1 Unity Blowup equations

The partition functions of generic 5d $\mathcal{N}=1$ gauge theories with hypermultiplets in R-representation in the Coulomb branch consist of classical action term, 1-loop term, and instanton partition functions.

$$Z(\epsilon_1,\epsilon_2,\vec{a},m_i,m_0) = Z_{\text{class}}(\epsilon_1,\epsilon_2,\vec{a},m_0)\, Z_{\text{1-loop}}(\epsilon_1,\epsilon_2,\vec{a},m_i)\, Z_{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a},m_i,m_0) \tag{1.1}$$

where

$$\begin{aligned}
Z_{\text{class}} &= \exp\Bigg[-\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}h_{ij}\phi^i\phi^j + \frac{1}{6}d_{ijk}\phi^i\phi^j\phi^k\right)\Bigg] \\
Z_{\text{1-loop}} &= \exp\Bigg[-\frac{1}{2\epsilon_1\epsilon_2}\Bigg(\sum_{\alpha\in\text{roots}}\left(\frac{1}{6}(\vec{a}\cdot\vec{\alpha})^3 - \frac{1}{4}(\epsilon_1+\epsilon_2)(\vec{a}\cdot\vec{\alpha})^2 + \frac{1}{12}((\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2)(\vec{a}\cdot\vec{\alpha})\right) \\
&\qquad + \sum_{\omega\in\rho(R)}\Bigg(\frac{1}{6}\Big(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\Big)^3 - \frac{\epsilon_1+\epsilon_2}{4}\Big(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\Big)^2 \\
&\qquad\qquad - \frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{24}\Big(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\Big)\Bigg)\Bigg)\Bigg] \\
&\quad\times \text{PE}\Bigg[\frac{1}{(1-p_1)(1-p_2)}\Bigg(-\sum_{\alpha\in\text{roots}} e^{\vec{a}\cdot\vec{\alpha}} + p_1^{1/2}p_2^{1/2}y_i\sum_{\omega\in\rho(R)} e^{\vec{a}\cdot\vec{\omega}}\Bigg)\Bigg].
\end{aligned} \tag{1.2}$$

Here $\vec{a}$ are Coulomb VEVs and $p_{1,2}=e^{\epsilon_{1,2}}$, $y_i = e^{m_i}$. Note that the normal exponential term saturates the zero-point energy of pletheystic exponential terms.[a]

The partition function satisfies so-called "Unity blowup equation"

$$\begin{aligned}
Z(\epsilon_1,\epsilon_2,\vec{a},m_i,m_0) = \sum_{\vec{k}\in\vec{\alpha}^\vee} & Z(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_1, m_i+r_i\,\epsilon_1, m_0+r_0\,\epsilon_1) \\
&\times\ Z(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_2, m_i+r_i\,\epsilon_2, m_0+r_0\,\epsilon_2)
\end{aligned} \tag{1.3}$$

for certain $\vec{r}_a$, r_i, r_0's. Here $\vec{\alpha}^\vee$ is the coroot lattice where the long root is normalized to have norm 2. The r_i's and r_0 are some numbers specifying the blowup equations.

Technically r_i's are constrained to be half integers since, for each single letter 1-loop partition functions

$$Z_{i,\vec{\omega}} = \text{PE}\left[\frac{p_1^{1/2}p_2^{1/2}}{(1-p_1)(1-p_2)}y_i\, e^{\vec{a}\cdot\vec{\omega}}\right], \tag{1.4}$$

[a] Technically, instead of considering this 1-loop prepotential terms, I inserted overall factors to the $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}}Z^{(1)}_{\text{1-loop}}/Z_{\text{1-loop}}$ so that it is written by Sinh terms.

the ratio between shifted ones and unshifted one is

$$
\begin{aligned}
l^{\vec{k}}_{i,\vec{\omega}} =& Z^{(1)}_{i,\vec{\omega}} Z^{(2)}_{i,\vec{\omega}} / Z_{i,\vec{\omega}} \\
=& \mathrm{PE}\left[\frac{p_1^{r_i} p_2^{1/2} y_i}{(1-p_1)(1-p_2/p_1)} p_1^{\vec{k}\cdot\vec{\omega}} e^{\vec{a}\cdot\vec{\omega}} + \frac{p_1^{1/2} p_2^{r_i} y_i}{(1-p_1/p_2)(1-p_2)} p_2^{\vec{k}\cdot\vec{\omega}} e^{\vec{a}\cdot\vec{\omega}} - \frac{p_1^{1/2} p_2^{1/2} y_i}{(1-p_1)(1-p_2)} e^{\vec{a}\cdot\vec{\omega}} \right] \\
=& \mathrm{PE}\left[\frac{p_1^{1/2} p_2^{1/2} y_i}{(1-p_1)(1-p_2)(p_1-p_2)} e^{\vec{a}\cdot\vec{\omega}} \left((1-p_2) p_1^{\vec{k}\cdot\vec{\omega}+r_i+1/2} - (1-p_1) p_2^{\vec{k}\cdot\vec{\omega}+r_i+1/2} \right) - (p_1-p_2) \right]
\end{aligned}
\tag{1.5}
$$

For the $l^{\vec{k}}_{i,\vec{\omega}}$ to be finite rational function, the plethystic exponent must be finite series. It can be satisfied only when r_i is a half integer.

2 Instanton partition functions from blowup equations

From blowup equations one can compute the partition functions as follows. Rewriting the blowup equation as

$$
1 = \sum_{\vec{k}\in\vec{\alpha}^\vee} f_{\vec{k}}\, l_{\vec{k}} \frac{Z^{(1)}_{\text{inst}} Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \tag{2.1}
$$

where $f_{\vec{k}} = Z^{(1)}_{\text{class}} Z^{(2)}_{\text{class}} / Z_{\text{class}}$ and $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}} Z^{(2)}_{\text{1-loop}} / Z_{\text{1-loop}}$ with abbreviated notation

$$
\begin{aligned}
Z^{(1)} &= Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\,\epsilon_1, m_i + r_i\,\epsilon_1, m_0 + r_0\,\epsilon_1) \\
Z^{(2)} &= Z(\epsilon_1 - \epsilon_2, \epsilon_2, \vec{a} + \vec{k}\,\epsilon_2, m_i + r_i\,\epsilon_2, m_0 + r_0\,\epsilon_2)
\end{aligned}
\tag{2.2}
$$

Here note that $l_{\vec{k}}$ is independent of $Q = e^{-m_0}$, and $f_{\vec{k}}$ is some overall factor in the order of $Q^{\vec{k}\cdot\vec{k}/2}$. Expanding the equation by instanton fugacity Q, then at each Q^n level the equation is written by

$$
\delta_{n,0} = p_1^{r_0} Z^{(1)}_n + p_2^{r_0} Z^{(2)}_n - Z_n + \sum_{\vec{k}\neq 0} f_{\vec{k},r_0} l_{\vec{k}} \left(\frac{Z^{(1)}_{\text{inst}} Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \right)\Bigg|_{O(Q^{n-\vec{k}\cdot\vec{k}/2})} . \tag{2.3}
$$

Since each Z_k and $Z^{(1,2)}_k$ are independent of r_0, one can solve (2.3) with three blowup equations with same r_i's but different r_0's.

The blowup equations for instanton partition functions of pure YM theory with generic gauge group were already studied in [3]. They are actually (1.1) with

$$
\vec{r}_a = 0, \qquad r_0 = d - h^\vee/2 \tag{2.4}
$$

where $d = 0, \cdots, h^\vee$. We extend these blowup equations to the theories with matters based on pure YM blowup equations. If one restrict the cases to $\vec{r}_a = 0$, as we explained in the previous section, the r_i's are technically required to be half intergers. Thus we look for the r_0's that provides the correct instanton partition functions by solving (2.3) while fixing $\vec{r}_a = 0$ and $r_i = 1/2$. Here are the results.

G	matter	r_0	d
$SU(N)_\kappa$	$N_f \times \boldsymbol{N}$	$d - N/2 - \kappa/2$	$0 \le d \le N - \lvert\kappa\rvert - 2N_f - 1$(?)
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \le d \le 6$
$SO(7)$	pure	$d - 5/2$	$0 \le d \le 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \le d \le 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \le d \le 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \le d \le 3$
G_2	pure	$d - 4/2$	$0 \le d \le 4$
G_2	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \le d \le 3$
F_4	pure	$d - 9/2$	$0 \le d \le 9$
F_4	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \le d \le 6$
F_4	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \le d \le 3$

They were tested by comparing the resulting instanton partition functions with the known results from [4]($SO(7)$ and G_2) and [5](F_4 with $N_{\mathbf{26}} = 2$). They were compared numerically, putting random numbers on the fugacities. Note that matters shift the r_0, each by one quarter of their Dynkin indices. It seems to differ from blowup formula for $SU(N)_\kappa + N_f$ instantons, where r_0 was affected only by its CS-level κ. However, one can rewrite the r_0 as

$$\begin{aligned} r_0 &= d - N/2 - \left(\kappa + \frac{1}{2} N_f\right)/2 + N_f/4 \\ &= d - N/2 - \kappa_{\text{eff}}/2 + N_f \times I_{\text{fund}}. \end{aligned} \tag{2.5}$$

Since fundamental matters shifts the effective CS-level, they cancel their index contributions and consequently the r_0 apparently looks independent of matters.

By above observations, we write the unity blowup equation for generic gauge groups and matter representations.

$$\begin{aligned} Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = \sum_{\vec{k} \in \vec{\alpha}^\vee} & Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\epsilon_1, m_i + \epsilon_1/2, m_0 + r_0\epsilon_1) \\ & \times\ Z(\epsilon_1 - \epsilon_2, \vec{a} + \vec{k}\epsilon_2, m_i + \epsilon_2/2, m_0 + r_0\epsilon_2) \end{aligned} \tag{2.6}$$

with

$$r_0 = d - h^\vee/2 - \kappa_{\text{eff}}/2 + N_{\boldsymbol{R}} \times I_{\boldsymbol{R}}. \tag{2.7}$$

Here $I_{\boldsymbol{R}}$ is the Dynkin index of $\boldsymbol{R}$ representation.

3 $SU(6)_3 + 1 \times \mathbf{20}$

As a non-trivial test, we consider the instanton partition function of the $SU(6)_3 + \mathbf{20}$ whose 5-brane realization was found recently [6]. Its web-diagram is given as figure.

(Written before computing the $SU(6)_3 + 20$ instanton partition function.)
Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the $SU(3) \times SU(3) \times U(1) \subset SU(6)$ where the $SU(6)$ multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}} &: \mathbf{35} \longrightarrow (\mathbf{8},1)_0 \oplus (1,\mathbf{8})_0 \oplus (\mathbf{3},\bar{\mathbf{3}})_2 \oplus (\bar{\mathbf{3}},\mathbf{3})_{-2} \oplus (1,1)_0, \\ \Phi_{ijk} &: \mathbf{20} \longrightarrow (\mathbf{3},\bar{\mathbf{3}})_{-1} \oplus (\bar{\mathbf{3}},\mathbf{3})_1 \oplus (1,1)_3 \oplus (1,1)_{-3}. \end{aligned} \tag{3.1}$$

Here to fit with the web-diagram, we set Φ_{156} and Φ_{234} are $(1,1)_3$ and $(1,1)_{-3}$. Once Φ_{156} and Φ_{234} get non-zero VEVs,

When $a_5 = -a_1 - a_6$, the web-diagram factorizes to two $SU(3)_3$ whose Coulomb VEVs are (a_1, a_5, a_6) and (a_2, a_3, a_4). In the gauge theory, it can be seen partly from prepotential. The prepotential of $S(6)_3 + 1 \times \mathbf{20}$ is

$$\mathcal{F} = \frac{1}{2} m_0 \sum_{i=1}^{6} a_i^2 + \frac{1}{2} \sum_{i=1}^{6} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3 - \frac{1}{6} \sum_{1<i<j} (a_1 + a_j + a_k)^3 \tag{3.2}$$

at the Weyl chamber $a_1 > \cdots > a_6$. As one sets the Coulomb VEV $a_6 = -a_1 - a_5$ and $a_4 = -a_2 - a_3$, one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \tag{3.3}$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2} m_0 \sum_{i=1}^{3} a_i^2 + \frac{1}{2} \sum_{i=1}^{3} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3. \tag{3.4}$$

It is Higgsed by

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