April 24, 2019

1 Prepotential

The effective prepotential on a Coulomb branch of a 5d gauge theory with a gauge group G and matter f in a representation R_f is given by [Seiberg]

$$\mathcal{F}(\phi) = \frac{m_0}{2} h_{ij}\phi_i\phi_j + \frac{\kappa}{6} d_{ijk}\phi_i\phi_j\phi_k + \frac{1}{12}\left(\sum_{r\in\text{roots}} |r\cdot\phi|^3 - \sum_f \sum_{w\in R_f} |w\cdot\phi - m_f|^3\right). \tag{1.1}$$

prepotential

Here, m_0 is the inverse of the squared gauge coupling, κ is the classical Chern-Simons level and m_f is a mass parameter for the matter f. r is a root of the Lie algebra $\mathfrak{g}$ associated to G and w is a weight of the representation R_f of $\mathfrak{g}$. We also defined $h_{ij} = \mathrm{Tr}(T_iT_j), d_{ijk} = \frac{1}{2}\mathrm{Tr}\,(T_i\{T_j, T_k\})$ where T_i are the Cartan generators of the Lie algebra $\mathfrak{g}$.

1.1 $SU(6)_\kappa + 1$**TAS**

The prepotential for $SU(6)_\kappa + 1\mathbf{TAS}$ is written as

$$\mathcal{F}^{SU(6)_\kappa}_{N_{\mathbf{TAS}}=1} = \frac{m_0}{2}\sum_{i=1}^{6} a_i^2 + \frac{\kappa}{6}\sum_{i=1}^{6} a_i^3 + \frac{1}{6}\left(\sum_{1\le i<j\le 6}(a_i - a_j)^3 - \sum_{2\le i<j\le 6}(a_1 + a_i + a_j)^3\right), \tag{1.2}$$

eq:preSU6TSA

where κ is the CS level.

Now we choose $\kappa = 3$. In Dynkin basis, one can re-express the prepotential for $SU(6)_3+$ $1\mathbf{TAS}$, using the relation

$$a_1 = \phi_1,\quad a_2 = \phi_2 - \phi_1,\quad a_3 = \phi_3 - \phi_2,\quad a_4 = \phi_4 - \phi_3,\quad a_5 = \phi_5 - \phi_4,\quad a_6 = -\phi_5, \tag{1.3}$$

orth2Dynkin

as follows:

$$\begin{aligned}\mathcal{F} =& m_0\left(\phi_1^2 - \phi_2\phi_1 + \phi_2^2 + \phi_3^2 + \phi_4^2 + \phi_5^2 - \phi_2\phi_3 - \phi_3\phi_4 - \phi_4\phi_5\right)\\ &+ \frac{\phi_1^3}{3} + 4\phi_2\phi_1^2 - 5\phi_2^2\phi_1 + 2\phi_2\phi_3\phi_1 - 2\left(\phi_3^2 - \phi_4\phi_3 + \phi_4^2 + \phi_5^2 - \phi_4\phi_5\right)\phi_1\\ &+ \frac{4\phi_2^3}{3} + \frac{4\phi_3^3}{3} + \frac{4\phi_4^3}{3} + \frac{4\phi_5^3}{3} - 2\phi_2\phi_3^2 - \phi_3\phi_4^2 + \phi_2^2\phi_3 - \phi_4^2\phi_5.\end{aligned} \tag{1.4}$$

eq:prep+SU6-3

One can obtain the monopole string tensions T_i from this prepotential.

1.2 5-brane web configuration

It follows from [arXiv:1902.04754] that a 5-brane web diagram for $SU(6)_3 + 1\mathbf{TAS}$ is depicted in Figure 1. It is straightforward to see that the area of the faces in the web diagram agrees with the monopole string tensions T_i from the prepotential (1.4):

$$T_1 = \textcircled{1} + 2\textcircled{2},\qquad T_2 = \textcircled{3},\qquad T_3 = \textcircled{4},\qquad T_4 = \textcircled{5},\qquad T_5 = \textcircled{6} + 2\textcircled{7}, \tag{1.5}$$

where the encircled numbers represent the area of apparent faces in Figure 1.

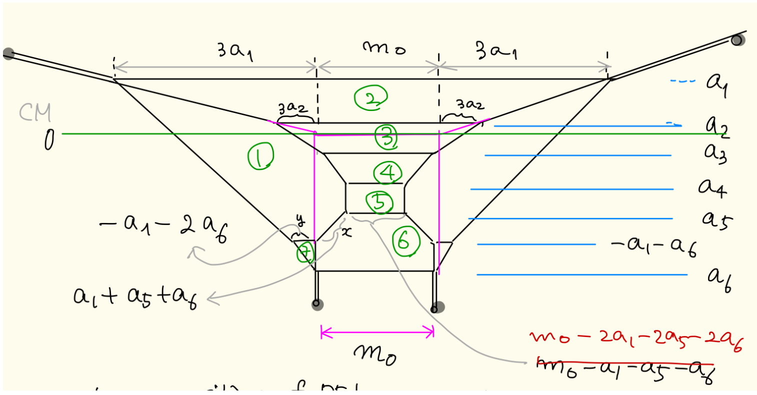

Figure 1. A 5-brane web for $SU(6)_3 + 1$**TAS** with massless **TAS**. fig:SU6-monopo

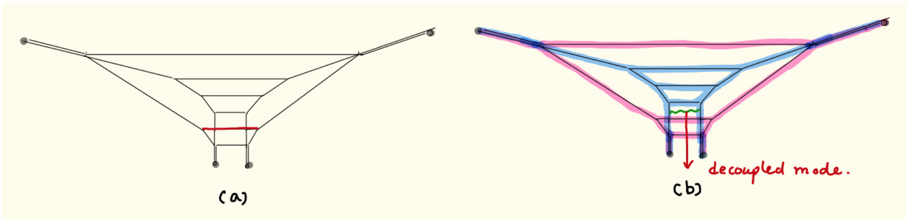

Figure 2. (a) A Higgsing from from $SU(6)_3 + 1$**TAS** to two $SU(3)_3$ theories. (a) Two $SU(3)_3$ theories are painted in blue and in pink respectively. A new decoupled mode emerges. fig:SU6-Higgsi

We note that there is a special Higgs phase that the $SU(6)_3 + 1$**TAS** can be expressed as a product of two pure $SU(3)_3$ theory. To this end, consider the following parameter relation

$$a_5 = -a_1 - a_6, \qquad \text{or} \qquad \phi_4 = \phi_1. \tag{1.6}$$

Higgsing

This is when the parameter x becomes 0 such that two D5-brane lined up. From usual Higgsing that branes disjoint and rejoin to other brane 2(a), one expects that the web diagram configuration is that of two pure $SU(3)_3$ theory, as depicted in Figure 2(b). The prepotential with this Higgsings is indeed written as a sum of two pure $SU(3)_3$ theories. From 1.1, the prepotential for $SU(3)_3$ is given by

$$\mathcal{F}_{SU(3)_3}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \frac{m_0}{2}\sum_{i=1}^{3} \tilde{a}_i^2 + \frac{1}{2}\sum_{i=1}^{3}\tilde{a}_i^3 + \frac{1}{6}\sum_{1\le i<j\le 3}(\tilde{a}_i - \tilde{a}_j)^3, \tag{1.7}$$

where $\sum_{i=1}^{3}\tilde{a}_i = 0$. From Figure 2, the Coulomb branch parameters for two $SU(3)_3$ theories read (a_1, a_5, a_6) and (a_2, a_3, a_4). It follows from the $SU(6)$ traceless condition $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0$ before the Higgsing, these two $SU(3)$ Coulomb branch parameters automatically satisfy the $SU(3)$ traceless condition with the Higgsing (1.6),

$$a_1 + a_5 + a_6 = 0, \qquad \text{and} \qquad a_2 + a_3 + a_4 = 0. \tag{1.8}$$

One then readily see that

$$\mathcal{F}^{SU(6)_3}_{N_{\mathbf{TAS}}=1}\Big|_{a_1+a_5+a_6\to 0} \to \mathcal{F}_{SU(3)_3}(a_1,a_5,a_6)+\mathcal{F}_{SU(3)_3}(a_2,a_3,a_4). \tag{1.9}$$

In terms of Dynkin label,

$$\mathcal{F}^{SU(6)_3}_{N_{\mathbf{TAS}}=1}\Big|_{\phi_4\to\phi_1} \to \mathcal{F}_{SU(3)_3}(\phi_1,\phi_5-\phi_1,-\phi_5)+\mathcal{F}_{SU(3)_3}(\phi_2-\phi_1,\phi_3-\phi_2,\phi_1-\phi_3). \tag{1.10}$$

This Higgsing may be understood as follows. Under the embedding

$$SU(6)\supset SU(3)\times SU(3)\times U(1), \tag{1.11}$$

the hypermultiplet in the rank-3 antisymmetric representation and the adjoint of $SU(6)$ are decomposed as

$$\begin{aligned}
&SU(6)\supset SU(3)\times SU(3)\times U(1),\\
&\quad \mathbf{35}=(\mathbf{1},\mathbf{1})_{\mathbf{0}}+(\mathbf{1},\mathbf{8})_{\mathbf{0}}+(\mathbf{8},\mathbf{1})_{\mathbf{0}}+(\mathbf{3},\bar{\mathbf{3}})_{\mathbf{2}}+(\bar{\mathbf{3}},\mathbf{3})_{-\mathbf{2}},\\
&\quad \mathbf{20}=(\mathbf{1},\mathbf{1})_{\mathbf{3}}+(\mathbf{1},\mathbf{1})_{-\mathbf{3}}+(\mathbf{3},\bar{\mathbf{3}})_{-\mathbf{1}}+(\bar{\mathbf{3}},\mathbf{3})_{\mathbf{1}}.
\end{aligned} \tag{1.12}$$

The fields which are charged under the $U(1)$, then get masses when the vev of the Coulomb branch modulus for the $U(1)$ is given. With large vev, the fields charged under the $U(1)$ decouple and the low energy effective theory becomes rank 2 pure $SU(3)$ theory, which is a pure $SU(3)_3\times SU(3)_3$ theory without bifundamental matter.

An immediate implication is that this Higgsing gives a consistency check for the blowup formula for $SU(6)_3+1\mathbf{TAS}$, that the partition function $Z_{SU(6)_3+1\mathbf{TAS}}$ has the Higgs phase

$$Z_{SU(6)_3+1\mathbf{TAS}}\Big|_{\text{Higgsing}} \quad\to\quad Z^{(1)}_{SU(3)_3}Z^{(2)}_{SU(3)_3}\,Z_{\text{decoupled}}\,, \tag{1.13}$$

where

$$\begin{aligned}
Z^{(1)}_{SU(3)_3}&=Z^{(1)}_{SU(3)_3}(\phi_1,\phi_5-\phi_1,-\phi_5),\\
Z^{(2)}_{SU(3)_3}&=Z^{(2)}_{SU(3)_3}(\phi_2-\phi_1,\phi_3-\phi_2,\phi_1-\phi_3),
\end{aligned} \tag{1.14}$$

and $Z_{\text{decoupled}}$ represents the extra term that does not depend on the Coulomb branch parameters, which would correspond to a new decoupled mode shown in Figure 2. This also means, we can see some partial information of the partition function for $SU(6)_3+1\mathbf{TAS}$, as the $Z_{SU(3)_3}$ partition function $Z_{SU(3)_3}$ can be easily computed, up to the extra term.

2 $SU(6)$ theory with a full rank-3 antisymmetric hypermultiplet

$$\begin{aligned}
Z_{\text{Nek}}=\sum_{\vec{Y}}&q^{\sum_{i=1}^{6}|Y_i|}(-A_1^6)^{|Y_1|}(-A_2^6)^{|Y_2|}(-A_2^2A_3^2A_4^2)^{|Y_3|}(-A_2^2A_3^4)^{|Y_4|+|Y_5|}\\
&\times f_{Y_1}(g)^5f_{Y_2}(g)^5f_{Y_3}(g)^3f_{Y_4}(g)f_{Y_5}(g)^{-1}f_{Y_6}(g)^2Z_{\text{left}}(\vec{Y})Z_{\text{right}}(\vec{Y}),
\end{aligned} \tag{2.1}$$

Znek1

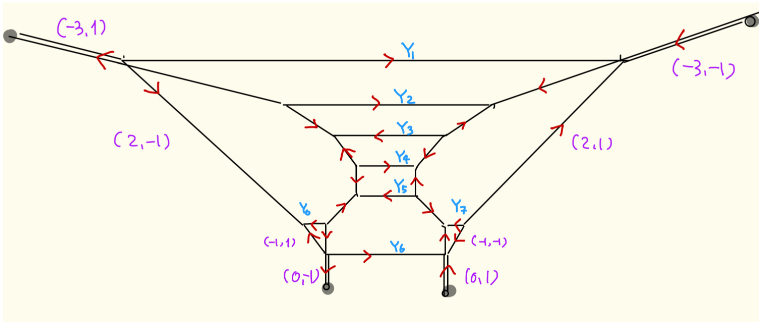

Figure 3. A labeling of Young diagrams assigned to the horizontal lines in Figure 1. fig:SU6young

where $\vec{Y} = (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$. $Z_{\text{left}}(\vec{Y})$ and $Z_{\text{right}}(\vec{Y})$ are given by

$$\begin{aligned} Z_{\text{left}}(\vec{Y}) = &\sum_{Y_0} (-{A_1}^{-1}{A_6}^{-2})^{|Y_0|} g^{\frac{||Y_0^t||^2+||Y_0||^2}{2}} \tilde{Z}_{Y_0}^2 f_{Y_0}^2(g) \prod_{i=1}^{6} g^{\frac{||Y_i||^2}{2}} \tilde{Z}_{Y_i} \\ &\times R^{-1}_{Y_1 Y_6^t}(A_1 {A_6}^{-1})\, R^{-1}_{Y_0 Y_6^t}({A_1}^{-1}{A_6}^{-2})\, R^{-1}_{Y_1 Y_0^t}(A_1^2 A_6) \\ &\times \prod_{2\le i<j\le 5} R^{-1}_{Y_i Y_j^t}(A_i {A_j}^{-1}) \prod_{i=2}^{5} R_{Y_0^t Y_i}(A_1 A_i A_6), \end{aligned} \tag{2.2}$$

and

$$Z_{\text{right}}(\vec{Y}) = Z_{\text{left}}(\vec{Y}) \,/.\, \{Y_0 \to Y_7\} \ . \tag{2.3}$$

Here,

$$A_6 = \prod_{i=1}^{5} {A_i}^{-1}, \quad \tilde{Z}_\lambda = \prod_{(i,j)\in\lambda} \frac{1}{1 - g^{\lambda_i + \lambda_j^t - i - j + 1}}, \quad R_{\lambda\mu}(Q) = \prod_{i.j=1}^{\infty} (1 - Q g^{i+j-\lambda_j-\mu_i-1}). \tag{2.4}$$

$f_Y(g)$ is the framing factor defined by

$$f_Y(g) = (-1)^{|Y|} g^{\frac{1}{2}(g^{||Y^t||^2 - ||Y||^2})}, \tag{2.5}$$

and the Coulomb branch parameters $A_i, (i = 1, \cdots, 6)$, the instanton fugacity q and the unrefined Ω-deformation parameter g are defined by

$$A_i = e^{-a_i}, \qquad q = e^{-m_0}, \qquad g = e^{-\epsilon}. \tag{2.6}$$

The partition function can be written as a sum of the instanton partition functions

$$Z_{\text{Nek}} = Z_{\text{pert}} \left(1 + \sum_{k=1}^{\infty} q^k Z_k \right), \tag{2.7}$$

where Z_{pert} represents the perturbative part of the partition function given by the order q^0 in (2.1), while Z_k stands for the k-instanton partition function.

The perturbative part. By setting $Y_1 = Y_2 = \cdots = Y_6 = \emptyset$ in (2.1), the partition part is obtained

$$
\begin{aligned}
Z_{\text{pert}} &= Z_{\text{left}}((\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset))Z_{\text{right}}((\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)) \\
&= \text{PE}\left[\frac{2g}{(1-g)^2}\left(A_1{A_6}^{-1} + {A_1}^{-1}{A_6}^{-2} + A_1^2 A_6 + \sum_{2\le i<j\le 5} A_i{A_j}^{-1} - \sum_{i=2}^{5} A_1 A_i A_6\right)\right] \\
&\quad \times \prod_{Y_k=\{Y_0,Y_7\}}\left(\sum_{Y_k}(-{A_1}^{-1}{A_6}^{-2})^{|Y_k|} g^{\frac{||Y_k^t||^2+||Y_0||^2}{2}} \tilde{Z}^2_{Y_k} f^2_{Y_k}(g)\right. \\
&\qquad \left. N^{-1}_{Y_k^t\emptyset}({A_1}^{-1}{A_6}^{-2}) N^{-1}_{Y_k\emptyset}(A_1^2 A_6)\prod_{i=2}^{5} N_{Y_k\emptyset}(A_1 A_i A_6)\right),
\end{aligned}
\tag{2.8}
$$

where we used the identity

$$
R_{\lambda\mu}(Q) = R_{\mu\lambda}(Q) = \text{PE}\left[-\frac{g}{(1-g)^2}Q\right] \times N_{\lambda^t\mu}(Q) \tag{2.9}
$$

with PE representing the Plethystic exponential and

$$
N_{\lambda\mu}(Q) = \prod_{(i,j)\in\lambda}\left(1 - Qg^{\lambda_i+\mu_j^t-i-j+1}\right)\prod_{(i,j)\in\mu}\left(1 - Qg^{-\lambda_j^t-\mu_i+i+j-1}\right). \tag{2.10}
$$

—— (liable up to here) —-

Note that in order to obtain the exact expression for the perturbative part we still need to sum over the Young diagram Y_0. We can still evaluate the summation in terms of an expansion by A_1. Namely when we sum over the Young diagram until $|Y_0| \le k$, the expression is exact until the order A_1^k. The summation of the Young diagram Y_0 until $|Y_0| = 7$ yields the expression

$$
Z_{\text{pert}} = \text{PE}\left[\frac{2\,g}{(1-g)^2}\left(\sum_{1\le i<j\le 6} A_i{A_j}^{-1} - \sum_{1=i<j<k\le 6} A_i A_j A_k + \mathcal{O}({A_1}^8)\right)\right]. \tag{2.11}
$$

Zpert1

We observed that the the series expansion by A_1 gives an expression which stops at the order A_1 inside the Plethystic exponential as far as we checked. Hence, we claim that $\mathcal{O}({A_1}^8)$ term is actually exactly zero. Indeed, the partition function (2.11) is exactly equal to the perturbative part of the partition function of the $SU(6)$ gauge theory with a half-hypermultiplet in the rank-3 antisymmetric representation. We can also see that the charge of the BPS states counted by the perturbative partition function agrees with the charge of the positive weights used in the prepotential computation for (**??**).

Next, we compute the 1-instanton part. The 1-instanton part can be read off from the coefficient of the q^1 order part in (2.1) divided by the perturbative part given in (2.11). Hence the order q^1 contribution is given by combinations where $|Y_i| = 1$ for one of the $Y_i, (i = 1, \cdots, 6)$ and the others are trivial. Furthermore, we still need to sum over Y_0 and evaluate the summation in terms of a series expansion by A_1[1]. A_1 is a good expansion parameter since the explicit summation of Young diagrams in (2.1) involves only positive powers of A_1. For example, the contribution from $|Y_1| = 1, |Y_j| = 0$ $(j = 2, 3, 4, 5, 6)$ to the 1-instanton part is given by, up to order $\mathcal{O}({A_1}^8)$,

$$-\frac{g}{(1-g)^2}\frac{A_1^6}{\prod_{i=2}^{6}(A_i - A_1)^2}\left(1 - \sum_{i=2}^{6} {A_i}^{-1} A_1 + \sum_{i=2}^{6} A_i A_1^2 - A_1^3\right)^2. \tag{2.12}$$

Z1inst1

We again observed that the stop of the series expansion by A_1 in the numerator of (2.12) and we claim that the $\mathcal{O}({A_1}^8)$ term is actually exactly zero. Similarly we can also compute the other combinations of the Young diagrams which contribute to the 1-instanton part. Summing up all the contributions from the other combinations of the Young diagrams which contribute to the 1-instanton part, we obtain

$$\begin{aligned} Z_1 &= -\sum_{\ell=1}^{6} \frac{g}{(1-g)^2}\frac{A_\ell^6}{\prod_{i\neq\ell}(A_i - A_\ell)^2}\left[1 - \sum_{i\neq\ell} {A_i}^{-1} A_\ell + \sum_{i\neq\ell} A_i A_\ell^2 - A_\ell^3\right]^2 \\ &= -\sum_{\ell=1}^{6} \frac{e^{-3a_\ell}}{(2\sinh\frac{\epsilon}{2})^2\prod_{i\neq\ell}(2\sinh\frac{a_i - a_\ell}{2})^2}\left[(2\sinh\frac{3a_\ell}{2}) - \sum_{i\neq\ell}(2\sinh\frac{2a_i + a_\ell}{2})\right]. \end{aligned} \tag{2.13}$$

Zinst1

This is the explicit expression for the 1-instanton part of the partition function for the $SU(6)$ gauge theory with $N_{\mathbf{TAS}} = 1$ and $\kappa = 3$.

References

[1]The expression (2.1) contains factors with A_1 in the denominator. We perform a series expansion by A_1 only for the numerator of (2.1).