Prepared for submission to JHEP KIAS-P19???, SNUTP19-???

Instantons from Blow-up

Joonho Kim,[a] **Sung-Soo Kim,**[b] **Ki-Hong Lee,**[c] **Kimyeong Lee,**[a] **and Jaewon Song**[a]

[a]*School of Physics, Korea Institute for Advanced Study, Seoul 02455, Korea*

[b]*School of Physics, University of Electronic Science and Technology of China, No.4, Section 2, North Jianshe Road, Chengdu, Sichuan 610054, China*

[c]*Department of Physics and Astronomy & Center for Theoretical Physics Seoul National University, Seoul 08826, Korea*

E-mail: joonhokim@kias.re.kr, sungsoo.kim@uestc.edu.cn, khlee11812@gmail.com, klee@kias.re.kr, jsong@kias.re.kr

Abstract: The Nekrasov partition function for 4d $\mathcal{N}=2$ or 5d $\mathcal{N}=1$ gauge theory on the blow up of a point $\hat{\mathbb{C}}^2$ can be written in terms of the partition function on the flat space $\mathbb{C}^2$. At the same time, the partition function on the blow up is identical to the partition function on a flat space for sufficiently large class of examples. This relation enables us to compute the instanton partition functions for 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ gauge theories for arbitrary gauge theory with large class of matter representations without knowing explicit construction of the instanton moduli space. Remarkably, the instanton partition function is completely determined by the perturbative part. We obtain the partition function for the previously unknown theories: exceptional gauge groups EFG with fundamental/spinor hypermultiplets and more. We also compute the case with $SU(6)$ with rank-3 antisymmetric tensor and compare with the topological vertex computation using the recently found 5-brane web configuration.

Contents

1 Introduction

The Seiberg-Witten prepotential provides a complete description for the low energy dynamics of 4d $\mathcal{N}=2$ or 5d $\mathcal{N}=1$ gauge theory in its Coulomb branch [1, 2]. It is a polynomial in the VEV of the vector multiplet scalar that parameterizes the Coulomb branch moduli space. Quantum correction to the prepotential is known to be one-loop exact, while there also exist non-perturbative corrections coming from Yang-Mills instantons.

An efficient way to compute the fully quantum corrected prepotential $\mathcal{F}$ is to study the Nekrasov partition function $\mathcal{Z}$ on Ω-deformed $\mathbb{C}^2$ or $\mathbb{C}^2\times S^1$, i.e., $\mathcal{F}=\lim_{\epsilon_i\to 0}\epsilon_1\epsilon_2\log\mathcal{Z}$ [3–5]. It can be written as the product of the classical, one-loop, and instanton contributions,

$$\mathcal{Z}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q})=\mathcal{Z}_{\text{class}}(\vec{a},\epsilon_1,\epsilon_2,\mathfrak{q})\ \mathcal{Z}_{\text{1-loop}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2)\ \mathcal{Z}_{\text{inst}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q}), \tag{1.1}$$

where the instanton piece is the fugacity sum over all multi-instanton contributions:

$$\mathcal{Z}_{\text{inst}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q})=1+\sum_{k=1}^{\infty}\mathfrak{q}^k\mathcal{Z}_k(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2). \tag{1.2}$$

Notation The equivariant parameters $\epsilon_1,\epsilon_2,\vec{a}$ are associated to the $U(1)^2\times U(1)^{|G|}$ action on the k-instanton moduli space $\mathcal{M}_{k,G}$. Additional parameters $\vec{\mathfrak{m}}$ are introduced if the gauge theory has an extra flavor symmetry. The instanton fugacity $\mathfrak{q}$ can be written as $\mathfrak{q}\equiv\Lambda^{b_0}$ (4d) or $\mathfrak{q}\equiv\exp(2\pi i\Lambda)$ (5d) where Λ denotes the bare coupling of the 4d/5d gauge theory.

Throughout Section 2, we will characterize the partition function in terms of $m_i\equiv\mathfrak{m}_i+\epsilon_+$ (where $\epsilon_+\equiv\frac{\epsilon_1+\epsilon_2}{2}$) as a reflection of the Donaldson-Witten twist by $SU(2)_R$ symmetry. Here

m_i corresponds to the 'physical' mass parameter of a matter hypermultiplet. We will also stick to the 'effective' instanton fugacity $q \equiv \mathfrak{q} e^{b_0 \epsilon_+/2}$, which absorbs the $SU(2)_R$ generated mass of fermionic instanton zero modes, in describing the 5d Nekrasov partition function.

2 Instanton Counting from Blow-up

The essential idea of using the blow-up of $\mathbb{C}^2$ for instanton counting is that the gauge theory partition function for a 4d $\mathcal{N} = 2$ (or 5d $\mathcal{N} = 1$) theory on a blow up of a point $\hat{\mathbb{C}}^2$ (or $S^1 \times \hat{\mathbb{C}}^2$) can be written in two different ways. This allows us to write a recursion relation for the instanton partiton function that can be solved easily [5–8].

Localization on the blow-up $\hat{\mathbb{C}}^2$ One of the expressions for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ comes from the Coulomb branch localization, which results that $\hat{\mathcal{Z}}$ can be obtained by patching together the flat-space partition function $\mathcal{Z}$ [9].

The blow-up of a point in $\mathbb{C}^2$ can be described as a subspace of $\mathbb{C}^2 \times \mathbb{P}^1$, defined as

$$\left\{ \big((x,y), [z:w] \big) \in \mathbb{C}^2 \times \mathbb{P}^1 \,\middle|\, xw = yz \right\} \;, \tag{2.1}$$

where $[z : w]$ represents the homogeneous coordinates on $\mathbb{P}^1$. Notice that this space is identical to $\mathbb{C}^2$ when $(x, y) \neq (0, 0)$, and the origin is replaced by complex projective plane. We will be considering the $U(1)^2$ equivariant partition function, with each $U(1)$'s acting as the rotation of a complex plane. This $U(1)^2$ action V acts on $\hat{\mathbb{C}}^2$ as

$$\big((x,y), [z:w] \big) \mapsto \big((e^{\epsilon_1} x, e^{\epsilon_2} y), [e^{\epsilon_1} z : e^{\epsilon_2} w] \big) \;. \tag{2.2}$$

Instantons are located at two fixed points of the $U(1)^2$ action, namely the north/south poles of the $\mathbb{P}^1$, whose coordinates are $((x, y), [z : w]) = ((0, 0), [1, 0])$ and $((0, 0), [0, 1])$ respectively. Around these fixed points, the weights of the $U(1)^2$ action V become:

$$(x,\, w/z) \mapsto (e^{\epsilon_1} x,\, e^{\epsilon_2 - \epsilon_1} w/z) \qquad \text{(near the north pole)} \tag{2.3}$$

$$(z/w,\, y) \mapsto (e^{\epsilon_1 - \epsilon_2} z/w,\, e^{\epsilon_2} y) \qquad \text{(near the south pole)} \tag{2.4}$$

The full partition function $\hat{\mathcal{Z}}$ on $\hat{\mathbb{C}}^2$, which includes both the perturbative and instanton contributions, can be obtained by performing the localization on the Coulomb branch. On the Coulomb branch, the gauge group is generically broken to $U(1)^r$ where r is the rank of the gauge group. The $U(1)^r$ equivariant parameters $\vec{a}$ naturally appear in the partition function. One needs to sum over all distinct field configurations with zero-sized instantons located at the north and south poles. All the inequivalent configurations are labeled by the vector $\vec{k}$ of the first Chern numbers, corresponding to different flux configurations on the divisor $\mathbb{P}^1$. Here we turn off any possible external flux that can be supported on the $\mathbb{P}^1$. Summing up, $\hat{\mathcal{Z}}$ can be expressed as the following formula [9–13]:

$$\hat{\mathcal{Z}}(\vec{a}, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} \mathcal{Z}(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1) \mathcal{Z}(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2) \tag{2.5}$$

where the flux sum is taken over the coweight lattice Λ of the gauge algebra. Notice that the Coulomb parameter $\vec{a}$ gets an appropriate shift at each fixed point p. This is because the $U(1)^r$ vector multiplet scalar $\langle\vec{\Phi}\rangle$ has the vacuum expectation value $\vec{a}+\vec{k}H|_p$ [14], where

$$H=\begin{cases}\dfrac{\epsilon_1|z|^2+(\epsilon_2-\epsilon_1)|w|^2}{|z|^2+|w|^2} & \text{(around the north pole),}\\ \dfrac{(\epsilon_1-\epsilon_2)|z|^2+\epsilon_2|w|^2}{|z|^2+|w|^2} & \text{(around the south pole).}\end{cases} \tag{2.6}$$

is the moment map $\iota_V\omega = dH$ for the $U(1)^2$ action V on the Kähler 2-form ω. It follows that $H|_{\mathrm{NP}}=\epsilon_1$ and $H|_{\mathrm{SP}}=\epsilon_2$.

Partition function on $\hat{\mathbb{C}}^2$ vs $\mathbb{C}^2$ An alternative fact for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ is that $\hat{\mathcal{Z}}$ is actually identical to the flat-space partition function $\mathcal{Z}$ [5–7].

The blow-up $\hat{\mathbb{C}}^2$ is identical to $\mathbb{C}^2$ except for the origin, which is replaced by the exceptional divisor $\mathbb{P}^1$. It can be identified with the total space of the bundle $\mathcal{O}(-1)\to\mathbb{P}^1$. Since the Nekrasov partition function gets contributions only from the small instantons localized at the fixed points of the $U(1)^2$ equivariant action V, the size of the divisor should not affect the partition function as we smoothly shrink it. So we expect that $\hat{\mathcal{Z}}=\mathcal{Z}$. More generally, if we consider the total space of $\mathcal{O}(-2)\to\mathbb{P}^1$ and shrink the base, we land on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ which is singular at the origin. There are extra states coming from the $\mathbb{P}^1$, which become massless as we shrink the size of the $\mathbb{P}^1$. However, they contribute to the orbifold partition function merely as the required twisted sector. We expect the Nekrasov partition function to remain the same by blowing up and down. This implies the following identity: [5–8, 15, 16]

$$\mathcal{Z}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}\mathcal{Z}(\vec{a}+\vec{k}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1)\mathcal{Z}(\vec{a}+\vec{k}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2)\ , \tag{2.7}$$

Background parameters The expected relation (2.7) is only a schematic expression. The partition function also depends on some background parameters, such as the gauge coupling Λ or the flavor chemical potentials $\vec{m}$. Those need to be modified appropriately at each fixed point p of the blow-up $\hat{\mathbb{C}}^2$. For example, the $\mathbb{C}^2\times S^1$ partition function of a free hypermultiplet with mass m can be written as

$$\exp\left[\frac{-2m^3+(\epsilon_+^2+\epsilon_-^2+12)\cdot m-24}{12\epsilon_1\epsilon_2}\right]\cdot\mathrm{PE}\left[\frac{e^{-m}\cdot e^{-\epsilon_+}}{(1-e^{-\epsilon_1})(1-e^{-\epsilon_2})}\right], \tag{2.8}$$

which can satisfy (2.7) only when accompanied by $m\to m+\frac{H}{2}|_p$. This shift keeps it unchanged the combination $(m+\epsilon_+)$ of background gauge fields, corresponding to the effective twisted mass of the hypermultiplet. We will denote it as $m_{\mathrm{phy}}\equiv(m+\epsilon_+)$ from here on.

Similarly, we investigate the $\mathbb{C}^2\times S^1$ partition function of the 5d $\mathcal{N}=1$ Maxwell theory. This is a product between the perturbative index of a free vector multiplet,

$$\exp\left[\frac{\epsilon_+(\epsilon_+^2-\epsilon_-^2-12)+24}{12\epsilon_1\epsilon_2}\right]\cdot\mathrm{PE}\left[-\frac{1}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\right] \tag{2.9}$$

satisfying (2.7) by itself, and the non-perturbative contribution from $U(1)$ multi-instantons

$$\mathrm{PE}\left[\frac{e^{\Lambda}\cdot e^{\epsilon_+}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\right] \tag{2.10}$$

which comply with (2.7) upon the appropriate shift of the gauge coupling $\Lambda \to \Lambda + \frac{H}{2}|_p$. Again, the effective twisted mass $(\Lambda + \epsilon_+)$ of an instanton stays invariant under this change.

We remark that the twisted mass of an instanton, in a general non-Abelian gauge theory with effective Chern-Simons level κ_{eff}, is $(\Lambda + b\cdot\epsilon_+)$ where $b \equiv h^\vee - \frac{1}{2}\sum_i I(\mathbf{R}_i) + \kappa_{\text{eff}}$. This is the supersymmetric Casimir energy of the ADHM quantum mechanics at one instanton. Here $h^\vee$ is the dual Coxeter number of the gauge group, $I_2(\mathbf{R})$ is the Dynkin index[1] of a given representation $\mathbf{R}$, and the sum is taken over all matter multiplets in the theory. The effective Chern-Simons level is $\kappa_{\text{eff}} \equiv \kappa_{\text{cl}} + \frac{1}{2}\sum_i \text{sgn}(m_i)\, I_3(\mathbf{R}_i)$ where $I_3(\mathbf{R})$ is the anomaly coefficient of an $SU(N)$ representation $\mathbf{R}$. Having identified the effective instanton mass, the proper shift of the gauge coupling at each fixed point p must be $\Lambda \to \Lambda + \frac{b}{2}H|_p$. In summary,

$$m|_p = m + \frac{H}{2}\bigg|_p \qquad \Lambda|_p = \Lambda + \frac{b}{2}H|_p. \tag{2.11}$$

Donaldson operator The equation (2.7) is not enough to determine the partition function since there are three unknown functions and only one relation. More independent relations can be found from insertion of topological operators. We want to consider the correlation functions of the topological operator $\mu(C)$ associated to the two-cycle C of the blow-up $\hat{\mathbb{C}}^2$.

Let us focus on the 4d Donaldson-twisted theory. The $\mathcal{Q}$-exact operator $\mu(C)$ is obtained by applying the descent procedure twice to the Casimir invariant $\text{Tr}(\Phi^2)$, which can be written in terms of the component fields as [17]

$$\mu(C) = \int \left\{\omega\wedge \text{Tr}\Big(\Phi F + \frac{1}{2}\psi\wedge\psi\Big) + \frac{H}{2}\,\text{Tr}\Big(F\wedge F\Big)\right\} . \tag{2.12}$$

Here ω and H are the Kähler two-form on C and the moment map (2.6) associated to V. We find it convenient to study the generating function $\langle e^{t\,\mu(C)}\rangle$ of the correlators $\langle \mu(C)\ldots\mu(C)\rangle$. The expectation value can be written as

$$\langle e^{t\,\mu(C)}\rangle = \sum_{\vec{k}\in\Lambda} \frac{\mathcal{Z}^{(N),t}(\vec{k})\cdot \mathcal{Z}^{(S),t}(\vec{k})}{\mathcal{Z}^{(0)}} , \tag{2.13}$$

where

$$\begin{aligned} \mathcal{Z}^{(N),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_1, \epsilon_1, \epsilon_2-\epsilon_1, \Lambda + (\tfrac{b}{2}+t)\epsilon_1, \vec{m}+\tfrac{1}{2}\epsilon_1) , \\ \mathcal{Z}^{(S),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2, \Lambda + (\tfrac{b}{2}+t)\epsilon_2, \vec{m}+\tfrac{1}{2}\epsilon_2) , \\ \mathcal{Z}^{(0)} &\equiv \mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, m). \end{aligned} \tag{2.14}$$

[1]The Dynkin index $I_2(\mathbf{R})$ is normalized such that $I_2(\mathbf{F}) = 1$ for the $SU(N)$ fundamental representation $\mathbf{F}$.

We notice that the insertion of the operator $\mu(C)$ has caused an extra shift of the effective instanton mass by $tH|_p$, at each fixed point p of the blow-up $\hat{\mathbb{C}}^2$ [5–7].

As we shrink the two-cycle C to recover the flat $\mathbb{C}^2$, the effect of inserting $\mu(C)^d$ turns out to give a vanishing contribution due to the conservation law. To see this, we recall that the instanton breaks the $U(1)_R$ symmetry to the a discrete subgroup $\mathbb{Z}_{2h^\vee - \sum_i I_2(\mathbf{R}_i)}$, under which the operator (2.16) carries a charge of $+2$. Unless the discrete charges add up to zero, the correlation function vanishes. Therefore, expanding in powers of t,

$$\langle e^{t\,\mu(C)} \rangle = 1 + \mathcal{O}(t^{2h^\vee - \sum_i I_2(\mathbf{R}_i)}) \ . \tag{2.15}$$

As long as the hypermultiplet representation is not too large, i.e., when $2h^\vee - \sum_i I_2(\mathbf{R}_i) > 2$, this allows us to write 3 independent relations for the 3 unknown varables, at every instanton number. Thus we can expand $\langle e^{t\,\mu(C)} \rangle$ up to order $\mathcal{O}(t^2)$ and recursively solve for the instanton partition function $\mathcal{Z}^{(0)}$ on $\mathbb{C}^2$. The instanton part of the partition function will be completely determined from the perturbative partition function, as we will illustrate in Section 2.1.

Turning to the 5d $\mathcal{N}=1$ gauge theory on S^1, the operator $\mu(C)$ is naturally uplifted via exponentiation as follows: [18]

$$\mu(C)^d = \exp\left[d \int \left\{ \omega \wedge \mathrm{Tr}\Big(\Phi F + \tfrac{1}{2} \psi \wedge \psi \Big) + \tfrac{H}{2}\, \mathrm{Tr}\Big(F \wedge F \Big) \right\} \right] \ . \tag{2.16}$$

This stems from the fact that $\mathrm{Tr}(\Phi^2)$ is not a well-defined operator in a 5d theory on $\mathcal{M}_4 \times S^1$. To take into account the periodicty boson, we need to exponentiate to obtain the 'Coulomb parameter'. The $\mu(C)$ operator has to be exponentiated accordingly to take into account the periodicity and the coefficient in front of the exponent has to be properly quantized. More precisely, the offset d_0 is set to be $d_0 \equiv \frac{h^\vee}{2} - \sum_i \frac{I_2(\mathbf{R_i})}{4}$ where the sum is taken over all matter multiplets, since (explain the reason). Then d is integer-valued.

we no longer have a discrete R-symmetry to impose the selection rule.

The natural uplift of the 4d operator is given via where the 4d complex scalar Φ_4 is decomposed into

$$\int \omega \wedge \mathrm{Tr}(\Phi_4 F) \ \longrightarrow\ \int \omega \wedge (\mathrm{CS}_3(A) + \mathrm{Tr}(\Phi_5 F)) \in \mu(C). \tag{2.17}$$

[TODO: Explain why $e^{\mu(C)}$ is good topological operator in 5d. Explain why inserting this does not alter the partition function.]

Therefore, we obtain

$$Z = \sum_{\vec{k} \in \Lambda} \exp\Big((d + d_0)\, \mu(\vec{k}) \Big)\, Z^{(N),d}(\vec{k}) Z^{(S),d}(\vec{k}) \qquad (d = 0, 1, \ldots, h^\vee - I(R)/2) \ , \tag{2.18}$$

for the 5d theories. We call the relation above as the blow-up equation. (up to here)

2.1 Instanton partition function from the blow-up equation

From the blow up equation (2.18), we can determine the instanton partition function for the 5d $\mathcal{N}=1$ theory on $S^1\times\mathbb{C}^2$. To see this, let us decompose the partition function in terms of classical, 1-loop and the instanton piece:

$$Z(\vec{a},m,\epsilon_1,\epsilon_2,q)=Z_{\text{class}}(\vec{a},\epsilon_1,\epsilon_2,q)Z_{\text{1-loop}}(\vec{a},m,\epsilon_1,\epsilon_2)Z_{\text{inst}}(\vec{a},m,\epsilon_1,\epsilon_2,q) \tag{2.19}$$

Then the blow-up equation can be rewritten as

$$Z_{\text{inst}}=\sum_{\vec{k}}\left[\exp\Big((d+d_0)\mu(\vec{k})\Big)\frac{Z^{(N),d}_{\text{pert}}Z^{(S),d}_{\text{pert}}}{Z_{\text{pert}}}\right]Z^{(N),d}_{\text{inst}}Z^{(S),d}_{\text{inst}}\ , \tag{2.20}$$

where $Z_{\text{pert}}\equiv Z_{\text{class}}Z_{\text{1-loop}}$ and the superscript (N/S) denotes appropriate shift of parameters in the north/south pole

$$\begin{aligned}Z^{(N),d}(\vec{k})&\equiv Z(\vec{a}+\vec{k}\epsilon_1,m,\epsilon_1,\epsilon_2-\epsilon_1,qe^{(d+d_0)\epsilon_1})\ ,\\ Z^{(S),d}(\vec{k})&\equiv Z(\vec{a}+\vec{k}\epsilon_2,m,\epsilon_1-\epsilon_2,\epsilon_2,qe^{(d+d_0)\epsilon_2})\ .\end{aligned} \tag{2.21}$$

Notice that there is no q-dependence in the 1-loop part, hence no d-dependence as well. The instanton piece can be expanded in terms of the instanton number as

$$Z_{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2,q)=\sum_{n\geq 0}q^nZ_n(\vec{a},\epsilon_1,\epsilon_2)\ . \tag{2.22}$$

Plugging in, we obtain

$$q^nZ_n=\sum_{\vec{k}}e^{(d+d_0)\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)\right)}q^{\frac{1}{2}\vec{k}\cdot\vec{k}}\left(\frac{Z^{(N)}_{\text{1-loop}}Z^{(S)}_{\text{1-loop}}}{Z_{\text{1-loop}}}\right)q^{\ell+m}e^{(d+d_0)(\ell\epsilon_1+m\epsilon_2)}Z^{(N)}_{\ell}Z^{(S)}_{m}\ , \tag{2.23}$$

where $q^{\frac{1}{2}\vec{k}\cdot\vec{k}}$ comes from the classical piece as we will see later. Collecting the terms with the same order, we obtain

$$Z_n=\sum_{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n}e^{(d+d_0)\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)+\ell\epsilon_1+m\epsilon_2\right)}g(\vec{k})Z^{(N)}_{\ell}(\vec{k})Z^{(S)}_{m}(\vec{k})\ , \tag{2.24}$$

where $g(\vec{k})\equiv\frac{Z^{(N)}_{\text{1-loop}}Z^{(S)}_{\text{1-loop}}}{Z_{\text{1-loop}}}$. Notice that the right-hand side of the above equation only involves the instanton parts with $\ell,m<n$. Therefore, one can determine the n-instanton partition function recursively from $Z_0=1$. To do this, let us separate n-instanton contribution from the above expression. We get

$$Z_n=e^{\epsilon_1(d+d_0)n}Z^{(N)}_n+e^{\epsilon_2(d+d_0)n}Z^{(S)}_n+I^{(d)}_n(\vec{a},m,\epsilon_1,\epsilon_2)\ , \tag{2.25}$$

where

$$I_n^{(d)}(\vec{a},m,\epsilon_1,\epsilon_2)=\sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n\\ \ell,m<n}} e^{(d+d_0)\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)+\ell\epsilon_1+m\epsilon_2\right)}g(\vec{k})Z_\ell^{(N)}(\vec{k})Z_m^{(S)}(\vec{k})\ . \tag{2.26}$$

Now, we can solve for $Z_n, Z_n^{(N)}, Z_n^{(S)}$ by using the blow-up equation for $d=0,1,2$ to obtain

$$Z_n(\vec{a},m,\epsilon_1,\epsilon_2)=\frac{e^{n(\epsilon_1+\epsilon_2)}I_n^{(0)}-(e^{n\epsilon_1}+e^{n\epsilon_2})I_n^{(1)}+I_n^{(2)}}{(1-e^{n\epsilon_1})(1-e^{n\epsilon_2})}\ . \tag{2.27}$$

Since $I_n^{(d)}$ only depends on Z_m with $m<n$ we can determine the instanton partition function recursively.

We now need to determine the $g(\vec{k})$, but this is also completely determined by the *1-loop part* of the partition function. Therefore we land on a remarkable conclusion: The non-perturbative partition function is completely fixed by the perturbative partition function! Notice that we have arrived at this conclusion not by requiring the perturbative series to be well-defined, as is often done in the resurgence analysis. Instead, we simply demand consistent answer upon deforming the spacetime geometry smoothly. This consistency condition is entirely non-perturbative, that the QFT has to be well-defined regardless of its spacetime.

In the remainder of this section we will give explicit formulae for 4d and 5d gauge theories. Especially, we will obtain a closed-form expression for the one-instanton partition function for a large class of gauge theories.

2.2 Instanton partition function for the 5d theory

Let us write the perturbative part of the partition function. The classical part is given as

$$Z_{\text{class}}=\exp\left[-\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}h_{ij}a^ia^j+\frac{1}{6}d_{ijk}a^ia^ja^k\right)\right] \tag{2.28}$$

where a_i denotes the Coulomb branch parameter and h_{ij}, d_{ijk} denotes the effective gauge couplings and Chern-Simons couplings in the Coulomb branch. The 1-loop part of the vector multiplet is given as

$$\begin{aligned}Z^V_{\text{1-loop}}=\exp&\left[-\frac{1}{2\epsilon_1\epsilon_2}\sum_{\alpha\in\Delta}\left(\frac{1}{6}(\vec{a}\cdot\vec{\alpha})^3-\frac{\epsilon_1+\epsilon_2}{4}(\vec{a}\cdot\vec{\alpha})^2+\frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{12}(\vec{a}\cdot\vec{\alpha})\right)\right]\\ &\times\ \text{PE}\left[-\frac{1}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\alpha\in\Delta}e^{\vec{a}\cdot\vec{\alpha}}\right]\ ,\end{aligned} \tag{2.29}$$

where Δ denotes the set of all roots. The 1-loop partition function of the hypermultiplet is given as

$$Z^H_{\text{1-loop}} = \exp\left[+ \frac{1}{2\epsilon_1\epsilon_2}\sum_{\omega\in R}\left(\frac{1}{6}\left(\vec{a}\cdot\vec{\omega}+m_i^{\rm phy}\right)^3 - \frac{\epsilon_1+\epsilon_2}{4}\left(\vec{a}\cdot\vec{\omega}+m_i^{\rm phy}\right)^2 \right.\right.$$
$$\left.\left. - \frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{24}\left(\vec{a}\cdot\vec{\omega}+m_i^{\rm phy}\right)\right)\right] \times \mathrm{PE}\left[\frac{e^{m_i^{\rm phy}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\omega\in R} e^{\vec{a}\cdot\vec{\omega}}\right] \tag{2.30}$$

where R is the set of all weight vectors in the representation of the hypermultiplet. We define the physical mass parameter as $m_i^{\rm phy} = m_i + \epsilon_+$ and $\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}$, since there is an effect of topological twisting that shifts the 'mass parameter' m_i by a unit of $SU(2)_R$ rotation [19].

Now, let us compute the $g(\vec{k})$ which is given by the ratio of the 1-loop part of the partition function. It turns out piece of the (2.29), (2.30) that are not inside the PE eventually cancel out due to $\vec{k}\to-\vec{k}$ and $\vec{a}\to-\vec{a}$ symmetry except for one single term for the hyper (change). We find

$$g(\vec{k})^V = e^{\frac{h^\vee}{2}(\vec{k}\cdot\vec{a})}\prod_{\vec{\alpha}\in\Delta}\mathrm{PE}\left[\frac{e^{\vec{a}\cdot\vec{\alpha}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} - \frac{e^{(\vec{a}+\vec{k}\epsilon_1)\cdot\vec{\alpha}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2-\epsilon_1})} - \frac{e^{(\vec{a}+\vec{k}\epsilon_2)\cdot\vec{\alpha}}}{(1-e^{\epsilon_1-\epsilon_2})(1-e^{\epsilon_2})}\right],$$
$$g(\vec{k})^H = e^{-\frac{I_2(\mathbf{R})}{4}(\vec{k}\cdot\vec{a})-\frac{I_2(\mathbf{R})}{4}(\vec{k}\cdot\vec{k})m_{\rm phy}}e^{-\frac{I_3(\mathbf{R})}{4}(d_{ijk}a_ik_jk_k)-\frac{I_3(\mathbf{R})}{6}\epsilon_+(d_{ijk}k_ik_jk_k)}$$
$$\times\prod_{\vec{w}\in R}\mathrm{PE}\left[-\frac{e^{\vec{a}\cdot\vec{w}+m^{\rm phy}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} + \frac{e^{(\vec{a}+\vec{k}\epsilon_1)\cdot\vec{w}+m^{\rm phy}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2-\epsilon_1})} + \frac{e^{(\vec{a}+\vec{k}\epsilon_2)\cdot\vec{w}+m^{\rm phy}}}{(1-e^{\epsilon_1-\epsilon_2})(1-e^{\epsilon_2})}\right] \tag{2.31}$$

for the vector and hypermultiplets respectively. Our notation is that $\mathrm{Tr}(T^iT^j) = I_2(\mathbf{R})\delta^{ij}$ and $\mathrm{Tr}(T^i\{T^j,T^k\}) = I_3(\mathbf{R})d_{ijk}$, where $d_{ijk}\neq 0$ only for $G=SU(N)$. The $SU(N)$ generators are normalized such that $I_3(\text{fund}) = 1$. Besides the zero-point energy contribution, the PE of $g(\vec{k})$ can be written as

$$\mathrm{PE}\left[e^{\vec{a}\cdot\vec{\alpha}}A(\vec{k}\cdot\vec{\alpha}, e^{\epsilon_1}e^{\epsilon_2})\right]\cdot\prod_i\prod_{\vec{w}\in R_i}\mathrm{PE}\left[-e^{\vec{a}\cdot\vec{w}+m_i^{\rm phy}}A(\vec{k}\cdot\vec{w}, e^{\epsilon_1}e^{\epsilon_2})\right] \tag{2.32}$$

where i runs over various hypermultiplets in the theory and $p_1 = e^{\epsilon_1}, p_2 = e^{\epsilon_2}$. Here the function A is given as

$$A(k,p_1,p_2) = \frac{1}{(1-p_1)(1-p_2)} - \frac{p_1^k}{(1-p_1)(1-p_2/p_1)} - \frac{p_2^k}{(1-p_1/p_2)(1-p_2)}\,. \tag{2.33}$$

We can easily see that $A(k,p_1,p_2)$ vanishes at $k=0,-1$. After some work, it is not hard to find that $A(k,p_1,p_2)$ can be written in terms of a finite sum as

$$A(k,p_1,p_2) = \begin{cases}\displaystyle\sum_{m+n\le k-1} p_1^m p_2^n & (k>0)\\ \displaystyle\sum_{m+n\le -k-2} p_1^{-m-1}p_2^{-n-1} & (k<-1)\\ 0 & (k=0,-1)\end{cases}\,. \tag{2.34}$$

Upon taking PE, we obtain

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) \equiv \mathrm{PE}\left[e^x A(k,e^{\epsilon_1},e^{\epsilon_2})\right] = \begin{cases} \displaystyle\prod_{\substack{m,n\geq 0\\ m+n\leq k-1}} \left(1-e^{x+m\epsilon_1+n\epsilon_2}\right) & (k>0) \\ \displaystyle\prod_{\substack{m,n\geq 0\\ m+n\leq -k-2}} \left(1-e^{x-(m+1)\epsilon_1-(n+1)\epsilon_2}\right) & (k<-1) \\ 1 & (k=0,-1) \end{cases} \tag{2.35}$$

Let us state our final answer:

$$Z_n(\vec{a},\vec{m}^{\mathrm{phy}},\epsilon_1,\epsilon_2) = \frac{e^{n(\epsilon_1+\epsilon_2)}I_n^{(0)} - (e^{n\epsilon_1}+e^{n\epsilon_2})I_n^{(1)} + I_n^{(2)}}{(1-e^{n\epsilon_1})(1-e^{n\epsilon_2})}\ , \tag{2.36}$$

with

$$\begin{aligned} I_n^{(d)} = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n\\ \ell,m<n}} \Bigg(&\exp\left[d\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)+\ell\epsilon_1+m\epsilon_2\right)\right] \\ &\times e^{d_0\left(\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)+\ell\epsilon_1+m\epsilon_2\right)} \times e^{-\frac{I_3(\mathbf{R})}{4}(d_{ijk}a_ik_jk_k)-\frac{I_3(\mathbf{R})}{6}\epsilon_+(d_{ijk}k_ik_jk_k)} \\ &\times \frac{\displaystyle\prod_i\prod_{\vec{w}\in R_i} e^{-\frac{(\vec{k}\cdot\vec{w})^2}{4}m^{\mathrm{phy}}}\mathcal{L}_{\vec{k}\cdot\vec{w}}(\vec{a}\cdot\vec{w}+m_i^{\mathrm{phy}},\epsilon_1,\epsilon_2)}{\displaystyle\prod_{\vec{\alpha}\in\Lambda}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)} \\ &\times Z_\ell(\vec{a}+\vec{k}\epsilon_1,\vec{m}^{\mathrm{phy}},\epsilon_1,\epsilon_2-\epsilon_1)Z_m(\vec{a}+\vec{k}\epsilon_2,\vec{m}^{\mathrm{phy}},\epsilon_1-\epsilon_2,\epsilon_2)\Bigg)\ , \end{aligned} \tag{2.37}$$

where i runs over all the hypermultiplets.

One-instanton partition function Let us compute the 1-instanton partition function using our formula. In this case, we only take the sum over the long roots having $\vec{k}\cdot\vec{k}=2$ and $\ell=m=0$. Therefore, (add the effect of $e^{-\frac{I_3(\mathbf{R})}{4}(d_{ijk}a_ik_jk_k)}$)

$$I_1^{(d)} = e^{d(\epsilon_1+\epsilon_2)}\sum_{\vec{\gamma}\in\Delta_l}\frac{e^{d\vec{\gamma}\cdot\vec{a}}M(\gamma)}{L(\gamma)}\ , \tag{2.38}$$

where Δ_l is the set of all long roots and

$$\begin{aligned} L(\gamma) &\equiv \prod_{\alpha\in\Delta}\mathcal{L}_{\gamma\cdot\alpha}(a\cdot\alpha,\epsilon_1,\epsilon_2) = \prod_{\gamma\cdot\alpha=\pm 2}\prod_{\gamma\cdot\alpha=1}\mathcal{L}_{\gamma\cdot\alpha}(a\cdot\alpha,\epsilon_1,\epsilon_2) \\ &= (1-e^{a_\gamma+\epsilon_1})(1-e^{a_\gamma+\epsilon_2})(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})\ , \end{aligned} \tag{2.39}$$

where $a_\gamma \equiv \vec{a}\cdot\vec{\gamma}$ with $a_\alpha \equiv \vec{a}\cdot\vec{\alpha}$ and

$$M(\gamma) \equiv \prod_{\vec{w}\in R} e^{-\frac{(\vec{\gamma}\cdot\vec{w})^2}{4}m^{\rm phy}}\mathcal{L}_{\gamma\cdot w}(a_w,\epsilon_1,\epsilon_2) = e^{-I(R)m^{\rm phy}/2}\prod_{\vec{w}\in R}\mathcal{L}_{\gamma\cdot w}(a_w,\epsilon_1,\epsilon_2)\ , \tag{2.40}$$

with $a_w \equiv \vec{a}\cdot\vec{w}$ and $I(R)$ is the Dynkin index of the representation R. Therefore, the one-instanton partition function can be written explicitly as

$$\begin{aligned} Z_1 &= \frac{e^{\epsilon_1+\epsilon_2}I_n^{(0)} - (e^{\epsilon_1}+e^{\epsilon_2})I_n^{(1)} + I_n^{(2)}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\gamma\in\Delta_l}\frac{(1-e^{a_\gamma+\epsilon_1})(1-e^{a_\gamma+\epsilon_2})M(\gamma)}{L(\gamma)}\ . \end{aligned} \tag{2.41}$$

For the case of pure SYM theory with no matters, $M(\gamma)=1$ so that

$$\begin{aligned} Z_1^{\rm SYM} &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\gamma\in\Delta_l}\frac{1}{(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\gamma\in\Delta_l}\frac{e^{(h^\vee-1)a_\gamma/2}}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})}\ , \end{aligned} \tag{2.42}$$

which is the one derived in [8, 20].

For the case with fundamental matters, we find that for the long root γ, there are only weights with $\gamma\cdot w = 0,\pm1$. Therefore, the $M(\gamma)$ can be simplified to give

$$M(\gamma) = \prod_{\gamma\cdot w=0,\pm1}\mathcal{L}_{\gamma\cdot w}(a_w,\epsilon_1,\epsilon_2) = \prod_{\gamma\cdot w=1}(1-e^{a_w+m^{\rm phy}})\ . \tag{2.43}$$

The one-instanton partition function is now given as

$$\begin{aligned} Z_1 &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\gamma\in\Delta_l}\frac{\prod_{\gamma\cdot w=1}(1-e^{a_w+m^{\rm phy}})}{(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\gamma\in\Delta_l}\frac{e^{(h^\vee-1)a_\gamma/2}\prod_{\gamma\cdot w=1}(1-e^{a_w+m^{\rm phy}})}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})}\ , \end{aligned} \tag{2.44}$$

which we conjecture to be true for all the hypermultiplets with the representation with $|\gamma\cdot w|\leq 1$ for all $w\in R$.

2.3 Instanton partition function for 4d gauge theory

Let us discuss the case for the 4d $\mathcal{N}=2$ theory. In principle, one can simply take the 4d limit of the 5d partition function by shrinking $\beta\to0$. Instead, let us write down the recursion formula directly to deduce the partition function. The perturbative partition functions for the vector and hypermultiplet are

$$Z^{\text{pert}}_{\text{vec}}(\vec{a},q)=\exp\left(-\sum_{\vec{\alpha}\in\Delta}\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{\alpha};q)\right)\ , \tag{2.45}$$

$$Z^{\text{pert}}_{\text{hyp}}(\vec{a},m,q)=\exp\left(\sum_{\vec{w}\in R}\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{w}-m;q)\right)\ . \tag{2.46}$$

Here the gamma function is defined as

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)=\left.\frac{d}{ds}\right|_{s=0}\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty\frac{dt}{t}t^s\frac{e^{-ts}}{(e^{\epsilon_1 t}-1)(e^{\epsilon_2 t}-1)}\ , \tag{2.47}$$

which is formally equivalent to

$$\log\left[\prod_{n,m\geq 0}\left(\frac{x-m\epsilon_1-n\epsilon_2}{\Lambda}\right)\right]\ . \tag{2.48}$$

If we write the equation (2.5) in terms of $Z=Z^{\text{pert}}Z^{\text{inst}}$, we get

$$Z^{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}\frac{Z^{(N),\text{pert}}(\vec{k})Z^{(S),\text{pert}}(\vec{k})}{Z^{\text{pert}}(\vec{a},\epsilon_1,\epsilon_2)}Z^{(N),\text{inst}}(\vec{k})Z^{(S),\text{inst}}(\vec{k}) \tag{2.49}$$

and here we omit the dependence on the Coulomb vev and the Omega deformation parameters. Now the factor in the middle can be explicitly worked out. Let us denote the ratio of the perturbative factor as $f(\vec{k})\equiv Z^{(N),\text{pert}}Z^{(S),\text{pert}}/Z^{\text{pert}}$. Then the ratio of the perturbative factor for the vector multiplet is given as

$$\begin{aligned}f(\vec{k})_{\text{vec}}&=\prod_{\vec{\alpha}\in\Delta}\exp\Big(\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{\alpha})-\gamma_{\epsilon_1,\epsilon_2-\epsilon_1}(\vec{a}\cdot\vec{\alpha}+\vec{k}\cdot\vec{\alpha}\epsilon_1)-\gamma_{\epsilon_1-\epsilon_2,\epsilon_2}(\vec{a}\cdot\vec{\alpha}+\vec{k}\cdot\vec{\alpha}\epsilon_2)\Big)\\&=\prod_{\vec{\alpha}\in\Delta}\frac{\Lambda^{(\vec{k}\cdot\vec{\alpha})^2/2}}{s(-\vec{k}\cdot\vec{\alpha},\vec{\alpha}\cdot\vec{a},\epsilon_1,\epsilon_2)}=\frac{(\Lambda^{2h^\vee})^{\vec{k}\cdot\vec{k}/2}}{\prod_{\vec{\alpha}\in\Delta}\ell^{\vec{k}}_{\vec{\alpha}}(\vec{a},\epsilon_1,\epsilon_2)}\ ,\end{aligned} \tag{2.50}$$

where $h^\vee$ refers to the dual Coxeter number of the gauge group. Notice that the beta function coefficient for the pure YM theory is given by $b_0=2h^\vee$ and the instanton parameter is given by $q\equiv\Lambda^{2h^\vee}$. The other symbols are given as

$$\ell^{\vec{k}}_{\vec{\alpha}}(\vec{a},\epsilon_1,\epsilon_2)=s(-\vec{k}\cdot\vec{\alpha},\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2) \tag{2.51}$$

$$s(k,x,\epsilon_1,\epsilon_2)=\begin{cases}\prod_{i,j\geq 0,i+j\leq k-1}(x-i\epsilon_1-je_2) & (k>0)\\ \prod_{i,j\geq 0,i+j\leq -k-2}(x+(i+1)\epsilon_1+(j+1)\epsilon_2) & (k<-1)\\ 1 & (k=0,-1)\end{cases} \tag{2.52}$$

The final identity of (2.50) involves a bit of work. This follows from the identity ([6], App. E.)

$$\begin{aligned}&\gamma_{\epsilon_1,\epsilon_2-\epsilon_1}(x+\epsilon_1 k;\Lambda)+\gamma_{\epsilon_1-\epsilon_2,\epsilon_2}(x+\epsilon_2 k;\Lambda)\\&\qquad\qquad=\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)+\log s(-k,x,\epsilon_1,\epsilon_2)-\frac{k(k-1)}{2}\log\Lambda\ .\end{aligned}\tag{2.53}$$

For the hypermultiplets we get,

$$\begin{aligned}f(\vec{k})_{\rm hyp}&=\prod_{\vec{w}\in R}\exp\Big(-\gamma_{\epsilon_1,\epsilon_2}(a_{w,m})+\gamma_{\epsilon_1,\epsilon_2-\epsilon_1}(a_{w,m}+k_w\epsilon_1)+\gamma_{\epsilon_1-\epsilon_2,\epsilon_2}(a_{w,m}+k_w\epsilon_2)\Big)\\&=\prod_{\vec{w}\in R}\Lambda^{-\frac{1}{2}k_w^2}s(-k_w,a_{w,m},\epsilon_1,\epsilon_2)=(\Lambda^{-2I(R)})^{\frac{1}{2}\vec{k}\cdot\vec{k}}\prod_{\vec{w}\in R}s(-k_w,a_{w,m},\epsilon_1,\epsilon_2)\ ,\end{aligned}\tag{2.54}$$

where we introduced the short-hand notation $k_w=\vec{k}\cdot\vec{w}$, $a_{w,m}=\vec{a}\cdot\vec{w}-m$ and $I(R)$ corresponds to the Dynkin index for the representation R. The Dynkin index appears in the beta function coefficients as $b_0=2h^\vee-I(R)$ for the hypermultiplets in the representation R. This gives the instanton parameter to be $q\equiv\Lambda^{b_0}=\Lambda^{2h^\vee-I(R)}$.

Now, for the SQCD, we obtain the following equation:

$$Z^{\rm inst}(\vec{a},m,\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}f(\vec{k})Z^{(N),{\rm inst}}(\vec{k})Z^{(S),{\rm inst}}(\vec{k})\ ,\tag{2.55}$$

with

$$f(\vec{k})=\frac{q^{\frac{1}{2}\vec{k}\cdot\vec{k}}\prod_i\prod_{\vec{w}\in R_i}s(-k_w,a_w-m_i,\epsilon_1,\epsilon_2)}{\prod_{\vec{\alpha}\in\Lambda}s(-k_\alpha,a_\alpha,\epsilon_1,\epsilon_2)}\ ,\tag{2.56}$$

where i runs over the charged hypermultiplets. Here $q=\Lambda^{2N_c-N_f}$ for the $SU(N_c)$ SQCD with N_f fundamental hypermultiplets. We have checked this expression explicitly for the $SU(2)$ gauge theory with $N_f=0,1$ hypermultiplets up to the first few order in instanton numbers.

Notice that in the Gottsche-Nakajima-Yoshioka [21, 22], the mass parameters and the instanton parameters (for the 5d) are also shifted when the contribution from North and South poles are computed. This is simply a reflection of the fact that they twist the instanton bundles by the half-Canonical bundle of the $\mathbb{C}^2$, which shift the mass parameters by $m\to m-\frac{\epsilon_1+\epsilon_2}{2}$. If we do not twist by this amount, we get a cleaner expression as above.

3 Examples

As a remarkable application of the blow-up equation, we construct the instanton partition function of various 5d gauge theories on $\mathbb{C}^2\times S^1$. The standard method to compute the multi-instanton correction to the partition function is to employ the ADHM construction of

Gauge group	Hypermultiplets
$SU(N)_\kappa$	x
$SO(2N+1)$	$n_{\mathbf{v}} \leq 2(N-1)$
$SO(2N)$	$n_{\mathbf{v}} \leq 2(N-2)$
$USp(2N)$	$n_{\mathbf{f}} \leq 2N$
$USp(2N)$	$n_{\Lambda^2} = 1$ and $n_{\mathbf{f}} \leq 2$

Table 1: asd

the instanton moduli space [3, 4, 23], and/or to apply the topological vertex formalism based on the geometric engineering of 5d $\mathcal{N}=1$ gauge theory [24, 25]. An alternative approach that comes from the consistency requirement for the blow-up $\hat{\mathbb{C}}^2$ will turn out to be very efficient for bootstrapping the $\mathbb{C}^2 \times S^1$ partition function of exceptional gauge theories [8, 15, 26].

There are multiple and distinct blow-up equations (2.18) at different values of d, ranged over $0 \leq d \leq d_{\max}$ where $d_{\max} \equiv h^\vee - \frac{1}{2}\sum_i I(\mathbf{R}_i)$. We need at least three distinct equations to solve for the instanton partition function, thus the bootstrap method is applicable only for gauge theories which satisfy $d_{\max} \geq 2$. Here we constrain our discussion to the theories having simple gauge groups, which are classified to have a 5d UV fixed point [27].

There are two

We will

The complete list of $\mathcal{N}=1$ gauge theories, satisfying $d_{\max} \geq 2$, is displayed in Table 1 and 2. All infinite families of theories, whose rank of the gauge group can be arbitrarily large, are summarized in Table 1. There are also finite families of theories, summarized in Table 2, which involve an exceptional gauge group or a spinor hypermultiplet.

(comparison with IMS bound, etc) (SCFT bound) Some theories in Table 2 do not belong to the

3.1 Known examples

3.2 Ki-Hong's note

Unity Blowup equations

Instanton partition functions from blowup equations From blowup equations one can compute the partition functions as follows. Rewriting the blowup equation as

$$1 = \sum_{\vec{k}\in\vec{\alpha}^\vee} f_{\vec{k}}\, l_{\vec{k}} \frac{Z^{(1)}_{\text{inst}} Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \tag{3.1}$$

where $f_{\vec{k}} = Z^{(1)}_{\text{class}} Z^{(2)}_{\text{class}}/Z_{\text{class}}$ and $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}} Z^{(2)}_{\text{1-loop}}/Z_{\text{1-loop}}$ with abbreviated notation

$$\begin{aligned} Z^{(1)} &= Z(\epsilon_1, \epsilon_2-\epsilon_1, \vec{a}+\vec{k}\,\epsilon_1, m_i + r_i\,\epsilon_1, m_0 + r_0\,\epsilon_1) \\ Z^{(2)} &= Z(\epsilon_1-\epsilon_2, \epsilon_2, \vec{a}+\vec{k}\,\epsilon_2, m_i + r_i\,\epsilon_2, m_0 + r_0\,\epsilon_2) \end{aligned} \tag{3.2}$$

Gauge group	Hypermultiplets
$SO(14)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 2$
$SO(13)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 2$
$SO(12)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=2$
$SO(12)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(11)$	$n_{\mathbf{s}}=2$
$SO(11)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(10)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=3$
$SO(10)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(10)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(9)$	$n_{\mathbf{s}}=3$
$SO(9)$	$n_{\mathbf{s}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(9)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=4$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=3$ and $n_{\mathbf{v}}=1$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 3$
$SO(7)$	$n_{\mathbf{s}}=4$
$SO(7)$	$n_{\mathbf{s}}=3$ and $n_{\mathbf{v}}=1$
$SO(7)$	$n_{\mathbf{s}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(7)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 3$
G_2	$n_{\mathbf{7}}\leq 2$
F_4	$n_{\mathbf{26}}\leq 2$
E_6	$n_{\mathbf{27}}+n_{\overline{\mathbf{27}}}\leq 3$
E_7	$n_{\mathbf{56}}\leq 2$
E_8	$\varnothing$

Table 2: asd

Here note that $l_{\vec{k}}$ is independent of $Q=e^{-m_0}$, and $f_{\vec{k}}$ is some overall factor in the order of $Q^{\vec{k}\cdot\vec{k}/2}$. Expanding the equation by instanton fugacity Q, then at each Q^n level the equation is written by

$$\delta_{n,0}=p_1^{r_0}Z_n^{(1)}+p_2^{r_0}Z_n^{(2)}-Z_n+\sum_{\vec{k}\neq 0}f_{\vec{k},r_0}l_{\vec{k}}\left(\frac{Z_{\text{inst}}^{(1)}Z_{\text{inst}}^{(2)}}{Z_{\text{inst}}}\right)\Bigg|_{O(Q^{n-\vec{k}\cdot\vec{k}/2})}. \tag{3.3}$$

Since each Z_k and $Z_k^{(1,2)}$ are independent of r_0, one can solve (3.3) with three blowup equations with same r_i's but different r_0's.

The blowup equations for instanton partition functions of pure YM theory with generic

G	matter	r_0	$d_{\max}$
$SU(N)_\kappa$	$N_f \times \boldsymbol{N}$	$d - N/2 - \kappa/2$	$0 \leq d \leq N - \lvert\kappa\rvert - 2N_f - 1$(?)
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \leq d \leq 6$
$SO(7)$	pure	$d - 5/2$	$0 \leq d \leq 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \leq d \leq 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \leq d \leq 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \leq d \leq 3$
G_2	pure	$d - 4/2$	$0 \leq d \leq 4$
G_2	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \leq d \leq 3$
F_4	pure	$d - 9/2$	$0 \leq d \leq 9$
F_4	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \leq d \leq 6$
F_4	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \leq d \leq 3$

Table 3: list of theories

gauge group were already studied in [8]. They are actually (**??**) with

$$\vec{r}_a = 0, \qquad r_0 = d - h^\vee/2 \tag{3.4}$$

where $d = 0, \cdots, h^\vee$. We extend these blowup equations to the theories with matters based on pure YM blowup equations. If one restrict the cases to $\vec{r}_a = 0$, as we explained in the previous section, the r_i's are technically required to be half intergers. Thus we look for the r_0's that provides the correct instanton partition functions by solving (3.3) while fixing $\vec{r}_a = 0$ and $r_i = 1/2$. Here are the results.

They were tested by comparing the resulting instanton partition functions with the known results from [28]($SO(7)$ and G_2) and [29](F_4 with $N_{\mathbf{26}} = 2$). They were compared numerically, putting random numbers on the fugacities. Note that matters shift the r_0, each by one quarter of their Dynkin indices. It seems to differ from blowup formula for $SU(N)_\kappa + N_f$ instantons, where r_0 was affected only by its CS-level κ. However, one can rewrite the r_0 as

$$\begin{aligned} r_0 &= d - N/2 - \left(\kappa + \frac{1}{2}N_f\right)/2 + N_f/4 \\ &= d - N/2 - \kappa_{\text{eff}}/2 + N_f \times I_{\text{fund}}. \end{aligned} \tag{3.5}$$

Since fundamental matters shifts the effective CS-level, they cancel their index contributions and consequently the r_0 apparently looks independent of matters.

By above observations, we write the unity blowup equation for generic gauge groups and matter representations.

$$\begin{aligned} Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = \sum_{\vec{k} \in \vec{\alpha}^\vee} & Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\epsilon_1, m_i + \epsilon_1/2, m_0 + r_0\epsilon_1) \\ & \times\ Z(\epsilon_1 - \epsilon_2, \vec{a} + \vec{k}\epsilon_2, m_i + \epsilon_2/2, m_0 + r_0\epsilon_2) \end{aligned} \tag{3.6}$$

with

$$r_0 = d - h^\vee/2 - \kappa_{\text{eff}}/2 + N_{\boldsymbol{R}} \times I_{\boldsymbol{R}}. \tag{3.7}$$

Here $I_{\boldsymbol{R}}$ is the Dynkin index of $\boldsymbol{R}$ representation.

$SU(6)_3 + 1 \times \mathbf{20}$ As a non-trivial test, we consider the instanton partition function of the $SU(6)_3 + \mathbf{20}$ whose 5-brane realization was found recently [30]. Its web-diagram is given as figure.

(Written before computing the $SU(6)_3 + 20$ instanton partition function.)
Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the $SU(3) \times SU(3) \times U(1) \subset SU(6)$ where the $SU(6)$ multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}} &: \mathbf{35} \longrightarrow (\mathbf{8}, 1)_0 \oplus (1, \mathbf{8})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_2 \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-2} \oplus (1,1)_0, \\ \Phi_{ijk} &: \mathbf{20} \longrightarrow (\mathbf{3}, \bar{\mathbf{3}})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{3})_1 \oplus (1,1)_3 \oplus (1,1)_{-3}. \end{aligned} \tag{3.8}$$

Here to fit with the web-diagram, we set Φ_{156} and Φ_{234} are $(1,1)_3$ and $(1,1)_{-3}$. Once Φ_{156} and Φ_{234} get non-zero VEVs,

When $a_5 = -a_1 - a_6$, the web-diagram factorizes to two $SU(3)_3$ whose Coulomb VEVs are (a_1, a_5, a_6) and (a_2, a_3, a_4). In the gauge theory, it can be seen partly from prepotential. The prepotential of $S(6)_3 + 1 \times \mathbf{20}$ is

$$\mathcal{F} = \frac{1}{2} m_0 \sum_{i=1}^{6} a_i^2 + \frac{1}{2} \sum_{i=1}^{6} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3 - \frac{1}{6} \sum_{1<i<j} (a_1 + a_j + a_k)^3 \tag{3.9}$$

at the Weyl chamber $a_1 > \cdots > a_6$. As one sets the Coulomb VEV $a_6 = -a_1 - a_5$ and $a_4 = -a_2 - a_3$, one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \tag{3.10}$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2} m_0 \sum_{i=1}^{3} a_i^2 + \frac{1}{2} \sum_{i=1}^{3} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3. \tag{3.11}$$

It is Higgsed by

3.3 $SU(6)_3$ with a rank-3 antisymmetric hyper

(describe blow-up computation)

It was recently found in [30] that the 5d $SU(6)_3$ gauge theory with a rank-3 antisymmetric hypermultiplet can be engineered from the 5-brane web configuration, depicted in Figure 1. Given a web diagram, we utilize the topological vertex method [25, 31] to compute all genus topological string amplitudes, which is the logarithm of the 5d Nekrasov partition function on Ω-deformed $\mathbf{R}^4 \times S^1$ [24]. We will check its agreement with the blowup partition function (eqn), providing a supporting evidence to the suggested blow-up equation (eqn).

By applying the topological vertex method to Figure 1, we find that the instanton partition function can be written as the following sum over all possible 6 Young diagrams:

$$\begin{aligned} Z_{\text{Nek}} = \sum_{(Y_1,\cdots,Y_6)} & q^{\sum_{i=1}^6 |Y_i|}(-A_1^6)^{|Y_1|}(-A_2^6)^{|Y_2|}(-A_2^2A_3^4)^{|Y_3|}(-A_2^2A_3^2A_4^2)^{|Y_4|+|Y_5|} \\ & \times f_{Y_1}(g)^5 f_{Y_2}(g)^5 f_{Y_3}(g)^3 f_{Y_4}(g) f_{Y_5}(g)^{-1} f_{Y_6}(g)^2 Z_{\text{half}}(Y_1,Y_2,Y_3,Y_4,Y_5,Y_6)^2. \end{aligned} \tag{3.12}$$

where the Kähler parameters can be identified as

$$A_i = e^{-a_i}, \qquad q = e^{-\frac{8\pi^2}{g^2}}, \qquad g = e^{-\epsilon_-}. \tag{3.13}$$

We briefly explain our notation: For a given Young diagram μ, $|\mu|$ denotes the total number of boxes. μ_i is the number of boxes in the i-th row of μ. μ^t is the transpose of μ. We also use

$$f_\mu(g) = (-1)^{|\mu|} g^{\frac{1}{2}(\|\mu^t\|^2 - \|\mu\|^2)}, \qquad \tilde{Z}_\lambda(g) = \prod_{(i,j)\in\lambda} (1 - g^{\lambda_i + \lambda_j^t - i - j + 1})^{-1} \tag{3.14}$$

with $\|\mu\|^2 = \sum_i \mu_i^2$. The factor $Z_{\text{half}}(Y_1,Y_2,Y_3,Y_4,Y_5,Y_6)$ involves a single summation over all possible Young diagrams, i.e.,

$$\begin{aligned} Z_{\text{left}}(\vec{Y}) = \sum_{Y_0} & (-A_1{}^{-1}A_6{}^{-2})^{|Y_0|} g^{\frac{\|Y_0^t\|^2 + \|Y_0\|^2}{2}} \tilde{Z}_{Y_0}(g)^2 f_{Y_0}(g)^2 \ \textstyle\prod_{i=1}^6 g^{\frac{\|Y_i^t\|^2}{2}} \tilde{Z}_{Y_i}(g) \\ & \times R^{-1}_{Y_1Y_6^t}(A_1A_6{}^{-1})\, R^{-1}_{Y_0Y_6^t}(A_1{}^{-1}A_6{}^{-2})\, R^{-1}_{Y_1Y_0^t}(A_1^2A_6) \\ & \times \textstyle\prod_{2\le i<j\le 5} R^{-1}_{Y_iY_j^t}(A_iA_j{}^{-1}) \ \prod_{i=2}^5 R_{Y_0^tY_i}(A_1A_iA_6), \end{aligned} \tag{3.15}$$

in which we introduce

$$R_{\lambda\mu}(Q) = R_{\mu\lambda}(Q) = \text{PE}\left[-\frac{g}{(1-g)^2}Q\right] \times N_{\lambda^t\mu}(Q) \tag{3.16}$$

with PE representing the Plethystic exponential and

$$N_{\lambda\mu}(Q) = \prod_{(i,j)\in\lambda} \left(1 - Qg^{\lambda_i + \mu_j^t - i - j + 1}\right) \prod_{(i,j)\in\mu} \left(1 - Qg^{-\lambda_j^t - \mu_i + i + j - 1}\right). \tag{3.17}$$

We also recall that the Nekrasov partition function is divided into the perturbative partition function Z_{pert} and the weighted sum of k-instanton partition function Z_k.

$$Z_{\text{Nek}} = Z_{\text{pert}}\Big(1 + \sum_{k=1}^{\infty} q^k Z_k\Big). \tag{3.18}$$

The perturbative part of the partition function Z_{pert} comes from the summand of (3.12) at empty Young diagrams, i.e., $(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) = (\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing)$. It is given by

$$
\begin{aligned}
Z_{\text{pert}} =& Z_{\text{half}}(\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing)^2 \\
=& \mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\frac{A_1}{A_6} + \frac{1}{A_1A_6^2} + A_1^2A_6 + \sum_{2\le i<j\le 5}\frac{A_i}{A_j} - \sum_{i=2}^{5} A_1A_iA_6\Big)\right] \\
&\times \Bigg(\sum_{Y_0}(-A_1{}^{-1}A_6{}^{-2})^{|Y_0|}\, g^{\frac{\|Y_0^t\|^2+\|Y_0\|^2}{2}}\, \tilde{Z}_{Y_0}(g)^2 f_{Y_0}^2(g) \\
&\qquad N^{-1}_{Y_0^t\varnothing}(A_1{}^{-1}A_6{}^{-2}) N^{-1}_{Y_0\varnothing}(A_1^2A_6)\textstyle\prod_{i=2}^5 N_{Y_0\varnothing}(A_1A_iA_6)\Bigg)^2 ,
\end{aligned}
\tag{3.19}
$$

where the last two lines can be combined into the following closed-form expression:

$$
\mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\sum_{i=2}^{5}\frac{A_1}{A_i} + \sum_{i=2}^{5}\frac{A_i}{A_6} - \frac{1}{A_1A_6^2} - A_1^2A_6 - \sum_{2\le i<j\le 5} A_1A_iA_j + \mathcal{O}(A_1^6)\Big)\right]. \tag{3.20}
$$

So (3.19) is manifestly consistent with the equivariant index [32] for 5d $SU(6)$ gauge theory with a hypermultiplet in the rank-3 antisymmetric representation **20**, i.e.,

$$
Z_{\text{pert}} = \mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\sum_{1\le i<j\le 6}\frac{A_i}{A_j} - \sum_{2\le i<j\le 6} A_1A_iA_j + \mathcal{O}(A_1^6)\Big)\right]. \tag{3.21}
$$

The 1-instanton partition function Z_1 can be obtained from the summands of (3.12) at Young diagrams satisfying $\sum_{i=1}^6 |Y_i| = 1$. There are 6 different profiles of Young diagrams. The configuration $|Y_i| = 1$ and $Y_{j\neq i} = \varnothing$ contributes to Z_1 by

$$
+\frac{g}{(1-g)^2}\frac{A_i^6}{\prod_{j\neq i}(A_i - A_j)^2}\Big(-A_i\sum_{j\neq i}A_j + \sum_{j\neq i}\frac{1}{A_j} - \frac{1}{A_i} + A_i^2\Big)^2. \tag{3.22}
$$

Summing over all six contributions, we find

$$
Z_1 = \sum_{i=1}^{6}\frac{g}{(1-g)^2}\frac{A_i^6}{\prod_{j\neq i}(A_i - A_j)^2}\Big(-A_i\sum_{j\neq i}A_j + \sum_{j\neq i}\frac{1}{A_j} - \frac{1}{A_i} + A_i^2\Big)^2. \tag{3.23}
$$

which is in agreement with the blowup partition function (eqn). (overall sign: looking at the GV invariant for single W-boson + single instanton (which should be -2), the topological vertex computation seems to be correct. blowup should have an overall sign issue)

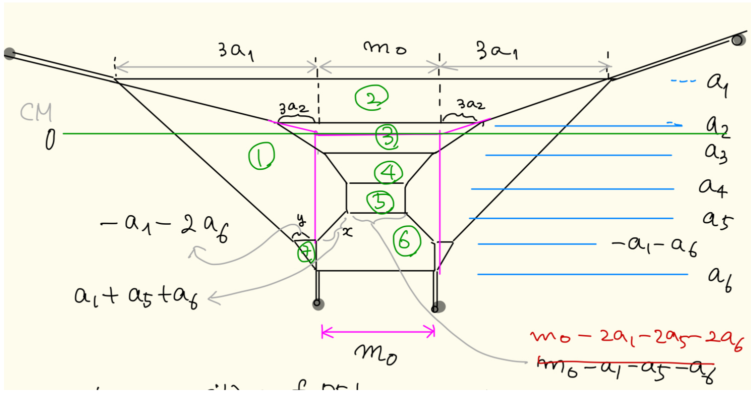

Figure 1: A 5-brane web for $SU(6)_3 + 1$**TAS** with massless **TAS**.

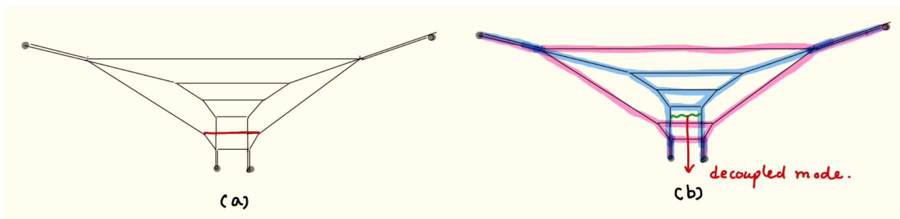

Figure 2: (a) A Higgsing from from $SU(6)_3 + 1$**TAS** to two $SU(3)_3$ theories. (a) Two $SU(3)_3$ theories are painted in blue and in pink respectively. A new decoupled mode emerges.

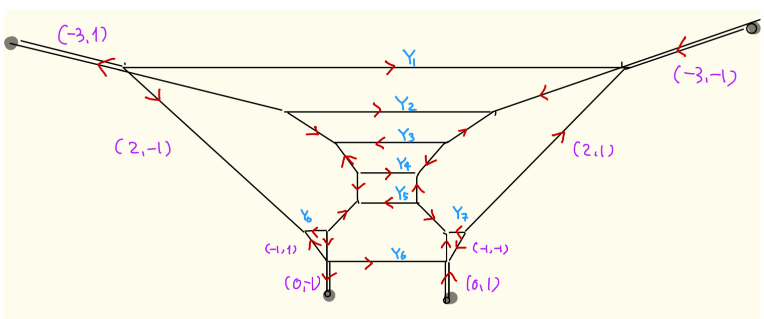

Figure 3: A labeling of Young diagrams assigned to the horizontal lines in Figure 1.

4 Conclusion

Acknowledgments

This work is supported in part by the UESTC Research Grant A03017023801317 (SSK), the National Research Foundation of Korea (NRF) Grants 2017R1D1A1B06034369 (KL, JS), and 2018R1A2B6004914 (KHL)

References

[1] N. Seiberg and E. Witten, *Electric–magnetic Duality, Monopole Condensation, and Confinement in $\mathcal{N}=2$ Supersymmetric Yang-Mills Theory*, Nucl. Phys. **B426** (1994) 19–52, [hep-th/9407087].

[2] N. Seiberg and E. Witten, *Monopoles, Duality and Chiral Symmetry Breaking in $\mathcal{N}=2$ Supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484–550, [hep-th/9408099].

[3] N. A. Nekrasov, *Seiberg-Witten Prepotential from Instanton Counting*, Adv. Theor. Math. Phys. **7** (2003) 831–864, [hep-th/0206161].

[4] N. Nekrasov and A. Okounkov, *Seiberg-Witten Theory and Random Partitions*, Prog. Math. **244** (2006) 525–596, [hep-th/0306238].

[5] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. 1.*, Invent. Math. **162** (2005) 313–355, [math/0306198].

[6] H. Nakajima and K. Yoshioka, *Lectures on instanton counting*, in *CRM Workshop on Algebraic Structures and Moduli Spaces Montreal, Canada, July 14-20, 2003*, 2003. math/0311058.

[7] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. II. K-theoretic partition function*, math/0505553.

[8] C. A. Keller and J. Song, *Counting Exceptional Instantons*, JHEP **07** (2012) 085, [1205.4722].

[9] N. A. Nekrasov, *Localizing Gauge Theories*, in *Mathematical Physics. Proceedings, 14Th International Congress, Icmp 2003, Lisbon, Portugal, July 28-August 2, 2003*, pp. 645–654, 2003.

[10] L. Gottsche, H. Nakajima and K. Yoshioka, *K-theoretic Donaldson invariants via instanton counting*, Pure Appl. Math. Quart. **5** (2009) 1029–1111, [math/0611945].

[11] L. Gottsche, H. Nakajima and K. Yoshioka, *Instanton counting and Donaldson invariants*, *J. Diff. Geom.* **80** (2008) 343–390, [math/0606180].

[12] E. Gasparim and C.-C. M. Liu, *The Nekrasov Conjecture for Toric Surfaces*, Commun. Math. Phys. **293** (2010) 661–700, [0808.0884].

[13] G. Bonelli, K. Maruyoshi, A. Tanzini and F. Yagi, $\mathcal{N}=2$ *Gauge Theories on Toric Singularities, Blow-Up Formulae and W-Algebrae*, JHEP **01** (2013) 014, [1208.0790].

[14] M. Bershtein, G. Bonelli, M. Ronzani and A. Tanzini, *Exact results for $\mathcal{N}=2$ supersymmetric gauge theories on compact toric manifolds and equivariant Donaldson invariants*, JHEP **07** (2016) 023, [1509.00267].

[15] J. Gu, B. Haghighat, K. Sun and X. Wang, *Blowup Equations for 6d SCFTs. I*, 1811.02577.

[16] J. Gu, A. Klemm, K. Sun and X. Wang, *Elliptic Blowup Equations for 6D Scfts. Ii: Exceptional Cases*, 1905.00864.

[17] E. Witten, *Topological Quantum Field Theory*, Commun. Math. Phys. **117** (1988) 353.

[18] L. Baulieu, A. Losev and N. Nekrasov, *Chern-Simons and Twisted Supersymmetry in Various Dimensions*, Nucl. Phys. **B522** (1998) 82–104, [hep-th/9707174].

[19] T. Okuda and V. Pestun, *On the instantons and the hypermultiplet mass of N=2* super Yang-Mills on S^4*, JHEP **03** (2012) 017, [1004.1222].

[20] C. A. Keller, N. Mekareeya, J. Song and Y. Tachikawa, *The Abcdefg of Instantons and W-Algebras*, *JHEP* **03** (2012) 045, [1111.5624].

[21] H. Nakajima and K. Yoshioka, *Perverse coherent sheaves on blowup, III: Blow-up formula from wall-crossing*, *Kyoto J. Math.* **51** (2011) 263–335, [0911.1773].

[22] L. Gottsche, H. Nakajima and K. Yoshioka, *Donaldson = Seiberg-Witten from Mochizuki's formula and instanton counting*, *Publ. Res. Inst. Math. Sci. Kyoto* **47** (2011) 307–359, [1001.5024].

[23] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, *Construction of Instantons*, *Phys. Lett.* **A65** (1978) 185–187.

[24] R. Gopakumar and C. Vafa, *M Theory and Topological Strings. 2.*, hep-th/9812127.

[25] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *The Topological Vertex*, *Commun. Math. Phys.* **254** (2005) 425–478, [hep-th/0305132].

[26] M.-x. Huang, K. Sun and X. Wang, *Blowup Equations for Refined Topological Strings*, *JHEP* **10** (2018) 196, [1711.09884].

[27] P. Jefferson, H.-C. Kim, C. Vafa and G. Zafrir, *Towards Classification of 5D Scfts: Single Gauge Node*, 1705.05836.

[28] H.-C. Kim, J. Kim, S. Kim, K.-H. Lee and J. Park, *6D Strings and Exceptional Instantons*, 1801.03579.

[29] M. Del Zotto and G. Lockhart, *Universal Features of BPS Strings in Six-Dimensional SCFTs*, *JHEP* **08** (2018) 173, [1804.09694].

[30] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, *Rank-3 Antisymmetric Matter on 5-Brane Webs*, 1902.04754.

[31] A. Iqbal, C. Kozçaz and C. Vafa, *The Refined Topological Vertex*, *JHEP* **10** (2009) 069, [hep-th/0701156].

[32] S. Shadchin, *On Certain Aspects of String Theory/Gauge Theory Correspondence*. PhD thesis, Orsay, LPT, 2005. hep-th/0502180.