the dual Coxeter number of the gauge group and $\mathbf{R}_i$ denotes the representation of the $i$-th hypermultiplet and $I_2(\mathbf{R})$ being the quardratic Dynkin index.[4] The first term of the operator $\mathcal{O}_{\mathbb{P}^1}$ (the two-form piece) carries $R$-charge $+2$. The correlation functions vanish unless the discrete charges add up to zero, modulo $4h^\vee - 2\sum_i I_2(\mathbf{R}_i)$. Therefore, expanding (2.10) in powers of $t$, we find

$$\boxed{\langle e^{t\mathcal{O}_{\mathbb{P}^1}} \rangle = \mathcal{Z} + \mathcal{O}(t^{2h^\vee - \sum_i I_2(\mathbf{R}_i)})} \ . \tag{2.12}$$

This is our blowup equation. The higher order terms in $t$ are generally non-trivial since $(\mathcal{O}_{\mathbb{P}_1})^n$ generally contains terms of $R$-charge between 0 and $2n$ and the instanton background carries (discretely broken) $R$-charges. **[JS: Need more explanation.]** As long as the hypermultiplet representation is not too large, i.e., when $b_0 = 2h^\vee - \sum_i I_2(\mathbf{R}_i) > 2$, this allows us to write 3 independent relations for the 3 unknown varables. One can expand $\langle e^{t\,\mathcal{O}_{\mathbb{P}_1}} \rangle$ until $\mathcal{O}(t^2)$ order and recursively solve for $\mathcal{Z}$ at each instanton number. So the instanton part of the partition function will be completely determined from the perturbative partition function. An explicit form of the recursion relation will be studied in Section 2.2.

**Correlation functions in 5d** We now turn to 5d $\mathcal{N}=1$ gauge theory wrapped on $S^1$. The Casimir invariant $\mathrm{Tr}(\Phi^2)$ and its descendants are no longer considered as well-defined observables. Instead, there are two types of $\mathcal{Q}$-invariant observables [38]. The first type of observables are constructed from the 5d Wilson loop on the $S^1$ by applying the descent procedure. The second type of observables introduce the 3d (Kähler) Chern-Simons term, which can be written as [38, 39]

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^1} = \exp\Bigg[\int\Bigg(&\omega\wedge\mathrm{Tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big)\\ &+\omega\wedge\Big(\phi\,F + \frac{1}{2}\psi\wedge\psi\Big)\wedge dt + H\,\mathrm{Tr}\Big(F\wedge F\Big)\wedge dt\Bigg)\Bigg]\ .\end{aligned} \tag{2.13}$$

It can be viewed as the natural $S^1$ uplift of (2.9) via exponentiation. The correlation function is now given by

$$\hat{\mathcal{Z}}^d \equiv \langle(\mathcal{O}_{\mathbb{P}^1})^d\rangle = \sum_{\vec{k}\in\Lambda} \mathcal{Z}^{(N),d}(\vec{k})\cdot\mathcal{Z}^{(S),d}(\vec{k})\ , \tag{2.14}$$

where

$$\begin{aligned}\mathcal{Z}^{(N),d}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_1, \epsilon_1, \epsilon_2-\epsilon_1, q\exp\left((d-\tfrac{b}{2})\epsilon_1\right), \vec{m}+\tfrac{1}{2}\epsilon_1)\ ,\\ \mathcal{Z}^{(S),d}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2, q\exp\left((d-\tfrac{b}{2})\epsilon_2\right), \vec{m}+\tfrac{1}{2}\epsilon_2)\ .\end{aligned} \tag{2.15}$$

Here the quantity $b$ is given as

$$b \equiv h^\vee - \frac{1}{2}\sum_i I_2(\mathbf{R}_i) - \kappa_{\text{eff}}\ , \quad \kappa_{\text{eff}} = \kappa - \frac{1}{2}\sum_i I_3(\mathbf{R}_i)\ , \tag{2.16}$$

[4] We normalize it so that $I_2(\mathbf{F}) = 1$ for the fundamental representation $\mathbf{F}$.

where $I_2(\mathbf{R})$ and $I_3(\mathbf{R})$ are quadratic and cubic Casimir invariants respectively. Notice that $d$ has to be an integer to be gauge-invariant.

The reason that the instanton parameter is furthered shifted by $\exp(\frac{b}{2}H|_p)$ is because the instanton mass parameter is twisted by $SU(2)_R$ as in the case of the hypermultiplet mass. Moreover, notice that the instanton soliton carries the $SU(2)_R$ twisted effective mass, given by $m_{\text{inst}} \equiv m_{0,\text{eff}} - \kappa_{\text{eff}}\,\epsilon_+$. In addition, the effective Chern-Simons coupling $\kappa_{\text{eff}}$ induces an electric charge to the instanton, contributing to its ground state energy as $E_0 = m_{\text{inst}} - \vec{a}\cdot\vec{\Pi}$, where $\vec{\Pi}$ is the $U(1)^r \subset G$ electric charge.[5] To keep it invariant the effective instanton mass $m_{\text{inst}}$ at a fixed point $p$ of the blow-up $\hat{\mathbb{C}}^2$, we require the shifted gauge coupling $m_0|_p$ to be

$$m_0|_p = m_0 + \frac{b}{2}H|_p \quad \text{with} \quad b \equiv h^\vee - \sum_i \frac{I_2(\boldsymbol{R}_i)}{2} - \kappa_{\text{eff}}. \tag{2.17}$$

For the case of 5d pure $\mathcal{N}=1$ SYM, the correlation function turns out to be

$$\boxed{\langle(\mathcal{O}_{\mathbb{P}^1})^d\rangle = \mathcal{Z} \quad \text{for} \quad 0 \le d \le d_{\max}}\,, \tag{2.18}$$

where $d_{\max} = h^\vee$.[6] We call (2.18) as the blowup equation. The value of $d_{\max}$ depends on the matter content and gauge group. For $d_{\max} \ge 2$, there are sufficient number of algebraic relations to determine the instanton partition function recursively in increasing order of instantons. This fact was utilized in [16] to compute instanton partition function for the gauge theories with exceptional gauge groups, for which the ADHM construction of instanton moduli space is unknown.

In this paper, we aim at developing the relation (2.18) for various 5d $\mathcal{N}=1$ gauge theories with hypermultiplets in various representations, and compute the instanton partition function. We will identify a certain bound on $d$ in Section 2.3 as the *necessary* condition for (2.18) for a large number of theories. We conjecture that the bound on $d$ we obtain is actually sufficient to obtain the blowup equation (2.18). While we do not attempt to prove this sufficiency, we compute $n$-instanton partition function $Z_n$, based on the recursion formula that will be derived shortly from (2.18), and confirm the agreement with the known result obtained from an alternative method.

We find a universal expression for the bound on $d$ when the gauge group is not of $SU$ or $Sp$ type:

$$d_{\max} = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) \quad \text{for } G \ne SU(N) \text{ or } Sp(N) \tag{2.19}$$

This is essentially identical condition as in $4d$ $\mathcal{N}=2$ gauge theory. But in 5d, some new effects come into play. For the $SU(N)$ case, we can have a Chern-Simons term generated at 1-loop, which alters the bound on $d$. When there is neither bare nor effective Chern-Simons coupling, the same bound holds for the $SU(N)$ case as well. The detailed condition will be given in section **??**. For the case of $Sp(N)$, one can turn on the discrete $\theta$-parameter and it turns out the bound on $d$ depends on this parameter.

[5] This agrees with the supersymmetric Casimir energy of the ADHM quantum mechanics.

[6] This was shown in [25] for the case of $G = SU(N)$.

### 2.2 Recursion formula for 5d instanton partition function

The blowup equation (2.18) can be translated to a recursion formula on the (5d) $n$-instanton contribution $Z_n$ to the full partition function $\mathcal{Z}$. To derive this, we decompose the partition function $\mathcal{Z}$ in terms of the classical, one-loop, and instanton pieces:

$$\mathcal{Z}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m}) = Z_{\text{class}}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m}) \cdot Z_{\text{1-loop}}(\vec{a},\epsilon_1,\epsilon_2,\vec{m}) \cdot Z_{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m}) \tag{2.20}$$

where $Z_{\text{inst}}$ can be further expanded in terms of the instanton fugacity $q$ as[7]

$$Z_{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m}) = \sum_{n\geq 0} q^n Z_n(\vec{a},\epsilon_1,\epsilon_2,m)\ . \tag{2.21}$$

Then the blowup equation (2.18) can be written as

$$\begin{aligned} Z_{\text{inst}} &= \sum_{\vec{k}} \left[ \frac{Z^{(N),d}_{\text{class}}(\vec{k})\, Z^{(S),d}_{\text{class}}(\vec{k})}{Z_{\text{class}}} \frac{Z^{(N),d}_{\text{1-loop}}(\vec{k})\, Z^{(S),d}_{\text{1-loop}}(\vec{k})}{Z_{\text{1-loop}}} \right] Z^{(N),d}_{\text{inst}}(\vec{k})\, Z^{(S),d}_{\text{inst}}(\vec{k}) \\ &\equiv \sum_{\vec{k}} f_d(\vec{k}) Z^{(N),d}_{\text{inst}}(\vec{k})\, Z^{(S),d}_{\text{inst}}(\vec{k})\ , \end{aligned} \tag{2.22}$$

where the superscript $(N/S), d$ implies the appropriate shift of parameters, specified in (2.11). The function $f_d(\vec{k})$ is determined only via perturbative part of the partition function.

We recall the known expressions for the classical and 1-loop partition function (in 5d) [3, 40, 41]:[8]

$$Z_{\text{class}} = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}m_0\, h_{ij}a_i a_j + \frac{\kappa}{6} d_{ijk}a^i a^j a^k\right)\right], \tag{2.23}$$

$$\begin{aligned} Z^{\text{vec}}_{\text{1-loop}} &= \exp\left[\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{\alpha}\in\Delta}\Big(\frac{(\vec{a}\cdot\vec{\alpha}+\epsilon_+)^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\alpha}+\epsilon_+)+1\Big)\right] \\ &\quad \times \text{PE}\left[-\frac{p_1p_2}{(1-p_1)(1-p_2)}\sum_{\vec{\alpha}\in\Delta} e^{-\vec{a}\cdot\vec{\alpha}}\right] \qquad \text{for the vector multiplet} \end{aligned} \tag{2.24}$$

$$\begin{aligned} Z^{\text{hyp},\ell}_{\text{1-loop}} &= \exp\left[-\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{\omega}\in\boldsymbol{R}_\ell}\Big(\frac{(\vec{a}\cdot\vec{\omega}+m_\ell)^3}{12}x - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\omega}+m_\ell)+1\Big)\right] \\ &\quad \times \text{PE}\left[+\frac{(p_1p_2)^{\frac{1}{2}}\cdot y_\ell}{(1-p_1)(1-p_2)}\sum_{\vec{\omega}\in\boldsymbol{R}_\ell} e^{-\vec{a}\cdot\vec{\omega}}\right] \qquad \text{for the } \ell\text{'th hypermultiplet} \end{aligned} \tag{2.25}$$

[7] Sometimes the instanton partition function is expanded in powers of the shifted instanton mass $q\exp(-b\frac{\epsilon_1+\epsilon_2}{2})$ instead of $q$ [16, 25, 28]. We expand it with the true instanton fugacity, which makes the symmetry property $\epsilon_{1,2}\to-\epsilon_{1,2}$ of $Z_n$ manifest. This is the one that we obtain using the ADHM quantum mechanics.

[8] The 'Casimir part' of the 1-loop free energy (the ones inside the exp) depends on the choice of ...

where $p_1 \equiv e^{-\epsilon_1}$, $p_2 \equiv e^{-\epsilon_2}$, $y_i \equiv e^{-m_i}, q \equiv e^{-m_0}$.[9] Also $\Delta$ is the set of all roots and $\mathbf{R}_\ell$ denotes the set of all weight vectors in representation $\mathbf{R}_\ell$. Here, PE represents the Plethystic exponential

$$\mathrm{PE}\left[f(\vec{a},\epsilon_1,\epsilon_2,m_0,\vec{m})\right] \equiv \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f(n\vec{a},n\epsilon_1,n\epsilon_2,nm_0,n\vec{m})\right). \tag{2.26}$$

We also set the radius of $S^1$ as $\beta = 1$. Also, the symbols $h_{ij}$ and $d_{ijk}$ are defined as

$$h_{ij} = \mathrm{Tr}(T_iT_j)\ , \quad d_{ijk} = \frac{1}{2}\mathrm{Tr}T_i\{T_j,T_k\}\ , \tag{2.27}$$

where $T_i$ are the generators of the gauge algebra. They satisfy the relations

$$\begin{aligned}\sum_{\vec{\omega}\in\boldsymbol{R}}(\vec{a}\cdot\vec{\omega})(\vec{b}\cdot\vec{\omega})(\vec{c}\cdot\vec{\omega}) &= I_3(\boldsymbol{R})\,d_{ijk}\,a^ib^jc^k,\\ \sum_{\vec{\omega}\in\boldsymbol{R}}(\vec{a}\cdot\vec{\omega})(\vec{b}\cdot\vec{\omega}) &= I_2(\boldsymbol{R})\,h_{ij}\,a^ib^j,\\ \sum_{\vec{\omega}\in\boldsymbol{R}}(\vec{a}\cdot\vec{\omega}) &= 0,\end{aligned} \tag{2.28}$$

where $I_2(\mathbf{R})$ and $I_3(\mathbf{R})$ are the quadratic and cubic Dynkin indices.

Substituting them to (2.22), we obtain the ratio of three different $Z$'s given as

$$f_d(\vec{k})_{\text{class}} = q^{\frac{\vec{k}\cdot\vec{k}}{2}}\,(p_1p_2)^{(\frac{b}{2}-d)(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa}{6}d_{ijk}\,k^ik^jk^k}\times e^{-(\frac{b}{2}-d)(\vec{a}\cdot\vec{k})}\,e^{-\frac{\kappa}{2}d_{ijk}\,a^ik^jk^k}\ , \tag{2.29}$$

$$f_d(\vec{k})^{\text{vec}}_{\text{1-loop}} = e^{\frac{h^\vee}{2}(\vec{a}\cdot\vec{k})}\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)^{-1}\ , \tag{2.30}$$

$$\begin{aligned}f_d(\vec{k})^{\text{hyp}}_{\text{1-loop}} &= e^{-\frac{I_2(\boldsymbol{R}_\ell)}{4}(\vec{a}\cdot\vec{k})+\frac{I_3(\boldsymbol{R}_\ell)}{4}d_{ijk}\,a^ik^jk^k}\,(p_1p_2)^{\frac{I_2(\boldsymbol{R}_\ell)}{8}(\vec{k}\cdot\vec{k})-\frac{I_3(\boldsymbol{R}_\ell)}{12}d_{ijk}\,k^ik^jk^k}\\ &\quad\times y_\ell^{-\frac{I_2(\boldsymbol{R}_\ell)}{4}(\vec{k}\cdot\vec{k})}\prod_{\omega\in\boldsymbol{R}_\ell}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\text{phy},\ell},\epsilon_1,\epsilon_2)\ ,\end{aligned} \tag{2.31}$$

where we split the $f_d(\vec{k})$ into classical and 1-loop pieces for vector and hypermultiplet. Here we used $I_2(\mathbf{adj}) = 2h^\vee$, $I_3(\mathbf{adj}) = 0$, and also the fact $h_{ij}$ and $d_{ijk}$ are totally symmetric. The function $\mathcal{L}_k(x,\epsilon_1,\epsilon_2)$ is introduced to denote concisely the combination of the PE parts:

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) \equiv \mathrm{PE}\left[e^{-x}\left(\frac{p_1^k\,p_2}{(1-p_1)(1-\frac{p_2}{p_1})}+\frac{p_1\,p_2^k}{(1-\frac{p_1}{p_2})(1-p_2)}-\frac{p_1p_2}{(1-p_1)(1-p_2)}\right)\right] \tag{2.32}$$

One can easily check that the expression inside the PE vanishes at $k = 0, 1$. After some work, it is not difficult to find that

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) = \begin{cases}\displaystyle\prod_{m+n\le k-2}(1-p_1^{m+1}p_2^{n+1}e^{-x}) & \text{for } k\ge +2\\ \displaystyle\prod_{m+n\le -k-1}(1-p_1^{-m}p_2^{-n}e^{-x}) & \text{for } k\le -1\\ 1 & \text{for } k=0,1.\end{cases}\ . \tag{2.33}$$

[9] We assume a particular Weyl chamber in the Coulomb branch, i.e., $0 < a_i < \epsilon_+ < m$ for all $i \in \{1,\cdots,r\}$.

Combining them all, the resursion formula on the $n$-instanton piece $Z_n$ can be written as

$$Z_n = \sum_{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n} \Bigg( (p_1p_2)^{(\frac{h^\vee}{2}-\frac{\kappa_{\rm eff}}{2}-d)(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa_{\rm eff}}{6}d_{ijk}\,k^ik^jk^k} e^{(d+\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\rm eff}}{2}d_{ijk}\,a^ik^jk^k} \\ \times \frac{\prod_l {y_l}^{-I_2(\boldsymbol{R}_l)(\frac{\vec{k}\cdot\vec{k}}{4})} \prod_{\omega\in\boldsymbol{R}_l} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{{\rm phy},l},\epsilon_1,\epsilon_2)}{\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)} \cdot p_1^{(\frac{b}{2}-d)\ell} p_2^{(\frac{b}{2}-d)m} Z^{(N)}_\ell(\vec{k}) Z^{(S)}_m(\vec{k}) \Bigg), \tag{2.34}$$

which is a generalization of the recursion formula found for the pure SYM case [25, 28]. Here $l$ runs over all hypermultiplets in the theory.

**Solving the recursion formulae** The recursion relation (2.34) can be rewritten as

$$Z_n = p_1^{n(\frac{b}{2}-d)} Z_n^{(N)} + p_2^{n(\frac{b}{2}-d)} Z_n^{(S)} + I_n^{(d)} \qquad \text{with an allowed range of } d, \tag{2.35}$$

where $I_n^{(d)}$ is defined as

$$I_n^{(d)} = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n\\ \ell,m\neq n}} \Bigg( (p_1p_2)^{(\frac{h^\vee}{2}-\frac{\kappa_{\rm eff}}{2}-d)(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa_{\rm eff}}{6}d_{ijk}\,k^ik^jk^k} e^{(d+\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\rm eff}}{2}d_{ijk}\,a^ik^jk^k} \\ \times \frac{\prod_l {y_l}^{-I_2(\boldsymbol{R}_l)(\vec{k}\cdot\vec{k}/4)} \prod_{\omega\in\boldsymbol{R}_l} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{{\rm phy},l},\epsilon_1,\epsilon_2)}{\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)} \cdot p_1^{(\frac{b}{2}-d)\ell} p_2^{(\frac{b}{2}-d)m} Z^{(N)}_\ell(\vec{k}) Z^{(S)}_m(\vec{k}) \Bigg), \tag{2.36}$$

where $y_{{\rm phy},l} \equiv e^{-m_{{\rm phy},l}} = y_l/\sqrt{p_1p_2}$. Notice that we have a set of equations labelled by the parameter $d$. If the blowup equation holds for at least 3 values of $d$, we can solve it for $Z_n$. The $n$-instanton partition function $Z_n$ is given as the solution to the three linear equations (2.35) with consecutive integers $\{d_0, d_0+1, d_0+2\}$.

$$Z_n(\vec{a},\epsilon_1,\epsilon_2,\vec{m}) = \frac{p_1^np_2^n\, I_n^{(d_0+2)} - (p_1^n+p_2^n)\, I_n^{(d_0+1)} + I_n^{(d_0)}}{(1-p_1^n)(1-p_2^n)} \tag{2.37}$$

Since $I_n^{(d)}$ only involves low-order instanton corrections, the $n$-instanton partition function $Z_n$ can be constructed from $Z_{m<n}$, allowing us to obtain the full non-perturbative part $Z_{\rm inst}$ in a recursive manner starting from $Z_0 = 1$.

Now, let us write the solution for 1-instanton explicitly. At one instanton level, the formula (2.36) can be written as

$$I_1^{(d)} = \sum_{\vec{k}\in\Delta_\ell} \Bigg( (p_1p_2)^{(\frac{b}{2}-d)} e^{(d+\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\rm eff}}{2}d_{ijk}\,a^ik^jk^k} \\ \times \frac{\prod_l y_{{\rm phy},l}^{-I_2(\boldsymbol{R}_l)/2} \prod_{\omega\in\boldsymbol{R}_l} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{{\rm phy},l},\epsilon_1,\epsilon_2)}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-p_1^{-1}e^{\vec{a}\cdot\vec{k}})(1-p_2^{-1}e^{\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})} \Bigg), \tag{2.38}$$

where $\Delta_\ell$ is the set of long roots ($\vec{k}\cdot\vec{k}=2$) and we used $Z_0=1$. It turns out to be more convenient to express $Z_1$ by decomposing $I_1^{(d)}$ into the flux sum, *i.e.*, $I_1^{(d)}\equiv\sum_{\vec{k}\in\Delta_\ell} i_1^{(d)}(\vec{k})$, where

$$i_1^{(d)}(\vec{k})\equiv(p_1p_2)^{(\frac{b}{2}-d)}\,e^{(d+\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})}e^{-\frac{\kappa_{\rm eff}}{2}d_{ijk}a^ik^jk^k} \times\frac{\prod_l(y_l^{\rm phy})^{-I_2(\boldsymbol{R}_l)/2}\prod_{\omega\in\boldsymbol{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{{\rm phy},l},\epsilon_1,\epsilon_2)}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})(1-p_1^{-1}e^{\vec{a}\cdot\vec{k}})(1-p_2^{-1}e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})}. \tag{2.39}$$

Using the property $i_1^{(d_0+\aleph)}(\vec{k})/i_1^{(d_0)}(\vec{k})=(p_1p_2)^{-\aleph}\,e^{\aleph(\vec{a}\cdot\vec{k})}$, the one-instanton partition function $Z_1$ can be written as

$$\begin{aligned}Z_1&=\sum_{\vec{k}\in\Delta_\ell}\frac{(1-p_1^{-1}e^{\vec{a}\cdot\vec{k}})(1-p_2^{-1}e^{\vec{a}\cdot\vec{k}})}{(1-p_1)(1-p_2)}\cdot i_1^{(d_0)}(\vec{k}) &(2.40)\\ &=\frac{(p_1p_2)^{(\frac{b}{2}-d_0)}\prod_l(y_l^{\rm phy})^{-\frac{I_2(\boldsymbol{R}_l)}{2}}}{(1-p_1)(1-p_2)}\sum_{\vec{k}\in\Delta_l}\frac{e^{(d_0+\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})-\frac{\kappa_{\rm eff}}{2}d_{ijk}a^ik^jk^k}\prod_{\omega\in\boldsymbol{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_l^{\rm phy})}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})}.\end{aligned}$$

Notice that there are multiple options for choosing $d_0$. However, we find that (2.40) is independent of a specific choice of $d_0$. Once we choose $d_0=0$, for instance, which works in most cases,[10] (2.40) becomes

$$\boxed{Z_1=\frac{(p_1p_2)^{\frac{b}{2}}\prod_l(y_l^{\rm phy})^{-\frac{I_2(\boldsymbol{R}_l)}{2}}}{(1-p_1)(1-p_2)}\sum_{\vec{k}\in\Delta_\ell}\frac{e^{\frac{\kappa_{\rm eff}}{2}(\vec{a}\cdot\vec{k}-d_{ijk}a^ik^jk^k)}\prod_{\omega\in\boldsymbol{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_l^{\rm phy})}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})}}\,. \tag{2.41}$$

When the hypermultiplets are in the representations with $|\vec{k}\cdot\vec{w}|\leq 1$ for all $\vec{w}\in\mathbf{R}$, we have

$$\prod_{\omega\in\boldsymbol{R}}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\rm phy},\epsilon_1,\epsilon_2)=\prod_{\vec{k}\cdot\vec{\omega}=-1}(1-y_{\rm phy}e^{-\vec{a}\cdot\vec{\omega}}). \tag{2.42}$$

The formula (2.40) indeed reduces to the pure YM partition function derived in [15, 16] upon removing hypermultiplets and Chern-Simons levels up to the overall factor $(p_1p_2)^{\frac{b}{2}}=e^{-\frac{h^\vee}{2}(\epsilon_1+\epsilon_2)}$ that accounts for the shift of instanton fugacity.

We claim that (2.40) is the closed-form expression for the one-instanton partition function, which holds *universally for any gauge theory* with $d_{\rm max}>2$. In section 2.3, we study the structure of the blowup equations to bound the number of possible independent equations.

### 2.3 Number of independent blowup equations

We are mainly interested in 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ gauge theories which are UV-complete. UV complete 4d $\mathcal{N}=2$ gauge theories are classified in [42]. In the case of 5d SCFTs that are UV complete as 5d SCFTs, the possible matter representations are restricted to [43]

[10] A numerical value of $d_0$ should be a half-integer for theories with $G=Sp(N)_{\theta=\pi}$.

- fundamental representation for $SU(N)$, $SO(N)$, $Sp(N)$, $G_2$, $F_4$, $E_6$, $E_7$
- antisymmetric representation for $SU(N)$, $Sp(N)$
- spinor representation for $SO(N)$ with $7 \leq N \leq 14$
- rank-3 antisymmetric representation for $Sp(3)$, $Sp(4)$, $SU(6)$, $SU(7)$
- symmetric representation for $SU(N)$.

when the gauge group is simple. In the case of 4d, we can also have following additional cases:

- adjoint representation for arbitrary group
- rank-3 antisymmetric for $SU(8)$
- **16** for $Sp(2)$ (half-hypermultiplet)

We are able to obtain the partition function for most of the cases in 5d using our formula. But we are not able to apply our formula for some cases including the one with adjoint hypremultiplets since the number of independent blowup equations is smaller than 3.

The formula (2.34) is valid only for a certain range of $d$, for which $\langle (\mathcal{O}_{\mathbb{P}^1})^d \rangle = \mathcal{Z}$. We want to narrow down the valid range of $d$ by performing a simple sanity check on the blowup equation for the one-instanton partition function:

$$Z_1 = p_1^{\frac{b}{2}-d} Z_1^{(N)} + p_2^{\frac{b}{2}-d} Z_1^{(S)} + I_1^{(d)} \qquad \text{with an allowed range of } d\ . \tag{2.43}$$

Specifically, we want to examine the expansion of each term in (2.43) with in powers of $p_1 p_2 \ll 1$. The leading exponent is identified as follows:

$$\begin{aligned} I_1^{(d)} \sim & \begin{cases} g_0(\vec{a}, \vec{m}_{\rm tw}) \cdot (p_1 p_2)^{\frac{b}{2}-d+1} + \cdots & \text{for } N_{\rm sym} = 0 \\ g_0(\vec{a}, \vec{m}_{\rm tw}) \cdot (p_1 p_2)^{\frac{b}{2}-d} + \cdots & \text{for } N_{\rm sym} = 1 \end{cases} \\ Z_1 \sim \ & g_1(\vec{a}, \vec{m}_{\rm tw}) \cdot (p_1 p_2)^{\frac{s}{2}} + \cdots \\ p_1^{\frac{b}{2}-d} Z_1^{(N)} \sim p_2^{\frac{b}{2}-d} Z_1^{(S)} \sim \ & g_2(\vec{a}, \vec{m}_{\rm tw}) \cdot (p_1 p_2)^{\frac{b}{4}-\frac{d}{2}+\frac{s}{4}} + \cdots\ . \end{aligned} \tag{2.44}$$

where $g_{0,1,2}(\vec{a}, \vec{m}_{\rm tw})$ are functions independent of $p_{1,2}$ and a numerical value of $s$ will be discussed shortly for a variety of gauge theories (Table 1) for the cases when ADHM-like construction is available. Notice that for the equation (2.43) to be true, some terms on the right-hand side should have the leading exponent less than or equal to that of $Z_1$. Therefore, the condition $d - \frac{b}{2} \geq -\frac{s}{2}$ is naturally imposed, setting a lower bound on $d$.

Similarly, an upper bound on $d$ can be found from an expansion of (2.43) with respect to $1/p_1p_2 \ll 1$.[11] Each single term in (2.43) can be written as

$$\begin{aligned} I_1^{(d)} &\sim \begin{cases} h_0(\vec{a},\vec{m}_{\rm tw})\cdot(1/p_1p_2)^{d-\frac{b}{2}+1}+\cdots & \text{for } N_{\rm sym}=0 \\ h_0(\vec{a},\vec{m}_{\rm tw})\cdot(1/p_1p_2)^{d-\frac{b}{2}}+\cdots & \text{for } N_{\rm sym}=1 \end{cases} \\ Z_1 &\sim \quad h_1(\vec{a},\vec{m}_{\rm tw})\cdot(1/p_1p_2)^{\frac{n'}{2}}+\cdots \\ p_1^{(\frac{b}{2}-d)}Z_1^{(N)} \sim p_2^{(\frac{b}{2}-d)}Z_1^{(S)} &\sim \quad h_2(\vec{a},\vec{m}_{\rm tw})\cdot(1/p_1p_2)^{\frac{d}{2}-\frac{b}{4}+\frac{n'}{4}}+\cdots \end{aligned} \tag{2.45}$$

Again, for (2.43) to be consistent, the leading exponent of $Z_1$ should be greater than or equal to those of some terms on the right-hand side. Such requirement imposes the upper bound on $d$, namely $\frac{n'}{2} \geq d-\frac{b}{2}$. Combining the two inequalities, one can identify the following range

$$-\frac{n}{2}+\frac{b}{2} \leq d \leq \frac{n'}{2}+\frac{b}{2}, \tag{2.46}$$

as a necessary condition for (2.43). Furthermore, we explicitly checked that the $k$-instanton partition function $Z_k$ actually satisfies all the $(\frac{n+n'}{2})$ recursion relations up to a certain value of $k$, for numerous examples (Table 1) whose $Z_k$ is already known from alternative methods, even though the bound (2.46) itself is merely a *necessary* condition found from one-instanton analysis. Recall that the simple selection rule, governed by $\mathbb{Z}_{4h^\vee-2\sum_i I_2(\mathbf{R}_i)} \subset U(1)_R$, supports the validity of (2.12) at all instanton orders in 4d Donaldson-twisted theories. Based on the empirical observations, we claim that the 5d recursion formulae (2.34) within the above range of $d$ are again true at all instanton orders. This implies that the non-perturbative partition function $Z_{\rm inst}$ is completely fixed by the perturbative partition function! Note that we have arrived at this statement not by requiring the perturbative series to be well-defined, as is often done in the resurgence analysis, but just by demanding consistency upon smooth deformation of the spacetime $\mathbb{C}^2\times S^1$. Such consistency condition turns out to be entirely non-perturbative.

Another remarkable thing is that a numerical value of $(n,n')$ exhibits the very simple pattern across a broad range of theories whose gauge group is not $SU(N)_\kappa$. (See Table 1.)

$$\begin{aligned} n = n' &= h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) && \text{for} \quad G \neq SU(N)_\kappa \text{ nor } Sp(N) \\ n = n' - 2\left\{\frac{N_f}{2}\right\} &= h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) && \text{for} \quad G = Sp(N)_{\theta=0} \\ n = n' + 2\left\{\frac{N_f}{2}\right\} &= h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l)+1 && \text{for} \quad G = Sp(N)_{\theta=\pi} \end{aligned} \tag{2.47}$$

[11]This is equivalent to assuming a different parameter regime $0 < a_i < -\epsilon_+ < m$ for all $1 \leq i \leq r$. In general, an explicit form of the 1-loop partition function (2.24)–(2.25) can change depending on a parameter regime, thus affecting (2.34). However, all the above expressions remain valid under flipping a sign of $\epsilon_+$, such that we can simply study the expansion of the single terms in (2.43) with respect to $1/p_1p_2 \ll 1$.

As the above numerical pattern (2.47) emerges for all $G \neq SU(N)_\kappa$ examples that we studied, we speculate that (2.47) is generally true, thereby taking the recursion formulae (2.34) with

$$\begin{aligned} 0 \le d \le h^\vee - \tfrac{1}{2}\textstyle\sum_l I_2(\mathbf{R}_l) \qquad & \text{for} \quad G \neq SU(N)_\kappa \text{ nor } Sp(N) \\ 0 \le d \le h^\vee - \tfrac{1}{2}\textstyle\sum_l I_2(\mathbf{R}_l) + \{\tfrac{N_f}{2}\} \qquad & \text{for} \quad G = Sp(N)_{\theta=0} \\ -\tfrac{1}{2} \le d \le h^\vee - \tfrac{1}{2}\textstyle\sum_l I_2(\mathbf{R}_l) + \tfrac{1}{2} - \{\tfrac{N_f}{2}\} \qquad & \text{for} \quad G = Sp(N)_{\theta=\pi} \end{aligned} \tag{2.48}$$

as a basic assumption to obtain the partition function $\mathcal{Z}$ for any $G \neq SU(N)_\kappa$ gauge theory. We will derive the general expression for the 1-instanton partition function $Z_1$, which applies for all $G \neq SU(N)_\kappa$ gauge theories having $h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) \geq 2$. It would be desirable to understand from the first principle the range (2.48) of $d$ for which (2.34) holds true.

On the contrary, it turns out to be more difficult to characterize a general pattern behind $(n, n')$ for $SU(N)_\kappa$ gauge theories, due to extra complication caused by the 5d Chern-Simons level $\kappa$. Here we consider two particular classes of $SU(N)_\kappa$ gauge theories as an illustration. In $SU(N)_\kappa + N_f\mathbf{F}$ gauge theory with $N_f + 2|\kappa| \leq 2N$, we find that

$$\begin{aligned} n &= \begin{cases} \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = N - N_f, \\ h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{F}) + |\kappa_{\text{eff}}| & \text{otherwise.} \end{cases} \\ n' &= \begin{cases} \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = -N, \\ h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{F}) + |\kappa_{\text{eff}} + \sum_l I_3(\mathbf{F})| & \text{otherwise.} \end{cases} \end{aligned} \tag{2.49}$$

Plugging these values in (2.46), the range of $d$ becomes

$$\begin{aligned} 0 \le d \le N \qquad & \text{if } \kappa = -N + \tfrac{N_f}{2} \\ 0 \le d \le N - \tfrac{N_f}{2} - \kappa \qquad & \text{if } \kappa \in (-N + \tfrac{N_f}{2}, -\tfrac{N_f}{2}] \\ 0 \le d \le N \qquad & \text{if } \kappa \in [-\tfrac{N_f}{2}, +\tfrac{N_f}{2}] \\ \tfrac{N_f}{2} - \kappa \le d \le N \qquad & \text{if } \kappa \in [\tfrac{N_f}{2}, N - \tfrac{N_f}{2}) \\ 0 \le d \le N \qquad & \text{if } \kappa = N - \tfrac{N_f}{2} \end{aligned} \tag{2.50}$$

including $0 \leq d \leq N$ in any case. Thus for $N \geq 2$, the recursion formula (2.34) determines the partition function $\mathcal{Z}$ completely. In $SU(N)_\kappa + N_f\mathbf{F} + 1\mathbf{AS}$ theory with $N_f + 2|\kappa| \leq N + 4$,

$$\begin{aligned} n &= \min\left(h^\vee - \tfrac{1}{2}\textstyle\sum_l I_2(\mathbf{R}_l) - (\kappa_{\text{eff}} - 2),\ h^\vee - \tfrac{1}{2}\sum_l I_2(\mathbf{R}_l) + 2\{\tfrac{\kappa_{\text{eff}}}{2}\}\right) \\ n' &= \min\left(h^\vee - \tfrac{1}{2}\textstyle\sum_l I_2(\mathbf{R}_l) + (\kappa_{\text{eff}} + \sum_l I_3(\mathbf{R}_l) + 2),\right. \\ &\qquad \left. h^\vee - \tfrac{1}{2}\textstyle\sum_l I_2(\mathbf{R}_l) + 2\{-\tfrac{\kappa_{\text{eff}}}{2} + \tfrac{1}{2}\sum_l I_3(\mathbf{R}_l)\}\right), \end{aligned} \tag{2.51}$$

from which one can identify the validity range of $d$ by making substitution to (2.46). So long as three distinct integers are allowed for $d$, at a given $(N, \kappa)$, the corresponding partition function $\mathcal{Z}$ can be solved from the recursion formula (2.34). We will also consider $SU(6)_\kappa + 1\mathbf{TAS}$ theory with $|\kappa| \leq 3$ in Section 3. This model can be Higgsed to two disjoint copies of $SU(3)_\kappa$

theory without a bifundamental hypermultiplet [44]. At the level of the partition function, the Higgsing is realized by turning off $m_{\text{phys}} = 0$ and imposing $SU(3)$ traceless conditions. As neither of them can modify $n$ and $n'$, the numerical value of $(n, n')$ must be identical to that of $SU(3)_\kappa$ gauge theory, which always contains $0 \leq d \leq 3$. Therefore, the recursion formula (2.34) sufficiently determines the full partition function $\mathcal{Z}$ for $SU(6)_\kappa + 1\mathbf{TAS}$ theory as well.

## 3 Examples

Recall that the recursion formula (2.34) for the $k$-instanton partition function, and especially the general expression (2.40) at one-instanton order, are widely applicable to 5d $\mathcal{N} = 1$ gauge theory whose $(n, n')$ satisfies $\frac{n+n'}{2} \geq 2$. Combined with the observation that $(n, n')$ follows (2.47) in most cases, they actually become a very efficient approach at obtaining the BPS partition function on $\mathbb{C}^2 \times S^1$, unless a matter representation is too large.

Conventionally, the instanton partition function has been computed by employing the ADHM construction of the instanton moduli space [3, 4, 45] or by applying the topological vertex formalism to the 5-brane web [46, 47]. Both are based on a certain UV realization of 5d $\mathcal{N} = 1$ gauge theory, motivated by its string theory engineering. Sometimes the UV completion of the gauge theory can be achieved only with specific string theory realizations, but not with the others. For instance, $SU(2)+N_f\mathbf{F}$ gauge theory with $N_f \geq 5$ must be embedded into D4-D8-O8 brane system, to be UV-completed as 5d $E_{N_f+1}$ Minahan-Nemeschansky SCFT, instead of a typical $(p, q)$ 5-brane web with colliding branes which indicate UV inconsistency. A sensible UV QFT observable can thus be obtained only through a proper embedding of the gauge theory into string theory. At some occasions, an extra factor entangled with a true QFT observable may appear during the above instanton computation, which is sensitive to the choice of a string theory embedding.

Our blow-up formula (2.34) implicitly assumes a particular string theory embedding of the gauge theory. There are a wide variety of 'exceptional' gauge theories, whose UV completion is found as M-theory wrapped on a singular Calabi-Yau 3-fold [48–51], but for which neither the ADHM quantum mechanics nor the topological vertex formalism has been developed. We will illustrate that bootstrapping the partition function $\mathcal{Z}$ based on the recursion formula (2.34) works well for those 'exceptional' theories, providing their BPS spectrum efficiently.

### 3.1 Theories with the known ADHM description

Let us first consider the 'standard' gauge theories with classical gauge groups, whose hypermultiplet admits UV realization as a perturbative string ending on non-perturbative branes. In these cases, the ADHM construction of the instanton moduli space is well-known [3, 41, 45]. As for the $k$-instanton partition function $Z_k$, the Witten index of the relevant ADHM quantum mechanics can be computed by SUSY localization [19, 52–54], ending up collecting all Jeffrey-Kirwan residues of a multi-dimensional contour integral. We will examine whether the recursion formula (2.34) actually produces the same result as the localization computation.

**SU(N)** The ADHM computation of the $k$-instanton partition function, for $SU(N)_\kappa + N_f\mathbf{F}$ theories with $N_f + 2|\kappa| \leq 2N$, can be summarized as the Young tableaux sum expression.

$$Z_k^{\rm ADHM} = \sum_{|\vec{Y}|=k} \prod_{i=1}^{N} \prod_{s\in Y_i} \frac{e^{-\kappa\phi(s)} \prod_{l=1}^{N_f} 2\sinh\frac{\phi(s)+m_l}{2}}{\prod_{j=1}^{N} 2\sinh\frac{E_{ij}}{2}\, 2\sinh\frac{E_{ij}-2\epsilon_+}{2}} \tag{3.1}$$

where

$$\begin{aligned} E_{ij}(s) &= a_i - a_j - \epsilon_1 h_i(s) + \epsilon_2(v_j(s)+1) \\ \varphi(s) &= a_i - \epsilon_+ - (n-1)\epsilon_1 - (m-1)\epsilon_2 \qquad\quad \text{for} \quad s=(m,n)\in Y_i. \end{aligned}$$

$h_i(s)$ denotes the distance from $s$ to the right end of the diagram $Y_i$ by moving right. $v_j(s)$ denotes the distance from $s$ to the bottom of the diagram $Y_j$ by moving down.

We checked that the instanton corrections $Z_1$ and $Z_2$ obtained from the recursion formulae (2.34) with (2.50) and the 1-instanton expression (2.40) precisely agree with the above $Z_{k=1,2}^{\rm ADHM}$ for $N = 2, 3, 4$. Recall that $Z_k^{\rm ADHM}$ often contains an additional factor $Z_{\rm extra}$ that captures the contribution from an extra branch of vacua. It is sensitive to the string theory embedding of gauge theory and regarded as spurious from the 5d QFT perspective. It is usually factorized from the true QFT partition function as

$$1 + \textstyle\sum_{k=1}^{\infty} q^k\, Z_k^{\rm ADHM}(\vec{a},\epsilon_1,\epsilon_2,\vec{m}) = Z_{\rm QFT}(\vec{a},\epsilon_1,\epsilon_2,\vec{m},\Lambda)\cdot Z_{\rm extra}(\epsilon_1,\epsilon_2,\vec{m},\Lambda). \tag{3.2}$$

A non-trivial $Z_{\rm extra} \neq 1$ appears in the above expression (3.1) if and only if $N_f + 2|\kappa| = 2N$. This factor can be identified as the contribution of D1-branes escaping from D5-branes which engineer the $SU(N)_\kappa + N_f\mathbf{F}$ gauge theory. Since $Z_k = Z_k^{\rm ADHM}$, the same factor $Z_{\rm extra}$ emerges from the recursion formula (2.34) as well. The 5-brane web construction of the gauge theory is thus indirectly reflected in the recursion formula.

A similar observation is that the 1-instanton expression (2.40) applied to $SU(2)_\kappa + N_f\mathbf{F}$ with $N_f \geq 5$ does not match the Witten index of the D0-D4-D8-O8$^-$ quantum mechanics, which is the correct 1-instanton partition function.[12] Instead, it coincides with the topological vertex computation applied to the 5-brane web with a colliding pair of branes, which engineers the $SU(2)$ gauge theory with $N_f \geq 5$ in IR, but behaves badly in UV. Again, this suggests that the recursion formula (2.34) is rooted in a specific string theory construction of the gauge theory, i.e., the web of $(p,q)$ 5-branes. It would be more satisfactory if one can trace at which step in our formulation of (2.34) the particular UV embedding of the gauge theory has been chosen. In fact, there must be a broader set of blow-up relations for the 5d Nekrasov partition function $\mathcal{Z}$, similar to those recently found for 6d SCFTs [32, 33, 56, 57]. We expect that some recursion formulae, derived from the generalized blow-up equations, may realize a different string theory embedding of the gauge theory. It would be very interesting if one can reveal the connection between the choice of UV embedding and the blow-up equations.

[12] The case with $SU(2) \simeq Sp(1)$ is an exception, which allows $N_f \leq 7$ fundamental hypermultiplets [55].

In $SU(N)_\kappa + N_f\mathbf{F} + 1\mathbf{AS}$ theory with $N_f + 2|\kappa| \le N+4$, the ADHM quantum mechanics is the worldvolume theory of D1-branes, probing the D5-NS5-D7-O7$^-$ brane configuration that realizes the gauge theory. Let us compute the Witten index for 1 and 2 D1-branes, then compare with the blow-up computation based on the recursion formula (2.34). For instance, the Witten index for the single D1-brane can be written as

$$\begin{aligned} Z_1^{\mathrm{ADHM}} = &-\sum_{i=1}^{N} \frac{e^{-\kappa(a_i-\epsilon_+)}}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}} \frac{\prod_{l=1}^{N_f} 2\sinh\frac{-\epsilon_+ + a_i + m_l}{2}}{2\sinh\frac{-3\epsilon_+ + 2a_i + m_a}{2}} \prod_{j\neq i} \frac{2\sinh\frac{a_i+a_j+m_a-\epsilon_+}{2}}{2\sinh\frac{a_i-a_j}{2}\,2\sinh\frac{2\epsilon_+ - a_i + a_j}{2}} \\ &-\frac{1}{2}\frac{e^{-\frac{\kappa}{2}(\epsilon_+ - m_a)}}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}} \left( \frac{\prod_{l=1}^{N_f} 2\sinh\frac{\epsilon_+ + 2m_l - m_a}{4}}{\prod_{i=1}^{N} 2\sinh\frac{3\epsilon_+ - M - 2a_i}{4}} - (-1)^{\kappa + (N-N_f)/2} \frac{\prod_{l=1}^{N_f} 2\cosh\frac{\epsilon_+ + 2m_l - m_a}{4}}{\prod_{i=1}^{N} 2\cosh\frac{3\epsilon_+ - M - 2a_i}{4}} \right). \end{aligned} \tag{3.3}$$

Note that $Z_k^{\mathrm{ADHM}}$ captures an extra factor $Z_{\mathrm{extra}} \neq 1$ if $N_f + 2|\kappa| = N+4$, displaying the BPS spectrum of D1-branes escaping from the D5-branes on which the gauge theory is supported. The appearance of $Z_{\mathrm{extra}} \neq 1$ is an artifact of the string theory embedding, spurious from the 5d QFT perspective. We checked the agreement between $Z_1^{\mathrm{ADHM}}$ and the 1-instanton formula (2.40) for the $SU(3)$, $SU(4)$, $SU(5)$ theories whose $(n, n')$ satisfies $\frac{n+n'}{2} \ge 2$. We confirmed $Z_2 = Z_2^{\mathrm{ADHM}}$ as well, where $Z_2$ is the solution of the recursion formulae (2.34) with (2.51). The same spurious factor $Z_{\mathrm{extra}}$ arises from the recursion formula, implying that our blowup equations are implicitly based on the D5-NS5-D7-O7$^-$ brane realization of the gauge theory.[13]

**Sp(N)** The $k$-instanton partition function for $Sp(N)_\theta + N_f\mathbf{F}$ theory with $N_f \le 2N+4$ can be computed from the ADHM quantum mechanics of D1-D5-NS5-O5 branes, which engineers the gauge theory and its instantons. The Witten index for a D1-brane can be written as

$$Z_1^{\mathrm{ADHM}} = \frac{1}{2}\frac{1}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}} \left( \frac{\prod_{l=1}^{N_f} 2\sinh\frac{m_l}{2}}{\prod_{i=1}^{N} 2\sinh\frac{\epsilon_+ \pm a_i}{2}} + e^{i\theta} \frac{\prod_{l=1}^{N_f} 2\cosh\frac{m_l}{2}}{\prod_{i=1}^{N} 2\cosh\frac{\epsilon_+ \pm a_i}{2}} \right). \tag{3.4}$$

Our 1-instanton formula (2.40) was checked to be in agreement with $Z_1^{\mathrm{ADHM}}$ for the $Sp(2)$ and $Sp(3)$ gauge theories satisfying $N-1 \ge \lfloor \frac{N_f}{2} \rfloor$ (at $\theta = 0$) and $N \ge \lceil \frac{N_f}{2} \rceil$ (at $\theta = \pi$). It was also confirmed that $Z_2^{\mathrm{ADHM}} = Z_2$, where $Z_2$ is the solution of the recursion formulae (2.34) with (2.48). Note that $Z_{\mathrm{extra}}$ does not appear in $Z_k^{\mathrm{ADHM}} = Z_k$ for the above theories.

In $Sp(N)_\theta + N_f\mathbf{F} + 1\mathbf{AS}$ theory with $N_f \le 7$, the relevant ADHM quantum mechanics is the worldvolume gauge theory of D0-branes which probe the D4-D8-O8 brane configuration. The QFT on D4-branes sees an enhanced $E_{N_f+1}$ flavor symmetry at the UV fixed point [55]. Let us consider the Witten index for one and two D0-branes [19, 58]. For a single D0-brane,

$$Z_1^{\mathrm{ADHM}} = \frac{1}{2}\frac{1}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}\,2\sinh\frac{m_a+\epsilon_+}{2}\,2\sinh\frac{m_a-\epsilon_+}{2}} \tag{3.5}$$

[13] An exceptional case is the $SU(2)$ gauge theory, in which the antisymmetric hypermultiplet decouples and never affects the recursion formula. The corresponding $Z_k$ is the same as the Young tableau formula (3.1).

$$\times\left(\frac{\prod_{i=1}^{N} 2\sinh\frac{m_a\pm a_i}{2}\prod_{l=1}^{N_f} 2\sinh\frac{m_l}{2}}{\prod_{i=1}^{N} 2\sinh\frac{\epsilon_+\pm a_i}{2}}+\frac{e^{i\theta}\prod_{i=1}^{N} 2\cosh\frac{m_a\pm a_i}{2}\prod_{l=1}^{N_f} 2\cosh\frac{m_l}{2}}{\prod_{i=1}^{N} 2\cosh\frac{\epsilon_+\pm a_i}{2}}\right).$$

We checked $Z_1^{\text{ADHM}}$ itself is *not* the same as the 1-instanton expression (2.40) for the $Sp(2)_\theta$, $Sp(3)_\theta$ theories with $N_f\leq 1$ (at $\theta=0$) and $N_f\leq 2$ (at $\theta=\pi$). Instead, the difference between $Z_1$ and $Z_1^{\text{ADHM}}$ can be identified as the BPS index of D0-branes moving away from the D4-D8-O8 brane system [19, 58]. Similarly, it was confirmed that the 2-instanton correction $Z_2$ captures the same 5d QFT spectrum as in $Z_2^{\text{ADHM}}$, while excluding the spurious contribution of escaping D0-branes. Our blow-up formulae seem to reflect a different UV embedding of the gauge theory, which does *not* allow an extra branch of vacua leading to $Z_{\text{extra}}\neq 1$.

**SO(N)** One can compute the instanton partition function of $SO(N)+N_v\mathbf{V}$ theory with $N_v\leq N-4$ using the ADHM quantum mechanics of the D1-D5-NS5-O5 brane system. At even $N$, the Witten index for a D1-brane can be written as

$$Z_1^{\text{ADHM}}=\sum_{i=1}^{N/2}\left(\frac{2\sinh(2\epsilon_+-a_i)\,2\sinh(a_i-\epsilon_+)\prod_{l=1}^{N_v}2\sinh\frac{m_l\pm(a_i-\epsilon_+)}{2}}{2\cdot 2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}\prod_{j\neq i}2\sinh\frac{a_i\pm a_j}{2}\,2\sinh\frac{2\epsilon_+-a_i\pm a_j}{2}}+(a_i\to -a_i)\right).\quad(3.6)$$

At odd $N$,

$$Z_1^{\text{ADHM}}=\sum_{i=1}^{\lfloor N/2\rfloor}\left(\frac{2\cosh\frac{2\epsilon_+-a_i}{2}\,2\sinh(a_i-\epsilon_+)\prod_{l=1}^{N_f}2\sinh\frac{m_l\pm(a_i-\epsilon_+)}{2}}{2\cdot 2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}\,2\sinh\frac{a_i}{2}\prod_{j\neq i}2\sinh\frac{a_i\pm a_j}{2}\,2\sinh\frac{2\epsilon_+-a_i\pm a_j}{2}}+(a_i\to -a_i)\right).\quad(3.7)$$

The general 1-instanton expression (2.40) and the recursion formula (2.34) are applicable for all $N_v\leq N-4$. We explicitly confirmed that $Z_k^{\text{ADHM}}=Z_k$, where $Z_1$ was written in (2.40) and $Z_2$ was obtained as the solution of the recursion formula (2.34), for $4\leq N\leq 9$. Notice that $Z_k^{\text{ADHM}}=Z_k$ involves a non-trivial extra factor $Z_{\text{extra}}\neq 1$ when $N_v=N-4$. This extra factor is associated to D1-branes moving away from the D5-NS5-O5 brane system, where the 5d QFT lives. It implies that the specific UV realization of the gauge theory, i.e., IIB string theory with D1-D5-NS5-O5, is implicit in our recursion formulae (2.34) with (2.48).

### 3.2 Theories with spinor hypermultiplets

So far, we have investigated the 'standard' gauge theories which have certain D-brane set-ups in IIA/IIB string theory to realize themselves and their non-perturbative solitons. Once there are a sufficient number of the blowup equations, the $k$-instanton partition function $Z_k$ can be obtained as the solution. Solving specifically at 1-instanton level, $Z_1$ is expressed as (2.40). All these results were found to be in agreement with the ADHM instanton calculus, sometimes modulo an extra factor $Z_{\text{extra}}$ sensitive to the string theory embedding of the gauge theory.

There are also 'exceptional' gauge theories, which involve exceptional gauge groups or matters that cannot be engineered from perturbative open strings ending on non-perturbative branes. Instead, their UV embedding is geometrically realized as M-theory wrapped on local

Calabi-Yau threefolds. Since the exceptional theories lack for the ADHM formalism, their instanton partition function $\mathcal{Z}$ has been studied on a case-by-case basis; Once the 5-brane web configuration engineering an exceptional theory is identified, the topological vertex formalism can be applied to compute the relevant partition function $\mathcal{Z}$. Alternatively, one can first construct the $\mathbb{C}^2 \times T^2$ partition function for a related 6d gauge theory, based on its modularity and anomaly, then take the circle reduction to obtain the 5d partition function $\mathcal{Z}$. Several interesting exceptional theories have been studied so far, based on the above two approaches.

We can take advantage of the universality of the blowup equation. Recall that the blow-up recursion formula (2.34) holds for a certain range of $d$, i.e., the set of all integers between $0 \leq d \leq h^\vee - \frac{1}{2}\sum_l I(\mathbf{R}_l)$, when the gauge group $G$ is neither $SU(N)_\kappa$ nor $Sp(N)_\theta$ so that an extra complication due to the Chern-Simons level $\kappa$ or the theta angle $\theta$ cannot occur. One can solve the recursion formulae for the $k$-instanton correction $Z_k$ to the partition function, as long as $h^\vee - \frac{1}{2}\sum_l I(\mathbf{R}_l) \geq 2$, even for the exceptional gauge theories. We assume that $Z_k$ solved from the recursion formulae would be the correct BPS data for UV-consistent 5d SCFTs, modulo an extra factor $Z_{\text{extra}}$ independent of the Coulomb VEV $\vec{a}$. This conjecture will be tested via comparison with [44, 59, 60] which compute $\mathcal{Z}$ for some exceptional cases.

In this section, we will focus on the $SO(N)$ gauge theories with spinor hypermultiplets. There are a sufficient number of recursion formulae (2.34) to fix the $k$-instanton partition function $Z_k$ of the $SO(N)$ gauge theory, if and only if

$$\begin{aligned} &N-4 \geq N_{\mathbf{v}} + 2^{\frac{N-7}{2}} \cdot N_{\mathbf{s}} && \text{for odd } N \\ &N-4 \geq N_{\mathbf{v}} + 2^{\frac{N-8}{2}} \cdot (N_{\mathbf{s}} + N_{\mathbf{c}}) && \text{for even } N \end{aligned} \tag{3.8}$$

where $N_{\mathbf{v}}$, $N_{\mathbf{s}}$, and $N_{\mathbf{c}}$ denote the number of hypermultiplets in the vector, spinor, and conjugate spinor representations, respectively. Our 1-instanton expression (2.40) is also applicable to the cases satisfying (3.8). Let us consider the special cases whose $Z_k$ is already known.

**SO(7)** The $k$-instanton contribution $Z_k$ of $SO(7) + N_{\mathbf{s}}\,\mathbf{S}$ theory can be obtained from the SUSY quantum mechanics proposed in [59], which can be summarized as the following $SU(4)$ Young tableau expression:

$$Z_k^{\text{YD}} = \sum_{|\vec{Y}|=k} \prod_{i=1}^{4} \prod_{s\in Y_i} \frac{2\sinh\left(\phi(s)\right)\, 2\sinh\left(\phi(s)-\epsilon_+\right)\, \prod_{l=1}^{N_{\mathbf{s}}} 2\sinh(\frac{m_l \pm \phi(s)}{2})}{\prod_{j=1}^{4} 2\sinh\frac{E_{ij}}{2}\, 2\sinh\frac{E_{ij}-2\epsilon_+}{2}\, 2\sinh\frac{\epsilon_+ - \phi(s) - a_j}{2}} \tag{3.9}$$

$$\times \prod_{i\leq j}^{4} \prod_{\substack{s_{i,j}\in Y_{i,j} \\ s_i < s_j}} \frac{2\sinh\frac{\phi(s_i)+\phi(s_j)}{2}\, 2\sinh\frac{\phi(s_i)+\phi(s_j)-2\epsilon_+}{2}}{2\sinh\frac{\epsilon_1 - \phi(s_i)-\phi(s_j)}{2}\, 2\sinh\frac{\epsilon_2-\phi(s_i)-\phi(s_j)}{2}} \tag{3.10}$$

We checked the agreement between $Z_1^{\text{YD}}$ and the 1-instanton formula $Z_1$ in (2.40) for $N_{\mathbf{s}} \leq 3$. At two instantons, it was also confirmed for $N_{\mathbf{s}} \leq 3$ that $Z_2^{\text{YD}} = Z_2$, where $Z_2$ is the solution of the recursion formula (2.34) with (2.48). Such explicit comparison implies that the blow-up recursion formula (2.34) actually works for the $SO(7) + N_{\mathbf{s}}\,\mathbf{S}$ theory.

The 1-instanton partition function of $SO(7)+4\mathbf{S}+1\mathbf{V}$ theory is written in (H.15) of [60]. We deduced the 1-instanton correction of $SO(7)+N_{\mathbf{s}}\,\mathbf{S}+N_{\mathbf{v}}\mathbf{V}$ theory with $(N_{\mathbf{s}},N_{\mathbf{v}})\le(2,1)$ from that, by taking several flavor chemical potentials to infinity. All of those were confirmed to be in agreement with our general 1-instanton expression (2.40) up to $(p_1p_2)^{13/2}$.

**SO(8)** The 1-instanton result of $SO(8)+1\mathbf{S}+1\mathbf{C}+1\mathbf{V}$ theory is found in (H.28) of [60]. That is expressed in terms of irreducible characters $\chi^S_{\mathbf{R}}$, whose superscript $S\in\{G,v,s,c\}$ means either the gauge symmetry ($G$) or the flavor symmery acting on the vector ($v$), spinor ($s$), or conjugate spinor ($c$) hypermultiplets. Their representation $\mathbf{R}$ is specified by the Dynkin label in the subscript. All irreducible characters for the flavor symmetry are assumed to be written in the orthogonal basis, to be compatible with our convention of mass parameters in (2.25), (2.34), (2.40). The mass parameters will be often distinguished by the superscript $S\in\{s,c,v\}$ according to the matter representation. The flavor symmetry is $Sp(N_{\mathbf{v}})_v\times Sp(N_{\mathbf{s}})_s\times Sp(N_{\mathbf{c}})_c$.

We obtained the 1-instanton partition function of $SO(8)+N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ theory with $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})\le(1,1,1)$ from (H.28) by sending appropriate mass parameters to infinity. All of those were consistent with our general 1-instanton expression (2.40) up to $t^{20}$ order, where $t\equiv\sqrt{p_1p_2}$. Furthermore, we could determine the unknown part of (H.28) of [60] as

$$\begin{aligned}\hat{Z}_1&=t^4+\sum_{n=0}^{\infty}t^{5+2n}(\chi^G_{(0n00)}\chi^v_{(1)}\chi^s_{(1)}\chi^c_{(1)}+\sum_{n=0}^{\infty}t^{6+2n}(\chi^G_{(1n00)}\chi^s_{(1)}\chi^c_{(1)}+\chi^G_{(0n10)}\chi^s_{(1)}\chi^v_{(1)}\\&\quad+\chi^G_{(0n01)}\chi^c_{(1)}\chi^v_{(1)})+\sum_{n=0}^{\infty}t^{7+2n}(\chi^G_{(1n10)}\chi^s_{(1)}+\chi^G_{(1n01)}\chi^c_{(1)}+\chi^G_{(0n11)}\chi^v_{(1)})-\sum_{n=0}^{\infty}t^{8+2n}\chi^G_{(1n11)},\end{aligned}\tag{3.11}$$

where $\hat{Z}_1\equiv(2\sinh\frac{\epsilon_{1,2}}{2})\cdot Z_1$ is the 1-instanton result after removing the center-of-mass factor.

Now we compare (2.40) with the 1-instanton partition function of $SO(8)+2\mathbf{S}+2\mathbf{C}+2\mathbf{V}$ theory, written in (H.19) of [60]. Our 1-instanton formula (2.40) applied to the $SO(8)$ theories having $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})\le(2,1,1),(1,2,2),(1,1,2),(2,2,0),(2,0,2),(0,2,2)$ was consistent with (H.19) up to $t^{20}$ order, after suitably setting some mass parameters in (H.19) to infinity. By comparison, we could further determine the unknown part of (H.19) of [60] as

$$\begin{aligned}\hat{Z}_1&=t^{-1}-t^3(\chi^v_{(01)}+\chi^s_{(01)}+\chi^c_{(01)})+t^5(\chi^G_{(1000)}\chi^s_{(10)}\chi^c_{(10)}+\chi^G_{(0010)}\chi^s_{(10)}\chi^v_{(10)}+\chi^G_{(0001)}\chi^c_{(10)}\chi^v_{(10)})\\&\quad-t^6(\chi^G_{(1010)}\chi^s_{(10)}+\chi^G_{(1001)}\chi^c_{(10)}+\chi^G_{(0011)}\chi^v_{(10)})+t^7\chi^G_{(1011)}-\sum_{n=0}^{\infty}\Big(t^{5+2n}\chi^G_{(0n00)}\chi^s_{(10)}\chi^c_{(10)}\chi^v_{(10)}\\&\quad+t^{6+2n}(\chi^G_{(1n00)}\chi^s_{(01)}\chi^c_{(01)}\chi^v_{(10)}+\chi^G_{(0n10)}\chi^s_{(01)}\chi^c_{(10)}\chi^v_{(01)}+\chi^G_{(0n01)}\chi^s_{(10)}\chi^c_{(01)}\chi^v_{(01)})\\&\quad-t^{7+2n}(\chi^G_{(1n10)}\chi^s_{(01)}\chi^c_{(10)}\chi^v_{(10)}+\chi^G_{(1n01)}\chi^s_{(10)}\chi^c_{(01)}\chi^v_{(10)}+\chi^G_{(0n11)}\chi^s_{(10)}\chi^c_{(10)}\chi^v_{(01)})\\&\quad+t^{8+2n}(\chi^G_{(2n10)}\chi^s_{(01)}\chi^c_{(10)}+\chi^G_{(2n01)}\chi^s_{(10)}\chi^c_{(01)}+\chi^G_{(1n20)}\chi^s_{(01)}\chi^v_{(10)}+\chi^G_{(1n02)}\chi^c_{(01)}\chi^v_{(10)}\\&\quad+\chi^G_{(0n21)}\chi^s_{(10)}\chi^v_{(01)}+\chi^G_{(0n12)}\chi^c_{(10)}\chi^v_{(01)})-t^{9+2n}(\chi^G_{(2n11)}\chi^s_{(10)}\chi^c_{(10)}+\chi^G_{(1n21)}\chi^s_{(10)}\chi^v_{(10)}\\&\quad+\chi^G_{(1n12)}\chi^c_{(10)}\chi^v_{(10)})+t^{10+2n}(\chi^G_{(2n21)}\chi^s_{(10)}+\chi^G_{(2n12)}\chi^c_{(10)}+\chi^G_{(1n22)}\chi^v_{(10)})-t^{11+2n}\chi^G_{(2n22)}\Big).\end{aligned}\tag{3.12}$$

Notice that (3.11) and (3.12) are manifestly invariant under the $SO(8)$ triality, transforming the $SO(8)$ representations as $(n_vn_an_cn_s)\to(n_sn_an_vn_c)$ along with $\chi^v_{\mathbf{R}}\to\chi^s_{\mathbf{R}}\to\chi^c_{\mathbf{R}}\to\chi^v_{\mathbf{R}}$.

It can be done by shuffling the Coulomb VEVs and renaming the flavor chemical potentials. We rearranged $Z_1$ in terms of the new variables $\vec{a}'$ or $\vec{a}''$,

$$\begin{aligned} (a_1', a_2', a_3', a_4') &= \left(\tfrac{-a_1+a_2+a_3-a_4}{2}, \tfrac{-a_1+a_2+a_3-a_4}{2}, \tfrac{-a_1+a_2+a_3-a_4}{2}, \tfrac{-a_1+a_2+a_3-a_4}{2}\right) \\ (a_1'', a_2'', a_3'', a_4'') &= \left(\tfrac{+a_1-a_2-a_3-a_4}{2}, \tfrac{-a_1+a_2-a_3-a_4}{2}, \tfrac{-a_1-a_2+a_3-a_4}{2}, \tfrac{+a_1+a_2+a_3-a_4}{2}\right), \end{aligned} \tag{3.13}$$

which exchanges the $SO(8)$ irreducible characters as

$$\chi_{(n_c n_a n_s n_v)}(\vec{a}) = \chi_{(n_v n_a n_c n_s)}(\vec{a}')|_{\vec{a}'\to\vec{a}}, \qquad \chi_{(n_s n_a n_v n_c)}(\vec{a}) = \chi_{(n_v n_a n_c n_s)}(\vec{a}'')|_{\vec{a}''\to\vec{a}}. \tag{3.14}$$

Dropping off primes from $Z_1(\vec{a}', \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v)$ or $Z_1(\vec{a}'', \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v)$, we confirmed

$$\begin{aligned} Z_1^{N_\mathbf{s}=N_\mathbf{c}=N_\mathbf{v}}(\vec{a}, \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v) &= Z_1^{N_\mathbf{s}=N_\mathbf{c}=N_\mathbf{v}}(\vec{a}', \epsilon_1, \epsilon_2; \vec{m}^v, \vec{m}^s, \vec{m}^c)|_{\vec{a}'\to\vec{a}} \\ Z_1^{N_\mathbf{s}=N_\mathbf{c}=N_\mathbf{v}}(\vec{a}, \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v) &= Z_1^{N_\mathbf{s}=N_\mathbf{c}=N_\mathbf{v}}(\vec{a}'', \epsilon_1, \epsilon_2; \vec{m}^c, \vec{m}^v, \vec{m}^s)|_{\vec{a}''\to\vec{a}}. \end{aligned} \tag{3.15}$$

Similarly, we also found the 1-instanton formula (2.40) applied to $SO(8)$ theories with $(N_\mathbf{s}, N_\mathbf{c}, N_\mathbf{v}) \le (4,0,0)$ or $(0,4,0)$ is compatible with the $SO(8)$ triality. Starting with the 1-instanton result $Z_1^{\text{ADHM}} = Z_1^{\text{ADHM}}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m})$ obtained from the relevant ADHM quantum mechanics for $SO(8) + N_\mathbf{v}\mathbf{V}$ theory with $N_\mathbf{v} \le 4$, we checked that

$$\begin{aligned} Z_1^{N_\mathbf{c},\, N_\mathbf{c}=N_\mathbf{v}=0}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) &= Z_1^{\text{ADHM}}(\vec{a}', \epsilon_1, \epsilon_2, \vec{m})|_{\vec{a}'\to\vec{a}} \\ Z_1^{N_\mathbf{s},\, N_\mathbf{s}=N_\mathbf{v}=0}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) &= Z_1^{\text{ADHM}}(\vec{a}'', \epsilon_1, \epsilon_2, \vec{m})|_{\vec{a}''\to\vec{a}}. \end{aligned} \tag{3.16}$$

**SO(9)** The 1-instanton formula (2.40) can be applied to $SO(9) + N_\mathbf{s}\mathbf{S} + N_\mathbf{v}\mathbf{V}$ theory with $(N_\mathbf{s}, N_\mathbf{v}) \le (1,3)$ or $(2,1)$, which has $Sp(N_\mathbf{s})_s \times Sp(N_\mathbf{v})_v$ flavor symmetry. It can be compared with the 1-instanton partition function of $SO(9) + 2\mathbf{S} + 3\mathbf{V}$ theory, which is written in (H.20) of [60] up to $t^7$ order, after appropriately taking some mass parameters to infinity. We checked all their consistency up to the given order. For example, the character expansion of $\hat{Z}_1$ for $SO(9) + 2\mathbf{S} + 1\mathbf{V}$ can be written as

$$\begin{aligned} \hat{Z}_1 = {} & t^4\chi^v_{(1)} + t^5\chi^s_{(20)} - t^6\chi^G_{(0001)}\chi^s_{(10)} + \sum_{n=0}^{\infty}\Big(t^{6+2n}\chi^G_{(0n00)}\chi^s_{(02)}\chi^v_{(1)} \\ & - t^{7+2n}(\chi^G_{(1n00)}\chi^s_{(02)} + \chi^G_{(0n01)}\chi^s_{(11)}\chi^v_{(1)}) \\ & + t^{8+2n}(\chi^G_{(1n01)}\chi^s_{(11)} + \chi^G_{(0n10)}\chi^s_{(20)}\chi^v_{(1)} + \chi^G_{(0n02)}\chi^s_{(01)}\chi^v_{(1)}) \\ & - t^{9+2n}(\chi^G_{(1n10)}\chi^s_{(20)} + \chi^G_{(1n02)}\chi^s_{(01)} + \chi^G_{(0n11)}\chi^s_{(10)}\chi^v_{(1)}) \\ & + t^{10+2n}(\chi^G_{(1n11)}\chi^s_{(10)} + \chi^G_{(0n20)}\chi^v_{(1)}) - t^{11+2n}\chi^G_{(1n20)}\Big), \end{aligned} \tag{3.17}$$

which was tested against the general formula (2.40) up to $t^{20}$ order. It is the same as (H.20) of [60] after reducing the $Sp(3)_v$ characters

$$\chi^v_{(001)} \to \chi^v_{(1)}, \qquad \chi^v_{(010)} \to 1, \qquad \chi^v_{(100)} \to 0, \qquad \chi^v_{(000)} \to 0. \tag{3.18}$$

**SO(10)** We apply our 1-instanton expression (2.40) to $SO(10)+N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ theory with $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})\leq(2,0,2),(1,1,2),(0,2,2),(1,0,4),(0,1,4)$. The relevant flavor symmetry is $U(N_{\mathbf{s}})\times U(N_{\mathbf{c}})\times Sp(N_{\mathbf{v}})$ because the $SO(10)$ (conjugate) spinor is a complex representation. Since the $SO(10)$ charge conjugation exchanges the spinor and conjugate spinor representations, i.e., $\chi^G_{(00001)}=(\chi^G_{(00010)})^*$, the instanton partition function for $SO(10)+(N_{\mathbf{s}}\mp 1)\mathbf{S}+(N_{\mathbf{c}}\pm 1)\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ must be identified with that of $SO(10)+N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ simply by flipping the sign of mass parameters for (conjugate) spinor hypermultiplets.

$$\begin{aligned}\hat{Z}_1^{N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}}}(m^s_{1,\cdots,N_{\mathbf{s}}};m^c_{1,\cdots,N_{\mathbf{c}}})&=\hat{Z}_1^{N_{\mathbf{s}}-1,N_{\mathbf{c}}+1,N_{\mathbf{v}}}(m^s_{1,\cdots,N_{\mathbf{s}}-1};m^c_{1,\cdots,N_{\mathbf{c}}+1})\big|_{m^c_{N_{\mathbf{c}}+1}=-m^s_{N_{\mathbf{s}}}}\\&=\hat{Z}_1^{N_{\mathbf{s}}+1,N_{\mathbf{c}}-1,N_{\mathbf{v}}}(m^s_{1,\cdots,N_{\mathbf{s}}+1};m^c_{1,\cdots,N_{\mathbf{c}}-1})\big|_{m^s_{N_{\mathbf{s}}+1}=-m^c_{N_{\mathbf{c}}}}\end{aligned}\tag{3.19}$$

This relation was explicitly confirmed in all above cases at 1-instanton order. We may want to compare (2.40) with the known 1-instanton partition function of $SO(10)+1\mathbf{S}+1\mathbf{C}+4\mathbf{V}$ theory, written in (H.21) of [60], after taking relevant mass parameters to infinity. However, (H.21) specifies $\hat{Z}_1$ only up to $\mathcal{O}(t^5)$, which leaves nothing for comparison once we reduce the mass parameters. Thus the consistency between two expressions could be only weakly tested. For instance, $\hat{Z}_1$ obtained from (2.40) for $SO(10)+N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+4\mathbf{V}$ theory with $N_{\mathbf{s}}+N_{\mathbf{c}}=2$ is displayed in (A.5), which turns out to be void until $t^4$ order.

**SO(12)** The 1-instanton partition function of $SO(12)+1\mathbf{S}+6\mathbf{V}$ theory is written in (H.22) of [60], up to $t^8$ order. It can be compared with our 1-instanton formula (2.40) applied to $SO(12)+N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ theory with $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})\leq(1,0,4)$ or $(0,1,4)$, whose flavor symmetry acting on matter multiplets is $SO(2N_{\mathbf{s}})_s\times SO(2N_{\mathbf{c}})_c\times Sp(N_{\mathbf{v}})_v$. For comparison, we need to appropriately decouple some mass parameters in (H.22) to infinity. It reduces the $Sp(6)_v$ characters in (H.22) to, e.g., the $Sp(4)_v$ irreducible characters as follows:

$$\begin{aligned}&\chi^v_{(000000)}\to 0, &&\chi^v_{(100000)}\to 0, &&\chi^v_{(010000)}\to 1,\\&\chi^v_{(001000)}\to\chi^v_{(1000)}, &&\chi^v_{(000100)}\to\chi^v_{(0100)}, &&\chi^v_{(000001)}\to\chi^v_{(0001)}\end{aligned}\tag{3.20}$$

The agreement between (H.22) and (2.40) was explicitly confirmed up to the given order, for $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(1,0,4)$. Moreover, we checked that the 1-instanton results $Z_1$ from (2.40) for $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(1,0,N_{\mathbf{v}})$ and $(0,1,N_{\mathbf{v}})$ could be interchanged as follows:

$$\hat{Z}_1^{N_{\mathbf{s}}=1,N_{\mathbf{c}}=0,N_{\mathbf{v}}}(a_1,a_2,a_3,a_4,a_5,a_6)=\hat{Z}_1^{N_{\mathbf{s}}=0,N_{\mathbf{c}}=1,N_{\mathbf{v}}}(a_1,a_2,a_3,a_4,a_5,-a_6).\tag{3.21}$$

We have compared so far the solution $Z_1$ of the recursion formulae (2.34) with the known 1-instanton partition function for various $SO(N)$ theories with spinor hypermultiplets. The comparison showed consistency for all the examples whose $Z_1$ had been computed [59, 60]. We also collect the character expansion of the 1-instanton partition function (2.40) in Appendix A for novel $SO(N)$ theories with spinor matters. See Table 3 for the list of character expansions.

### 3.3 Theories with an exceptional gauge group

, and especially the general 1-instaon expression $Z_1$ indeed produces

We also obtain various

[16, 32, 57].

We propose a general blow-up formula for general gauge theory with arbitrary representations, under the condition that the matter representation is not too large.

### 3.4 Theories with a rank-3 antisymmetic hypermultiplet

We will show the novel results for the $SO(N)$ gauge theories with spinor hypermultiplets in Section 3.2, for the gauge theories with exceptional groups in Section 3.3, and for the $SU(6)$ gauge theories with rank-3 antisymmetric hypermultiplets in Section 3.4. (See Table ???)

especially for 'exceptional' gauge theories

## 4 Conclusion

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## A One-instanton partition functions

This appendix collects the character expansion of the 1-instanton partition function $Z_1$ for a variety of 5d $\mathcal{N}=1$ gauge theories. For simplicity, we display the $\hat{Z}_1 \equiv (2\sinh\frac{\epsilon_{1,2}}{2})\cdot Z_1$ which takes off the center-of-mass factor. They are written in terms of irreducible characters $\chi^S_{\mathbf{R}}$, whose superscript $S \in \{G, v, s, c\}$ indicates the gauge symmetry ($G$) or the flavor symmery acting on the vector ($v$), spinor ($s$), or conjugate spinor ($c$) hypermultiplets. The representation $\mathbf{R}$ of an irreducible character $\chi^S_{\mathbf{R}}$ is specified by its Dynkin label.[14] An irreducible character for the flavor symmetry is assumed to be in the orthogonal basis, such that it can be consistent with the mass parameters $m_\ell$ introduced in Section 2. We will often distinguish the mass parameters by the superscript $S \in \{s, c, v\}$ according to the matter representation.

**SO(8)** The flavor symmetry acting on $N_\mathbf{s}\mathbf{S} + N_\mathbf{c}\mathbf{C} + N_\mathbf{v}\mathbf{V}$ matter multiplets is given by $Sp(N_\mathbf{s})_s \times Sp(N_\mathbf{c})_c \times Sp(N_\mathbf{v})_v$. For $(N_\mathbf{s}, N_\mathbf{c}, N_\mathbf{v}) = (3,1,0)$, the character expansion of the 1-instanton result $\hat{Z}_1$ is

$$\begin{aligned}\hat{Z}_1 = \textstyle\sum_{n=0}^{\infty} \Big(& t^{5+2n}\chi^G_{(0n00)}\chi^s_{(001)}\chi^v_{(1)} - t^{6+2n}(\chi^G_{(0n01)}\chi^s_{(010)}\chi^v_{(1)} + \chi^G_{(1n00)}\chi^s_{(001)}) \\ &+ t^{7+2n}(\chi^G_{(1n01)}\chi^s_{(010)} + \chi^G_{(0n02)}\chi^s_{(100)}\chi^v_{(1)}) \\ &- t^{8+2n}(\chi^G_{(1n02)}\chi^s_{(100)} + \chi^G_{(0n03)}\chi^v_{(1)}) + t^{9+2n}\chi^G_{(1n03)}\Big).\end{aligned} \tag{A.1}$$

which was compared with the closed-form expression (2.40) up to $t^{20}$ order. We checked that the 1-instanton partition functions $Z_1$ from (2.40) for $(N_\mathbf{s}, N_\mathbf{c}, N_\mathbf{v}) = (3,1,0)$ and $(1,3,0)$ could be interchanged as follows:

$$Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(1,3,0)}(a_1,a_2,a_3,a_4) = Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(3,1,0)}(a_1,a_2,a_3,-a_4). \tag{A.2}$$

The $SO(8)$ triality (3.13) was also confirmed as in Section 3.2. Namely, we found that

$$\begin{aligned}Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(3,1,0)}(\vec{a},\epsilon_1,\epsilon_2;\vec{m}^s,\vec{m}^c,0) &= Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(0,3,1)}(\vec{a}',\epsilon_1,\epsilon_2;0,\vec{m}^s,\vec{m}^c)|_{\vec{a}'\to\vec{a}} \\ &= Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(1,0,3)}(\vec{a}'',\epsilon_1,\epsilon_2;\vec{m}^c,0,\vec{m}^s)|_{\vec{a}''\to\vec{a}}\end{aligned} \tag{A.3}$$

$$\begin{aligned}Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(1,3,0)}(\vec{a},\epsilon_1,\epsilon_2;\vec{m}^s,\vec{m}^c,0) &= Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(0,1,3)}(\vec{a}',\epsilon_1,\epsilon_2;0,\vec{m}^s,\vec{m}^c)|_{\vec{a}'\to\vec{a}} \\ &= Z_1^{(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(3,0,1)}(\vec{a}'',\epsilon_1,\epsilon_2;\vec{m}^c,0,\vec{m}^s)|_{\vec{a}''\to\vec{a}}\end{aligned} \tag{A.4}$$

The character expansion for other $SO(8)$ theories with less number of hypermultiplets can be obtained from (A.1) by decoupling some mass parameters to infinity. It was checked that the general 1-instanton formula (2.40) agrees with that.

**SO(10)** The flavor symmetry acting on $N_\mathbf{s}\mathbf{S} + N_\mathbf{c}\mathbf{C} + N_\mathbf{v}\mathbf{V}$ hypermultiplets is $U(N_\mathbf{s})_s \times U(N_\mathbf{c})_c \times Sp(N_\mathbf{v})_v$, reflecting that the $SO(10)$ (conjugate) spinor representation is complex. For $N_\mathbf{s} + N_\mathbf{c} = 2$ and $N_\mathbf{v} = 2$, the character expansion of $\hat{Z}_1$ is given by

$$\hat{Z}_1 = t^5(\chi^s_{(2)_0} + \chi^v_{(01)}) + t^6(\chi^s_{(2)_{-2}} + \chi^s_{(2)_2}) - t^7(\chi^G_{(00010)}\chi^s_{(1)_{-1}}\chi^v_{(10)} + \chi^G_{(00001)}\chi^s_{(1)_1}\chi^v_{(10)} + \chi^G_{(01000)}$$

[14] In this paper, we follow the convention of LieART [61] to denote the Dynkin label of a representation $\mathbf{R}$.

$$
\begin{aligned}
&+\chi^G_{(10000)}(\chi^s_{(2)_{-2}}+\chi^s_{(2)_2}))+t^8(\chi^G_{(00100)}\chi^v_{(10)}+\chi^G_{(10010)}\chi^s_{(1)_{-1}}+\chi^G_{(10001)}\chi^s_{(1)_1})-t^9\chi^G_{(10100)}\\
&+\textstyle\sum_{n=0}^\infty\Big(t^{7+2n}\chi^G_{(0n000)}(\chi^s_{(0)_{-4}}+\chi^s_{(0)_4}+\chi^s_{(4)_0})\chi^v_{(01)}-t^{8+2n}(\chi^G_{(0n010)}(\chi^s_{(1)_{-3}}+\chi^s_{(3)_1})\chi^v_{(01)}\\
&\qquad\qquad+\chi^G_{(0n001)}(\chi^s_{(1)_3}+\chi^s_{(3)_{-1}})\chi^v_{(01)}+\chi^G_{(1n000)}(\chi^s_{(0)_{-4}}+\chi^s_{(0)_4}+\chi^s_{(4)_0})\chi^v_{(10)})\\
&\qquad+t^{9+2n}(\chi^G_{(0n100)}(\chi^s_{(2)_{-2}}+\chi^s_{(2)_2})\chi^v_{(01)}+\chi^G_{(0n020)}\chi^s_{(0)_{-2}}\chi^v_{(01)}+\chi^G_{(0n002)}\chi^s_{(0)_2}\chi^v_{(01)}\\
&\qquad\qquad+\chi^G_{(0n011)}\chi^s_{(2)_0}\chi^v_{(01)}+\chi^G_{(1n010)}(\chi^s_{(1)_{-3}}+\chi^s_{(3)_1})\chi^v_{(10)}+\chi^G_{(1n001)}(\chi^s_{(1)_3}+\chi^s_{(3)_{-1}})\chi^v_{(10)}\\
&\qquad\qquad+\chi^G_{(2n000)}(\chi^s_{(0)_{-4}}+\chi^s_{(0)_4}+\chi^s_{(4)_0}))\\
&\qquad-t^{10+2n}(\chi^G_{(0n110)}\chi^s_{(1)_{-1}}\chi^v_{(01)}+\chi^G_{(0n101)}\chi^s_{(1)_1}\chi^v_{(01)}+\chi^G_{(1n100)}(\chi^s_{(2)_{-2}}+\chi^s_{(2)_2})\chi^v_{(10)}\\
&\qquad\qquad+\chi^G_{(1n020)}\chi^s_{(0)_{-2}}\chi^v_{(10)}+\chi^G_{(1n002)}\chi^s_{(0)_2}\chi^v_{(10)}+\chi^G_{(1n011)}\chi^s_{(2)_0}\chi^v_{(10)}\\
&\qquad\qquad+\chi^G_{(2n010)}(\chi^s_{(1)_{-3}}+\chi^s_{(3)_1})+\chi^G_{(2n001)}(\chi^s_{(1)_3}+\chi^s_{(3)_{-1}})\\
&\qquad+t^{11+2n}(\chi^G_{(1n200)}\chi^v_{(01)}+\chi^G_{(1n110)}\chi^s_{(1)_{-1}}\chi^v_{(10)}+\chi^G_{(1n101)}\chi^s_{(1)_1}\chi^v_{(10)}+\chi^G_{(2n100)}(\chi^s_{(2)_{-2}}+\chi^s_{(2)_2})\\
&\qquad\qquad+\chi^G_{(2n020)}\chi^s_{(0)_{-2}}+\chi^G_{(2n002)}\chi^s_{(0)_2}+\chi^G_{(2n011)}\chi^s_{(2)_0})\\
&\qquad-t^{12+2n}(\chi^G_{(1n200)}\chi^v_{(10)}+\chi^G_{(2n110)}\chi^s_{(1)_{-1}}+\chi^G_{(2n101)}\chi^s_{(1)_1})+t^{13+2n}\chi^G_{(2n200)}\Big)
\end{aligned}\tag{A.5}
$$

where the $U(2)$ character $\chi^s_{(j)_b}$ is defined as (with $y_{s,i}\equiv e^{-m^s_i}$ and $y_{c,i}\equiv e^{-m^c_i}$ understood)

$$
\chi^s_{(j)_b}=\begin{cases}(y_{s,1}\,y_{s,2})^{b/2}\cdot\sum_{a=0}^j\big(y_{s,1}/y_{s,2}\big)^{-j/2+a} & \text{for}\quad (N_{\mathbf{s}},N_{\mathbf{c}})=(2,0)\\ (y_{s,1}/y_{c,1})^{b/2}\cdot\sum_{a=0}^j\big(y_{s,1}\,y_{c,1}\big)^{-j/2+a} & \text{for}\quad (N_{\mathbf{s}},N_{\mathbf{c}})=(1,1)\\ (y_{c,1}\,y_{c,2})^{-b/2}\cdot\sum_{a=0}^j\big(y_{c,1}/y_{c,2}\big)^{-j/2+a} & \text{for}\quad (N_{\mathbf{s}},N_{\mathbf{c}})=(0,2).\end{cases}\tag{A.6}
$$

Similarly, for $N_{\mathbf{s}}+N_{\mathbf{c}}=3$ and $N_{\mathbf{v}}=0$, the character expansion of $\hat{Z}_1$ is given by

$$
\begin{aligned}
\hat{Z}_1=\;&t^5(\chi^G_{(10000)}+\chi^s_{(20)_{-2}}+\chi^s_{(02)_2})-t^6(\chi^s_{(10)_{-1}}+\chi^s_{(01)_1})+t^7\chi^G_{(00100)}\\
&+\textstyle\sum_{n=0}^\infty\Big(t^{7+2n}(\chi^G_{(0n000)}(\chi^s_{(00)_{-6}}+\chi^s_{(00)_6}+\chi^s_{(04)_{-2}}+\chi^s_{(40)_2}))\\
&\qquad-t^{8+2n}(\chi^G_{(0n010)}(\chi^s_{(01)_{-5}}+\chi^s_{(30)_3}+\chi^s_{(13)_{-1}})+\chi^G_{(0n001)}(\chi^s_{(10)_5}+\chi^s_{(03)_{-3}}+\chi^s_{(31)_1}))\\
&\qquad+t^{9+2n}(\chi^G_{(0n100)}(\chi^s_{(02)_{-4}}+\chi^s_{(20)_4}+\chi^s_{(22)_0})+\chi^G_{(0n011)}(\chi^s_{(12)_{-2}}+\chi^s_{(21)_2})\\
&\qquad\qquad+\chi^G_{(0n020)}(\chi^s_{(10)_{-4}}+\chi^s_{(03)_0})+\chi^G_{(0n002)}(\chi^s_{(01)_4}+\chi^s_{(30)_0}))\\
&\qquad-t^{10+2n}(\chi^G_{(0n110)}(\chi^s_{(11)_{-3}}+\chi^s_{(12)_1})+\chi^G_{(0n101)}(\chi^s_{(11)_3}+\chi^s_{(21)_{-1}})\\
&\qquad\qquad+\chi^G_{(0n030)}\chi^s_{(00)_{-3}}+\chi^G_{(0n003)}\chi^s_{(00)_3}+\chi^G_{(0n021)}\chi^s_{(02)_{-1}}+\chi^G_{(0n012)}\chi^s_{(20)_1})\\
&\qquad+t^{11+2n}(\chi^G_{(0n200)}(\chi^s_{(20)_{-2}}+\chi^s_{(02)_2})+\chi^G_{(0n120)}\chi^s_{(01)_{-2}}+\chi^G_{(0n102)}\chi^s_{(10)_2}+\chi^G_{0n111)}\chi^s_{(11)_0})\\
&\qquad-t^{12+2n}(\chi^G_{(0n210)}\chi^s_{(10)_{-1}}+\chi^G_{(0n201)}\chi^s_{(01)_1})+t^{13+2n}\chi_{(0n300)}\Big)
\end{aligned}\tag{A.7}
$$

where the $U(3)$ character $\chi^s_{(mn)_c}$ is defined as

$$
\chi^s_{(mn)_c}=(w_1w_2w_3)^{\frac{c-m+n}{3}}\cdot\Bigg(\sum_{\substack{1\le i_1\le\cdots\le i_m\le 3\\1\le j_1\le\cdots\le j_n\le 3}}\frac{w_{i_1}\cdots w_{i_m}}{w_{j_1}\cdots w_{j_n}}-\sum_{\substack{1\le i_1\le\cdots\le i_{m-1}\le 3\\1\le j_1\le\cdots\le j_{n-1}\le 3}}\frac{w_{i_1}\cdots w_{i_{m-1}}}{w_{j_1}\cdots w_{j_{n-1}}}\Bigg)\tag{A.8}
$$

with

$$(w_1, w_2, w_3) = \begin{cases} (y_{s,1}, y_{s,2}, y_{s,3}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (3,0) \\ (y_{s,1}, y_{s,2}, y_{c,1}^{-1}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (2,1) \\ (y_{s,1}, y_{c,1}^{-1}, y_{c,2}^{-1}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (1,2) \\ (y_{c,1}^{-1}, y_{c,2}^{-1}, y_{c,3}^{-1}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (0,3) \end{cases} \tag{A.9}$$

Again, (A.5) and (A.7) was tested against the closed-form expression (2.40) up to $t^{20}$ order.

**SO(11)** The flavor symmetry acting on $N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{v}}\mathbf{V}$ hypermultiplets is $SO(2N_{\mathbf{s}})_s \times Sp(N_{\mathbf{v}})_v$. For $N_{\mathbf{s}} = 1$ and $N_{\mathbf{v}} = 3$, the character expansion of $\hat{Z}_1$ can be written as

$$\begin{aligned}
\hat{Z}_1 = \; & t^5 + t^6(\chi^v_{(001)} + (y_s^2 + y_s^{-2})\chi^v_{(100)}) + t^7((y_s^2 + y_s^{-2} + 1)\chi^v_{(010)} - (y_s^2 + y_s^{-2})\chi^G_{(10000)}) \\
& - t^8((y_s + y_s^1)\chi^G_{(00001)}\chi^v_{(010)} + (y_s^2 + y_s^{-2} + 1)\chi^G_{(10000)}\chi^v_{(100)} + \chi^G_{(01000)}\chi^v_{(100)}) \\
& + t^9(\chi^G_{(00100)}\chi^v_{(010)} + \chi^G_{(10001)}(y_s + y_s^{-1})\chi^v_{(100)} + \chi^G_{(20000)}(y_s^2 + y_s^{-2} + 1) + \chi^G_{(11000)}) \\
& - t^{10}(\chi^G_{(10100)}\chi^v_{(100)} + \chi^G_{(20001)}(y_s + y_s^{-1})) + t^{11}\chi^G_{(20100)} \\
& + \textstyle\sum_{n=0}^{\infty} \Big( t^{8+2n}(\chi^G_{(0n000)}(y_s^4 + y_s^{-4} + 1)\chi^v_{(001)}) \\
& \quad - t^{9+2n}(\chi^G_{(0n001)}(y_s^3 + y_s^{-3})\chi^v_{(001)} + \chi^G_{(0n001)}(y_s + y_s^{-1})\chi^v_{(001)} + \chi^G_{(1n000)}(y_s^4 + y_s^{-4} + 1)\chi^v_{(010)}) \\
& \quad + t^{10+2n}(\chi^G_{(0n010)}(y_s^2 + y_s^{-2})\chi^v_{(001)} + \chi^G_{(0n100)}(y_s^2 + y_s^{-2} + 1)\chi^v_{(001)} + \chi^G_{(0n002)}\chi^v_{(001)} \\
& \qquad + \chi^G_{(1n001)}(y_s^3 + y_s + y_s^{-1} + y_s^{-3})\chi^v_{(010)} + \chi^G_{(2n000)}(y_s^4 + y_s^{-4} + 1)\chi^v_{(100)}) \\
& \quad - t^{11+2n}(\chi^G_{(0n101)}(y_s + y_s^{-1})\chi^v_{(001)} + \chi^G_{(1n100)}(y_s^2 + y_s^{-2} + 1)\chi^v_{(010)} + \chi^G_{(1n010)}(y_s^2 + y_s^{-2})\chi^v_{(010)} \\
& \qquad + \chi^G_{(1n002)}\chi^v_{(010)} + \chi^G_{(2n001)}(y_s^3 + y_s + y_s^{-1} + y_s^{-3})\chi^v_{(100)} + \chi^G_{(3n000)}(y_s^4 + y_s^{-4} + 1)) \\
& \quad + t^{12+2n}(\chi^G_{(0n200)}\chi^v_{(001)} + \chi^G_{(1n101)}(y_s + y_s^{-1})\chi^v_{(010)} + \chi^G_{(2n100)}(y_s^2 + y_s^{-2} + 1)\chi^v_{(100)} \\
& \qquad + \chi^G_{(2n010)}(y_s^2 + y_s^{-2})\chi^v_{(100)} + \chi^G_{(2n002)}\chi^v_{(100)} + \chi^G_{(3n001)}(y_s^3 + y_s + y_s^{-1} + y_s^{-3})) \\
& \quad - t^{13+2n}(\chi^G_{(1n200)}\chi^v_{(010)} + \chi^G_{(2n101)}(y_s + y_s^{-1})\chi^v_{(100)} + \chi^G_{(3n100)}(y_s^2 + y_s^{-2} + 1) \\
& \qquad + \chi^G_{(3n010)}(y_s^2 + y_s^{-2}) + \chi^G_{(3n002)}) \\
& \quad + t^{14+2n}(\chi^G_{(2n200)}\chi^v_{(100)} + \chi^G_{(3n101)}(y_s + y_s^{-1})) - t^{15+2n}\chi^G_{(3n200)} \Big)
\end{aligned} \tag{A.10}$$

which was compared with the closed-form expression (2.40) up to $t^{20}$ order.

**SO(12)** The flavor symmetry acting on $N_{\mathbf{s}}\mathbf{S} + N_{\mathbf{c}}\mathbf{C} + N_{\mathbf{v}}\mathbf{V}$ hypermultiplets is $SO(2N_{\mathbf{s}})_s \times SO(2N_{\mathbf{c}})_c \times Sp(N_{\mathbf{v}})_v$. Here we turn off the Coulomb VEV $\vec{a} = 0$ for simplicity. The character expansion of $\hat{Z}_1$ at $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (2,0,0)$ can be written as

$$\begin{aligned}
\hat{Z}_1 = \; & \frac{t^{18}}{(1-t^2)^{18}} \Big( - 96096\,(\chi^s_{(13)} + \chi^s_{(31)}) \cdot (7t^4 + 42t^2 + 72 + 42t^{-2} + 7t^{-4}) \\
& + 10010\,(\chi^s_{(24)} + \chi^s_{(42)}) \cdot (9t^5 + 88t^3 + 243t + 243t^{-1} + 88t^{-3} + 9t^{-5}) \\
& - 352\,(\chi^s_{(15)} + \chi^s_{(51)}) \cdot (25t^6 + 474t^4 + 2169t^2 + 3504 + 2169t^{-2} + 474t^{-4} + 25t^{-6})
\end{aligned} \tag{A.11}$$

$$- 2464\,(\chi^s_{(35)} + \chi^s_{(53)}) \cdot (2t^6 + 27t^4 + 108t^2 + 168 + 108t^{-2} + 27t^{-4} + 2t^{-6})$$
$$+ 11\,(\chi^s_{(06)} + \chi^s_{(60)}) \cdot (42t^7 + 1194t^5 + 8451t^3 + 21253t + 21253t^{-1} + \cdots + 42t^{-7})$$
$$+ 11\,(\chi^s_{(26)} + \chi^s_{(62)}) \cdot (45t^7 + 1101t^5 + 6983t^3 + 16623t + 16623t^{-1} + \cdots + 45t^{-7})$$
$$- 32\,(\chi^s_{(17)} + \chi^s_{(71)}) \cdot (t^8 + 36t^6 + 336t^4 + 1176t^2 + 17641 + 1176t^{-2} + \cdots + t^{-8})$$
$$+ 99\,\chi^s_{(22)} \cdot (5t^9 - 90t^7 + 1623t^5 + 26743t^3 + 83103t + (t \to t^{-1}))$$
$$+ 462\,(\chi^s_{(02)} + \chi^s_{(20)}) \cdot (t^9 - 18t^7 + 153t^5 + 4059t^3 + 13485t + (t \to t^{-1}))$$
$$- 32\,\chi^s_{(33)} \cdot (t^{10} - 18t^8 + 450t^6 + 13340t^4 + 66977t^2 + 110772 + 66977t^{-2} + \cdots + t^{-10})$$
$$+ \chi^s_{(44)} \cdot (t^{11} - 18t^9 + 615t^7 + 26332t^5 + 187749t^3 + 466001t^1 + 466001t^{-1} + \cdots + t^{-11})$$
$$+ (\chi^s_{(04)} + \chi^s_{(40)}) \cdot (t^{13} - 18t^{11} + 153t^9 - 816t^7 + 58115t^5 + 730170t^3 + 2129595t^1 + (t \to t^{-1}))$$
$$- 352\,\chi^s_{(11)} \cdot (t^{10} - 4t^8 - 99t^6 + 2496t^4 + 18246t^2 + 32976 + 18246t^{-2} + \cdots + t^{-10})$$
$$+ (\chi^s_{(08)} + \chi^s_{(80)}) \cdot (t^9 + 48t^7 + 603t^5 + 2898t^3 + 6174t + (t \to t^{-1}))$$
$$+ (t^{17} - 18t^{15} + 153t^{13} - 739t^{11} + 3753t^9 - 20195t^7 + 49881t^5 + 1203597t^3 + 4481279t^1 + (t \to t^{-1}))\Big)$$

It was explicitly checked that the 1-instanton partition function $Z_1$ at $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (0,2,0)$ could be identified with the above as

$$Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(0,2,0)}(a_1,a_2,a_3,a_4,a_5,a_6) = Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(2,0,0)}(a_1,a_2,a_3,a_4,a_5,-a_6). \tag{A.12}$$

Similarly, the character expansion of $\hat{Z}_1$ at $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (1,1,0)$ can be displayed as follows:

$$\begin{aligned}
\hat{Z}_1 = \frac{t^{18}}{(1-t^2)^{18}} \sum_{\pm} \Big( & - 2462\,(y_s^{\pm1}y_c^{\pm4} + y_s^{\pm4}y_c^{\pm1}) \cdot (2t^6 + 27t^4 + 108t^2 + 168 + 108t^{-2} + 27t^{-4} + 2t^{-6}) \\
& + 11\,(y_s^{\pm2}y_c^{\pm4} + y_s^{\pm4}y_c^{\pm2}) \cdot (45t^7 + 1101t^5 + 6983t^3 + 16623t + (t \to t^{-1})) \\
& + 44\,(y_s^{\pm3}y_c^{\pm3}) \cdot (23t^7 + 587t^5 + 3925t^3 + 9609t + (t \to t^{-1})) \\
& + 44\,(y_s^{\pm1}y_c^{\pm3} + y_s^{\pm3}y_c^{\pm1}) \cdot (23t^7 + 2927t^5 + 26025t^3 + 70033t + (t \to t^{-1})) \\
& - 32\,(y_s^{\pm3}y_c^{\pm4} + y_s^{\pm4}y_c^{\pm3}) \cdot (t^8 + 36t^6 + 336t^4 + 1176t^2 + 1764 + 1176t^{-2} + \cdots + t^{-8}) \\
& - 32\,(y_s^{\pm2}y_c^{\pm3} + y_s^{\pm3}y_c^{\pm2}) \cdot (t^8 + 465t^6 + 7629t^4 + 33351t^2 + 53244 + 33351t^{-2} + \cdots + t^{-8}) \\
& + (y_s^{\pm4}y_c^{\pm4}) \cdot (t^9 + 48t^7 + 603t^5 + 2898t^3 + 6174t + (t \to t^{-1})) \\
& - 32\,(y_s^{\pm1}y_c^{\pm2} + y_s^{\pm2}y_c^{\pm1}) \cdot (t^{10} - 17t^8 + 1069t^6 + 44069t^4 + 234770t^2 + 393168 + 234770t^{-2} + \cdots + t^{-10}) \\
& - 32\,(y_s^{\pm3} + y_c^{\pm3}) \cdot (t^{10} - 17t^8 + 750t^6 + 17526t^4 + 83553t^2 + 136714 + 83358t^{-2} + \cdots + t^{-10}) \\
& - 32\,(y_s^{\pm1} + y_c^{\pm1}) \cdot (13t^{10} - 79t^8 + 408t^6 + 97724t^4 + 587351t^2 + 1011546 + 587351t^{-2} + \cdots + 13t^{-10})) \\
& + (y_s^{\pm4} + y_c^{\pm4}) \cdot (t^{11} - 18t^9 + 615t^7 + 26332t^5 + 187749t^3 + 466001t + (t \to t^{-1})) \\
& + (y_s^{\pm2} + y_c^{\pm2}) \cdot (t^{11} + 477t^9 - 7305t^7 + 391411t^5 + 4750692t^3 + 13923764t^1 + (t \to t^{-1})) \\
& + 4\,(y_s^{\pm1}y_c^{\pm1}) \cdot (3t^{11} + 199t^9 + 132676t^7 + 1864041t^5 + 5630341t^3 + 5630341t^{+}(t \to t^{-1})) \\
& + (y_s^{\pm2}y_c^{\pm2})(t^{13} - 17t^{11} + 1136t^9 + 804t^7 + 200385t^5 + 1971471t^3 + 5450836t + (t \to t^{-1})) \\
& + (t^{15} - 17t^{13} + 214t^{11} + 1414t^9 - 33152t^7 + 704404t^5 + 11381979t^3 + 35592757t + (t \to t^{-1}))\Big)
\end{aligned} \tag{A.13}$$

in which $\sum_\pm$ notation is understood as follows: $\sum_\pm x^{\pm1}y^{\pm1} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1}$, $\sum_\pm x^{\pm1} = x + x^{-1}$, and $\sum_\pm 1 = 1$.

**SO(13)** The flavor symmetry on $N_\mathbf{s}\mathbf{S} + N_\mathbf{v}\mathbf{V}$ matter multiplets is $SO(2N_\mathbf{s})_s \times Sp(N_\mathbf{v})_v$. The character expansion of $\hat{Z}_1$ at $(N_\mathbf{s}, N_\mathbf{v}) = (1, 1)$ is written below, after setting the Coulomb VEV $\vec{a} = 0$ to keep the expression concise.

$$\begin{aligned}
\hat{Z}_1 = \frac{t^{20}}{(1-t^2)^{20}} \sum_\pm \Big( & y_s^{\pm 8}\chi^V_{(1)} \cdot (t^{10} + 58t^8 + 905t^6 + 5580t^4 + 15876t^2 + 22344 + +15876t^{-2} \cdots + t^{-10}) \\
& - 64\, y_s^{\pm 7}\chi^V_{(1)} \cdot (t^9 + 45t^7 + 540t^5 + 2520t^3 + 5292t + (t \to t^{-1})) \\
& + 26\, y_s^{\pm 6}\chi^V_{(1)} \cdot (77t^8 + 2541t^6 + 22226t^4 + 74811t^2 + 110770 + 74811t^{-2} + \cdots + 77t^{-8}) \\
& - 5824\, y_s^{\pm 5}\chi^V_{(1)} \cdot (7t^7 + 154t^5 + 924t^3 + 2145t + (t \to t^{-1})) \\
& + y_s^{\pm 4}\chi^V_{(1)} \cdot (t^{14} - 19t^{12} + 170t^{10} + 766t^8 + 576628t^6 + 7601283t^4 + 29870761t^2 \\
& \qquad + 46175700 + 29870761t^{-2} + 7601283t^{-4} + \cdots + t^{-14}) \\
& - 64\, y_s^{\pm 3}\chi^V_{(1)} \cdot (t^{11} - 20t^9 + 1256t^7 + 83074t^5 + 628311t^3 + 1580032t + (t \to t^{-1})) \\
& + 2002\, y_s^{\pm 2}\chi^V_{(1)} \cdot (t^{10} - 19t^8 + 756t^6 + 15006t^4 + 66051t^2 + 105146 + 66051t^{-2} + \cdots + t^{-10}) \\
& - 64\, y_s^{\pm 1}\chi^V_{(1)} \cdot (13t^{11} - 51t^9 - 436t^7 + 182670t^5 + 1603925t^3 + 4218449t + (t \to t^{-1})) \\
& + \chi^V_{(1)} \cdot (t^{16} - 19t^{14} + 274t^{12} + 3185t^{10} - 73808t^8 + 1918679t^6 + 46355974t^4 + 212905247t^2 \\
& \qquad + 342439014 + 212905247t^{-2} + 46355974t^{-4} + \cdots + t^{-16}) \\
& - 13\, y_s^{\pm 8} \cdot (t^9 + 35t^7 + 365t^5 + 1575t^3 + 3192t + (t \to t^{-1})) \\
& + 256\, y_s^{\pm 7} \cdot (3t^8 + 80t^6 + 630t^4 + 2016t^2 + 2940 + 2016t^{-2} + \cdots + 3t^{-8}) \\
& - 26\, y_s^{\pm 6} \cdot (847t^7 + 15989t^5 + 89887t^3 + 203357t + (t \to t^{-1})) \\
& - 64\, y_s^{\pm 5} \cdot (t^{10} - 20t^8 - 6180t^6 - 75228t^4 - 286725t^2 - 439416 - 286725t^{-2} + \cdots + t^{-10}) \\
& + y_s^{\pm 4} \cdot (t^{13} - 20t^{11} + 2907t^9 - 74785t^7 - 4557934t^5 - 33690015t^3 - 83955034t + (t \to t^{-1})) \\
& - 64\, y_s^{\pm 3} \cdot (t^{12} - 19t^{10} + 807t^8 - 24636t^6 - 510121t^4 - 2255129t^2 - 3592422 \\
& \qquad - 2255129t^{-2} - 510121t^{-4} - 24636t^{-6} + \cdots + t^{-12}) \\
& + 2\, y_s^{\pm 2} \cdot (7t^{13} - 140t^{11} + 3189t^9 + 86972t^7 - 7685485t^5 - 71293018t^3 - 190116261t + (t \to t^{-1})) \\
& - 64\, y_s^{\pm 1} \cdot (t^{12} - 84t^{10} + 2667t^8 - 36526t^6 - 1227485t^4 - 5926190t^2 - 9643046 \\
& \qquad - 5926190t^{-2} - 1227485t^{-4} - 36526t^{-6} + \cdots + t^{-12}) \\
& + (t^{15} - 20t^{13} - 602t^{11} + 5691t^9 + 495005t^7 - 22183672t^5 - 225823570t^3 - 617150913t + (t \to t^{-1}))\Big)
\end{aligned} \tag{A.14}$$

**SO(14)** The classical flavor symmetry on $N_\mathbf{s}\mathbf{S} + N_\mathbf{c}\mathbf{C} + N_\mathbf{v}\mathbf{V}$ hypermultiplets is $U(N_\mathbf{s})_s \times U(N_\mathbf{c})_c \times Sp(N_\mathbf{v})_v$. The character expansion of $\hat{Z}_1$ at $(N_\mathbf{s}, N_\mathbf{c}, N_\mathbf{v}) = (1, 0, 2)$ is written below, after turning off the $SO(14)$ Coulomb VEV $\vec{a} = 0$.

$$
\begin{aligned}
\hat{Z}_1 = \frac{t^{22}}{(1-t^2)^{22}} \sum_{\pm} \Big( & y_s^{\pm 8} \chi^V_{(01)} \cdot (t^{11} + 69t^9 + 1309t^7 + 10065t^5 + 36828t^3 + 69300t + (t \to t^{-1})) \\
& - 64\, y_s^{\pm 7} \chi^V_{(01)} \cdot (t^{10} + 55t^8 + 825t^6 + 4950t^4 + 13860t^2 + 19404 + 13860t^{-2} + \cdots + t^{-10}) \\
& + 26\, y_s^{\pm 6} \chi^V_{(01)} \cdot (77t^9 + 3234t^7 + 36667t^5 + 164401t^3 + 338261t + (t \to t^{-1})) \\
& - 5824\, y_s^{\pm 5} \chi^V_{(01)} \cdot (7t^8 + 210t^6 + 1694t^4 + 5434t^2 + 7920 + 5434t^{-2} + \cdots + 7t^{-8}) \\
& + y_s^{\pm 4} \chi^V_{(01)} \cdot (t^{15} - 22t^{13} + 231t^{11} - 1540t^9 + 614558t^7 \\
& \qquad\qquad + 11510191t^5 + 62671224t^3 + 139186397t + (t \to t^{-1})) \\
& - 832\, y_s^{\pm 3} \chi^V_{(01)} \cdot (33t^8 + 7744t^6 + 83776t^4 + 300104t^2 + 451192 + 300104t^{-2} + \cdots + 33t^{-8}) \\
& + 2002\, y_s^{\pm 2} \chi^V_{(01)} \cdot (t^{11} - 22t^9 + 621t^7 + 21262t^5 + 134245t^3 + 314181t + (t \to t^{-1})) \\
& - 832\, y_s^{\pm 1} \chi^V_{(01)} \cdot (t^{12} - t^{10} - 231t^8 + 15631t^6 + 206987t^4 + 790240t^2 + 1207976 + \\
& \qquad\qquad + 790240t^{-2} + 206987t^{-4} + 15631t^{-6} + \cdots + t^{-12}) \\
& + \chi^V_{(01)} \cdot (t^{17} - 22t^{15} + 335t^{13} + 3179t^{11} - 84595t^9 + 1320011t^7 \\
& \qquad\qquad + 63966077t^5 + 427850621t^3 + 1020096033t + (t \to t^{-1})) \\
& - 14\, y_s^{\pm 8} \chi^V_{(10)} \cdot (t^{10} + 42t^8 + 539t^6 + 2948t^4 + 7854t^2 + 10824 + 7854t^{-2} + \cdots + t^{-10}) \\
& + 832\, y_s^{\pm 7} \chi^V_{(10)} \cdot (t^9 + 33t^7 + 330t^5 + 1386t^3 + 2772t + (t \to t^{-1})) \\
& - 2184\, y_s^{\pm 6} \chi^V_{(10)} \cdot (11t^8 + 270t^6 + 2002t^4 + 6182t^2 + 8910 + 6182t^{-2} + \cdots + 11t^{-8}) \\
& + 5824\, y_s^{\pm 5} \chi^V_{(10)} \cdot (77t^7 + 1281t^5 + 6677t^3 + 14575t + (t \to t^{-1})) \\
& + 52\, y_s^{\pm 4} \chi^V_{(10)} \cdot (33t^{10} - 726t^8 - 109153t^6 - 1133396t^4 - 3996580t^2 - 5980436 \\
& \qquad\qquad - 3996580t^{-2} - 1133396t^{-4} - 109153t^{-6} + \cdots + 33t^{-10}) \\
& - 64\, y_s^{\pm 3} \chi^V_{(10)} \cdot (t^{13} - 22t^{11} + 868t^9 - 20559t^7 - 726341t^5 - 4583956t^3 - 10718569t + (t \to t^{-1})) \\
& + 8008\, y_s^{\pm 2} \chi^V_{(10)} \cdot (49t^8 - 2102t^6 - 29678t^4 - 115094t^2 - 176638 - 115094t^{-2} + \cdots + 49t^{-8}) \\
& + 4928\, y_s^{\pm 1} \chi^V_{(10)} \cdot (t^{11} - 35t^9 + 217t^7 + 21505t^5 + 152866t^3 + 371316t + (t \to t^{-1})) \\
& - 8\, \chi^V_{(10)} \cdot t^{10} (112t^{12} - 189t^{10} - 104258t^8 + 2855160t^6 + 46213090t^4 + 185620270t^2 \\
& \qquad\qquad + 287407450 + 185620270t^{-2} + 46213090t^{-4} + \cdots + 112t^{-12}) \\
& + 13\, y_s^{\pm 8} \cdot (8t^9 + 229t^7 + 2101t^5 + 8393t^3 + 16401t + (t \to t^{-1})) \\
& - 5824\, y_s^{\pm 7} \cdot (t^8 + 22t^6 + 154t^4 + 462t^2 + 660 + 462t^{-2} + \cdots + t^{-8}) \\
& + 26\, y_s^{\pm 6} \cdot (6075t^7 + 95425t^5 + 483483t^3 + 1042937t + (t \to t^{-1})) \\
& - 64\, y_s^{\pm 5} \cdot (t^{12} - 22t^{10} + 231t^8 + 41580t^6 + 427575t^4 + 1498244t^2 + 2237312 \\
& \qquad\qquad + 1498244t^{-2} + 427575t^{-4} + 41580t^{-6} + \cdots + t^{-12}) \\
& + 91\, y_s^{\pm 4} \cdot (11t^{11} - 473t^9 + 7623t^7 + 312675t^5 + 2010490t^3 + 4723994t + (t \to t^{-1})) \\
& + 5824\, y_s^{\pm 3} \cdot (77t^8 - 2046t^6 - 32546t^4 - 129768t^2 - 200508 - 129768t^{-2} + \cdots + 77t^{-8})
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\, y_s^{\pm 2} \cdot (7t^{15} - 154t^{13} + 2475t^{11} - 93720t^9 - 257649t^7 \\
&\qquad\qquad\qquad\qquad + 50128782t^5 + 390072133t^3 + 972422990t + (t \to t^{-1})) \\
&- 64\, y_s^{\pm 1} \cdot (t^{14} - 22t^{12} + 231t^{10} - 24927t^8 + 317625t^6 + 7227990t^4 + 31070743t^2 + 48912688+ \\
&\qquad\qquad\qquad\qquad + 31070743t^{-2} + 7227990t^{-4} + 317625t^{-6} + \cdots + t^{-14}) \\
&+ 154\,(20t^{11} - 1740t^9 - 16109t^7 + 958563t^5 + 8046291t^3 + 20489955t + (t \to t^{-1}))
\end{aligned}
\tag{A.15}
$$

We also confirmed that the 1-instanton partition function $Z_1$ for $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (0, 1, 2)$ could be identified with the above as follows:

$$
Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(0,1,2)}(\vec{a}, \epsilon_1, \epsilon_2, m^c, \vec{m}^v) = Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(1,0,2)}(\vec{a}, \epsilon_1, \epsilon_2, m^s, \vec{m}^v)|_{m^s \to -m^c}. \tag{A.16}
$$

# B SU(N) theory

We find that

$$
\begin{cases}
n = \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = N - N_f, \\
n = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + |\kappa_{\text{eff}}| & \text{otherwise.}
\end{cases}
\tag{B.1}
$$

for $SU(N)_\kappa + N_f\mathbf{F}$ theory with $N_f + 2|\kappa| \leq 2N$,

$$
\begin{cases}
n = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) - (\kappa_{\text{eff}} - 2) & \text{if } N\text{: odd and } \kappa_{\text{eff}} \geq 3 - \frac{N_f}{2}, \\
n = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) - (\kappa_{\text{eff}} - 2) & \text{if } N\text{: even and } \kappa_{\text{eff}} > 3 - \frac{N_f}{2}, \\
n \geq h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + 2\{\frac{\kappa_{\text{eff}}}{2}\} & \text{otherwise.}
\end{cases}
\tag{B.2}
$$

for $SU(N)_\kappa + N_f\mathbf{F} + 1\mathbf{A}$ theory with $N_f + 2|\kappa| \leq N + 4$.

$$
\begin{cases}
n' = \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = -N, \\
n' = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + \left|\kappa + \frac{1}{2}\sum_l I_2(\mathbf{R}_l)\right| & \text{otherwise.}
\end{cases}
$$

for $SU(N)_\kappa + N_f\mathbf{F}$ theory with $N_f + 2|\kappa| \leq 2N$,

$$
\begin{cases}
n' = h^\vee + \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + \kappa_{\text{eff}} & \text{if } N\text{: odd and } \kappa_{\text{eff}} \leq 1 - N - \frac{N_f}{2}, \\
n' = h^\vee + \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + \kappa_{\text{eff}} & \text{if } N\text{: even and } \kappa_{\text{eff}} < 1 - N - \frac{N_f}{2}, \\
n' \geq h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + 2\{-\frac{\kappa_{\text{eff}} + N + N_f}{2}\} & \text{otherwise.}
\end{cases}
\tag{B.3}
$$

for $SU(N)_\kappa + N_f\mathbf{F} + 1\mathbf{A}$ theory with $N_f + 2|\kappa| \leq N + 4$.

i.e., and

In other words, This suggests that, as long as $\frac{b+b'}{2} > 2$, we can solve for the instanton partition function $Z_n$

In summary, there are $(b+b')/2$ blowup equations

we have found $n = n' = b = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l)$. Thus the range of $d$ is constrained to

$$0 \le d \le h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) \qquad \text{for the theories with } G \neq SU(N). \tag{B.4}$$

See Appendix B for the pattern on $n$ in two classes of

Especially when $\kappa_{\text{eff}} = 0$, the value of $n$ is same as the above.

where we observe that

$$n' = h^\vee - \tfrac{1}{2}\textstyle\sum_l I_2(\mathbf{R}_l) \tag{B.5}$$

for the theories with $G \neq SU(N)$,

We need to be a bit careful when dealing with $SU(N)_\kappa$ theories. For $SU(N)_\kappa + N_f\mathbf{F}$ theory, the validity range of $d$ becomes

$$\begin{cases} d \in [0, N] & \text{for} \quad \kappa = -N + \frac{N_f}{2} \\ d \in [0, N - \frac{N_f}{2} - \kappa] & \text{for} \quad \kappa \in (-N + \frac{N_f}{2}, -\frac{N_f}{2}] \\ d \in [0, N] & \text{for} \quad \kappa \in [-\frac{N_f}{2}, +\frac{N_f}{2}] \\ d \in [\frac{N_f}{2} - \kappa, N] & \text{for} \quad \kappa \in [+\frac{N_f}{2}, N - \frac{N_f}{2}) \\ d \in [0, N] & \text{for} \quad \kappa = +N - \frac{N_f}{2} \end{cases} \tag{B.6}$$

which always includes $0 \le d \le N$. For $SU(N)_\kappa + N_f\mathbf{F} + 1\mathbf{A}$ theory,

Substituting $(n, n')$ to (2.46), $d$ takes a value in the following range:

$$\begin{cases} d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N] & \text{for} \quad \kappa \in [-2 - \frac{N}{2} + \frac{N_f}{2}, 1 - N - \frac{N_f}{2}] \\ d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for} \quad \kappa \in (1 - N - \frac{N_f}{2}, 3 - \frac{N_f}{2}) \\ d \in [-1, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for} \quad \kappa \in [3 - \frac{N_f}{2}, +\frac{N_f}{2}, 2 + \frac{N}{2} - \frac{N_f}{2}] \end{cases} \tag{B.7}$$

with odd $N$, and

$$\begin{cases} d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N] & \text{for} \quad \kappa \in [-2 - \frac{N}{2} + \frac{N_f}{2}, 1 - N - \frac{N_f}{2}] \\ d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for} \quad \kappa \in (1 - N - \frac{N_f}{2}, 3 - \frac{N_f}{2}) \\ d \in [-1, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for} \quad \kappa \in [3 - \frac{N_f}{2}, +\frac{N_f}{2}, 2 + \frac{N}{2} - \frac{N_f}{2}] \end{cases} \tag{B.8}$$

with even $N$, where $a$ and $b$ are defined by $a = 2\kappa - N - N_f$ and $b = 2\kappa + N + N_f$. Although these patterns look

$$-\frac{\kappa_{\text{eff}}}{2} - \{\frac{\kappa_{\text{eff}}}{2}\} \le d \le N \qquad \text{for } -N \le \kappa_{\text{eff}} \le 1 - N - \frac{N_f}{2} \tag{B.9}$$

$$-\frac{\kappa_{\text{eff}}}{2} - \{\frac{\kappa_{\text{eff}}}{2}\} \leq d \leq N + 1 - \frac{\kappa_{\text{eff}} + N + N_f}{2} + \{-\frac{\kappa_{\text{eff}} + N + N_f}{2}\} \tag{B.10}$$

$$-1 \leq d \leq N + 1 - \frac{\kappa_{\text{eff}} + N + N_f}{2} + \{-\frac{\kappa_{\text{eff}} + N + N_f}{2}\} \tag{B.11}$$

For $SU(N)_{|\kappa|<N}$ SYM without a hypermultiplet, $n = h^\vee + |\kappa|$ so that [28]

$$-\tfrac{\kappa+|\kappa|}{2} \leq d \leq h^\vee + \tfrac{|\kappa|-\kappa}{2}. \tag{B.12}$$

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| $G$ | matter | $r_0$ | $d_{\max}$ |
|---|---|---|---|
| $SU(N)_\kappa$ | $N_f \times \mathbf{N}$ | $d - N/2 - \kappa/2$ | $0 \leq d \leq N - \lvert\kappa\rvert - 2N_f - 1$(?) |
| $SU(6)_3$ | $1 \times \mathbf{20}$ | $d - 6/2 - 3/2 + 3/2$ | $1 \leq d \leq 6$ |
| $SO(7)$ | pure | $d - 5/2$ | $0 \leq d \leq 5$ |
| $SO(7)$ | $1 \times \mathbf{8}$ | $d - 5/2 + 1/2$ | $0 \leq d \leq 4$ |
| $SO(7)$ | $1 \times \mathbf{7}$ | $d - 5/2 + 1 \times 1/2$ | $0 \leq d \leq 4$ |
| $SO(7)$ | $2 \times \mathbf{7}$ | $d - 5/2 + 2 \times 1/2$ | $0 \leq d \leq 3$ |
| $G_2$ | pure | $d - 4/2$ | $0 \leq d \leq 4$ |
| $G_2$ | $1 \times \mathbf{7}$ | $d - 4/2 + 1/2$ | $0 \leq d \leq 3$ |
| $F_4$ | pure | $d - 9/2$ | $0 \leq d \leq 9$ |
| $F_4$ | $1 \times \mathbf{26}$ | $d - 9/2 + 1 \times 3/2$ | $0 \leq d \leq 6$ |
| $F_4$ | $2 \times \mathbf{26}$ | $d - 9/2 + 2 \times 3/2$ | $0 \leq d \leq 3$ |

**Table 1**: list of theories

| Gauge group | Hypermultiplets |
|---|---|
| $SO(14)$ | $n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 2$ |
| $SO(13)$ | $n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 2$ |
| $SO(12)$ | $n_{\mathbf{s}} + n_{\mathbf{c}} = 2$ |
| $SO(12)$ | $n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 4$ |
| $SO(11)$ | $n_{\mathbf{s}} = 2$ |
| $SO(11)$ | $n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 4$ |
| $SO(10)$ | $n_{\mathbf{s}} + n_{\mathbf{c}} = 3$ |
| $SO(10)$ | $n_{\mathbf{s}} + n_{\mathbf{c}} = 2$ and $n_{\mathbf{v}} \leq 2$ |
| $SO(10)$ | $n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 4$ |
| $SO(9)$ | $n_{\mathbf{s}} = 3$ |
| $SO(9)$ | $n_{\mathbf{s}} = 2$ and $n_{\mathbf{v}} \leq 2$ |
| $SO(9)$ | $n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 4$ |
| $SO(8)$ | $n_{\mathbf{s}} + n_{\mathbf{c}} + n_{\mathbf{v}} = 4$ |
| $SO(7)$ | $n_{\mathbf{s}} + n_{\mathbf{v}} = 3$ |

**Table 2**: asd

| Gauge Group | Hypermultiplets | Equation No. | Attachment |
|---|---|---|---|
| $SO(8)$ | $1\mathbf{S}+1\mathbf{C}+1\mathbf{V}$ | (3.11) | |
| $SO(8)$ | $2\mathbf{S}+2\mathbf{C}+2\mathbf{V}$ | (3.12) | |
| $SO(9)$ | $2\mathbf{S}+1\mathbf{V}$ | (3.17) | |
| $SO(10)$ | $2\mathbf{S}+4\mathbf{V}$ | (A.5) | |
| $SO(10)$ | $1\mathbf{S}+1\mathbf{C}+4\mathbf{V}$ | (A.5) | |
| $SO(10)$ | $2\mathbf{C}+4\mathbf{V}$ | (A.5) | |
| $SO(8)$ | $3\mathbf{S}+1\mathbf{C}$ | | |
| $SO(8)$ | $3\mathbf{C}+1\mathbf{V}$ | | |
| $SO(8)$ | $3\mathbf{C}+1\mathbf{V}$ | | |

**Table 3**: Character expansion of $SO(N)$ theory with spinor hypermultiplets