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Exceptional Instantons from Blow-up

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ABSTRACT:

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The essential idea for us is that the gauge theory partition function for a 4d $\mathcal{N}=2$ theory (or 5d $\mathcal{N}=1$) on a blow up of a point $\hat{\mathbb{C}}^2$ (or $S^1\times\hat{\mathbb{C}}^2$) is identical to that on the flat space $\mathbb{C}^2$ (or $S^1\times\mathbb{C}^2$) [1–3]. This can be argued as follows: The blow up $\hat{\mathbb{C}}^2$ is identical to $\mathbb{C}^2$ except for the origin, which is replaced by $\mathbb{P}^1$. There can be massive states coming from $\mathbb{P}^1$. As we shrink the size of $\mathbb{P}^1$, this may become massless which might in principle contribute to a new state. This happens when we have a singularity as we blow-down. For example, if we consider the total space of $\mathcal{O}(-2)\to\mathbb{P}^1$ and shrink the base, we land on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ which is singular at the origin. In our case, we do not obtain any singularity as we blow down the sphere. Therefore, as long as we keep the size of blow up small (and do not turn on extra flux through the sphere), the physical degree of freedom should be identical to that of the flat $\mathbb{C}^2$. (In some sense, the partition function is a birational invariant. But there can be wall-crossing.)

The partition function on a blow up can be written as a sum over a product of the partition function on $\mathbb{C}^2$ as follows (if we turn off any external flux that can be supported on the blow up):

$$\hat{Z}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}Z(\vec{a}+\vec{k}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1)Z(\vec{a}+\vec{k}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2) \tag{2.1}$$

Here Λ is the weight lattice of the gauge group and the vector $\vec{k}$ labels different flux configurations on the divisor of the blow-up classified by the first Chern numbers.

2.1 4d gauge theory

Building blocks for the 4d

$$Z_{\text{vec}}^{\text{pert}}(\vec{a},q)=\exp\left(-\sum_{\vec{\alpha}\in\Delta}\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{\alpha};q)\right) \tag{2.2}$$

$$Z_{\text{hyp}}^{\text{pert}}(\vec{a},m,q)=\exp\left(\sum_{\vec{w}\in R}\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{w}-m;q)\right) \tag{2.3}$$

Here the gamma function is defined as

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)=\left.\frac{d}{ds}\right|_{s=0}\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty\frac{dt}{t}t^s\frac{e^{-ts}}{(e^{\epsilon_1 t}-1)(e^{\epsilon_2 t}-1)}\ , \tag{2.4}$$

which is formally equivalent to

$$\log\left[\prod_{n,m\geq 0}\left(\frac{x-m\epsilon_1-n\epsilon_2}{\Lambda}\right)\right]\ . \tag{2.5}$$

If we write the equation (2.1) in terms of $Z=Z^{\text{pert}}Z^{\text{inst}}$, we get

$$Z^{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}\frac{Z^{(N),\text{pert}}(\vec{k})Z^{(S),\text{pert}}(\vec{k})}{Z^{\text{pert}}(\vec{a},\epsilon_1,\epsilon_2)}Z^{(N),\text{inst}}(\vec{k})Z^{(S),\text{inst}}(\vec{k}) \tag{2.6}$$

where

$$Z^{(N)}(\vec{k})=Z(\vec{a}+\epsilon_1\vec{k},\epsilon_1,\epsilon_2-\epsilon_1) \tag{2.7}$$

$$Z^{(S)}(\vec{k})=Z(\vec{a}+\epsilon_2\vec{k},\epsilon_1-\epsilon_2,\epsilon_2) \tag{2.8}$$

and here we omit the dependence on the Coulomb vev and the Omega deformation parameters. Now the factor in the middle can be explicitly worked out. Let us denote the ratio of the perturbative factor as $f(\vec{k})\equiv Z^{(N),\text{pert}}Z^{(S),\text{pert}}/Z^{\text{pert}}$. Then the ratio of the perturbative factor for the vector multiplet is given as

$$f(\vec{k})_{\text{vec}}=\prod_{\alpha\in\Delta}\exp\Big(\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{\alpha})-\gamma_{\epsilon_1,\epsilon_2-\epsilon_1}(\vec{a}\cdot\vec{\alpha}+\vec{k}\cdot\vec{\alpha}\epsilon_1)-\gamma_{\epsilon_1-\epsilon_2,\epsilon_2}(\vec{a}\cdot\vec{\alpha}+\vec{k}\cdot\vec{\alpha}\epsilon_2)\Big) \tag{2.9}$$

$$=\prod_{\vec{\alpha}\in\Delta}\frac{\Lambda^{(\vec{k}\cdot\vec{\alpha})^2/2}}{s(-\vec{k}\cdot\vec{\alpha},\vec{\alpha}\cdot\vec{a},\epsilon_1,\epsilon_2)}=\frac{(\Lambda^{2h^\vee})^{\vec{k}\cdot\vec{k}/2}}{\prod_{\vec{\alpha}\in\Delta}\ell^{\vec{k}}_{\vec{\alpha}}(\vec{a},\epsilon_1,\epsilon_2)} \tag{2.10}$$

where $h^\vee$) refers to the dual Coxeter number of the gauge group. Notice that ithe beta function coefficient for the pure YM theory is given by $b_0=2h^\vee$ and the instanton number is given as $q \equiv \Lambda^{2h^\vee}$. The other symbols are given as

$$\ell^{\vec{k}}_{\vec{\alpha}}(\vec{a},\epsilon_1,\epsilon_2) = s(-\vec{k}\cdot\vec{\alpha},\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2) \tag{2.11}$$

$$s(k,x,\epsilon_1,\epsilon_2) = \begin{cases} \displaystyle\prod_{i,j\geq 0, i+j\leq k-1} (x - i\epsilon_1 - je_2) & (k>0) \\ \displaystyle\prod_{i,j\geq 0, i+j\leq -k-2} (x+(i+1)\epsilon_1+(j+1)\epsilon_2) & (k<-1) \\ 1 & (k=0,-1) \end{cases} \tag{2.12}$$

The final identity of (2.9) involves a bit of work. This follows from the identity ([2], App. E.)

$$\gamma_{\epsilon_1,\epsilon_2-\epsilon_1}(x+\epsilon_1 k;\Lambda) + \gamma_{\epsilon_1-\epsilon_2,\epsilon_2}(x+\epsilon_2 k;\Lambda) = \gamma_{\epsilon_1,\epsilon_2}(x;\Lambda) + \log s(-k,x,\epsilon_1,\epsilon_2) - \frac{k(k-1)}{2}\log\Lambda \ . \tag{2.13}$$

For the hypermultiplets we get,

$$\begin{aligned} f(\vec{k})_{\text{hyp}} &= \prod_{\vec{w}\in R} \exp\Big(-\gamma_{\epsilon_1,\epsilon_2}(a_w - m) + \gamma_{\epsilon_1,\epsilon_2-\epsilon_1}(a_w + k_w\epsilon_1 - m) + \gamma_{\epsilon_1-\epsilon_2,\epsilon_2}(a_w+k_w\epsilon_2 - m)\Big) \\ &= \prod_{\vec{w}\in R} \Lambda^{-\frac{1}{2}k_w^2} s(-k_w, a_w - m,\epsilon_1,\epsilon_2) = (\Lambda^{-2C_2(R)})^{\vec{k}\cdot\vec{k}/2} \prod_{\vec{w}\in R} s(-k_w,a_w-m,\epsilon_1,\epsilon_2) \ , \end{aligned} \tag{2.14}$$

where we introduced the short-hand notation $k_w = \vec{k}\cdot\vec{w}$, $a_w = \vec{a}\cdot\vec{w}$ and $C_2(R)$ corresponds to the second Casimir invariant for the representation R. This invariant appears in the beta function coefficients as $b_0 = 2h^\vee - 2\sum_R C_2(R)$ for the hypermultiplets in irrep R. This gives the instanton parameter to be $q \equiv \Lambda^{b_0} = \Lambda^{2h^\vee - 2\sum_R C_2(R)}$.

Now, for the SQCD, we obtain the following equation:

$$Z^{\text{inst}}(\vec{a},m,\epsilon_1,\epsilon_2) = \sum_{\vec{k}\in\Lambda} f(\vec{k}) Z^{(N),\text{inst}}(\vec{k}) Z^{(S),\text{inst}}(\vec{k}) \ , \tag{2.15}$$

with

$$f(\vec{k}) = \frac{q^{\frac{1}{2}\vec{k}\cdot\vec{k}} \displaystyle\prod_i \prod_{\vec{w}\in R_i} s(-k_w, a_w - m_i,\epsilon_1,\epsilon_2)}{\displaystyle\prod_{\vec{\alpha}\in\Lambda} s(-k_\alpha, a_\alpha,\epsilon_1,\epsilon_2)} \ , \tag{2.16}$$

where i runs over the charged hypermultiplets. Here $q = \Lambda^{2N_c - N_f}$ for the $SU(N_c)$ SQCD with N_f fundamental hypermultiplets. We have checked this expression explicitly for the $SU(2)$ gauge theory with $N_f = 0, 1$ hypermultiplets up to the first few order in instanton numbers.

Notice that in the Gottsche-Nakajima-Yoshioka [4, 5], the mass parameters and the instanton parameters (for the 5d) are also shifted when the contribution from North and South poles are computed. This is simply a reflection of the fact that they twist the instanton bundles by the half-Canonical bundle of the $\mathbb{C}^2$, which shift the mass parameters by $m \to m - \frac{\epsilon_1+\epsilon_2}{2}$. If we do not twist by this amount, we get a cleaner expression as above.

2.1.1 Correlation functions

On the blow-up, we have a non-trivial 2-cycle. This allows us to insert the equivariant version of the Donaldson operator, which can be written in terms of the twisted fields as

$$\left\langle \exp\left[t\int d^4x\left(\omega\wedge\phi F+\frac{1}{2}\psi\wedge\psi+HF\wedge F\right)\right]\right\rangle \ . \tag{2.17}$$

Here H is the moment map for the $U(1)^2_{\epsilon_1,\epsilon_2}$ action so that $dH=\iota_V\omega$. Upon localization, it is easy to see that it can be written as

$$\hat{Z}^{\text{inst}}(\vec{a},m,\epsilon_1,\epsilon_2,t)=\sum_{\vec{k}\in\Lambda}e^{t\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)\right)}f(\vec{k})Z^{\text{inst}}(\vec{k};qe^{t\epsilon_1})Z^{(S),\text{inst}}(\vec{k};qe^{t\epsilon_2})\ . \tag{2.18}$$

Notice that we shift the instanton parameter by equivariant parameters. This is due to the term $HF\wedge F$ in (2.17) which shifts the instanton parameter by $e^{t(\epsilon_1+\epsilon_2)}$. We need to compensate this part by shifting q. The insertion (2.17) inside the exponent has the conserved charge $U=+2$, and the instanton generates $4h^\vee$ which means it counts the number of fermion zero modes. Therefore, when expanding in powers of t,

$$\hat{Z}=Z+\mathcal{O}(t^{2h^\vee-2\sum_R C_2(R)})\ . \tag{2.19}$$

Therefore, when $N_f<2N_c-2$ for the $SU(N_c)$ SQCD, we have enough number of relations to fix the instanton partition function.

2.2 Blowup Equation for 5d gauge theory

We consider a 5d $\mathcal{N}=1$ gauge theory with a gauge group G. It has the Coulomb branch moduli space, parametrized by the vacuum expectation value $\alpha_i\equiv\langle\Phi_{ii}\rangle$ of the vector multiplet scalar Φ. The gauge symmetry G is spontaeously broken to its Abelian subgroup $U(1)^{|G|}$ on the Coulomb branch. The low energy Abelian theory is described by the effective prepotential, which can be written as [6]

$$\mathcal{F}_{\text{classical}}=\frac{1}{2g^2}h_{ij}\alpha_i\alpha_j+\frac{\kappa}{6}d_{ijk}\alpha_i\alpha_j\alpha_k \tag{2.20}$$

at the classical level. It was found in [7, 8] that the fully quantum corrected prepotential can be obtained from the BPS instanton partition function $\mathcal{Z}$ on Ω-deformed $\mathbf{R}^4\times S^1$, i.e.,

$$\mathcal{F}=\lim_{\epsilon_{1,2}\to 0}\epsilon_1\epsilon_2\log\mathcal{Z}=\mathcal{F}_{\text{classical}}+\mathcal{F}_{\text{quantum}}. \tag{2.21}$$

The BPS partition function $\mathcal{Z}$ is defined with equivariant parameters $\epsilon_1,\epsilon_2,\alpha_1,\cdots,\alpha_{|G|}$ associated to the $U(1)^2\times U(1)^{|G|}$ action on the k-instanton moduli space $\mathcal{M}_{k,G}$ [7, 8]. Additional equivariant parameters $m_1,\cdots,m_{|F|}$ can be introduced if a given theory has the flavor symmetry F. It takes the form of

$$\mathcal{Z}=\exp\left(F_0\right)\cdot\mathcal{I} \tag{2.22}$$

where $\mathcal{I}$ is the Witten index counting the BPS bound states of fundamental particles and/or non-perturbative instanton solitons. More precisely,

$$\mathcal{I} \equiv \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F e^{-\frac{8\pi^2}{g^2}k} e^{-\epsilon_1(J_1+\frac{R}{2})} e^{-\epsilon_2(J_2+\frac{R}{2})} \prod_{i=1}^{|G|} e^{-\alpha_i Q_i} \prod_{l=1}^{|F|} e^{-m_l F_l}\right] \tag{2.23}$$

where (J_1, J_2) are the angular momenta associated to the two $\mathbf{R}^2$ planes, R is the Cartan generator of $SU(2)_R$ symmetry, $(Q_1, \cdots, Q_{|G|})$ are the electric charges of $U(1)^{|G|} \subset G$, and $(F_1, \cdots, F_{|F|})$ are the Cartan generators of the flavor symmetry group F. We also frequently use the notation $\epsilon_+ \equiv \frac{\epsilon_1+\epsilon_2}{2}$, $\epsilon_- \equiv \frac{\epsilon_1-\epsilon_2}{2}$ and $J_l = \frac{J_1-J_2}{2}$, $J_r = \frac{J_1+J_2}{2}$, generating self-dual and anti-self-dual rotation inside the $\mathbf{R}^4$. The fugacity variables used throughout this paper are

$$p_1 = e^{-\epsilon_1},\ p_2 = e^{-\epsilon_2},\ \omega_i = e^{-\alpha_i},\ y_l = e^{-m_l},\ Q = e^{-8\pi^2/g^2},\ t = \sqrt{p_1 p_2},\ u = \sqrt{p_1/p_2}. \tag{2.24}$$

Each multiplet

3 Examples

3.1 Ki-Hong's note

Unity Blowup equations The partition functions of generic 5d $\mathcal{N} = 1$ gauge theories with hypermultiplets in R-representation in the Coulomb branch consist of classical action term, 1-loop term, and instanton partition functions.

$$Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = Z_{\text{class}}(\epsilon_1, \epsilon_2, \vec{a}, m_0)\, Z_{\text{1-loop}}(\epsilon_1, \epsilon_2, \vec{a}, m_i)\, Z_{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) \tag{3.1}$$

where

$$\begin{aligned}
Z_{\text{class}} &= \exp\left[-\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}h_{ij}\phi^i\phi^j + \frac{1}{6}d_{ijk}\phi^i\phi^j\phi^k\right)\right] \\
Z_{\text{1-loop}} &= \exp\Bigg[-\frac{1}{2\epsilon_1\epsilon_2}\Bigg(\sum_{\alpha\in\text{roots}}\left(\frac{1}{6}(\vec{a}\cdot\vec{\alpha})^3 - \frac{1}{4}(\epsilon_1+\epsilon_2)(\vec{a}\cdot\vec{\alpha})^2 + \frac{1}{12}((\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2)(\vec{a}\cdot\vec{\alpha})\right) \\
&\qquad + \sum_{\omega\in\rho(R)}\Bigg(\frac{1}{6}\left(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\right)^3 - \frac{\epsilon_1+\epsilon_2}{4}\left(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\right)^2 \\
&\qquad\qquad - \frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{24}\left(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\right)\Bigg)\Bigg)\Bigg] \\
&\quad \times \mathrm{PE}\left[\frac{1}{(1-p_1)(1-p_2)}\left(-\sum_{\alpha\in\text{roots}} e^{\vec{a}\cdot\vec{\alpha}} + p_1^{1/2}p_2^{1/2}y_i\sum_{\omega\in\rho(R)} e^{\vec{a}\cdot\vec{\omega}}\right)\right].
\end{aligned} \tag{3.2}$$

Here $\vec{a}$ are Coulomb VEVs and $p_{1,2} = e^{\epsilon_{1,2}}$, $y_i = e^{m_i}$. Note that the normal exponential term saturates the zero-point energy of pletheystic exponential terms.[1]

[1]Technically, instead of considering this 1-loop prepotential terms, I inserted overall factors to the $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}}Z^{(1)}_{\text{1-loop}}/Z_{\text{1-loop}}$ so that it is written by Sinh terms.

The partition function satisfies so-called "Unity blowup equation"

$$Z(\epsilon_1,\epsilon_2,\vec{a},m_i,m_0)=\sum_{\vec{k}\in\vec{\alpha}^\vee} Z(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_1,m_i+r_i\,\epsilon_1,m_0+r_0\,\epsilon_1)$$
$$\times\; Z(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_2,m_i+r_i\,\epsilon_2,m_0+r_0\,\epsilon_2) \qquad (3.3)$$

for certain $\vec{r}_a$, r_i, r_0's. Here $\vec{\alpha}^\vee$ is the coroot lattice where the long root is normalized to have norm 2. The r_i's and r_0 are some numbers specifying the blowup equations.

Technically r_i's are constrained to be half integers since, for each single letter 1-loop partition functions

$$Z_{i,\vec{\omega}}=\mathrm{PE}\left[\frac{p_1^{1/2}p_2^{1/2}}{(1-p_1)(1-p_2)}y_i\,e^{\vec{a}\cdot\vec{\omega}}\right], \qquad (3.4)$$

the ratio between shifted ones and unshifted one is

$$\begin{aligned} l^{\vec{k}}_{i,\vec{\omega}} &= Z^{(1)}_{i,\vec{\omega}}Z^{(2)}_{i,\vec{\omega}}/Z_{i,\vec{\omega}} \\ &= \mathrm{PE}\left[\frac{p_1^{r_i}p_2^{1/2}y_i}{(1-p_1)(1-p_2/p_1)}p_1^{\vec{k}\cdot\vec{\omega}}e^{\vec{a}\cdot\vec{\omega}}+\frac{p_1^{1/2}p_2^{r_i}y_i}{(1-p_1/p_2)(1-p_2)}p_2^{\vec{k}\cdot\vec{\omega}}e^{\vec{a}\cdot\vec{\omega}}-\frac{p_1^{1/2}p_2^{1/2}y_i}{(1-p_1)(1-p_2)}e^{\vec{a}\cdot\vec{\omega}}\right] \\ &= \mathrm{PE}\left[\frac{p_1^{1/2}p_2^{1/2}y_i}{(1-p_1)(1-p_2)(p_1-p_2)}e^{\vec{a}\cdot\vec{\omega}}\Big((1-p_2)p_1^{\vec{k}\cdot\vec{\omega}+r_i+1/2}-(1-p_1)p_2^{\vec{k}\cdot\vec{\omega}+r_i+1/2}\Big)-(p_1-p_2)\right] \end{aligned} \qquad (3.5)$$

For the $l^{\vec{k}}_{i,\vec{\omega}}$ to be finite rational function, the plethystic exponent must be finite series. It can be satisfied only when r_i is a half integer.

Instanton partition functions from blowup equations From blowup equations one can compute the partition functions as follows. Rewriting the blowup equation as

$$1=\sum_{\vec{k}\in\vec{\alpha}^\vee} f_{\vec{k}}\,l_{\vec{k}}\frac{Z^{(1)}_{\text{inst}}Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \qquad (3.6)$$

where $f_{\vec{k}}=Z^{(1)}_{\text{class}}Z^{(2)}_{\text{class}}/Z_{\text{class}}$ and $l_{\vec{k}}=Z^{(1)}_{\text{1-loop}}Z^{(2)}_{\text{1-loop}}/Z_{\text{1-loop}}$ with abbreviated notation

$$\begin{aligned} Z^{(1)} &= Z(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\vec{k}\,\epsilon_1,m_i+r_i\,\epsilon_1,m_0+r_0\,\epsilon_1) \\ Z^{(2)} &= Z(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\vec{k}\,\epsilon_2,m_i+r_i\,\epsilon_2,m_0+r_0\,\epsilon_2) \end{aligned} \qquad (3.7)$$

Here note that $l_{\vec{k}}$ is independent of $Q=e^{-m_0}$, and $f_{\vec{k}}$ is some overall factor in the order of $Q^{\vec{k}\cdot\vec{k}/2}$. Expanding the equation by instanton fugacity Q, then at each Q^n level the equation is written by

$$\delta_{n,0}=p_1^{r_0}Z_n^{(1)}+p_2^{r_0}Z_n^{(2)}-Z_n+\sum_{\vec{k}\neq 0} f_{\vec{k},r_0}\,l_{\vec{k}}\left(\frac{Z^{(1)}_{\text{inst}}Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}}\right)\Bigg|_{O(Q^{n-\vec{k}\cdot\vec{k}/2})}. \qquad (3.8)$$

Since each Z_k and $Z_k^{(1,2)}$ are independent of r_0, one can solve (3.8) with three blowup equations with same r_i's but different r_0's.

The blowup equations for instanton partition functions of pure YM theory with generic gauge group were already studied in [9]. They are actually (3.1) with

$$\vec{r}_a = 0, \qquad r_0 = d - h^\vee/2 \tag{3.9}$$

where $d = 0, \cdots, h^\vee$. We extend these blowup equations to the theories with matters based on pure YM blowup equations. If one restrict the cases to $\vec{r}_a = 0$, as we explained in the previous section, the r_i's are technically required to be half intergers. Thus we look for the r_0's that provides the correct instanton partition functions by solving (3.8) while fixing $\vec{r}_a = 0$ and $r_i = 1/2$. Here are the results.

G	matter	r_0	d
$SU(N)_\kappa$	$N_f \times \boldsymbol{N}$	$d - N/2 - \kappa/2$	$0 \le d \le N - \lvert\kappa\rvert - 2N_f - 1$(?)
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \le d \le 6$
$SO(7)$	pure	$d - 5/2$	$0 \le d \le 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \le d \le 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \le d \le 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \le d \le 3$
G_2	pure	$d - 4/2$	$0 \le d \le 4$
G_2	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \le d \le 3$
F_4	pure	$d - 9/2$	$0 \le d \le 9$
F_4	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \le d \le 6$
F_4	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \le d \le 3$

They were tested by comparing the resulting instanton partition functions with the known results from [10]($SO(7)$ and G_2) and [11](F_4 with $N_{\mathbf{26}} = 2$). They were compared numerically, putting random numbers on the fugacities. Note that matters shift the r_0, each by one quarter of their Dynkin indices. It seems to differ from blowup formula for $SU(N)_\kappa + N_f$ instantons, where r_0 was affected only by its CS-level κ. However, one can rewrite the r_0 as

$$\begin{aligned} r_0 &= d - N/2 - \left(\kappa + \frac{1}{2}N_f\right)/2 + N_f/4 \\ &= d - N/2 - \kappa_{\text{eff}}/2 + N_f \times I_{\text{fund}}. \end{aligned} \tag{3.10}$$

Since fundamental matters shifts the effective CS-level, they cancel their index contributions and consequently the r_0 apparently looks independent of matters.

By above observations, we write the unity blowup equation for generic gauge groups and matter representations.

$$\begin{aligned} Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = \sum_{\vec{k} \in \vec{\alpha}^\vee} & Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\epsilon_1, m_i + \epsilon_1/2, m_0 + r_0\epsilon_1) \\ & \times Z(\epsilon_1 - \epsilon_2, \vec{a} + \vec{k}\epsilon_2, m_i + \epsilon_2/2, m_0 + r_0\epsilon_2) \end{aligned} \tag{3.11}$$

with

$$r_0 = d - h^\vee/2 - \kappa_{\text{eff}}/2 + N_{\boldsymbol{R}} \times I_{\boldsymbol{R}}. \tag{3.12}$$

Here $I_{\boldsymbol{R}}$ is the Dynkin index of $\boldsymbol{R}$ representation.

$SU(6)_3 + 1 \times \mathbf{20}$ As a non-trivial test, we consider the instanton partition function of the $SU(6)_3 + \mathbf{20}$ whose 5-brane realization was found recently [12]. Its web-diagram is given as figure.

(Written before computing the $SU(6)_3 + 20$ instanton partition function.)
Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the $SU(3) \times SU(3) \times U(1) \subset SU(6)$ where the $SU(6)$ multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}} :\ & \mathbf{35} \longrightarrow (\mathbf{8}, 1)_0 \oplus (1, \mathbf{8})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_2 \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-2} \oplus (1,1)_0, \\ \Phi_{ijk} :\ & \mathbf{20} \longrightarrow (\mathbf{3}, \bar{\mathbf{3}})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{3})_1 \oplus (1,1)_3 \oplus (1,1)_{-3}. \end{aligned} \tag{3.13}$$

Here to fit with the web-diagram, we set Φ_{156} and Φ_{234} are $(1,1)_3$ and $(1,1)_{-3}$. Once Φ_{156} and Φ_{234} get non-zero VEVs,

When $a_5 = -a_1 - a_6$, the web-diagram factorizes to two $SU(3)_3$ whose Coulomb VEVs are (a_1, a_5, a_6) and (a_2, a_3, a_4). In the gauge theory, it can be seen partly from prepotential. The prepotential of $S(6)_3 + 1 \times \mathbf{20}$ is

$$\mathcal{F} = \frac{1}{2} m_0 \sum_{i=1}^{6} a_i^2 + \frac{1}{2} \sum_{i=1}^{6} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3 - \frac{1}{6} \sum_{1<i<j} (a_1 + a_j + a_k)^3 \tag{3.14}$$

at the Weyl chamber $a_1 > \cdots > a_6$. As one sets the Coulomb VEV $a_6 = -a_1 - a_5$ and $a_4 = -a_2 - a_3$, one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \tag{3.15}$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2} m_0 \sum_{i=1}^{3} a_i^2 + \frac{1}{2} \sum_{i=1}^{3} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3. \tag{3.16}$$

It is Higgsed by

4 Conclusion

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