

# Instantons from Blow-up

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**ABSTRACT:** The Nekrasov partition function for 4d  $\mathcal{N} = 2$  or 5d  $\mathcal{N} = 1$  gauge theory on the blow up of a point  $\hat{\mathbb{C}}^2$  can be written in terms of the partition function on the flat space  $\mathbb{C}^2$ . At the same time, the partition function on the blow up is identical to the partition function on a flat space for sufficiently large class of examples. This relation enables us to compute the instanton partition functions for 4d  $\mathcal{N} = 2$  and 5d  $\mathcal{N} = 1$  gauge theories for arbitrary gauge theory with large class of matter representations without knowing explicit construction of the instanton moduli space. Remarkably, the instanton partition function is completely determined by the perturbative part. We obtain the partition function for the previously unknown theories: exceptional gauge groups  $EFG$  with fundamental/spinor hypermultiplets and more. We also compute the case with  $SU(6)$  with rank-3 antisymmetric tensor and compare with the topological vertex computation using the recently found 5-brane web configuration.

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## 1 Introduction

The Seiberg-Witten prepotential provides a complete description for the low energy dynamics of 4d  $\mathcal{N} = 2$  or 5d  $\mathcal{N} = 1$  gauge theory in its Coulomb branch [1, 2]. It is a function of the vacuum expectation value (VEV) of the scalar in the vector multiplet that parameterizes the Coulomb branch moduli space. Quantum correction to the prepotential is known to be one-loop exact, while there also exist non-perturbative corrections coming from Yang-Mills instantons.

An efficient way to compute the fully quantum corrected prepotential  $\mathcal{F}$  is to study the Nekrasov partition function  $\mathcal{Z}$  on  $\Omega$ -deformed  $\mathbb{C}^2$  or  $\mathbb{C}^2 \times S^1$ . It can be written as the product of the classical, one-loop, and instanton contributions,

$$\mathcal{Z}(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2, \mathbf{q}) = \mathcal{Z}_{\text{class}}(\vec{a}, \epsilon_1, \epsilon_2, \mathbf{q}) \mathcal{Z}_{\text{1-loop}}(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2) \mathcal{Z}_{\text{inst}}(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2, \mathbf{q}), \quad (1.1)$$

where the instanton piece is the fugacity sum over all multi-instanton contributions:

$$\mathcal{Z}_{\text{inst}}(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2, \mathbf{q}) = 1 + \sum_{k=1}^{\infty} \mathbf{q}^k \mathcal{Z}_k(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2). \quad (1.2)$$

Once the Nekrasov partition function is known, one can extract the prepotential via taking  $\epsilon_i \rightarrow 0$  limit as  $\mathcal{F} = \lim_{\epsilon_i \rightarrow 0} \epsilon_1 \epsilon_2 \log \mathcal{Z}$  [3–6].

The instanton part of the partition function in the  $\Omega$ -background can be computed once we know appropriate instanton moduli space. For the classical gauge group, the ADHM construction of the moduli space provides a direct way compute the instanton partition function. The ADHM construction can be understood as the quantum mechanics described by the  $Dp$ - $D(p+4)$  system. The Higgs branch moduli space of the  $Dp$  system gives the desired moduli space. Matter fields can be also introduced by considering the world-volume theory on the D0-branes of the D0-D4-D8 system. By using the localization on the 1d system on the D0-branes or its dimensional reduction [7, 8], the partition function has been obtained for variety of cases: classical gauge groups [9–13], exceptional gauge groups [14–17]. 3d  $N=4$  Coulomb branch realization of the instanton moduli space: [18] The precise choice of the Contour of the ADHM integral has been derived in [19–21] following the Jeffrey-Kirwan residue formula in 2d elliptic genus [22, 23].

But there is no ADHM type construction for the exceptional gauge groups or generic type of matter fields. String-theoretic picture implies that they require strong-coupling dynamics or non-Lagrangian field theories to realize instanton moduli space of exceptional group as a vacuum moduli space. There has been a few results regarding the exceptional instantons.

In this paper, we generalize the approach of Nakajima-Yoshioka (NY) [5, 24, 25] for the pure YM theory. In [16], the NY blow-up formula were used to compute the instanton partition function for exceptional gauge group and tested against the superconformal index of 4d SCFT where the Higgs branch is given by the instanton moduli space [26]. We propose a general blow-up formula for general gauge theory with arbitrary representations, under the condition that the matter representation is not too large.<sup>1</sup> This enables us to compute the instanton partition functions for numerous gauge theories that have not been known before, without relying on to the explicit construction of the moduli space.

The basic idea is as follows: Let us consider a one-point blow up  $\hat{\mathbb{C}}^2$  of the flat space  $\mathbb{C}^2$ . The *full* partition function on  $\hat{\mathbb{C}}^2$  can be written in terms of the products of the *full* partition function of  $\mathbb{C}^2$ . But at the same time, the partition function on the blowup is identical to that of the flat space since we can smoothly blow-down  $\hat{\mathbb{C}}^2$  to  $\mathbb{C}^2$  as long as the matter representation is not ‘too large’. This provides us a functional relation for the partition function, which turns out to be sufficient to determine the instanton partition function itself. Remarkably, this relation is completely determined by the perturbative part of the partition function. Therefore we arrive at a surprising conclusion: The *perturbative* physics determine the *non-perturbative* physics!

For example, we find the following universal expression for the 1-instanton partition function with arbitrary gauge group and matters:

$$Z_1 = \frac{e^{\epsilon_1 + \epsilon_2}}{(1 - e^{\epsilon_1})(1 - e^{\epsilon_2})} \sum_{\gamma \in \Delta_l} \frac{e^{(h^\vee - 1)a_\gamma/2} \prod_{\gamma \cdot w = 1} (1 - e^{a_w + m^{\text{phy}}})}{(e^{a_\gamma/2} - e^{-a_\gamma/2})(1 - e^{a_\gamma - \epsilon_1 - \epsilon_2}) \prod_{\gamma \cdot \alpha = 1} (e^{a_\alpha/2} - e^{-a_\alpha/2})}, \quad (1.3)$$

where ...

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<sup>1</sup>This has to do with the 1-loop beta function coefficient as we will discuss in detail.

**Notation** The equivariant parameters  $\epsilon_1, \epsilon_2, \vec{a}$  are associated to the  $U(1)^2 \times U(1)^{|G|}$  action on the  $k$ -instanton moduli space  $\mathcal{M}_{k,G}$ . Additional parameters  $\vec{m}$  are introduced if the gauge theory has an extra flavor symmetry. The instanton fugacity  $q$  can be written as  $q \equiv \Lambda^{b_0}$  (4d) or  $q \equiv \exp(2\pi i \Lambda)$  (5d) where  $\Lambda$  denotes the bare coupling of the 4d/5d gauge theory.

Throughout Section 2, we will characterize the partition function in terms of  $m_i \equiv \mathbf{m}_i + \epsilon_+$  (where  $\epsilon_+ \equiv \frac{\epsilon_1 + \epsilon_2}{2}$ ) as a reflection of the Donaldson-Witten twist by  $SU(2)_R$  symmetry. Here  $m_i$  corresponds to the ‘physical’ mass parameter of a matter hypermultiplet. We will also stick to the ‘effective’ instanton fugacity  $q \equiv q e^{b_0 \epsilon_+}$ , which absorbs the  $SU(2)_R$  generated mass of fermionic instanton zero modes, in describing the 5d Nekrasov partition function.

## 2 Instanton Counting from Blow-up

The essential idea of using the blow-up of  $\mathbb{C}^2$  for instanton counting is that the gauge theory partition function for a 4d  $\mathcal{N} = 2$  (or 5d  $\mathcal{N} = 1$ ) theory on the blow-up of a point  $\hat{\mathbb{C}}^2$  (or  $S^1 \times \hat{\mathbb{C}}^2$ ) can be written in two different ways. This will allow us to write a recursion relation for the instanton partition function that can be solved rather easily [5, 16, 24, 25].

**Localization on the blow-up  $\hat{\mathbb{C}}^2$**  One of the expressions for the partition function  $\hat{\mathcal{Z}}$  on the blow-up  $\hat{\mathbb{C}}^2$  comes from the Coulomb branch localization, which results that  $\hat{\mathcal{Z}}$  can be obtained by patching together the flat-space partition function  $\mathcal{Z}$  [27].

The blow-up  $\hat{\mathbb{C}}^2$  is constructed from  $\mathbb{C}^2$  by replacing the origin with the compact 2-cycle. In particular, the geometry near the  $\mathbb{P}^1$  is identical to a line bundle of degree  $(-1)$  on the  $\mathbb{P}^1$ . One can parametrize  $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$  using three homogeneous coordinates  $(z_0, z_1, z_2)$ , satisfying the projective condition  $(z_0, z_1, z_2) \sim (\lambda^{-1} z_0, \lambda^1 z_1, \lambda^1 z_2)$  for any  $\lambda \in \mathbb{C}^*$ , where the two-cycle  $\mathbb{P}^1 \subset \hat{\mathbb{C}}^2$  corresponds to the locus  $z_0 = 0$ . We are interested in the  $U(1)^2$  equivariant partition function, with the  $U(1)^2$  action  $V$  rotating the complex coordinates  $(z_0, z_1, z_2)$  as follows:

$$(z_0, z_1, z_2) \mapsto (z_0, e^{\epsilon_1} z_1, e^{\epsilon_2} z_2). \quad (2.1)$$

Instantons are located at two fixed points of the  $U(1)^2$  action, i.e., the north/south poles of the  $\mathbb{P}^1$ , whose coordinates are  $(z_0, z_1, z_2) = (0, 1, 0)$  and  $(0, 0, 1)$ . Around these fixed points, the local weights under the  $U(1)^2$  action  $V$  are:

$$(z_0 z_1, z_2/z_1) \mapsto (e^{\epsilon_1} z_0 z_1, e^{\epsilon_2 - \epsilon_1} z_2/z_1) \quad (\text{near the north pole}) \quad (2.2)$$

$$(z_0 z_2, z_1/z_2) \mapsto (e^{\epsilon_2} z_0 z_2, e^{\epsilon_1 - \epsilon_2} z_1/z_2) \quad (\text{near the south pole}) \quad (2.3)$$

The full partition function  $\hat{\mathcal{Z}}$  on  $\hat{\mathbb{C}}^2$ , which includes both the perturbative and instanton contributions, can be obtained by performing the localization on the Coulomb branch. On the Coulomb branch, the gauge group is generically broken to  $U(1)^r$  where  $r$  is the rank of the gauge group. The  $U(1)^r$  equivariant parameters  $\vec{a}$  naturally appear in the partition function. One needs to sum over all distinct field configurations with zero-sized instantons located at the north and south poles. All the inequivalent configurations are labeled by the vector  $\vec{k}$  of

the first Chern numbers, corresponding to different flux configurations on the two-cycle  $\mathbb{P}^1$ . Here we turn off any possible external flux that can be supported on the  $\mathbb{P}^1$ . Summing up,  $\hat{\mathcal{Z}}$  can be expressed as the following formula [27–31]:

$$\hat{\mathcal{Z}}(\vec{a}, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} \mathcal{Z}(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1) \mathcal{Z}(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2) \quad (2.4)$$

where the flux sum is taken over the co-root lattice  $\Lambda$  of the gauge algebra. Notice that the Coulomb parameter  $\vec{a}$  gets an appropriate shift at each fixed point  $p$ , induced by the non-trivial magnetic flux  $\vec{k}$  on the blown-up  $\mathbb{P}^1$ , with the proportionality constant  $H|_p$ . Values of the moment map  $H$  for the  $U(1)^2$  action  $V$ , i.e.,  $dH = \iota_V \omega$ , at the north and south poles are

$$H|_{\text{NP}} = \epsilon_1 \text{ and } H|_{\text{SP}} = \epsilon_2. \quad (2.5)$$

**Partition function on  $\hat{\mathbb{C}}^2$  vs  $\mathbb{C}^2$**  An alternative fact for the partition function  $\hat{\mathcal{Z}}$  on the blow-up  $\hat{\mathbb{C}}^2$  is that  $\hat{\mathcal{Z}}$  is actually identical to the flat-space partition function  $\mathcal{Z}$  [5, 24, 25].

The blow-up  $\hat{\mathbb{C}}^2$  is identical to  $\mathbb{C}^2$  except for the origin, which is replaced by the blown-up sphere  $\mathbb{P}^1$ . Since the Nekrasov partition function gets contributions only from the small instantons localized at the fixed points of the  $U(1)^2$  equivariant action  $V$ , the size of the divisor should not affect the partition function as we smoothly shrink it. So we expect that  $\hat{\mathcal{Z}} = \mathcal{Z}$ . This implies the following relation: [5, 16, 24, 25, 32, 33]

$$\mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} \mathcal{Z}(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1) \mathcal{Z}(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2), \quad (2.6)$$

The blow-up identity can also be generalized to orbifold partition functions [31, 34, 35]. For example, if we consider the total space of the bundle  $\mathcal{O}(-2) \rightarrow \mathbb{P}^1$  and shrink the base, we land on the orbifold  $\mathbb{C}^2/\mathbb{Z}_2$  with singularity at the origin. There are certain states originated from wrapped D-branes on the  $\mathbb{P}^1$  which become massless by shrinking the  $\mathbb{P}^1$ . They constitute to the required twisted sector of the orbifold partition function. In any case, we expect that the Nekrasov partition function still remains the same by blowing up and down.<sup>2</sup>

**Background parameters** The expected relation (2.6) is only a schematic expression. The partition function also depends on some background parameters, such as the gauge coupling  $\Lambda$  and flavor chemical potentials  $m$ . They need to be appropriately shifted at each fixed point  $p$  of the blow-up  $\hat{\mathbb{C}}^2$ , keeping invariant the effective mass parameters twisted by  $SU(2)_R$ .

One can identify the shifted parameters  $\Lambda|_p$  and  $m|_p$  at a fixed point  $p$ , by examining the 1-loop effective free energy of the 5d Nekrasov partition function  $\mathcal{Z}$ . For a general 5d  $\mathcal{N} = 1$

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<sup>2</sup>This simple picture does not necessarily hold when there are too many hypermultiplets, due to some subtle scheme dependence related to wall-crossing [28, 29, 35]. We will discuss about this issue in Section ??.

gauge theory with the Chern-Simons level  $\kappa$  and/or some hypermultiplets,<sup>3</sup>

$$\begin{aligned} \log \mathcal{Z} = & \frac{1}{\epsilon_1 \epsilon_2} \left[ \frac{1}{2} \Lambda h_{ij} a_i a_j + \frac{\kappa}{6} d_{ijk} a^i a^j a^k + \sum_{\vec{\alpha} \in \Delta} \left( \frac{(\vec{\alpha} \cdot \vec{\alpha} + \epsilon_+)^3}{12} - \frac{\epsilon_1^2 + \epsilon_2^2 + 24}{48} (\vec{\alpha} \cdot \vec{\alpha} + \epsilon_+) + 1 \right) \right. \\ & \left. - \sum_i \sum_{\vec{\omega} \in \mathbf{R}_i} \left( \frac{(\vec{\alpha} \cdot \vec{\omega} + m_i)^3}{12} - \frac{\epsilon_1^2 + \epsilon_2^2 + 24}{48} (\vec{\alpha} \cdot \vec{\omega} + m_i) + 1 \right) \right] + \sum_{n=1}^{\infty} \frac{f(n\vec{\alpha}, n\epsilon_{1,2}, nm_i)}{n} \quad (2.7) \end{aligned}$$

in which the first 2 terms inside the square brackets constitute the classical prepotential  $\mathcal{F}_{\text{cl}}$ , whereas all the other terms come in at one-loop level. The set of all roots is denoted by  $\Delta$ . The  $\vec{\omega}$  runs over all weight vectors in the representation  $\mathbf{R}_i$  of the  $i$ -th hypermultiplet. First, let us consider a 5d free hypermultiplet with mass  $m$ , whose corresponding free energy is

$$\log \mathcal{Z} = -\frac{1}{\epsilon_1 \epsilon_2} \left( \frac{m^3}{12} - \frac{\epsilon_1^2 + \epsilon_2^2 + 24}{48} m + 1 \right) + \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-nm} \cdot e^{-n\epsilon_+}}{(1 - e^{-n\epsilon_1})(1 - e^{-n\epsilon_2})}. \quad (2.8)$$

This satisfies the relation (2.6) only when it is accompanied by  $m \rightarrow m \pm \frac{H}{2}|_p$ . Each  $\pm$  shift preserves the combination  $(m \pm \epsilon_+)$  respectively, corresponding to the  $SU(2)_R$  twisted mass of the hypermultiplet [36]. Throughout this paper, we take  $m_{\text{phy}} \equiv (m - \epsilon_+)$  as the invariant  $SU(2)_R$  twisted mass, identifying that

$$m|_p = m - \frac{H}{2}|_p \quad (2.9)$$

at a fixed point  $p$ . Second, the twisted instanton mass can be easily read off after simplifying (2.7) based on the following identities:

$$\begin{aligned} \sum_{\vec{\omega} \in \mathbf{R}} (\vec{\alpha} \cdot \vec{\omega}) (\vec{b} \cdot \vec{\omega}) (\vec{c} \cdot \vec{\omega}) &= I_3(\mathbf{R}) d_{ijk} a^i b^j c^k, \\ \sum_{\vec{\omega} \in \mathbf{R}} (\vec{\alpha} \cdot \vec{\omega}) (\vec{b} \cdot \vec{\omega}) &= I_2(\mathbf{R}) h_{ij} a^i b^j, \\ \sum_{\vec{\omega} \in \mathbf{R}} (\vec{\alpha} \cdot \vec{\omega}) &= 0. \end{aligned} \quad (2.10)$$

Here  $I_2(\mathbf{R})$  and  $I_3(\mathbf{R})$  are the quadratic and cubic Dynkin indices<sup>4</sup> of a given representation  $\mathbf{R}$ . In particular,  $I_2(\mathbf{adj}) = h^\vee$  is the dual Coxeter number of the gauge algebra. Speaking explicitly, the rational part of the one-loop free energy (2.7) can be written as

$$\log \mathcal{Z} = \frac{1}{\epsilon_1 \epsilon_2} \left[ \frac{1}{2} \Lambda_{\text{eff}} h_{ij} a^i a^j + \frac{\kappa_{\text{eff}}}{6} d_{ijk} a^i a^j a^k + (\text{independent of } \vec{a}) \right], \quad (2.11)$$

where the one-loop corrected parameters are

$$\Lambda_{\text{eff}} = \Lambda + \left( h^\vee - \sum_i \frac{I_2(\mathbf{R}_i)}{2} \right) \epsilon_+ - \sum_i \frac{I_2(\mathbf{R}_i)}{2} m_i^{\text{phy}}, \quad \kappa_{\text{eff}} = \kappa - \sum_i \frac{I_3(\mathbf{R}_i)}{2} \quad (2.12)$$

<sup>3</sup>We assume a particular Weyl chamber in the Coulomb branch, i.e.,  $0 < a_i < \epsilon_+ < m$  for all  $i \in \{1, \dots, r\}$ .

<sup>4</sup>The Dynkin indices are normalized such that  $I_2(\mathbf{F}) = 1$  for the  $SU(N)$  fundamental representation  $\mathbf{F}$ .

in which the sum is taken over all the matter multiplets. We notice that the instanton soliton carries the  $SU(2)_R$  twisted effective mass, given by  $m_{\text{inst}} \equiv \Lambda_{\text{eff}} - \kappa_{\text{eff}} \epsilon_+$ . In addition, the effective Chern-Simons coupling  $\kappa_{\text{eff}}$  induces an electric charge to the instanton, contributing to its ground state energy as  $E_0 = m_{\text{inst}} - \vec{a} \cdot \vec{\Pi}$ , where  $\vec{\Pi}$  is the  $U(1)^r \subset G$  electric charge.<sup>5</sup> To keep it invariant the effective instanton mass  $m_{\text{inst}}$  at a fixed point  $p$  of the blow-up  $\hat{\mathbb{C}}^2$ , we require the shifted gauge coupling  $\Lambda|_p$  to be

$$\Lambda|_p = \Lambda + \frac{b}{2}H|_p \quad \text{with} \quad b \equiv h^\vee - \sum_i \frac{I_2(\mathbf{R}_i)}{2} - \kappa_{\text{eff}}. \quad (2.13)$$

For example, the  $\mathbb{C}^2 \times S^1$  partition function of  $\mathcal{N} = 1$  free Maxwell theory, given as a product between the perturbative index of a free vector multiplet,

$$\exp \left[ +\frac{1}{\epsilon_1 \epsilon_2} \left( \frac{\epsilon_1 \epsilon_2 (\epsilon_1 + \epsilon_2)}{48} - \frac{\epsilon_1 + \epsilon_2}{4} + 1 \right) \right] \cdot \text{PE} \left[ -\frac{e^{-\epsilon_1 - \epsilon_2}}{(1 - e^{-\epsilon_1})(1 - e^{-\epsilon_2})} \right], \quad (2.14)$$

and the non-perturbative correction from  $U(1)$  multi-instantons,

$$\text{PE} \left[ \frac{e^{-\Lambda} \cdot e^{-\epsilon_+}}{(1 - e^{-\epsilon_1})(1 - e^{-\epsilon_2})} \right], \quad (2.15)$$

will comply with the relation (2.6) by shifting the gauge coupling as  $\Lambda \rightarrow \Lambda|_p = \Lambda + \frac{H}{2}|_p$ .

In summary, the blow-up relation (2.6) should always be understood with (2.9) and (2.13).

**Correlation functions** In fact, (2.6) is not enough to fix the partition function completely, since there are 3 unknown functions and only one relation. More independent relations would be found from insertion of non-trivial  $\mathcal{Q}$ -closed operators [5, 25]. So we consider the correlation functions of that operator  $\mathcal{O}_{\mathbb{P}^1}$  associated to the two-cycle  $\mathbb{P}^1$  of the blow-up  $\hat{\mathbb{C}}^2$ .

In the 4d Donaldson-twisted theory, the  $\mathcal{Q}$ -invariant observable  $\mathcal{O}_{\mathbb{P}^1}$  can be constructed by applying the descent procedure twice to the Casimir invariant  $\text{Tr}(\Phi^2)$ . It can be written in terms of the component fields as follows: [37]

$$\mathcal{O}_{\mathbb{P}^1} = \int \left\{ \omega \wedge \text{Tr} \left( \Phi F + \frac{1}{2} \psi \wedge \psi \right) + H \text{Tr} (F \wedge F) \right\}. \quad (2.16)$$

Here  $\omega$  and  $H$  are the Kähler two-form on the  $\mathbb{P}^1$  and the moment map  $\iota_V \omega = dH$ , respectively. We find it convenient to study the generating function  $\langle e^{t \mathcal{O}_{\mathbb{P}^1}} \rangle$  of the correlators  $\langle \mathcal{O}_{\mathbb{P}^1} \dots \mathcal{O}_{\mathbb{P}^1} \rangle$ . In particular, the insertion of  $\exp(t \mathcal{O}_{\mathbb{P}^1})$  causes an extra shift of the instanton mass by  $2tH|_p$  at a fixed point  $p$  of the blow-up  $\hat{\mathbb{C}}^2$  [5, 24, 25]. This modifies (2.13) to  $\Lambda|_p = \Lambda + (\frac{b}{2} - t)H|_p$ . Therefore the expectation value of the generating function is given by

$$\langle e^{t \mathcal{O}_{\mathbb{P}^1}} \rangle = \sum_{\vec{k} \in \Lambda} \frac{\mathcal{Z}^{(N),t}(\vec{k}) \cdot \mathcal{Z}^{(S),t}(\vec{k})}{\mathcal{Z}^{(0)}} , \quad (2.17)$$

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<sup>5</sup>This agrees with the supersymmetric Casimir energy of the ADHM quantum mechanics.

where

$$\begin{aligned}
\mathcal{Z}^{(N),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, \Lambda + (\tfrac{b}{2} - t)\epsilon_1, \vec{m} + \tfrac{1}{2}\epsilon_1) , \\
\mathcal{Z}^{(S),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, \Lambda + (\tfrac{b}{2} - t)\epsilon_2, \vec{m} + \tfrac{1}{2}\epsilon_2) , \\
\mathcal{Z}^{(0)} &\equiv \mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, \vec{m}) .
\end{aligned} \tag{2.18}$$

As we shrink the two-cycle  $\mathbb{P}^1$  to recover the flat  $\mathbb{C}^2$ , the effect of inserting  $\mathcal{O}_{\mathbb{P}^1}^d$  turns out to give a vanishing contribution due to the selection rule. We recall that the instanton breaks the  $U(1)_R$  symmetry to the discrete subgroup  $\mathbb{Z}_{4h^\vee - 2\sum_i I_2(\mathbf{R}_i)}$  under which the operator  $\mathcal{O}_{\mathbb{P}^1}^d$  carries a charge of  $+2$ . The correlation functions vanish unless the discrete charges add up to zero, modulo  $4h^\vee - 2\sum_i I_2(\mathbf{R}_i)$ . Therefore, expanding (2.17) in powers of  $t$ , we find

$$\langle e^{t\mathcal{O}_{\mathbb{P}^1}} \rangle = 1 + \mathcal{O}(t^{2h^\vee - \sum_i I_2(\mathbf{R}_i)}) . \tag{2.19}$$

As long as the hypermultiplet representation is not too large, i.e., when  $2h^\vee - \sum_i I_2(\mathbf{R}_i) > 2$ , this allows us to write 3 independent relations for the 3 unknown variables. One can expand  $\langle e^{t\mu^{(C)}} \rangle$  until  $\mathcal{O}(t^2)$  order and recursively solve for  $\mathcal{Z}^{(0)}$  at every instanton number. So the instanton part of the partition function will be completely determined from the perturbative partition function. An explicit form of the recursion relation will be displayed in Section 2.1.

We now turn to 5d  $\mathcal{N} = 1$  gauge theory wrapped on  $S^1$ . The Casimir invariant  $\text{Tr}(\Phi^2)$  and its descendants are no longer considered as well-defined observables. Instead, there are two types of  $\mathcal{Q}$ -invariant observables [38]. The first type of observables are constructed from the 5d Wilson loop on the  $S^1$  by applying the descent procedure. The second type of observables introduce the 3d (Kähler) Chern-Simons term, which can be written as [38, 39]

$$\begin{aligned}
\mathcal{O}_{\mathbb{P}^1}^d = \exp \Bigg[ d \int \Bigg( \omega \wedge \text{Tr} \Big( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big) \\
+ \omega \wedge \Big( \phi F + \frac{1}{2} \psi \wedge \psi \Big) \wedge dt + H \text{Tr} (F \wedge F) \Bigg) \Bigg] .
\end{aligned} \tag{2.20}$$

It can be viewed as the natural  $S^1$  uplift of (2.16) via exponentiation, where the coefficient  $d$  is integer-valued as required by gauge invariance. The correlation function is given by

$$\langle \mathcal{O}_{\mathbb{P}^1}^d \rangle = \sum_{\vec{k} \in \Lambda} \frac{\mathcal{Z}^{(N),d}(\vec{k}) \cdot \mathcal{Z}^{(S),d}(\vec{k})}{\mathcal{Z}^{(0)}} \tag{2.21}$$

with the definitions (2.18) understood. For the case of 5d pure  $\mathcal{N} = 1$  SYM, the correlation function turns out to be [25]

$$\langle \mathcal{O}_{\mathbb{P}^1}^d \rangle = 1 \quad \text{for} \quad 0 \leq d \leq d_{\max} \tag{2.22}$$

where  $d_{\max} = h^\vee$ . For  $d_{\max} = h^\vee \geq 2$ , there are sufficient number of algebraic relations to determine the instanton partition function  $\mathcal{Z}$  iteratively, in increasing the order of instantons. This fact was utilized in [16] to compute  $\mathcal{Z}$  in exceptional gauge theories, for which the ADHM



construction of instanton moduli space is not known. In this paper, we aim at developing the relation (2.22) for various 5d  $\mathcal{N} = 1$  gauge theories with hypermultiplets, then writing down the corresponding  $\mathcal{Z}$  based on that. We will identify a certain bound on  $d$  in Section 2.2 as the *necessary* condition on (2.22) for the wide array of theories which have UV completion. See Table 1 for the list of all considered examples. The bound can be summarized as follows:

$$h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) \textcolor{red}{(add)} \quad (2.23)$$

And also, we conjecture that the bound on  $d$  is sufficient for (2.22). While we do not attempt to prove this sufficiency, we will compute an  $n$ -instanton partition function  $Z_n$ , based on the recursion formula that will be derived shortly from (2.22), and confirm the agreement with the known result obtained from an alternative method. We will call (2.22) the blowup equation.

## 2.1 Recursion formula

The blowup equation (2.22) can be translated to useful recursion formulae on the  $k$ -instanton contribution  $Z_k$  to the entire partition function  $\mathcal{Z}$ . To derive this, we decompose the partition function  $\mathcal{Z}$  in terms of the classical, one-loop, and instanton pieces:

$$\mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, \vec{m}) = Z_{\text{class}}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, \vec{m}) \cdot Z_{1\text{-loop}}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) \cdot Z_{\text{inst}}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, \vec{m}) \quad (2.24)$$

where  $Z_{\text{inst}}$  can be further expanded in terms of the instanton fugacity  $q \equiv e^{-\Lambda}$  as

$$Z_{\text{inst}}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, \vec{m}) = 1 + \sum_{k \geq 1} q^k Z_k(\vec{a}, \epsilon_1, \epsilon_2, m) . \quad (2.25)$$

Then the blowup equation (2.22) can be written as

$$Z_{\text{inst}} = \sum_{\vec{k}} \left[ \frac{Z_{\text{class}}^{(N),d}(\vec{k}) Z_{\text{class}}^{(S),d}(\vec{k})}{Z_{\text{class}}} \frac{Z_{1\text{-loop}}^{(N),d}(\vec{k}) Z_{1\text{-loop}}^{(S),d}(\vec{k})}{Z_{1\text{-loop}}} \right] Z_{\text{inst}}^{(N),d}(\vec{k}) Z_{\text{inst}}^{(S),d}(\vec{k}) , \quad (2.26)$$

where the superscript  $(N/S), d$  implies the appropriate shift of parameters, specified in (2.18).

We recall the known expressions for the classical and 1-loop partition function [3, 40, 41]:

$$Z_{\text{class}} = \exp \left[ \frac{1}{\epsilon_1 \epsilon_2} \left( \frac{1}{2} \Lambda h_{ij} a_i a_j + \frac{\kappa}{6} d_{ijk} a^i a^j a^k \right) \right] , \quad (2.27)$$

$$\begin{aligned} Z_{1\text{-loop}}^{\text{vec}} &= \exp \left[ \frac{1}{\epsilon_1 \epsilon_2} \sum_{\vec{\alpha} \in \Delta} \left( \frac{(\vec{a} \cdot \vec{\alpha} + \epsilon_+)^3}{12} - \frac{\epsilon_1^2 + \epsilon_2^2 + 24}{48} (\vec{a} \cdot \vec{\alpha} + \epsilon_+) + 1 \right) \right] \\ &\times \text{PE} \left[ - \frac{p_1 p_2}{(1-p_1)(1-p_2)} \sum_{\vec{\alpha} \in \Delta} e^{-\vec{a} \cdot \vec{\alpha}} \right] \quad \text{for the vector multiplet} \end{aligned} \quad (2.28)$$

$$\begin{aligned} Z_{1\text{-loop}}^{\text{hyp},i} &= \exp \left[ - \frac{1}{\epsilon_1 \epsilon_2} \sum_{\vec{\omega} \in \mathbf{R}_\ell} \left( \frac{(\vec{a} \cdot \vec{\omega} + m_\ell)^3}{12} - \frac{\epsilon_1^2 + \epsilon_2^2 + 24}{48} (\vec{a} \cdot \vec{\omega} + m_\ell) + 1 \right) \right] \\ &\times \text{PE} \left[ + \frac{(p_1 p_2)^{\frac{1}{2}} \cdot y_\ell}{(1-p_1)(1-p_2)} \sum_{\vec{\omega} \in \mathbf{R}_\ell} e^{-\vec{a} \cdot \vec{\omega}} \right] \quad \text{for the } \ell\text{'th hypermultiplet} \end{aligned} \quad (2.29)$$

where  $p_1 \equiv e^{-\epsilon_1}$ ,  $p_2 \equiv e^{-\epsilon_2}$ ,  $y_i \equiv e^{-m_i}$  and the PE represents the Plethystic exponential

$$\text{PE}[f(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, \vec{m})] \equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(n\vec{a}, n\epsilon_1, n\epsilon_2, n\Lambda, n\vec{m})\right). \quad (2.30)$$

Substituting them to (2.26), the combinations of three different  $Z$ 's are given by

$$\begin{aligned} \frac{Z^{(N),d}(\vec{k}) Z^{(S),d}(\vec{k})}{Z} &= q^{\frac{\vec{k} \cdot \vec{k}}{2}} (p_1 p_2)^{(\frac{b}{2}-d)(\frac{\vec{k} \cdot \vec{k}}{2}) + \frac{\kappa}{6} d_{ijk} k^i k^j k^k} \\ &\quad \times e^{-(\frac{b}{2}-d)(\vec{a} \cdot \vec{k})} e^{-\frac{\kappa}{2} d_{ijk} a^i k^j k^k} \quad \text{for } Z = Z_{\text{class}}, \end{aligned} \quad (2.31)$$

$$\frac{Z^{(N),d}(\vec{k}) Z^{(S),d}(\vec{k})}{Z} = e^{\frac{h^\vee}{2}(\vec{a} \cdot \vec{k})} \prod_{\alpha \in \Delta} \mathcal{L}_{\vec{k} \cdot \vec{\alpha}}(\vec{a} \cdot \vec{\alpha}, \epsilon_1, \epsilon_2)^{-1} \quad \text{for } Z = Z_{1\text{-loop}}^{\text{vec}}, \quad (2.32)$$

$$\begin{aligned} \frac{Z^{(N),d}(\vec{k}) Z^{(S),d}(\vec{k})}{Z} &= e^{-\frac{I_2(\mathbf{R}_\ell)}{4}(\vec{a} \cdot \vec{k}) + \frac{I_3(\mathbf{R}_\ell)}{4} d_{ijk} a^i k^j k^k} (p_1 p_2)^{\frac{I_2(\mathbf{R}_\ell)}{8}(\vec{k} \cdot \vec{k}) - \frac{I_3(\mathbf{R}_\ell)}{12} d_{ijk} k^i k^j k^k} \\ &\quad \times y_\ell^{-\frac{I_2(\mathbf{R}_\ell)}{4}(\vec{k} \cdot \vec{k})} \prod_{\omega \in \mathbf{R}_\ell} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_{\text{phy},\ell}, \epsilon_1, \epsilon_2) \quad \text{for } Z = Z_{1\text{-loop}}^{\text{hyp},\ell}. \end{aligned} \quad (2.33)$$

Here we used  $I_2(\mathbf{adj}) = 2h^\vee$ ,  $I_3(\mathbf{adj}) = 0$ , and also the fact  $h_{ij}$  and  $d_{ijk}$  are totally symmetric. The function  $\mathcal{L}_k(x, \epsilon_1, \epsilon_2)$  is introduced to denote concisely the combination of the PE parts:

$$\mathcal{L}_k(x, \epsilon_1, \epsilon_2) \equiv \text{PE} \left[ \left( \frac{p_1^k p_2}{(1-p_1)(1-p_2/p_1)} + \frac{p_1 p_2^k}{(1-p_1/p_2)(1-p_2)} - \frac{p_1 p_2}{(1-p_1)(1-p_2)} \right) \times e^{-x} \right] \quad (2.34)$$

One can easily check that the expression inside the PE vanishes at  $k = 0, 1$ . After some work, it is not difficult to find that

$$\mathcal{L}_k(x, \epsilon_1, \epsilon_2) = \begin{cases} \prod_{m+n \leq k-2} (1 - p_1^{m+1} p_2^{n+1} e^{-x}) & \text{for } k \geq +2 \\ \prod_{m+n \leq -k-1} (1 - p_1^{-m} p_2^{-n} e^{-x}) & \text{for } k \leq -1 \\ 1 & \text{for } k = 0, 1. \end{cases} \quad (2.35)$$

Combining them all, the resursion formula on the  $n$ -instanton piece  $Z_n$  can be written as

$$\begin{aligned} Z_n &= \sum_{\frac{1}{2}\vec{k} \cdot \vec{k} + \ell + m = n} \left( (p_1 p_2)^{(\frac{h^\vee}{2} - \frac{\kappa_{\text{eff}}}{2} - d)(\frac{\vec{k} \cdot \vec{k}}{2}) + \frac{\kappa_{\text{eff}}}{6} d_{ijk} k^i k^j k^k} e^{(d + \frac{\kappa_{\text{eff}}}{2})(\vec{a} \cdot \vec{k})} e^{-\frac{\kappa_{\text{eff}}}{2} d_{ijk} a^i k^j k^k} \right. \\ &\quad \times \frac{\prod_l y_l^{-I_2(\mathbf{R}_l)(\vec{k} \cdot \vec{k}/4)} \prod_{\omega \in \mathbf{R}_l} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_{\text{phy},l}, \epsilon_1, \epsilon_2)}{\prod_{\alpha \in \Delta} \mathcal{L}_{\vec{k} \cdot \vec{\alpha}}(\vec{a} \cdot \vec{\alpha}, \epsilon_1, \epsilon_2)} \cdot p_1^{(\frac{b}{2}-d)\ell} p_2^{(\frac{b}{2}-d)m} Z_\ell^{(N)}(\vec{k}) Z_m^{(S)}(\vec{k}) \Big), \end{aligned} \quad (2.36)$$

which is generalization of the recursion formula found for the pure SYM case [25, 28]. The right-hand side of the recursion relation (2.36) only involves the instanton parts with  $\ell, m < n$ . Being so, one can determine the  $n$ -instanton correction  $Z_n$  recursively from  $Z_0 = 1$ .

## 2.2 General 1-instanton partition function

At one instanton level, the formula (2.36) can be written as follows:

$$Z_1 = p_1^{(\frac{b}{2}-d)} Z_1^{(N)} + p_2^{(\frac{b}{2}-d)} Z_1^{(S)} + I_1^{(d)}, \quad (2.37)$$

where (with the fugacity variable  $y_{\text{phy},l} \equiv e^{-m_{\text{phy},l}} = y_l/\sqrt{p_1 p_2}$  understood)

$$I_1^{(d)} = \sum_{\vec{k} \in \Delta_l} \left( (p_1 p_2)^{(\frac{b}{2}-d)} e^{(d+\frac{\kappa_{\text{eff}}}{2})(\vec{a} \cdot \vec{k})} e^{-\frac{\kappa_{\text{eff}}}{2} d_{ijk} a^i k^j k^k} \right. \\ \left. \times \frac{\prod_l y_{\text{phy},l}^{-I_2(\mathbf{R}_l)/2} \prod_{\omega \in \mathbf{R}_l} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_{\text{phy},l}, \epsilon_1, \epsilon_2)}{(1 - p_1 p_2 e^{-\vec{a} \cdot \vec{k}})(1 - p_1^{-1} e^{\vec{a} \cdot \vec{k}})(1 - p_2^{-1} e^{\vec{a} \cdot \vec{k}})(1 - e^{\vec{a} \cdot \vec{k}}) \prod_{\vec{\alpha} \cdot \vec{k} = -1} (1 - e^{-\vec{a} \cdot \vec{\alpha}})} \right). \quad (2.38)$$

We are interested in 5d  $\mathcal{N} = 1$  gauge theories which are UV-complete as 5d  $\mathcal{N} = 1$  SCFTs. Then the set of possible matter representation should be restricted to [42]

- fundamental representation for  $SU(N)$ ,  $SO(N)$ ,  $Sp(N)$ ,  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$
- antisymmetric representation for  $SU(N)$ ,  $Sp(N)$
- spinor representation for  $SO(N)$  with  $7 \leq N \leq 14$
- rank-3 antisymmetric representation for  $Sp(3)$ ,  $Sp(4)$ ,  $SU(6)$ ,  $SU(7)$
- symmetric representation for  $SU(N)$ .

For the  $SU(N)$  symmetric representation, the inner product  $\vec{k} \cdot \vec{\omega}$  can be either 0,  $\pm 1$ , or  $\pm 2$ .

$$\prod_{\omega \in \text{sym}} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_{\text{phy}}, \epsilon_1, \epsilon_2) = \prod_{\vec{k} \cdot \vec{\omega} = 2} (1 - y_{\text{phy}} p_1 p_2 e^{-\vec{a} \cdot \vec{\omega}}) \times \prod_{\vec{k} \cdot \vec{\omega} = -1} (1 - y_{\text{phy}} e^{-\vec{a} \cdot \vec{\omega}}) \\ \times \prod_{\vec{k} \cdot \vec{\omega} = -2} (1 - y_{\text{phy}} p_1^{-1} e^{-\vec{a} \cdot \vec{\omega}})(1 - y_{\text{phy}} p_2^{-1} e^{-\vec{a} \cdot \vec{\omega}}). \quad (2.39)$$

For the other cases, the inner product  $\vec{k} \cdot \vec{\omega}$  can be either 0,  $\pm 1$ , such that

$$\prod_{\omega \in \mathbf{R}} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_{\text{phy}}, \epsilon_1, \epsilon_2) = \prod_{\vec{k} \cdot \vec{\omega} = -1} (1 - y_{\text{phy}} e^{-\vec{a} \cdot \vec{\omega}}). \quad (2.40)$$

**Number of the blowup equations** Prior to solving the equations for  $Z_1$ , we recall that the formula (2.36) is valid only in a certain range of  $d$ , for which  $\langle \mathcal{O}_{\mathbb{P}^1}^d \rangle = 1$ . We want to narrow down the validity range of  $d$  by doing a simple sanity check on (2.37). Specifically, we want to examine the expansion of all single terms in (2.37) with respect to  $\sqrt{p_1 p_2} \ll 1$ , identifying the leading exponent as follows:

$$I_1^{(d)} \sim \begin{cases} g_0(\vec{a}, \vec{m}_{\text{phy}}) \cdot (p_1 p_2)^{\frac{b}{2}-d+1} + \dots & \text{for } N_{\text{sym}} = 0 \\ g_0(\vec{a}, \vec{m}_{\text{phy}}) \cdot (p_1 p_2)^{\frac{b}{2}-d} + \dots & \text{for } N_{\text{sym}} = 1 \end{cases} \\ Z_1 \sim g_1(\vec{a}, \vec{m}_{\text{phy}}) \cdot (p_1 p_2)^{\frac{n}{2}} + \dots$$

$$p_1^{(\frac{b}{2}-d)} Z_1^{(N)} \sim p_2^{(\frac{b}{2}-d)} Z_1^{(S)} \sim g_2(\vec{a}, \vec{m}_{\text{phy}}) \cdot (p_1 p_2)^{\frac{b}{4}-\frac{d}{2}+\frac{n}{4}} + \dots \quad (2.41)$$

where a numerical value of  $n$  will be discussed shortly for a variety of gauge theories (Table 1). Notice that for the equation (2.37) to be true, some terms on the right-hand side should have the leading exponent less than or equal to that of  $Z_1$ . Therefore, the condition  $d - \frac{b}{2} \geq -\frac{n}{2}$  is naturally imposed, setting a lower bound on  $d$ .

Similarly, an upper bound on  $d$  can be found from an expansion of (2.37) with respect to  $1/\sqrt{p_1 p_2} \ll 1$ .<sup>6</sup> Each single term in (2.37) can be written as

$$\begin{aligned} I_1^{(d)} &\sim \begin{cases} h_0(\vec{a}, \vec{m}_{\text{phy}}) \cdot (1/p_1 p_2)^{d-\frac{b}{2}+1} + \dots & \text{for } N_{\text{sym}} = 0 \\ h_0(\vec{a}, \vec{m}_{\text{phy}}) \cdot (1/p_1 p_2)^{d-\frac{b}{2}} + \dots & \text{for } N_{\text{sym}} = 1 \end{cases} \\ Z_1 &\sim h_1(\vec{a}, \vec{m}_{\text{phy}}) \cdot (1/p_1 p_2)^{\frac{n'}{2}} + \dots \\ p_1^{(\frac{b}{2}-d)} Z_1^{(N)} \sim p_2^{(\frac{b}{2}-d)} Z_1^{(S)} &\sim h_2(\vec{a}, \vec{m}_{\text{phy}}) \cdot (1/p_1 p_2)^{\frac{d}{2}-\frac{b}{4}+\frac{n'}{4}} + \dots \end{aligned} \quad (2.42)$$

Again, for (2.37) to be consistent, the leading exponent of  $Z_1$  should be greater than or equal to those of some terms on the right-hand side. Such requirement imposes the upper bound on  $d$ , namely  $\frac{n'}{2} \geq d - \frac{b}{2}$ . Combining the two inequalities, one can identify the following range

$$-\frac{n}{2} + \frac{b}{2} \leq d \leq \frac{n'}{2} + \frac{b}{2}, \quad (2.43)$$

as a necessary condition for (2.37). Furthermore, we explicitly checked that the  $k$ -instanton partition function  $Z_k$  actually satisfies all the  $(\frac{n+n'}{2})$  recursion relations up to a certain value of  $k$ , for numerous examples (Table 1) whose  $Z_k$  is already known from alternative methods, even though the bound (2.43) itself is merely a *necessary* condition found from one-instanton analysis. Recall that the simple selection rule, governed by  $\mathbb{Z}_{4h^\vee-2\sum_i I_2(\mathbf{R}_i)} \subset U(1)_R$ , supports the validity of (2.19) at all instanton orders in 4d Donaldson-twisted theories. Based on the empirical observations, we claim that the 5d recursion formulae (2.36) within the above range of  $d$  are again true at all instanton orders. This implies that the non-perturbative partition function  $Z_{\text{inst}}$  is completely fixed by the perturbative partition function! Note that we have arrived at this statement not by requiring the perturbative series to be well-defined, as is often done in the resurgence analysis, but just by demanding consistency upon smooth deformation of the spacetime  $\mathbb{C}^2 \times S^1$ . Such consistency condition turns out to be entirely non-perturbative.

Another remarkable thing is that a numerical value of  $(n, n')$  exhibits the very simple pattern across a broad range of theories whose gauge group is not  $SU(N)_\kappa$ . (See Table 1.)

$$n = n' \quad = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) \quad \text{for } G \neq SU(N)_\kappa \text{ nor } Sp(N)$$

---

<sup>6</sup>This is equivalent to assuming a different parameter regime  $0 < a_i < -\epsilon_+ < m$  for all  $1 \leq i \leq r$ . In general, an explicit form of the 1-loop partition function (2.28)–(2.29) can change depending on a parameter regime, thus affecting (2.36). However, all the above expressions remain valid under flipping a sign of  $\epsilon_+$ , such that we can simply study the expansion of the single terms in (2.37) with respect to  $1/\sqrt{p_1 p_2} \ll 1$ .

$$\begin{aligned}
n &= n' - 2\{\frac{N_f}{2}\} = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) & \text{for } G = Sp(N)_{\theta=0} \\
n &= n' + 2\{\frac{N_f}{2}\} = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + 1 & \text{for } G = Sp(N)_{\theta=\pi}
\end{aligned} \tag{2.44}$$

As the above numerical pattern (2.44) emerges for all  $G \neq SU(N)_\kappa$  examples that we studied, we speculate that (2.44) is generally true, thereby taking the recursion formulae (2.36) with

$$\begin{aligned}
0 \leq d &\leq h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) & \text{for } G \neq SU(N)_\kappa \text{ nor } Sp(N) \\
0 \leq d &\leq h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + \{\frac{N_f}{2}\} & \text{for } G = Sp(N)_{\theta=0} \\
-\frac{1}{2} \leq d &\leq h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + \frac{1}{2} - \{\frac{N_f}{2}\} & \text{for } G = Sp(N)_{\theta=\pi}
\end{aligned} \tag{2.45}$$

as a basic assumption to obtain the partition function  $\mathcal{Z}$  for any  $G \neq SU(N)_\kappa$  gauge theory. We will derive the general expression for the 1-instanton partition function  $Z_1$ , which applies for all  $G \neq SU(N)_\kappa$  gauge theories having  $h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) \geq 2$ . It would be desirable to understand from the first principle the range (2.45) of  $d$  for which (2.36) holds true.

On the contrary, it turns out to be more difficult to characterize a general pattern behind  $(n, n')$  for  $SU(N)_\kappa$  gauge theories, due to extra complication caused by the 5d Chern-Simons level  $\kappa$ . Here we consider two particular classes of  $SU(N)_\kappa$  gauge theories as an illustration. In  $SU(N)_\kappa + N_f \mathbf{F}$  gauge theory with  $N_f + 2|\kappa| \leq 2N$ , we find that

$$\begin{aligned}
n &= \begin{cases} \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = N - N_f, \\ h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{F}) + |\kappa_{\text{eff}}| & \text{otherwise.} \end{cases} \\
n' &= \begin{cases} \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = -N, \\ h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{F}) + |\kappa_{\text{eff}} + \sum_l I_3(\mathbf{F})| & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.46}$$

Plugging these values in (2.43), the range of  $d$  becomes

$$\begin{aligned}
0 \leq d &\leq N & \text{if } \kappa = -N + \frac{N_f}{2} \\
0 \leq d &\leq N - \frac{N_f}{2} - \kappa & \text{if } \kappa \in (-N + \frac{N_f}{2}, -\frac{N_f}{2}] \\
0 \leq d &\leq N & \text{if } \kappa \in [-\frac{N_f}{2}, +\frac{N_f}{2}] \\
\frac{N_f}{2} - \kappa \leq d &\leq N & \text{if } \kappa \in [\frac{N_f}{2}, N - \frac{N_f}{2}) \\
0 \leq d &\leq N & \text{if } \kappa = N - \frac{N_f}{2}
\end{aligned} \tag{2.47}$$

including  $0 \leq d \leq N$  in any case. Thus for  $N \geq 2$ , the recursion formula (2.36) determines the partition function  $\mathcal{Z}$  completely. In  $SU(N)_\kappa + N_f \mathbf{F} + 1\mathbf{AS}$  theory with  $N_f + 2|\kappa| \leq N + 4$ ,

$$\begin{aligned}
n &= \min \left( h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) - (\kappa_{\text{eff}} - 2), h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + 2\{\frac{\kappa_{\text{eff}}}{2}\} \right) \\
n' &= \min \left( h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + (\kappa_{\text{eff}} + \sum_l I_3(\mathbf{R}_l) + 2), \right. \\
&\quad \left. h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + 2\{-\frac{\kappa_{\text{eff}}}{2} + \frac{1}{2} \sum_l I_3(\mathbf{R}_l)\} \right),
\end{aligned} \tag{2.48}$$

from which one can identify the validity range of  $d$  by making substitution to (2.43). So long as three distinct integers are allowed for  $d$ , at a given  $(N, \kappa)$ , the corresponding partition function

$\mathcal{Z}$  can be solved from the recursion formula (2.36). We will also consider  $SU(6)_\kappa + 1\mathbf{TAS}$  theory with  $|\kappa| \leq 3$  in Section 3. This model can be Higgsed to two disjoint copies of  $SU(3)_\kappa$  theory without a bifundamental hypermultiplet [43]. At the level of the partition function, the Higgsing is realized by turning off  $m_{\text{phys}} = 0$  and imposing  $SU(3)$  traceless conditions. As neither of them can modify  $n$  and  $n'$ , the numerical value of  $(n, n')$  must be identical to that of  $SU(3)_\kappa$  gauge theory, which always contains  $0 \leq d \leq 3$ . Therefore, the recursion formula (2.36) sufficiently determines the full partition function  $\mathcal{Z}$  for  $SU(6)_\kappa + 1\mathbf{TAS}$  theory as well.

**Solving the recursion formulae** We now turn to solve the recursion formulae (2.36) for the one-instanton partition function  $Z_1$ . As long as there are at least 3 consecutive integers  $\{d_0, d_0 + 1, d_0 + 2\}$  for  $d$ , the solution  $Z_1$  to the three linear equations (2.37) is given by

$$Z_1(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) = \frac{p_1 p_2 I_1^{(d_0+2)} - (p_1 + p_2) I_1^{(d_0+1)} + I_1^{(d_0)}}{(1 - p_1)(1 - p_2)} \quad (2.49)$$

with  $I_1^{(d)}$  defined in (2.38). It turns out to be more convenient to express (2.49) by decomposing  $I_1^{(d)}$  into the flux sum, i.e.,  $I_1^{(d)} \equiv \sum_{\vec{k} \in \Delta_l} i_1^{(d)}(\vec{k})$ , where

$$i_1^{(d)}(\vec{k}) \equiv (p_1 p_2)^{(\frac{b}{2}-d)} e^{(d+\frac{\kappa_{\text{eff}}}{2})(\vec{a} \cdot \vec{k})} e^{-\frac{\kappa_{\text{eff}}}{2} d_{ijk} a^i k^j k^k} \quad (2.50)$$

$$\times \frac{\prod_l (y_l^{\text{phy}})^{-I_2(\mathbf{R}_l)/2} \prod_{\omega \in \mathbf{R}_l} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_{\text{phys}, l}, \epsilon_1, \epsilon_2)}{(1 - p_1 p_2 e^{-\vec{a} \cdot \vec{k}})(1 - e^{\vec{a} \cdot \vec{k}})(1 - p_1^{-1} e^{\vec{a} \cdot \vec{k}})(1 - p_2^{-1} e^{\vec{a} \cdot \vec{k}}) \prod_{\vec{\alpha} \cdot \vec{k} = -1} (1 - e^{-\vec{a} \cdot \vec{\alpha}})}.$$

Using the property  $i_1^{(d_0+\aleph)}(\vec{k})/i_1^{(d_0)}(\vec{k}) = (p_1 p_2)^{-\aleph} e^{\aleph(\vec{a} \cdot \vec{k})}$ , the above  $Z_1$  can be written as

$$Z_1 = \sum_{\vec{k} \in \Delta_l} \frac{(1 - p_1^{-1} e^{\vec{a} \cdot \vec{k}})(1 - p_2^{-1} e^{\vec{a} \cdot \vec{k}})}{(1 - p_1)(1 - p_2)} \cdot i_1^{(d_0)}(\vec{k}) \quad (2.51)$$

where the factor  $\prod_{\omega \in \mathbf{R}_l} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_l^{\text{phy}}, \epsilon_1, \epsilon_2)$  in (2.50) is explicitly written in (2.39) and (2.40). Notice that there are multiple options for choosing  $d_0$  in case of  $\frac{n+n'}{2} > 2$ . However, it was observed that (2.51) is independent of a specific choice of  $d_0$ . Once we choose  $d_0 = 0$ , for instance, which is an available choice in most cases,<sup>7</sup> (2.51) becomes

$$Z_1 = \frac{(p_1 p_2)^{\frac{b}{2}} \prod_l (y_l^{\text{phy}})^{-I_2(\mathbf{R}_l)/2}}{(1 - p_1)(1 - p_2)} \sum_{\vec{k} \in \Delta_l} \frac{e^{\frac{\kappa_{\text{eff}}}{2}(\vec{a} \cdot \vec{k}) - \frac{\kappa_{\text{eff}}}{2} d_{ijk} a^i k^j k^k} \prod_{\omega \in \mathbf{R}_l} \mathcal{L}_{\vec{k} \cdot \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_l^{\text{phy}}, \epsilon_{1,2})}{(1 - p_1 p_2 e^{-\vec{a} \cdot \vec{k}})(1 - e^{\vec{a} \cdot \vec{k}}) \prod_{\vec{\alpha} \cdot \vec{k} = -1} (1 - e^{-\vec{a} \cdot \vec{\alpha}})}. \quad (2.52)$$

We claim that (2.51) is the closed-form expression for the one-instanton partition function, which *universally* holds for any gauge theory having  $\frac{n+n'}{2} \geq 2$ .

More generally, the  $k$ -instanton correction  $Z_k$  can be determined by solving the following set of equations:

$$Z_k = p_1^{k(\frac{b}{2}-d)} Z_k^{(N)} + p_2^{k(\frac{b}{2}-d)} Z_k^{(S)} + I_k^{(d)} \quad \text{with an allowed range of } d, \quad (2.53)$$

---

<sup>7</sup>A numerical value of  $d_0$  should be a half-integer for theories with  $G = Sp(N)_{\theta=\pi}$ .

where  $I_k^{(d)}$  is defined as

$$I_k^{(d)} = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=k \\ \ell, m \neq k}} \left( (p_1 p_2)^{\left(\frac{h^\vee}{2} - \frac{\kappa_{\text{eff}}}{2} - d\right)\left(\frac{\vec{k}\cdot\vec{k}}{2}\right) + \frac{\kappa_{\text{eff}}}{6} d_{ijk} k^i k^j k^k} e^{(d + \frac{\kappa_{\text{eff}}}{2})(\vec{a}\cdot\vec{k})} e^{-\frac{\kappa_{\text{eff}}}{2} d_{ijk} a^i k^j k^k} \right. \\ \left. \times \frac{\prod_l y_l^{-I_2(\mathbf{R}_l)(\vec{k}\cdot\vec{k}/4)} \prod_{\omega \in \mathbf{R}_l} \mathcal{L}_{\vec{k}, \vec{\omega}}(\vec{a} \cdot \vec{\omega} + m_{\text{phy}, l}, \epsilon_1, \epsilon_2)}{\prod_{\alpha \in \Delta} \mathcal{L}_{\vec{k}, \vec{\alpha}}(\vec{a} \cdot \vec{\alpha}, \epsilon_1, \epsilon_2)} \cdot p_1^{(\frac{b}{2}-d)\ell} p_2^{(\frac{b}{2}-d)m} Z_\ell^{(N)}(\vec{k}) Z_m^{(S)}(\vec{k}) \right). \quad (2.54)$$

The  $k$ -instanton partition function  $Z_k$  is given as the solution to the three linear equations (2.53) with consecutive integers  $\{d_0, d_0 + 1, d_0 + 2\}$ .

$$Z_k(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) = \frac{p_1^k p_2^k I_k^{(d_0+2)} - (p_1^k + p_2^k) I_k^{(d_0+1)} + I_k^{(d_0)}}{(1 - p_1^k)(1 - p_2^k)} \quad (2.55)$$

Since  $I_k^{(d)}$  only involves low-order instanton corrections, the  $k$ -instanton partition function  $Z_k$  can be constructed from  $Z_{m < k}$ , allowing us to obtain the full non-perturbative part  $Z_{\text{inst}}$  in a recursive manner. Again, this approach is applicable for any gauge theory having  $\frac{n+n'}{2} \geq 2$ .

### 3 Examples

Recall that the recursion formula (2.36) for the  $k$ -instanton partition function, and especially the general expression (2.51) at one-instanton order, are widely applicable to 5d  $\mathcal{N} = 1$  gauge theory whose  $(n, n')$  satisfies  $\frac{n+n'}{2} \geq 2$ . Combined with the observation that  $(n, n')$  follows (2.44) in most cases, they actually become a very efficient approach at obtaining the BPS partition function on  $\mathbb{C}^2 \times S^1$ , unless a matter representation is too large.

Conventionally, the instanton partition function has been computed by employing the ADHM construction of the instanton moduli space [3, 4, 44] or by applying the topological vertex formalism to the 5-brane web [45, 46]. Both are based on a certain UV realization of 5d  $\mathcal{N} = 1$  gauge theory, motivated by its string theory engineering. Sometimes the UV completion of the gauge theory can be achieved only with specific string theory realizations, but not with the others. For instance,  $SU(2) + N_f \mathbf{F}$  gauge theory with  $N_f \geq 5$  must be embedded into D4-D8-O8 brane system, to be UV-completed as 5d  $E_{N_f+1}$  Minahan-Nemeschansky SCFT, instead of a typical  $(p, q)$  5-brane web with colliding branes which indicate UV inconsistency. A sensible UV QFT observable can thus be obtained only through a proper embedding of the gauge theory into string theory. At some occasions, an extra factor entangled with a true QFT observable may appear during the above instanton computation, which is sensitive to the choice of a string theory embedding.

Our blow-up formula (2.36) implicitly assumes a particular string theory embedding of the gauge theory. There are a wide variety of ‘exceptional’ gauge theories, whose UV completion is found as M-theory wrapped on a singular Calabi-Yau 3-fold [47–50], but for which neither the ADHM quantum mechanics nor the topological vertex formalism has been developed. We will illustrate that bootstrapping the partition function  $\mathcal{Z}$  based on the recursion formula (2.36) works well for those ‘exceptional’ theories, providing their BPS spectrum efficiently.

### 3.1 Theories with the known ADHM description

Let us first consider the ‘standard’ gauge theories with classical gauge groups, whose hypermultiplet admits UV realization as a perturbative string ending on non-perturbative branes. In these cases, the ADHM construction of the instanton moduli space is well-known [3, 41, 44]. As for the  $k$ -instanton partition function  $Z_k$ , the Witten index of the relevant ADHM quantum mechanics can be computed by SUSY localization [19, 51–53], ending up collecting all Jeffrey-Kirwan residues of a multi-dimensional contour integral. We will examine whether the recursion formula (2.36) actually produces the same result as the localization computation.

**SU(N)** The ADHM computation of the  $k$ -instanton partition function, for  $SU(N)_\kappa + N_f \mathbf{F}$  theories with  $N_f + 2|\kappa| \leq 2N$ , can be summarized as the Young tableaux sum expression.

$$Z_k^{\text{ADHM}} = \sum_{|\vec{Y}|=k} \prod_{i=1}^N \prod_{s \in Y_i} \frac{e^{-\kappa \phi(s)} \prod_{l=1}^{N_f} 2 \sinh \frac{\phi(s) + m_l}{2}}{\prod_{j=1}^N 2 \sinh \frac{E_{ij}}{2} 2 \sinh \frac{E_{ij} - 2\epsilon_+}{2}} \quad (3.1)$$

where

$$\begin{aligned} E_{ij}(s) &= a_i - a_j - \epsilon_1 h_i(s) + \epsilon_2 (v_j(s) + 1) \\ \varphi(s) &= a_i - \epsilon_+ - (n-1)\epsilon_1 - (m-1)\epsilon_2 \quad \text{for } s = (m, n) \in Y_i. \end{aligned}$$

$h_i(s)$  denotes the distance from  $s$  to the right end of the diagram  $Y_i$  by moving right.  $v_j(s)$  denotes the distance from  $s$  to the bottom of the diagram  $Y_j$  by moving down.

We checked that the instanton corrections  $Z_1$  and  $Z_2$  obtained from the recursion formulae (2.36) with (2.47) and the 1-instanton expression (2.51) precisely agree with the above  $Z_{k=1,2}^{\text{ADHM}}$  for  $N = 2, 3, 4$ . Recall that  $Z_k^{\text{ADHM}}$  often contains an additional factor  $Z_{\text{extra}}$  that captures the contribution from an extra branch of vacua. It is sensitive to the string theory embedding of gauge theory and regarded as spurious from the 5d QFT perspective. It is usually factorized from the true QFT partition function as

$$1 + \sum_{k=1}^{\infty} q^k Z_k^{\text{ADHM}}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) = Z_{\text{QFT}}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}, \Lambda) \cdot Z_{\text{extra}}(\epsilon_1, \epsilon_2, \vec{m}, \Lambda). \quad (3.2)$$

A non-trivial  $Z_{\text{extra}} \neq 1$  appears in the above expression (3.1) if and only if  $N_f + 2|\kappa| = 2N$ . This factor can be identified as the contribution of D1-branes escaping from D5-branes which engineer the  $SU(N)_\kappa + N_f \mathbf{F}$  gauge theory. Since  $Z_k = Z_k^{\text{ADHM}}$ , the same factor  $Z_{\text{extra}}$  emerges from the recursion formula (2.36) as well. The 5-brane web construction of the gauge theory is thus indirectly reflected in the recursion formula.

A similar observation is that the 1-instanton expression (2.51) applied to  $SU(2)_\kappa + N_f \mathbf{F}$  with  $N_f \geq 5$  does not match the Witten index of the D0-D4-D8-O8<sup>−</sup> quantum mechanics, which is the correct 1-instanton partition function.<sup>8</sup> Instead, it coincides with the topological vertex computation applied to the 5-brane web with a colliding pair of branes, which engineers the  $SU(2)$  gauge theory with  $N_f \geq 5$  in IR, but behaves badly in UV. Again, this suggests

<sup>8</sup>The case with  $SU(2) \simeq Sp(1)$  is an exception, which allows  $N_f \leq 7$  fundamental hypermultiplets [54].



that the recursion formula (2.36) is rooted in a specific string theory construction of the gauge theory, i.e., the web of  $(p, q)$  5-branes. It would be more satisfactory if one can trace at which step in our formulation of (2.36) the particular UV embedding of the gauge theory has been chosen. In fact, there must be a broader set of blow-up relations for the 5d Nekrasov partition function  $\mathcal{Z}$ , similar to those recently found for 6d SCFTs [32, 33, 55, 56]. We expect that some recursion formulae, derived from the generalized blow-up equations, may realize a different string theory embedding of the gauge theory. It would be very interesting if one can reveal the connection between the choice of UV embedding and the blow-up equations.

In  $SU(N)_\kappa + N_f \mathbf{F} + 1 \mathbf{AS}$  theory with  $N_f + 2|\kappa| \leq N + 4$ , the ADHM quantum mechanics is the worldvolume theory of D1-branes, probing the D5-NS5-D7-O7<sup>-</sup> brane configuration that realizes the gauge theory. Let us compute the Witten index for 1 and 2 D1-branes, then compare with the blow-up computation based on the recursion formula (2.36). For instance, the Witten index for the single D1-brane can be written as

$$Z_1^{\text{ADHM}} = - \sum_{i=1}^N \frac{e^{-\kappa(a_i - \epsilon_+)}}{2 \sinh \frac{\epsilon_1}{2} 2 \sinh \frac{\epsilon_2}{2}} \frac{\prod_{l=1}^{N_f} 2 \sinh \frac{-\epsilon_+ + a_i + m_l}{2}}{2 \sinh \frac{-3\epsilon_+ + 2a_i + m_a}{2}} \prod_{j \neq i} \frac{2 \sinh \frac{a_i + a_j + m_a - \epsilon_+}{2}}{2 \sinh \frac{a_i - a_j}{2} 2 \sinh \frac{2\epsilon_+ - a_i + a_j}{2}} \\ - \frac{1}{2} \frac{e^{-\frac{\kappa}{2}(\epsilon_+ - m_a)}}{2 \sinh \frac{\epsilon_1}{2} 2 \sinh \frac{\epsilon_2}{2}} \left( \frac{\prod_{l=1}^{N_f} 2 \sinh \frac{\epsilon_+ + 2m_l - m_a}{4}}{\prod_{i=1}^N 2 \sinh \frac{3\epsilon_+ - M - 2a_i}{4}} - (-1)^{\kappa + (N - N_f)/2} \frac{\prod_{l=1}^{N_f} 2 \cosh \frac{\epsilon_+ + 2m_l - m_a}{4}}{\prod_{i=1}^N 2 \cosh \frac{3\epsilon_+ - M - 2a_i}{4}} \right). \quad (3.3)$$

Note that  $Z_k^{\text{ADHM}}$  captures an extra factor  $Z_{\text{extra}} \neq 1$  if  $N_f + 2|\kappa| = N + 4$ , displaying the BPS spectrum of D1-branes escaping from the D5-branes on which the gauge theory is supported. The appearance of  $Z_{\text{extra}} \neq 1$  is an artifact of the string theory embedding, spurious from the 5d QFT perspective. We checked the agreement between  $Z_1^{\text{ADHM}}$  and the 1-instanton formula (2.51) for the  $SU(3)$ ,  $SU(4)$ ,  $SU(5)$  theories whose  $(n, n')$  satisfies  $\frac{n+n'}{2} \geq 2$ . We confirmed  $Z_2 = Z_2^{\text{ADHM}}$  as well, where  $Z_2$  is the solution of the recursion formulae (2.36) with (2.48). The same spurious factor  $Z_{\text{extra}}$  arises from the recursion formula, implying that our blowup equations are implicitly based on the D5-NS5-D7-O7<sup>-</sup> brane realization of the gauge theory.<sup>9</sup>

**Sp(N)** The  $k$ -instanton partition function for  $Sp(N)_\theta + N_f \mathbf{F}$  theory with  $N_f \leq 2N + 4$  can be computed from the ADHM quantum mechanics of D1-D5-NS5-O5 branes, which engineers the gauge theory and its instantons. The Witten index for a D1-brane can be written as

$$Z_1^{\text{ADHM}} = \frac{1}{2} \frac{1}{2 \sinh \frac{\epsilon_1}{2} 2 \sinh \frac{\epsilon_2}{2}} \left( \frac{\prod_{l=1}^{N_f} 2 \sinh \frac{m_l}{2}}{\prod_{i=1}^N 2 \sinh \frac{\epsilon_+ \pm a_i}{2}} + e^{i\theta} \frac{\prod_{l=1}^{N_f} 2 \cosh \frac{m_l}{2}}{\prod_{i=1}^N 2 \cosh \frac{\epsilon_+ \pm a_i}{2}} \right). \quad (3.4)$$

Our 1-instanton formula (2.51) was checked to be in agreement with  $Z_1^{\text{ADHM}}$  for the  $Sp(2)$  and  $Sp(3)$  gauge theories satisfying  $N - 1 \geq \lfloor \frac{N_f}{2} \rfloor$  (at  $\theta = 0$ ) and  $N \geq \lceil \frac{N_f}{2} \rceil$  (at  $\theta = \pi$ ). It was also confirmed that  $Z_2^{\text{ADHM}} = Z_2$ , where  $Z_2$  is the solution of the recursion formulae (2.36) with (2.45). Note that  $Z_{\text{extra}}$  does not appear in  $Z_k^{\text{ADHM}} = Z_k$  for the above theories.

<sup>9</sup>An exceptional case is the  $SU(2)$  gauge theory, in which the antisymmetric hypermultiplet decouples and never affects the recursion formula. The corresponding  $Z_k$  is the same as the Young tableau formula (3.1).

In  $Sp(N)_\theta + N_f \mathbf{F} + 1 \mathbf{AS}$  theory with  $N_f \leq 7$ , the relevant ADHM quantum mechanics is the worldvolume gauge theory of D0-branes which probe the D4-D8-O8 brane configuration. The QFT on D4-branes sees an enhanced  $E_{N_f+1}$  flavor symmetry at the UV fixed point [54]. Let us consider the Witten index for one and two D0-branes [19, 57]. For a single D0-brane,

$$Z_1^{\text{ADHM}} = \frac{1}{2} \frac{1}{2 \sinh \frac{\epsilon_1}{2} 2 \sinh \frac{\epsilon_2}{2} 2 \sinh \frac{m_a + \epsilon_+}{2} 2 \sinh \frac{m_a - \epsilon_+}{2}} \quad (3.5)$$

$$\times \left( \frac{\prod_{i=1}^N 2 \sinh \frac{m_a \pm a_i}{2} \prod_{l=1}^{N_f} 2 \sinh \frac{m_l}{2}}{\prod_{i=1}^N 2 \sinh \frac{\epsilon_+ \pm a_i}{2}} + \frac{e^{i\theta} \prod_{i=1}^N 2 \cosh \frac{m_a \pm a_i}{2} \prod_{l=1}^{N_f} 2 \cosh \frac{m_l}{2}}{\prod_{i=1}^N 2 \cosh \frac{\epsilon_+ \pm a_i}{2}} \right).$$

We checked  $Z_1^{\text{ADHM}}$  itself is *not* the same as the 1-instanton expression (2.51) for the  $Sp(2)_\theta$ ,  $Sp(3)_\theta$  theories with  $N_f \leq 1$  (at  $\theta = 0$ ) and  $N_f \leq 2$  (at  $\theta = \pi$ ). Instead, the difference between  $Z_1$  and  $Z_1^{\text{ADHM}}$  can be identified as the BPS index of D0-branes moving away from the D4-D8-O8 brane system [19, 57]. Similarly, it was confirmed that the 2-instanton correction  $Z_2$  captures the same 5d QFT spectrum as in  $Z_2^{\text{ADHM}}$ , while excluding the spurious contribution of escaping D0-branes. Our blow-up formulae seem to reflect a different UV embedding of the gauge theory, which does *not* allow an extra branch of vacua leading to  $Z_{\text{extra}} \neq 1$ .

**SO(N)** One can compute the instanton partition function of  $SO(N) + N_v \mathbf{V}$  theory with  $N_v \leq N - 4$  using the ADHM quantum mechanics of the D1-D5-NS5-O5 brane system. At even  $N$ , the Witten index for a D1-brane can be written as

$$Z_1^{\text{ADHM}} = \sum_{i=1}^{N/2} \left( \frac{2 \sinh(2\epsilon_+ - a_i) 2 \sinh(a_i - \epsilon_+) \prod_{l=1}^{N_v} 2 \sinh \frac{m_l \pm (a_i - \epsilon_+)}{2}}{2 \cdot 2 \sinh \frac{\epsilon_1}{2} 2 \sinh \frac{\epsilon_2}{2} \prod_{j \neq i} 2 \sinh \frac{a_i \pm a_j}{2} 2 \sinh \frac{2\epsilon_+ - a_i \pm a_j}{2}} + (a_i \rightarrow -a_i) \right). \quad (3.6)$$

At odd  $N$ ,

$$Z_1^{\text{ADHM}} = \sum_{i=1}^{\lfloor N/2 \rfloor} \left( \frac{2 \cosh \frac{2\epsilon_+ - a_i}{2} 2 \sinh(a_i - \epsilon_+) \prod_{l=1}^{N_f} 2 \sinh \frac{m_l \pm (a_i - \epsilon_+)}{2}}{2 \cdot 2 \sinh \frac{\epsilon_1}{2} 2 \sinh \frac{\epsilon_2}{2} 2 \sinh \frac{a_i}{2} \prod_{j \neq i} 2 \sinh \frac{a_i \pm a_j}{2} 2 \sinh \frac{2\epsilon_+ - a_i \pm a_j}{2}} + (a_i \rightarrow -a_i) \right). \quad (3.7)$$

The general 1-instanton expression (2.51) and the recursion formula (2.36) are applicable for all  $N_v \leq N - 4$ . We explicitly confirmed that  $Z_k^{\text{ADHM}} = Z_k$ , where  $Z_1$  was written in (2.51) and  $Z_2$  was obtained as the solution of the recursion formula (2.36), for  $4 \leq N \leq 9$ . Notice that  $Z_k^{\text{ADHM}} = Z_k$  involves a non-trivial extra factor  $Z_{\text{extra}} \neq 1$  when  $N_v = N - 4$ . This extra factor is associated to D1-branes moving away from the D5-NS5-O5 brane system, where the 5d QFT lives. It implies that the specific UV realization of the gauge theory, i.e., IIB string theory with D1-D5-NS5-O5, is implicit in our recursion formulae (2.36) with (2.45).

### 3.2 Theories with spinor hypermultiplets

So far, we have investigated the ‘standard’ gauge theories which have certain D-brane set-ups in IIA/IIB string theory to realize themselves and their non-perturbative solitons. Once there

are a sufficient number of the blowup equations, the  $k$ -instanton partition function  $Z_k$  can be obtained as the solution. Solving specifically at 1-instanton level,  $Z_1$  is expressed as (2.51). All these results were found to be in agreement with the ADHM instanton calculus, sometimes modulo an extra factor  $Z_{\text{extra}}$  sensitive to the string theory embedding of the gauge theory.

There are also ‘exceptional’ gauge theories, which involve exceptional gauge groups or matters that cannot be engineered from perturbative open strings ending on non-perturbative branes. Instead, their UV embedding is geometrically realized as M-theory wrapped on local Calabi-Yau threefolds. Since the exceptional theories lack for the ADHM formalism, their instanton partition function  $\mathcal{Z}$  has been studied on a case-by-case basis; Once the 5-brane web configuration engineering an exceptional theory is identified, the topological vertex formalism can be applied to compute the relevant partition function  $\mathcal{Z}$ . Alternatively, one can first construct the  $\mathbb{C}^2 \times T^2$  partition function for a related 6d gauge theory, based on its modularity and anomaly, then take the circle reduction to obtain the 5d partition function  $\mathcal{Z}$ . Several interesting exceptional theories have been studied so far, based on the above two approaches.

We can take advantage of the universality of the blowup equation. Recall that the blow-up recursion formula (2.36) holds for a certain range of  $d$ , i.e., the set of all integers between  $0 \leq d \leq h^\vee - \frac{1}{2} \sum_l I(\mathbf{R}_l)$ , when the gauge group  $G$  is neither  $SU(N)_\kappa$  nor  $Sp(N)_\theta$  so that an extra complication due to the Chern-Simons level  $\kappa$  or the theta angle  $\theta$  cannot occur. One can solve the recursion formulae for the  $k$ -instanton correction  $Z_k$  to the partition function, as long as  $h^\vee - \frac{1}{2} \sum_l I(\mathbf{R}_l) \geq 2$ , even for the exceptional gauge theories. We assume that  $Z_k$  solved from the recursion formulae would be the correct BPS data for UV-consistent 5d SCFTs, modulo an extra factor  $Z_{\text{extra}}$  independent of the Coulomb VEV  $\vec{a}$ . This conjecture will be tested through comparison with [43, 58, 59] which compute  $\mathcal{Z}$  for certain exceptional theories. We will show the novel results for the  $SO(N)$  gauge theories with spinor hypermultiplets in Section 3.2, for the gauge theories with exceptional groups in Section 3.3, and for the  $SU(6)$  gauge theories with rank-3 antisymmetric hypermultiplets in Section 3.4. (See Table ???)

There are a sufficient number of recursion formulae (2.36) to fix the  $k$ -instanton partition function  $Z_k$  of the  $SO(N)$  gauge theory, if and only if

$$\begin{aligned} N - 4 &\geq n_{\mathbf{v}} + 2^{\frac{N-7}{2}} \cdot n_{\mathbf{s}} && \text{for odd } N \\ N - 4 &\geq n_{\mathbf{v}} + 2^{\frac{N-8}{2}} \cdot (n_{\mathbf{s}} + n_{\mathbf{c}}) && \text{for even } N \end{aligned} \tag{3.8}$$

where  $n_{\mathbf{v}}$ ,  $n_{\mathbf{s}}$ , and  $n_{\mathbf{c}}$  denote the number of hypermultiplets in the vector, spinor, and conjugate spinor representations, respectively.

**SO(7)** condition  $h^\vee - \frac{1}{2} \sum_l I(\mathbf{R}_l) \geq 2$  is equivalent to

Our 1-instanton expression (2.51) and the recursion formulae (??) is valid for

**SO(8)**

**SO(9)**

**SO(10)**

**SO(11)**

**SO(12)**

**SO(13)**

**SO(14)** As a remarkable application of the blow-up equation, we construct the instanton partition function of various 5d gauge theories on  $\mathbb{C}^2 \times S^1$ . The standard method to compute the multi-instanton correction to the partition function is to , and/or to apply the topological vertex formalism . An alternative approach that comes from the consistency requirement for the blow-up  $\hat{\mathbb{C}}^2$  will turn out to be very efficient for bootstrapping the  $\mathbb{C}^2 \times S^1$  partition function of exceptional gauge theories [16, 32, 56].

We propose a general blow-up formula for general gauge theory with arbitrary representations, under the condition that the matter representation is not too large.

### 3.3 Theories with an exceptional gauge group

### 3.4 Theories with a rank-3 antisymmetric hypermultiplet

especially for ‘exceptional’ gauge theories

There are multiple and distinct blow-up equations (??) at different values of  $d$ , ranged over  $0 \leq d \leq d_{\max}$  where  $d_{\max} \equiv h^\vee - \frac{1}{2} \sum_i I(\mathbf{R}_i)$ . We need at least three distinct equations to solve for the instanton partition function, thus the bootstrap method is applicable only for gauge theories which satisfy  $d_{\max} \geq 2$ . Here we constrain our discussion to the theories having simple gauge groups, which are classified to have a 5d UV fixed point [42].

There are two

We will

The complete list of  $\mathcal{N} = 1$  gauge theories, satisfying  $d_{\max} \geq 2$ , is displayed in Table 3 and 4. All infinite families of theories, whose rank of the gauge group can be arbitrarily large, are summarized in Table 3. There are also finite families of theories, summarized in Table 4, which involve an exceptional gauge group or a spinor hypermultiplet.

(comparison with IMS bound, etc) (SCFT bound) Some theories in Table 4 do not belong to the

$SU(6)_3 + 1 \times \mathbf{20}$  As a non-trivial test, we consider the instanton partition function of the  $SU(6)_3 + \mathbf{20}$  whose 5-brane realization was found recently [43]. Its web-diagram is given as figure.

(Written before computing the  $SU(6)_3 + \mathbf{20}$  instanton partition function.)

Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the  $SU(3) \times SU(3) \times U(1) \subset SU(6)$  where the  $SU(6)$  multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}} : \mathbf{35} &\longrightarrow (\mathbf{8}, 1)_0 \oplus (1, \mathbf{8})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_2 \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-2} \oplus (1, 1)_0, \\ \Phi_{ijk} : \mathbf{20} &\longrightarrow (\mathbf{3}, \bar{\mathbf{3}})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{3})_1 \oplus (1, 1)_3 \oplus (1, 1)_{-3}. \end{aligned} \tag{3.9}$$

Here to fit with the web-diagram, we set  $\Phi_{156}$  and  $\Phi_{234}$  are  $(1, 1)_3$  and  $(1, 1)_{-3}$ . Once  $\Phi_{156}$  and  $\Phi_{234}$  get non-zero VEVs,

When  $a_5 = -a_1 - a_6$ , the web-diagram factorizes to two  $SU(3)_3$  whose Coulomb VEVs are  $(a_1, a_5, a_6)$  and  $(a_2, a_3, a_4)$ . In the gauge theory, it can be seen partly from prepotential. The prepotential of  $S(6)_3 + 1 \times \mathbf{20}$  is

$$\mathcal{F} = \frac{1}{2}m_0 \sum_{i=1}^6 a_i^2 + \frac{1}{2} \sum_{i=1}^6 a_i^3 + \frac{1}{6} \sum_{i < j} (a_i - a_j)^3 - \frac{1}{6} \sum_{1 < i < j} (a_1 + a_j + a_k)^3 \quad (3.10)$$

at the Weyl chamber  $a_1 > \dots > a_6$ . As one sets the Coulomb VEV  $a_6 = -a_1 - a_5$  and  $a_4 = -a_2 - a_3$ , one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \quad (3.11)$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2}m_0 \sum_{i=1}^3 a_i^2 + \frac{1}{2} \sum_{i=1}^3 a_i^3 + \frac{1}{6} \sum_{i < j} (a_i - a_j)^3. \quad (3.12)$$

It is Higgsed by

It was recently found in [43] that the 5d  $SU(6)_3$  gauge theory with a rank-3 antisymmetric hypermultiplet can be engineered from the 5-brane web configuration, depicted in Figure 1. Given a web diagram, we utilize the topological vertex method [45, 46] to compute all genus topological string amplitudes, which is the logarithm of the 5d Nekrasov partition function on  $\Omega$ -deformed  $\mathbf{R}^4 \times S^1$  [60]. We will check its agreement with the blowup partition function (eqn), providing a supporting evidence to the suggested blow-up equation (eqn).

By applying the topological vertex method to Figure 1, we find that the instanton partition function can be written as the following sum over all possible 6 Young diagrams:

$$\begin{aligned} Z_{\text{Nek}} = & \sum_{(Y_1, \dots, Y_6)} q^{\sum_{i=1}^6 |Y_i|} (-A_1^6)^{|Y_1|} (-A_2^6)^{|Y_2|} (-A_2^2 A_3^4)^{|Y_3|} (-A_2^2 A_3^2 A_4^2)^{|Y_4| + |Y_5|} \\ & \times f_{Y_1}(g)^5 f_{Y_2}(g)^5 f_{Y_3}(g)^3 f_{Y_4}(g) f_{Y_5}(g)^{-1} f_{Y_6}(g)^2 Z_{\text{half}}(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)^2. \end{aligned} \quad (3.13)$$

where the Kähler parameters can be identified as

$$A_i = e^{-a_i}, \quad q = e^{-\frac{8\pi^2}{g^2}}, \quad g = e^{-\epsilon_-}. \quad (3.14)$$

We briefly explain our notation: For a given Young diagram  $\mu$ ,  $|\mu|$  denotes the total number of boxes.  $\mu_i$  is the number of boxes in the  $i$ -th row of  $\mu$ .  $\mu^t$  is the transpose of  $\mu$ . We also use

$$f_\mu(g) = (-1)^{|\mu|} g^{\frac{1}{2}(\|\mu^t\|^2 - \|\mu\|^2)}, \quad \tilde{Z}_\lambda(g) = \prod_{(i,j) \in \lambda} (1 - g^{\lambda_i + \lambda_j^t - i - j + 1})^{-1} \quad (3.15)$$

with  $\|\mu\|^2 = \sum_i \mu_i^2$ . The factor  $Z_{\text{half}}(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$  involves a single summation over all possible Young diagrams, i.e.,

$$\begin{aligned} Z_{\text{left}}(\vec{Y}) = & \sum_{Y_0} (-A_1^{-1} A_6^{-2})^{|Y_0|} g^{\frac{\|Y_0^t\|^2 + \|Y_0\|^2}{2}} \tilde{Z}_{Y_0}(g)^2 f_{Y_0}(g)^2 \prod_{i=1}^6 g^{\frac{\|Y_i\|^2}{2}} \tilde{Z}_{Y_i}(g) \\ & \times R_{Y_1 Y_6^t}^{-1}(A_1 A_6^{-1}) R_{Y_0 Y_6^t}^{-1}(A_1^{-1} A_6^{-2}) R_{Y_1 Y_0^t}^{-1}(A_1^2 A_6) \\ & \times \prod_{2 \leq i < j \leq 5} R_{Y_i Y_j^t}^{-1}(A_i A_j^{-1}) \prod_{i=2}^5 R_{Y_0^t Y_i}(A_1 A_i A_6), \end{aligned} \quad (3.16)$$

in which we introduce

$$R_{\lambda\mu}(Q) = R_{\mu\lambda}(Q) = \text{PE} \left[ -\frac{g}{(1-g)^2} Q \right] \times N_{\lambda^t \mu}(Q) \quad (3.17)$$

with PE representing the Plethystic exponential and

$$N_{\lambda\mu}(Q) = \prod_{(i,j) \in \lambda} \left( 1 - Q g^{\lambda_i + \mu_j^t - i - j + 1} \right) \prod_{(i,j) \in \mu} \left( 1 - Q g^{-\lambda_j^t - \mu_i + i + j - 1} \right). \quad (3.18)$$

We also recall that the Nekrasov partition function is divided into the perturbative partition function  $Z_{\text{pert}}$  and the weighted sum of  $k$ -instanton partition function  $Z_k$ .

$$Z_{\text{Nek}} = Z_{\text{pert}} \left( 1 + \sum_{k=1}^{\infty} q^k Z_k \right). \quad (3.19)$$

The perturbative part of the partition function  $Z_{\text{pert}}$  comes from the summand of (3.13) at empty Young diagrams, i.e.,  $(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$ . It is given by

$$\begin{aligned} Z_{\text{pert}} = & Z_{\text{half}}(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)^2 \\ = & \text{PE} \left[ \frac{2g}{(1-g)^2} \left( \frac{A_1}{A_6} + \frac{1}{A_1 A_6^2} + A_1^2 A_6 + \sum_{2 \leq i < j \leq 5} \frac{A_i}{A_j} - \sum_{i=2}^5 A_1 A_i A_6 \right) \right] \\ & \times \left( \sum_{Y_0} (-A_1^{-1} A_6^{-2})^{|Y_0|} g^{\frac{\|Y_0^t\|^2 + \|Y_0\|^2}{2}} \tilde{Z}_{Y_0}(g)^2 f_{Y_0}^2(g) \right. \\ & \left. N_{Y_0^t \emptyset}^{-1}(A_1^{-1} A_6^{-2}) N_{Y_0 \emptyset}^{-1}(A_1^2 A_6) \prod_{i=2}^5 N_{Y_0 \emptyset}(A_1 A_i A_6) \right)^2, \end{aligned} \quad (3.20)$$

where the last two lines can be combined into the following closed-form expression:

$$\text{PE} \left[ \frac{2g}{(1-g)^2} \left( \sum_{i=2}^5 \frac{A_1}{A_i} + \sum_{i=2}^5 \frac{A_i}{A_6} - \frac{1}{A_1 A_6^2} - A_1^2 A_6 - \sum_{2 \leq i < j \leq 5} A_1 A_i A_j + \mathcal{O}(A_1^6) \right) \right]. \quad (3.21)$$

So (3.20) is manifestly consistent with the equivariant index [41] for 5d  $SU(6)$  gauge theory with a hypermultiplet in the rank-3 antisymmetric representation **20**, i.e.,

$$Z_{\text{pert}} = \text{PE} \left[ \frac{2g}{(1-g)^2} \left( \sum_{1 \leq i < j \leq 6} \frac{A_i}{A_j} - \sum_{2 \leq i < j \leq 6} A_1 A_i A_j + \mathcal{O}(A_1^6) \right) \right]. \quad (3.22)$$

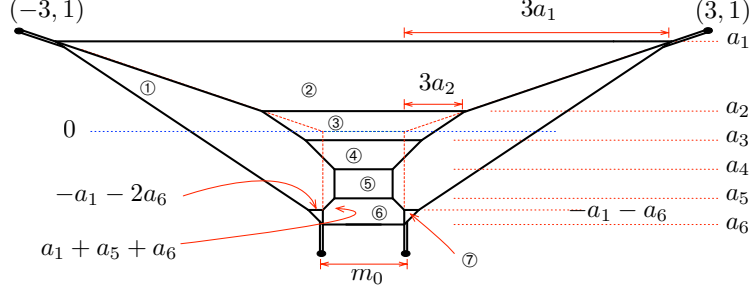
The 1-instanton partition function  $Z_1$  can be obtained from the summands of (3.13) at Young diagrams satisfying  $\sum_{i=1}^6 |Y_i| = 1$ . There are 6 different profiles of Young diagrams. The configuration  $|Y_i| = 1$  and  $Y_{j \neq i} = \emptyset$  contributes to  $Z_1$  by

$$+\frac{g}{(1-g)^2} \frac{A_i^6}{\prod_{j \neq i} (A_i - A_j)^2} \left( -A_i \sum_{j \neq i} A_j + \sum_{j \neq i} \frac{1}{A_j} - \frac{1}{A_i} + A_i^2 \right)^2. \quad (3.23)$$

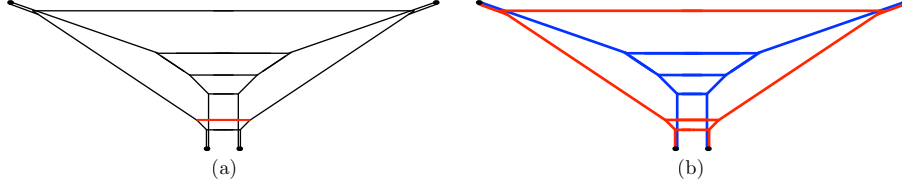
Summing over all six contributions, we find

$$Z_1 = \sum_{i=1}^6 \frac{g}{(1-g)^2} \frac{A_i^6}{\prod_{j \neq i} (A_i - A_j)^2} \left( -A_i \sum_{j \neq i} A_j + \sum_{j \neq i} \frac{1}{A_j} - \frac{1}{A_i} + A_i^2 \right)^2. \quad (3.24)$$

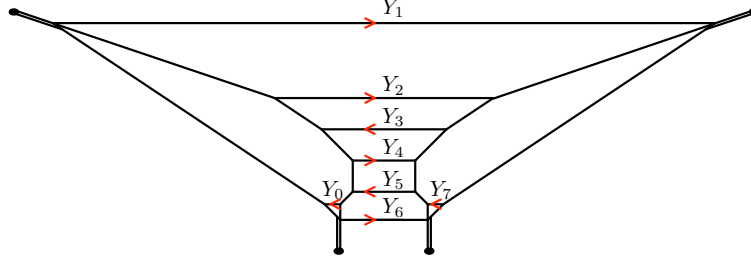
which is in agreement with the blowup partition function (eqn). (overall sign: looking at the GV invariant for single W-boson + single instanton (which should be -2), the topological vertex computation seems to be correct. blowup should have an overall sign issue)



**Figure 1:** A 5-brane web for  $SU(6)_3 + 1\text{TAS}$  with massless **TAS**.



**Figure 2:** (a) A Higgsing from  $SU(6)_3 + 1\text{TAS}$  to two  $SU(3)_3$  theories. (a) Two  $SU(3)_3$  theories are painted in blue and in pink respectively. A new decoupled mode emerges.



**Figure 3:** A labeling of Young diagrams assigned to the horizontal lines in Figure 1.

## 4 Conclusion

### Acknowledgments

This work is supported in part by the UESTC Research Grant A03017023801317 (SSK), the National Research Foundation of Korea (NRF) Grants 2017R1D1A1B06034369 (KL, JS), and 2018R1A2B6004914 (KHL)

### A $SU(N)$ theory

We find that

$$\begin{cases} n = \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = N - N_f, \\ n = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + |\kappa_{\text{eff}}| & \text{otherwise.} \end{cases} \quad (\text{A.1})$$



for  $SU(N)_\kappa + N_f \mathbf{F}$  theory with  $N_f + 2|\kappa| \leq 2N$ ,

$$\begin{cases} n = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) - (\kappa_{\text{eff}} - 2) & \text{if } N: \text{ odd and } \kappa_{\text{eff}} \geq 3 - \frac{N_f}{2}, \\ n = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) - (\kappa_{\text{eff}} - 2) & \text{if } N: \text{ even and } \kappa_{\text{eff}} > 3 - \frac{N_f}{2}, \\ n \geq h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + 2\{\frac{\kappa_{\text{eff}}}{2}\} & \text{otherwise.} \end{cases} \quad (\text{A.2})$$

for  $SU(N)_\kappa + N_f \mathbf{F} + 1\mathbf{A}$  theory with  $N_f + 2|\kappa| \leq N + 4$ .

$$\begin{cases} n' = \frac{N_f}{2} & \text{if } \kappa_{\text{eff}} = -N, \\ n' = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + |\kappa + \frac{1}{2} \sum_l I_2(\mathbf{R}_l)| & \text{otherwise.} \end{cases}$$

for  $SU(N)_\kappa + N_f \mathbf{F}$  theory with  $N_f + 2|\kappa| \leq 2N$ ,

$$\begin{cases} n' = h^\vee + \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + \kappa_{\text{eff}} & \text{if } N: \text{ odd and } \kappa_{\text{eff}} \leq 1 - N - \frac{N_f}{2}, \\ n' = h^\vee + \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + \kappa_{\text{eff}} & \text{if } N: \text{ even and } \kappa_{\text{eff}} < 1 - N - \frac{N_f}{2}, \\ n' \geq h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) + 2\{-\frac{\kappa_{\text{eff}} + N + N_f}{2}\} & \text{otherwise.} \end{cases} \quad (\text{A.3})$$

for  $SU(N)_\kappa + N_f \mathbf{F} + 1\mathbf{A}$  theory with  $N_f + 2|\kappa| \leq N + 4$ .

i.e., and

In other words, This suggests that, as long as  $\frac{b+b'}{2} > 2$ , we can solve for the instanton partition function  $Z_n$

In summary, there are  $(b+b')/2$  blowup equations

we have found  $n = n' = b = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l)$ . Thus the range of  $d$  is constrained to

$$0 \leq d \leq h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) \quad \text{for the theories with } G \neq SU(N). \quad (\text{A.4})$$

See Appendix A for the pattern on  $n$  in two classes of

Especially when  $\kappa_{\text{eff}} = 0$ , the value of  $n$  is same as the above.

where we observe that

$$n' = h^\vee - \frac{1}{2} \sum_l I_2(\mathbf{R}_l) \quad (\text{A.5})$$

for the theories with  $G \neq SU(N)$ ,

We need to be a bit careful when dealing with  $SU(N)_\kappa$  theories. For  $SU(N)_\kappa + N_f \mathbf{F}$  theory, the validity range of  $d$  becomes

$$\begin{cases} d \in [0, N] & \text{for } \kappa = -N + \frac{N_f}{2} \\ d \in [0, N - \frac{N_f}{2} - \kappa] & \text{for } \kappa \in (-N + \frac{N_f}{2}, -\frac{N_f}{2}] \\ d \in [0, N] & \text{for } \kappa \in [-\frac{N_f}{2}, +\frac{N_f}{2}] \\ d \in [\frac{N_f}{2} - \kappa, N] & \text{for } \kappa \in [+ \frac{N_f}{2}, N - \frac{N_f}{2}) \\ d \in [0, N] & \text{for } \kappa = +N - \frac{N_f}{2} \end{cases} \quad (\text{A.6})$$

which always includes  $0 \leq d \leq N$ . For  $SU(N)_\kappa + N_f \mathbf{F} + 1\mathbf{A}$  theory,

Substituting  $(n, n')$  to (2.43),  $d$  takes a value in the following range:

$$\begin{cases} d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N] & \text{for } \kappa \in [-2 - \frac{N}{2} + \frac{N_f}{2}, 1 - N - \frac{N_f}{2}] \\ d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for } \kappa \in (1 - N - \frac{N_f}{2}, 3 - \frac{N_f}{2}) \\ d \in [-1, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for } \kappa \in [3 - \frac{N_f}{2}, +\frac{N_f}{2}, 2 + \frac{N}{2} - \frac{N_f}{2}] \end{cases} \quad (\text{A.7})$$

with odd  $N$ , and

$$\begin{cases} d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N] & \text{for } \kappa \in [-2 - \frac{N}{2} + \frac{N_f}{2}, 1 - N - \frac{N_f}{2}] \\ d \in [-1 - \frac{a}{4} - \{\frac{a}{4}\}, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for } \kappa \in (1 - N - \frac{N_f}{2}, 3 - \frac{N_f}{2}) \\ d \in [-1, N - \frac{b}{2} - \{-\frac{b}{2}\}] & \text{for } \kappa \in [3 - \frac{N_f}{2}, +\frac{N_f}{2}, 2 + \frac{N}{2} - \frac{N_f}{2}] \end{cases} \quad (\text{A.8})$$

with even  $N$ , where  $a$  and  $b$  are defined by  $a = 2\kappa - N - N_f$  and  $b = 2\kappa + N + N_f$ . Although these patterns look

$$-\frac{\kappa_{\text{eff}}}{2} - \{\frac{\kappa_{\text{eff}}}{2}\} \leq d \leq N \quad \text{for } -N \leq \kappa_{\text{eff}} \leq 1 - N - \frac{N_f}{2} \quad (\text{A.9})$$

$$-\frac{\kappa_{\text{eff}}}{2} - \left\{ \frac{\kappa_{\text{eff}}}{2} \right\} \leq d \leq N + 1 - \frac{\kappa_{\text{eff}} + N + N_f}{2} + \left\{ -\frac{\kappa_{\text{eff}} + N + N_f}{2} \right\} \quad (\text{A.10})$$

$$-1 \leq d \leq N + 1 - \frac{\kappa_{\text{eff}} + N + N_f}{2} + \left\{ -\frac{\kappa_{\text{eff}} + N + N_f}{2} \right\} \quad (\text{A.11})$$

For  $SU(N)_{|\kappa| < N}$  SYM without a hypermultiplet,  $n = h^\vee + |\kappa|$  so that [28]

$$-\frac{\kappa + |\kappa|}{2} \leq d \leq h^\vee + \frac{|\kappa| - \kappa}{2}. \quad (\text{A.12})$$

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$G$	matter	$r_0$	$d_{\max}$
$SU(N)_\kappa$	$N_f \times \mathbf{N}$	$d - N/2 - \kappa/2$	$0 \leq d \leq N -  \kappa  - 2N_f - 1(?)$
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \leq d \leq 6$
$SO(7)$	pure	$d - 5/2$	$0 \leq d \leq 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \leq d \leq 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \leq d \leq 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \leq d \leq 3$
$G_2$	pure	$d - 4/2$	$0 \leq d \leq 4$
$G_2$	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \leq d \leq 3$
$F_4$	pure	$d - 9/2$	$0 \leq d \leq 9$
$F_4$	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \leq d \leq 6$
$F_4$	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \leq d \leq 3$

**Table 1:** list of theories

Gauge group	Hypermultiplets
$SO(14)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 2$
$SO(13)$	$n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 2$
$SO(12)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 2$
$SO(12)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(11)$	$n_{\mathbf{s}} = 2$
$SO(11)$	$n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(10)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 3$
$SO(10)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 2$ and $n_{\mathbf{v}} \leq 2$
$SO(10)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(9)$	$n_{\mathbf{s}} = 3$
$SO(9)$	$n_{\mathbf{s}} = 2$ and $n_{\mathbf{v}} \leq 2$
$SO(9)$	$n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(8)$	$n_{\mathbf{s}} + n_{\mathbf{c}} + n_{\mathbf{v}} = 4$
$SO(7)$	$n_{\mathbf{s}} + n_{\mathbf{v}} = 3$

**Table 2:** asd

Gauge group	Hypermultiplets
$SU(N)_\kappa$	x
$SO(2N+1)$	$n_{\mathbf{v}} \leq 2(N-1)$
$SO(2N)$	$n_{\mathbf{v}} \leq 2(N-2)$
$USp(2N)$	$n_{\mathbf{f}} \leq 2N$
$USp(2N)$	$n_{\Lambda^2} = 1$ and $n_{\mathbf{f}} \leq 2$

**Table 3:** asd

Gauge group	Hypermultiplets
$SO(14)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 2$
$SO(13)$	$n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 2$
$SO(12)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 2$
$SO(12)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(11)$	$n_{\mathbf{s}} = 2$
$SO(11)$	$n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(10)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 3$
$SO(10)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 2$ and $n_{\mathbf{v}} \leq 2$
$SO(10)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(9)$	$n_{\mathbf{s}} = 3$
$SO(9)$	$n_{\mathbf{s}} = 2$ and $n_{\mathbf{v}} \leq 2$
$SO(9)$	$n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 4$
$SO(8)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 4$
$SO(8)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 3$ and $n_{\mathbf{v}} = 1$
$SO(8)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 2$ and $n_{\mathbf{v}} \leq 2$
$SO(8)$	$n_{\mathbf{s}} + n_{\mathbf{c}} = 1$ and $n_{\mathbf{v}} \leq 3$
$SO(7)$	$n_{\mathbf{s}} = 4$
$SO(7)$	$n_{\mathbf{s}} = 3$ and $n_{\mathbf{v}} = 1$
$SO(7)$	$n_{\mathbf{s}} = 2$ and $n_{\mathbf{v}} \leq 2$
$SO(7)$	$n_{\mathbf{s}} = 1$ and $n_{\mathbf{v}} \leq 3$
$G_2$	$n_{\mathbf{7}} \leq 2$
$F_4$	$n_{\mathbf{26}} \leq 2$
$E_6$	$n_{\mathbf{27}} + n_{\overline{\mathbf{27}}} \leq 3$
$E_7$	$n_{\mathbf{56}} \leq 2$
$E_8$	$\emptyset$

**Table 4:** asd