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Instantons from Blow-up

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ABSTRACT: The Nekrasov partition function for 4d $\mathcal{N}=2$ or 5d $\mathcal{N}=1$ gauge theory on the blow up of a point $\hat{\mathbb{C}}^2$ can be written in terms of the partition function on the flat space $\mathbb{C}^2$. At the same time, the partition function on the blow up is identical to the partition function on a flat space for sufficiently large class of examples. This relation enables us to compute the instanton partition functions for 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ gauge theories for arbitrary gauge theory with large class of matter representations without knowing explicit construction of the instanton moduli space. Remarkably, the instanton partition function is completely determined by the perturbative part. We obtain the partition function for the previously unknown theories: exceptional gauge groups EFG with fundamental/spinor hypermultiplets and more. We also compute the case with $SU(6)$ with rank-3 antisymmetric tensor and compare with the topological vertex computation using the recently found 5-brane web configuration.

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1 Introduction

The Seiberg-Witten prepotential provides a complete description for the low energy dynamics of 4d $\mathcal{N}=2$ or 5d $\mathcal{N}=1$ gauge theory in its Coulomb branch [1, 2]. It is a function of the vacuum expectation value (VEV) of the scalar in the vector multiplet that parameterizes the Coulomb branch moduli space. Quantum correction to the prepotential is known to be one-loop exact, while there also exist non-perturbative corrections coming from Yang-Mills instantons.

An efficient way to compute the fully quantum corrected prepotential $\mathcal{F}$ is to study the Nekrasov partition function $\mathcal{Z}$ on Ω-deformed $\mathbb{C}^2$ or $\mathbb{C}^2 \times S^1$. It can be written as the product of the classical, one-loop, and instanton contributions,

$$\mathcal{Z}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q}) = \mathcal{Z}_{\text{class}}(\vec{a},\epsilon_1,\epsilon_2,\mathfrak{q})\ \mathcal{Z}_{\text{1-loop}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2)\ \mathcal{Z}_{\text{inst}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q}), \tag{1.1}$$

where the instanton piece is the fugacity sum over all multi-instanton contributions:

$$\mathcal{Z}_{\text{inst}}(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2,\mathfrak{q}) = 1 + \sum_{k=1}^{\infty} \mathfrak{q}^k \mathcal{Z}_k(\vec{a},\vec{\mathfrak{m}},\epsilon_1,\epsilon_2). \tag{1.2}$$

Once the Nekrasov partition function is known, one can extract the prepotential via taking $\epsilon_i \to 0$ limit as $\mathcal{F} = \lim_{\epsilon_i\to 0} \epsilon_1\epsilon_2 \log \mathcal{Z}$ [3–6].

The instanton part of the partition function in the Ω-background can be computed once we know appropriate instanton moduli space. For the classical gauge group, the ADHM construction of the moduli space provides a direct way compute the instanton partition function. The ADHM construction can be understood as the quantum mechanics described by the Dp-D(p+4) system. The Higgs branch moduli space of the Dp system gives the desired moduli space. Matter fields can be also introduced by considering the world-volume theory on the D0-branes of the D0-D4-D8 system. By using the localization on the 1d system on the D0-branes or its dimensional reduction [7, 8], the partition function has been obtained for variety of cases: classical gauge groups [9–13], exceptional gauge groups [14–17]. 3d N=4 Coulomb branch realization of the instanton moduli space: [18] The precise choice of the Contour of the ADHM integral has been derived in [19–21] following the Jeffrey-Kirwan residue formula in 2d elliptic genus [22, 23].

But there is no ADHM type construction for the exceptional gauge groups or generic type of matter fields. String-theoretic picture implies that they require strong-coupling dynamics or non-Lagrangian field theories to realize instanton moduli space of exceptional group as a vacuum moduli space. There has been a few results regarding the exceptional instantons.

In this paper, we generalize the approach of Nakajima-Yoshioka (NY) [5, 24, 25] for the pure YM theory. In [16], the NY blow-up formula were used to compute the instanton partition function for exceptional gauge group and tested against the superconformal index of 4d SCFT where the Higgs branch is given by the instanton moduli space [26]. We propose a general blow-up formula for general gauge theory with arbitrary representations, under the condition that the matter representation is not too large.[1] This enables us to compute the instanton partition functions for numerous gauge theories that have not been known before, without relying on to the explicit construction of the moduli space.

The basic idea is as follows: Let us consider a one-point blow up $\hat{\mathbb{C}}^2$ of the flat space $\mathbb{C}^2$. The *full* partition function on $\hat{\mathbb{C}}^2$ can be written in terms of the products of the *full* partition function of $\mathbb{C}^2$. But at the same time, the partition function on the blowup is identical to that of the flat space since we can smoothly blow-down $\hat{\mathbb{C}}^2$ to $\mathbb{C}^2$ as long as the matter representation is not 'too large'. This provides us a functional relation for the partition function, which turns out to be sufficient to determine the instanton partition function itself. Remarkably, this relation is completely determined by the perturbative part of the partition function. Therefore we arrive at a surprising conclusion: The *perturbative* physics determine the *non-perturbative* physics!

For example, we find the following universal expression for the 1-instanton partition function with arbitrary gauge group and matters:

$$Z_1 = \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma \in \Delta_l} \frac{e^{(h^\vee-1)a_\gamma/2} \prod_{\gamma\cdot w=1}(1-e^{a_w+m^{\rm phy}})}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-e_2})\prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})} , \quad (1.3)$$

where ...

[1] This has to do with the 1-loop beta function coefficient as we will discuss in detail.

Notation The equivariant parameters $\epsilon_1, \epsilon_2, \vec{a}$ are associated to the $U(1)^2 \times U(1)^{|G|}$ action on the k-instanton moduli space $\mathcal{M}_{k,G}$. Additional parameters $\vec{\mathfrak{m}}$ are introduced if the gauge theory has an extra flavor symmetry. The instanton fugacity $\mathfrak{q}$ can be written as $\mathfrak{q} \equiv \Lambda^{b_0}$ (4d) or $\mathfrak{q} \equiv \exp(2\pi i \Lambda)$ (5d) where Λ denotes the bare coupling of the 4d/5d gauge theory.

Throughout Section 2, we will characterize the partition function in terms of $m_i \equiv \mathfrak{m}_i + \epsilon_+$ (where $\epsilon_+ \equiv \frac{\epsilon_1+\epsilon_2}{2}$) as a reflection of the Donaldson-Witten twist by $SU(2)_R$ symmetry. Here m_i corresponds to the 'physical' mass parameter of a matter hypermultiplet. We will also stick to the 'effective' instanton fugacity $q \equiv \mathfrak{q} e^{b_0 \epsilon_+}$, which absorbs the $SU(2)_R$ generated mass of fermionic instanton zero modes, in describing the 5d Nekrasov partition function.

2 Instanton Counting from Blow-up

The essential idea of using the blow-up of $\mathbb{C}^2$ for instanton counting is that the gauge theory partition function for a 4d $\mathcal{N}=2$ (or 5d $\mathcal{N}=1$) theory on the blow-up of a point $\hat{\mathbb{C}}^2$ (or $S^1 \times \hat{\mathbb{C}}^2$) can be written in two different ways. This will allow us to write a recursion relation for the instanton partiton function that can be solved rather easily [5, 16, 24, 25].

Localization on the blow-up $\hat{\mathbb{C}}^2$ One of the expressions for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ comes from the Coulomb branch localization, which results that $\hat{\mathcal{Z}}$ can be obtained by patching together the flat-space partition function $\mathcal{Z}$ [27].

The blow-up $\hat{\mathbb{C}}^2$ is constructed from $\mathbb{C}^2$ by replacing the origin with the compact 2-cycle. In particular, the geometry near the $\mathbb{P}^1$ is identical to a line bundle of degree (-1) on the $\mathbb{P}^1$. One can parametrize $\mathcal{O}(-1) \to \mathbb{P}^1$ using three homogeneous coordinates (z_0, z_1, z_2), satisfying the projective condition $(z_0, z_1, z_2) \sim (\lambda^{-1} z_0, \lambda^1 z_1, \lambda^1 z_2)$ for any $\lambda \in \mathbb{C}^*$, where the two-cycle $\mathbb{P}^1 \subset \hat{\mathbb{C}}^2$ corresponds to the locus $z_0 = 0$. We are interested in the $U(1)^2$ equivariant partition function, with the $U(1)^2$ action V rotating the complex coordinates (z_0, z_1, z_2) as follows:

$$(z_0, z_1, z_2) \mapsto (z_0, e^{\epsilon_1} z_1, e^{\epsilon_2} z_2). \tag{2.1}$$

Instantons are located at two fixed points of the $U(1)^2$ action, i.e., the north/south poles of the $\mathbb{P}^1$, whose coordinates are $(z_0, z_1, z_2) = (0,1,0)$ and $(0,0,1)$. Around these fixed points, the local weights under the $U(1)^2$ action V are:

$$(z_0 z_1,\ z_2/z_1) \mapsto (e^{\epsilon_1} z_0 z_1,\ e^{\epsilon_2 - \epsilon_1} z_2/z_1) \qquad \text{(near the north pole)} \tag{2.2}$$

$$(z_0 z_2,\ z_1/z_2) \mapsto (e^{\epsilon_2} z_0 z_2,\ e^{\epsilon_1 - \epsilon_2} z_1/z_2) \qquad \text{(near the south pole)} \tag{2.3}$$

The full partition function $\hat{\mathcal{Z}}$ on $\hat{\mathbb{C}}^2$, which includes both the perturbative and instanton contributions, can be obtained by performing the localization on the Coulomb branch. On the Coulomb branch, the gauge group is generically broken to $U(1)^r$ where r is the rank of the gauge group. The $U(1)^r$ equivariant parameters $\vec{a}$ naturally appear in the partition function. One needs to sum over all distinct field configurations with zero-sized instantons located at the north and south poles. All the inequivalent configurations are labeled by the vector $\vec{k}$ of the first Chern numbers, corresponding to different flux configurations on the two-cycle $\mathbb{P}^1$. Here we turn off any possible external flux that can be supported on the $\mathbb{P}^1$. Summing up, $\hat{\mathcal{Z}}$ can be expressed as the following formula [27–31]:

$$\hat{\mathcal{Z}}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}\mathcal{Z}(\vec{a}+\vec{k}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1)\mathcal{Z}(\vec{a}+\vec{k}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2) \tag{2.4}$$

where the flux sum is taken over the co-root lattice Λ of the gauge algebra. Notice that the Coulomb parameter $\vec{a}$ gets an appropriate shift at each fixed point p, induced by the non-trivial magnetic flux $\vec{k}$ on the blown-up $\mathbb{P}^1$, with the proportionality constant $H|_p$. Values of the moment map H for the $U(1)^2$ action V, i.e., $dH=\iota_V\omega$, at the north and south poles are

$$H|_{\rm NP}=\epsilon_1 \text{ and } H|_{\rm SP}=\epsilon_2. \tag{2.5}$$

Partition function on $\hat{\mathbb{C}}^2$ vs $\mathbb{C}^2$ An alternative fact for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ is that $\hat{\mathcal{Z}}$ is actually identical to the flat-space partition function $\mathcal{Z}$ [5, 24, 25].

The blow-up $\hat{\mathbb{C}}^2$ is identical to $\mathbb{C}^2$ except for the origin, which is replaced by the blown-up sphere $\mathbb{P}^1$. Since the Nekrasov partition function gets contributions only from the small instantons localized at the fixed points of the $U(1)^2$ equivariant action V, the size of the divisor should not affect the partition function as we smoothly shrink it. So we expect that $\hat{\mathcal{Z}}=\mathcal{Z}$. This implies the following relation: [5, 16, 24, 25, 32, 33]

$$\mathcal{Z}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}\mathcal{Z}(\vec{a}+\vec{k}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1)\mathcal{Z}(\vec{a}+\vec{k}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2)\ , \tag{2.6}$$

The blow-up identity can also be generalized to orbifold partition functions [31, 34, 35]. For example, if we consider the total space of the bundle $\mathcal{O}(-2)\to\mathbb{P}^1$ and shrink the base, we land on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ with singularity at the origin. There are certain states originated from wrapped D-branes on the $\mathbb{P}^1$ which become massless by shrinking the $\mathbb{P}^1$. They constitute to the required twisted sector of the orbifold partition function. In any case, we expect that the Nekrasov partition function still remains the same by blowing up and down.[2]

Background parameters The expected relation (2.6) is only a schematic expression. The partition function also depends on some background parameters, such as the gauge coupling Λ and flavor chemical potentials m. They need to be appropriately shifted at each fixed point p of the blow-up $\hat{\mathbb{C}}^2$, keeping invariant the effective mass parameters twisted by $SU(2)_R$.

One can identify the shifted parameters $\Lambda|_p$ and $m|_p$ at a fixed point p, by examining the 1-loop effective free energy of the 5d Nekrasov partition function $\mathcal{Z}$. For a general 5d $\mathcal{N}=1$

[2]This simple picture does not necessarily hold when there are too many hypermultiplets, due to some subtle scheme dependence related to wall-crossing [28, 29, 35]. We will discuss about this issue in Section **??**.

gauge theory with the Chern-Simons level κ and/or some hypermultiplets, [3]

$$\log \mathcal{Z} = \frac{1}{\epsilon_1\epsilon_2}\left[\frac{1}{2}\Lambda\, h_{ij}a_i a_j + \frac{\kappa}{6}d_{ijk}a^i a^j a^k + \sum_{\vec{\alpha}\in\Delta}\left(\frac{(\vec{a}\cdot\vec{\alpha}+\epsilon_+)^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\alpha}+\epsilon_+)+1\right)\right.$$
$$\left. - \sum_i \sum_{\vec{\omega}\in\boldsymbol{R}_i}\left(\frac{(\vec{a}\cdot\vec{\omega}+m_i)^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\omega}+m_i)+1\right)\right] + \sum_{n=1}^{\infty}\frac{f(n\vec{a}, n\epsilon_{1,2}, nm_i)}{n} \quad (2.7)$$

in which the first 2 terms inside the square brackets constitute the classical prepotential $\mathcal{F}_{\text{cl}}$, whereas all the other terms come in at one-loop level. The set of all roots is denoted by Δ. The $\vec{\omega}$ runs over all weight vectors in the representation $\mathbf{R}_i$ of the i-th hypermultiplet. First, let us consider a 5d free hypermultiplet with mass m, whose corresponding free energy is

$$\log \mathcal{Z} = -\frac{1}{\epsilon_1\epsilon_2}\left(\frac{m^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}m+1\right) + \sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-nm}\cdot e^{-n\epsilon_+}}{(1-e^{-n\epsilon_1})(1-e^{-n\epsilon_2})}. \quad (2.8)$$

This satisfies the relation (2.6) only when it is accompanied by $m \to m \pm \frac{H}{2}|_p$. Each $\pm$ shift preserves the combination $(m \pm \epsilon_+)$ respectively, corresponding to the $SU(2)_R$ twisted mass of the hypermultiplet [36]. Throughout this paper, we take $m_{\text{phy}} \equiv (m - \epsilon_+)$ as the invariant $SU(2)_R$ twisted mass, identifying that

$$m|_p = m - \frac{H}{2}|_p \quad (2.9)$$

at a fixed point p. Second, the twisted instanton mass can be easily read off after simplifying (2.7) based on the following identities:

$$\begin{aligned}\textstyle\sum_{\vec{\omega}\in\boldsymbol{R}}(\vec{a}\cdot\vec{\omega})(\vec{b}\cdot\vec{\omega})(\vec{c}\cdot\vec{\omega}) &= I_3(\boldsymbol{R})\, d_{ijk}\, a^i b^j c^k,\\ \textstyle\sum_{\vec{\omega}\in\boldsymbol{R}}(\vec{a}\cdot\vec{\omega})(\vec{b}\cdot\vec{\omega}) &= I_2(\boldsymbol{R})\, h_{ij}\, a^i b^j,\\ \textstyle\sum_{\vec{\omega}\in\boldsymbol{R}}(\vec{a}\cdot\vec{\omega}) &= 0.\end{aligned} \quad (2.10)$$

Here $I_2(\mathbf{R})$ and $I_3(\mathbf{R})$ are the quadratic and cubic Dynkin indices[4] of a given representation $\mathbf{R}$. In particular, $I_2(\mathbf{adj}) = h^\vee$ is the dual Coxeter number of the gauge algebra. Speaking explicitly, the rational part of the one-loop free energy (2.7) can be written as

$$\log \mathcal{Z} = \frac{1}{\epsilon_1\epsilon_2}\left[\frac{1}{2}\Lambda_{\text{eff}}\, h_{ij}\, a^i a^j + \frac{\kappa_{\text{eff}}}{6}d_{ijk}\, a^i a^j a^k + (\text{independent of } \vec{a})\right], \quad (2.11)$$

where the one-loop corrected parameters are

$$\Lambda_{\text{eff}} = \Lambda + \left(h^\vee - \sum_i \frac{I_2(\boldsymbol{R}_i)}{2}\right)\epsilon_+ - \sum_i \frac{I_2(\boldsymbol{R}_i)}{2}m_i^{\text{phy}}, \quad \kappa_{\text{eff}} = \kappa - \sum_i \frac{I_3(\boldsymbol{R}_i)}{2} \quad (2.12)$$

[3] We assume a particular Weyl chamber in the Coulomb branch, i.e., $0 < a_i < \epsilon_+ < m$ for all $i \in \{1, \cdots, r\}$.
[4] The Dynkin indices are normalized such that $I_2(\mathbf{F}) = 1$ for the $SU(N)$ fundamental representation $\mathbf{F}$.

in which the sum is taken over all the matter multiplets. We notice that the instanton soliton carries the $SU(2)_R$ twisted effective mass, given by $m_{\text{inst}} \equiv \Lambda_{\text{eff}} - \kappa_{\text{eff}}\,\epsilon_+$. In addition, the effective Chern-Simons coupling κ_{eff} induces an electric charge to the instanton, contributing to its ground state energy as $E_0 = m_{\text{inst}} - \vec{a}\cdot\vec{\Pi}$, where $\vec{\Pi}$ is the $U(1)^r \subset G$ electric charge.[5] To keep it invariant the effective instanton mass m_{inst} at a fixed point p of the blow-up $\hat{\mathbb{C}}^2$, we require the shifted gauge coupling $\Lambda|_p$ to be

$$\Lambda|_p = \Lambda + \frac{b}{2}H|_p \quad \text{with} \quad b \equiv h^\vee - \sum_i \frac{I_2(\boldsymbol{R}_i)}{2} - \kappa_{\text{eff}}. \tag{2.13}$$

For example, the $\mathbb{C}^2 \times S^1$ partition function of $\mathcal{N}=1$ free Maxwell theory, given as a product between the perturbative index of a free vector multiplet,

$$\exp\left[+\frac{1}{\epsilon_1\epsilon_2}\left(\frac{\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)}{48} - \frac{\epsilon_1+\epsilon_2}{4} + 1\right)\right]\cdot \mathrm{PE}\left[-\frac{e^{-\epsilon_1-\epsilon_2}}{(1-e^{-\epsilon_1})(1-e^{-\epsilon_2})}\right], \tag{2.14}$$

and the non-perturbative correction from $U(1)$ multi-instantons,

$$\mathrm{PE}\left[\frac{e^{-\Lambda}\cdot e^{-\epsilon_+}}{(1-e^{-\epsilon_1})(1-e^{-\epsilon_2})}\right], \tag{2.15}$$

will comply with the relation (2.6) by shifting the gauge coupling as $\Lambda \to \Lambda|_p = \Lambda + \frac{H}{2}|_p$.

In summary, the blow-up relation (2.6) should always be understood with (2.9) and (2.13).

Correlation functions In fact, (2.6) is not enough to fix the partition function completely, since there are 3 unknown functions and only one relation. More independent relations would be found from insertion of non-trivial $\mathcal{Q}$-closed operators [5, 25]. So we consider the correlation functions of that operator $\mathcal{O}_{\mathbb{P}^1}$ associated to the two-cycle $\mathbb{P}^1$ of the blow-up $\hat{\mathbb{C}}^2$.

In the 4d Donaldson-twisted theory, the $\mathcal{Q}$-invariant observable $\mathcal{O}_{\mathbb{P}^1}$ can be constructed by applying the descent procedure twice to the Casimir invariant $\mathrm{Tr}(\Phi^2)$. It can be written in terms of the component fields as follows: [37]

$$\mathcal{O}_{\mathbb{P}^1} = \int \left\{\omega\wedge \mathrm{Tr}\Big(\Phi F + \frac{1}{2}\psi\wedge\psi\Big) - H\,\mathrm{Tr}\Big(F\wedge F\Big)\right\} \,. \tag{2.16}$$

Here ω and H are the Kähler two-form on the $\mathbb{P}^1$ and the moment map $\iota_V\omega = dH$, respectively. We find it convenient to study the generating function $\langle e^{t\,\mathcal{O}_{\mathbb{P}^1}}\rangle$ of the correlators $\langle \mathcal{O}_{\mathbb{P}^1}\ldots\mathcal{O}_{\mathbb{P}^1}\rangle$. In particular, the insertion of $\exp(t\mathcal{O}_{\mathbb{P}^1})$ causes an extra shift of the instanton mass by $2tH|_p$ at a fixed point p of the blow-up $\hat{\mathbb{C}}^2$ [5, 24, 25]. This modifies (2.13) to $\Lambda|_p = \Lambda + (\frac{b}{2}+t)H|_p$. Therefore the expectation value of the generating function is given by

$$\langle e^{t\mathcal{O}_{\mathbb{P}^1}}\rangle = \sum_{\vec{k}\in\Lambda} \frac{\mathcal{Z}^{(N),t}(\vec{k})\cdot\mathcal{Z}^{(S),t}(\vec{k})}{\mathcal{Z}^{(0)}} \;, \tag{2.17}$$

[5]This agrees with the supersymmetric Casimir energy of the ADHM quantum mechanics.

where

$$
\begin{aligned}
\mathcal{Z}^{(N),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_1, \epsilon_1, \epsilon_2-\epsilon_1, \Lambda+(\tfrac{b}{2}+t)\epsilon_1, \vec{m}+\tfrac{1}{2}\epsilon_1)\ , \\
\mathcal{Z}^{(S),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2, \Lambda+(\tfrac{b}{2}+t)\epsilon_2, \vec{m}+\tfrac{1}{2}\epsilon_2)\ , \\
\mathcal{Z}^{(0)} &\equiv \mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda, \vec{m}).
\end{aligned}
\tag{2.18}
$$

As we shrink the two-cycle $\mathbb{P}^1$ to recover the flat $\mathbb{C}^2$, the effect of inserting $\mathcal{O}^d_{\mathbb{P}^1}$ turns out to give a vanishing contribution due to the selection rule. We recall that the instanton breaks the $U(1)_R$ symmetry to the discrete subgroup $\mathbb{Z}_{4h^\vee-2\sum_i I_2(\mathbf{R}_i)}$ under which the operator $\mathcal{O}^d_{\mathbb{P}^1}$ carries a charge of $+2$. The correlation functions vanish unless the discrete charges add up to zero, modulo $4h^\vee-2\sum_i I_2(\mathbf{R}_i)$. Therefore, expanding (2.17) in powers of t, we find

$$
\langle e^{t\mathcal{O}_{\mathbb{P}^1}}\rangle = 1+\mathcal{O}(t^{2h^\vee-\sum_i I_2(\mathbf{R}_i)})\ . \tag{2.19}
$$

As long as the hypermultiplet representation is not too large, i.e., when $2h^\vee-\sum_i I_2(\mathbf{R}_i)>2$, this allows us to write 3 independent relations for the 3 unknown varables. One can expand $\langle e^{t\,\mu(C)}\rangle$ until $\mathcal{O}(t^2)$ order and recursively solve for $\mathcal{Z}^{(0)}$ at every instanton number. So the instanton part of the partition function will be completely determined from the perturbative partition function. An explicit form of the recursion relation will be displayed in Section 2.1.

We now turn to 5d $\mathcal{N}=1$ gauge theory wrapped on S^1. The Casimir invariant $\mathrm{Tr}(\Phi^2)$ and its descendants are no longer considered as well-defined observables. Instead, there are two types of $\mathcal{Q}$-invariant observables [38]. The first type of observables are constructed from the 5d Wilson loop on the S^1 by applying the descent procedure. The second type of observables introduce the 3d (Kähler) Chern-Simons term, which can be written as [38, 39]

$$
\begin{aligned}
\mathcal{O}^d_{\mathbb{P}^1} = \exp\Bigg[d\int\Big(&\omega\wedge\mathrm{Tr}\Big(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\Big) \\
&+\omega\wedge\Big(\phi\,F+\frac{1}{2}\psi\wedge\psi\Big)\wedge dt - H\,\mathrm{Tr}\Big(F\wedge F\Big)\Big)\Bigg]\ .
\end{aligned}
\tag{2.20}
$$

It can be viewed as the natural S^1 uplift of (2.16) via exponentiation, where the coefficient d is integer-valued as required by gauge invariance. The correlation function is given by

$$
\langle\mathcal{O}^d_{\mathbb{P}^1}\rangle = \sum_{\vec{k}\in\Lambda}\frac{\mathcal{Z}^{(N),d}(\vec{k})\cdot\mathcal{Z}^{(S),d}(\vec{k})}{\mathcal{Z}^{(0)}} \tag{2.21}
$$

with the definitions (2.18) understood. For the case of 5d pure $\mathcal{N}=1$ SYM, with a possible non-zero Chern-Simons level κ, the correlation function turns out to be [25, 29]

$$
\langle\mathcal{O}^d_{\mathbb{P}^1}\rangle = 1 \quad \text{for} \quad 0\le d\le d_{\max} \tag{2.22}
$$

where $d_{\max}=h^\vee$. For $d_{\max}=h^\vee\ge 2$, there are sufficient number of algebraic relations to determine the instanton partition function $\mathcal{Z}$ iteratively, in increasing the order of instantons. This fact was utilized in [16] to compute $\mathcal{Z}$ in exceptional gauge theories, for which the ADHM construction of instanton moduli space is not known. In this paper, we aim at developing the relation (2.22) for various 5d $\mathcal{N}=1$ gauge theories with hypermultiplets, then writing down the corresponding $\mathcal{Z}$ based on that. We will shortly see the following *necessary* condition for (2.22) to hold true:

$$d_{\max} = \begin{cases} h^\vee - \frac{1}{2}\sum_i I_2(\mathbf{R}_i) & \text{if} \quad G \neq SU(N) \\ h^\vee - \frac{1}{2}\sum_i I_2(\mathbf{R}_i) & \text{if} \quad G = SU(N) \textit{change} \end{cases} \tag{2.23}$$

Furthermore, we conjecture that (2.23) is also sufficient for (2.22). While we do not attempt to prove this sufficiency, we will bring a wide variety of 5d $\mathcal{N}=1$ SCFTs later in Section 3, whose $\mathcal{Z}$ is known from an alternative method, and confirm that (2.22) with (2.23) is indeed true for the considered set of examples. We call the relation (2.22) the blowup equation.

2.1 Recursion formula

The blowup equation (2.22) can be translated to useful recursion formulae on the k-instanton contribution Z_k to the entire partition function $\mathcal{Z}$. To derive this, we decompose the partition function $\mathcal{Z}$ in terms of the classical, one-loop, and instanton pieces:

$$\mathcal{Z}(\vec{a},\epsilon_1,\epsilon_2,\Lambda,\vec{m}) = Z_{\text{class}}(\vec{a},\epsilon_1,\epsilon_2,\Lambda,\vec{m}) \cdot Z_{\text{1-loop}}(\vec{a},\epsilon_1,\epsilon_2,\vec{m}) \cdot Z_{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2,\Lambda,\vec{m}) \tag{2.24}$$

where Z_{inst} can be further expanded in terms of the instanton fugacity $q \equiv e^{-\Lambda}$ as

$$Z_{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2,\Lambda,\vec{m}) = 1 + \sum_{k\geq 1} q^n Z_k(\vec{a},\epsilon_1,\epsilon_2,m) \ . \tag{2.25}$$

Then the blowup equation (2.22) can be written as

$$Z_{\text{inst}} = \sum_{\vec{k}} \left[\frac{Z^{(N),d}_{\text{class}}(\vec{k})\, Z^{(S),d}_{\text{class}}(\vec{k})}{Z_{\text{class}}} \frac{Z^{(N),d}_{\text{1-loop}}(\vec{k})\, Z^{(S),d}_{\text{1-loop}}(\vec{k})}{Z_{\text{1-loop}}} \right] Z^{(N),d}_{\text{inst}}(\vec{k})\, Z^{(S),d}_{\text{inst}}(\vec{k}) \ , \tag{2.26}$$

where the superscript $(N/S), d$ implies the appropriate shift of parameters, specified in (2.18).

We recall the known expressions for the classical and 1-loop partition function [3, 40, 41]:

$$Z_{\text{class}} = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}\Lambda\, h_{ij}a_i a_j + \frac{\kappa}{6} d_{ijk} a^i a^j a^k\right)\right], \tag{2.27}$$

$$\begin{aligned} Z^{\text{vec}}_{\text{1-loop}} &= \exp\left[\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{\alpha}\in\Delta}\Big(\frac{(\vec{a}\cdot\vec{\alpha}+\epsilon_+)^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\alpha}+\epsilon_+)+1\Big)\right] \\ &\quad \times \text{PE}\left[-\frac{p_1p_2}{(1-p_1)(1-p_2)}\sum_{\vec{\alpha}\in\Delta} e^{-\vec{a}\cdot\vec{\alpha}}\right] \qquad \text{for the vector multiplet} \end{aligned} \tag{2.28}$$

$$\begin{aligned} Z^{\text{hyp},i}_{\text{1-loop}} &= \exp\left[-\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{\omega}\in\boldsymbol{R}_\ell}\Big(\frac{(\vec{a}\cdot\vec{\omega}+m_\ell)^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\omega}+m_\ell)+1\Big)\right] \\ &\quad \times \text{PE}\left[+\frac{(p_1p_2)^{\frac{1}{2}}\cdot y_\ell}{(1-p_1)(1-p_2)}\sum_{\vec{\omega}\in\boldsymbol{R}_\ell} e^{-\vec{a}\cdot\vec{\omega}}\right] \qquad \text{for the } \ell\text{'th hypermultiplet} \end{aligned} \tag{2.29}$$

where $p_1 \equiv e^{-\epsilon_1}$, $p_2 \equiv e^{-\epsilon_2}$, $y_i \equiv e^{-m_i}$ and the PE represents the Plethystic exponential

$$\mathrm{PE}\left[f(\vec{a},\epsilon_1,\epsilon_2,\Lambda,\vec{m})\right] \equiv \exp\Big(\textstyle\sum_{n=1}^{\infty}\frac{1}{n}f(n\vec{a},n\epsilon_1,n\epsilon_2,n\Lambda,n\vec{m})\Big). \tag{2.30}$$

Substituting them to (2.26), the combinations of three different Z's are given by

$$\begin{aligned}
\frac{Z^{(N),d}(\vec{k})\,Z^{(S),d}(\vec{k})}{Z} &= q^{\frac{\vec{k}\cdot\vec{k}}{2}}\,(p_1p_2)^{(d+\frac{b}{2})(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa}{6}d_{ijk}\,k^ik^jk^k} \\
&\quad\times e^{-(d+\frac{b}{2})(\vec{a}\cdot\vec{k})}\,e^{-\frac{\kappa}{2}d_{ijk}\,a^ik^jk^k} \qquad\qquad \text{for } Z=Z_{\text{class}}, & (2.31)\\
\frac{Z^{(N),d}(\vec{k})\,Z^{(S),d}(\vec{k})}{Z} &= e^{\frac{h^\vee}{2}(\vec{a}\cdot\vec{k})}\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)^{-1} \qquad \text{for } Z=Z^{\text{vec}}_{\text{1-loop}}, & (2.32)\\
\frac{Z^{(N),d}(\vec{k})\,Z^{(S),d}(\vec{k})}{Z} &= e^{-\frac{I_2(\boldsymbol{R}_\ell)}{4}(\vec{a}\cdot\vec{k})+\frac{I_3(\boldsymbol{R}_\ell)}{4}d_{ijk}\,a^ik^jk^k}\,(p_1p_2)^{\frac{I_2(\boldsymbol{R}_\ell)}{8}(\vec{k}\cdot\vec{k})-\frac{I_3(\boldsymbol{R}_\ell)}{12}d_{ijk}\,k^ik^jk^k} \\
&\quad\times y_\ell^{-\frac{I_2(\boldsymbol{R}_\ell)}{4}(\vec{k}\cdot\vec{k})}\prod_{\omega\in\boldsymbol{R}_\ell}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\text{phy},\ell},\epsilon_1,\epsilon_2) \quad \text{for } Z=Z^{\text{hyp},\ell}_{\text{1-loop}}. & (2.33)
\end{aligned}$$

Here we used $I_2(\mathbf{adj}) = 2h^\vee$, $I_3(\mathbf{adj}) = 0$, and also the fact h_{ij} and d_{ijk} are totally symmetric. The function $\mathcal{L}_k(x,\epsilon_1,\epsilon_2)$ is introduced to denote concisely the combination of the PE parts:

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) \equiv \mathrm{PE}\left[\left(\frac{p_1^k\,p_2}{(1-p_1)(1-p_2/p_1)}+\frac{p_1\,p_2^k}{(1-p_1/p_2)(1-p_2)}-\frac{p_1p_2}{(1-p_1)(1-p_2)}\right)\times e^{-x}\right] \tag{2.34}$$

One can easily check that the expression inside the PE vanishes at $k=0,1$. After some work, it is not difficult to find that

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) = \begin{cases} \displaystyle\prod_{m+n\le k-2}(1-p_1^{m+1}p_2^{n+1}e^{-x}) & \text{for } k\ge +2 \\ \displaystyle\prod_{m+n\le -k-1}(1-p_1^{-m}p_2^{-n}e^{-x}) & \text{for } k\le -1 \\ 1 & \text{for } k=0,1. \end{cases} \tag{2.35}$$

Combining them all, the resursion formula on the n-instanton piece Z_n can be written as

$$\begin{aligned}
Z_n = \sum_{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n}&\Bigg((p_1p_2)^{(d+\frac{h^\vee}{2}-\frac{\kappa_{\text{eff}}}{2})(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa_{\text{eff}}}{6}d_{ijk}\,k^ik^jk^k}\,e^{-(d-\frac{\kappa_{\text{eff}}}{2})(\vec{a}\cdot\vec{k})}\,e^{-\frac{\kappa_{\text{eff}}}{2}d_{ijk}\,a^ik^jk^k} \\
&\times\frac{\prod_l y_l^{-I_2(\boldsymbol{R}_l)(\vec{k}\cdot\vec{k}/4)}\prod_{\omega\in\boldsymbol{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\text{phy},l},\epsilon_1,\epsilon_2)}{\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)}\cdot p_1^{(d+\frac{b}{2})\ell}\,p_2^{(d+\frac{b}{2})m}\,Z^{(N)}_\ell(\vec{k})Z^{(S)}_m(\vec{k})\Bigg),
\end{aligned} \tag{2.36}$$

which is generalization of the recursion formula found for the pure SYM case [25, 28].

2.2 One instanton partition function

The right-hand side of the recursion relation (2.36) only involves the instanton parts with $\ell, m < n$. Being so, one can determine the n-instanton correction Z_n recursively from $Z_0 = 1$. At one instanton level, the formula (2.36) can be written as follows:

$$Z_1 = p_1^{(d+\frac{b}{2})} Z_1^{(N)} + p_2^{(d+\frac{b}{2})} Z_1^{(S)} + I_1^{(d)} \ , \tag{2.37}$$

where (with the fugacity variable $y_{\mathrm{phy},l} \equiv e^{-m_{\mathrm{phy},l}} = y_l/\sqrt{p_1 p_2}$ understood)

$$\begin{aligned} I_1^{(d)} = \sum_{\vec{k}\in\Delta_l} \Bigg(& (p_1p_2)^{(d+\frac{b}{2})+\frac{\kappa_{\mathrm{eff}}}{6} d_{ijk}\, k^i k^j k^k} e^{-(d-\frac{\kappa_{\mathrm{eff}}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\mathrm{eff}}}{2} d_{ijk}\, a^i k^j k^k} \\ & \times \frac{\prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \prod_{\omega\in\boldsymbol{R}_l} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega} + m_{\mathrm{phy},l}, \epsilon_1, \epsilon_2)}{(1-p_1p_2 e^{-\vec{a}\cdot\vec{k}})(1-p_1^{-1}e^{\vec{a}\cdot\vec{k}})(1-p_2^{-1}e^{\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})} \Bigg). \end{aligned} \tag{2.38}$$

We are interested in 5d $\mathcal{N}=1$ gauge theories which are UV-complete as 5d $\mathcal{N}=1$ SCFTs. Then the set of possible matter representation should be restricted to [42]

- fundamental representation for $SU(N)$, $SO(N)$, $Sp(N)$, G_2, F_4, E_6, E_7
- antisymmetric representation for $SU(N)$, $Sp(N)$
- spinor representation for $SO(N)$ with $7 \le N \le 14$
- rank-3 antisymmetric representation for $Sp(3)$, $Sp(4)$, $SU(6)$, $SU(7)$
- symmetric representation for $SU(N)$.

For the $SU(N)$ symmetric representation, the inner product $\vec{k}\cdot\vec{\omega}$ can be either 0, ±1, or ±2.

$$\begin{aligned} \textstyle\prod_{\omega\in\mathbf{sym}} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\mathrm{phy}},\epsilon_1,\epsilon_2) = & \textstyle\prod_{\vec{k}\cdot\vec{\omega}=2}(1-y_{\mathrm{phy}}\, p_1p_2 e^{-\vec{a}\cdot\vec{\omega}}) \times \prod_{\vec{k}\cdot\vec{\omega}=-1}(1-y_{\mathrm{phy}} e^{-\vec{a}\cdot\vec{\omega}}) \\ & \times \textstyle\prod_{\vec{k}\cdot\vec{\omega}=-2}(1-y_{\mathrm{phy}}\, p_1^{-1} e^{-\vec{a}\cdot\vec{\omega}})(1-y_{\mathrm{phy}}\, p_2^{-1} e^{-\vec{a}\cdot\vec{\omega}}). \end{aligned} \tag{2.39}$$

For the other cases, the inner product $\vec{k}\cdot\vec{\omega}$ can be either 0, ±1, such that

$$\textstyle\prod_{\omega\in\boldsymbol{R}} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\mathrm{phy}},\epsilon_1,\epsilon_2) = \prod_{\vec{k}\cdot\vec{\omega}=-1}(1-y_{\mathrm{phy}} e^{-\vec{a}\cdot\vec{\omega}}). \tag{2.40}$$

Range of d Prior to solving the equations for Z_1, we recall that the formula (2.36) is valid only in a certain range of d, for which $\langle \mathcal{O}^d_{\mathbb{P}^1}\rangle = 1$. We want to narrow down the validity range of d by doing a simple sanity check on (2.37). Specifically, we examine the double expansion of (2.37) with respect to $y_{\mathrm{phy},l} \ll \sqrt{p_1p_2} \ll 1$, identifying an exponent in $\sqrt{p_1p_2}$ of the leading term. The expansion of the single terms in (2.37) can be written as

$$\begin{aligned} I_1^{(d)} &\sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \Big((p_1p_2)^{d+\frac{b}{2}+1} g_0(\vec{a}) + \cdots \Big) + \cdots \\ Z_1 &\sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \Big((p_1p_2)^{\frac{n}{2}} g_1(\vec{a}) + \cdots \Big) + \cdots \end{aligned}$$

$$p_1^{(d+\frac{b}{2})} Z_1^{(N)} \sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{\frac{d}{2}+\frac{b}{4}+\frac{n}{4}} g_2(\vec{a}) + \cdots \right) + \cdots$$

$$p_2^{(d+\frac{b}{2})} Z_1^{(S)} \sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{\frac{d}{2}+\frac{b}{4}+\frac{n}{4}} g_3(\vec{a}) + \cdots \right) + \cdots . \tag{2.41}$$

where

$$n = \begin{cases} h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) & \text{for } G \neq SU(N) \\ h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) - \kappa_{\mathrm{eff}} & \text{for } G = SU(N). \end{cases} \tag{2.42}$$

except for the special case of $SU(N)_{|\kappa|<N}$ gauge theory without a hypermultiplet, whose n enhances to $n = h^\vee + |\kappa|$ [28]. We note that for the equation (2.37) to hold, some terms on the right-hand side should have the leading exponent with respect to $\sqrt{p_1 p_2}$ less than or equal to that of Z_1. Thus the following condition is naturally imposed:

$$\frac{n}{2} \geq d + \frac{b}{2}. \tag{2.43}$$

which sets an upper bound on d. Similarly, a lower bound on d can be found from another expansion of (2.37), with respect to $y_{\mathrm{phy},l} \ll 1/\sqrt{p_1 p_2} \ll 1$.[6] Each term in (2.37) becomes

$$\begin{aligned} I_1^{(d)} &\sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \left((1/p_1 p_2)^{-d-\frac{b}{2}+1} h_0(\vec{a}) + \cdots \right) + \cdots \\ &\quad + \prod_l y_{\mathrm{phy},l}^{+I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{-d-\frac{b}{2}+1} h_1(\vec{a}) + \cdots \right) \\ Z_1 &\sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{\frac{n'}{2}+\frac{1}{2}\sum_l I_2(\mathbf{R}_l)} h_2(\vec{a}) + \cdots \right) + \cdots \\ &\quad + \prod_l y_{\mathrm{phy},l}^{+I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{\frac{n'}{2}} h_3(\vec{a}) + \cdots \right) \\ p_1^{(d+\frac{b}{2})} Z_1^{(N)} &\sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{-\frac{d}{2}-\frac{b}{4}+\frac{n'}{4}+\frac{1}{4}\sum_l I_2(\mathbf{R}_l)} h_4(\vec{a}) + \cdots \right) + \cdots \\ &\quad + \prod_l y_{\mathrm{phy},l}^{+I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{-\frac{d}{2}-\frac{b}{4}+\frac{n'}{4}} h_5(\vec{a}) + \cdots \right) \\ p_2^{(d+\frac{b}{2})} Z_1^{(S)} &\sim \prod_l y_{\mathrm{phy},l}^{-I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{-\frac{d}{2}-\frac{b}{4}+\frac{n'}{4}+\frac{1}{4}\sum_l I_2(\mathbf{R}_l)} h_6(\vec{a}) + \cdots \right) + \cdots \\ &\quad + \prod_l y_{\mathrm{phy},l}^{+I_2(\boldsymbol{R}_l)/2} \left((p_1 p_2)^{-\frac{d}{2}-\frac{b}{4}+\frac{n'}{4}} h_5(\vec{a}) + \cdots \right) \end{aligned} \tag{2.44}$$

where

$$n' = \begin{cases} h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) & \text{for } G \neq SU(N) \\ h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) - \kappa_{\mathrm{eff}} & \text{for } G = SU(N). \end{cases} \tag{2.45}$$

[6] This is equivalent to assuming a different parameter regime $0 < a_i < -\epsilon_+ < m$ for all $1 \leq i \leq r$. In general, an explicit form of the 1-loop partition function (2.28)–(2.29) can change depending on a parameter regime, thus affecting (2.36). However, all the above expressions remain valid under flipping a sign of ϵ_+, such that we can simply study the double expansion of the single terms in (2.37) with respect to $y_{\mathrm{phy},l} \ll 1/\sqrt{p_1 p_2} \ll 1$.

So actually the stronger bound on d

$$Z_n = p_1^{(d+\frac{b}{2})n} Z_n^{(N)} + p_2^{(d+\frac{b}{2})n} Z_n^{(S)} + I_n^{(d)} \ , \tag{2.46}$$

with

$$I_n^{(d)} = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n \\ \ell,m<n}} \Bigg((p_1p_2)^{(d+\frac{h^\vee}{2}-\frac{\kappa_{\rm eff}}{2})(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa_{\rm eff}}{6}d_{ijk}\,k^ik^jk^k} e^{-(d-\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\rm eff}}{2}d_{ijk}\,a^ik^jk^k} \tag{2.47}$$
$$\times \frac{\prod_l y_l^{-I_2(\boldsymbol{R}_l)(\vec{k}\cdot\vec{k}/4)} \prod_{\omega\in\boldsymbol{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{{\rm phy},l},\epsilon_1,\epsilon_2)}{\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)} \cdot p_1^{(d+\frac{b}{2})\ell} p_2^{(d+\frac{b}{2})m} Z_\ell^{(N)}(\vec{k}) Z_m^{(S)}(\vec{k}) \Bigg).$$

Let us state our final answer:

$$Z_n(\vec{a},\vec{m}^{\rm phy},\epsilon_1,\epsilon_2) = \frac{e^{n(\epsilon_1+\epsilon_2)}I_n^{(0)} - (e^{n\epsilon_1}+e^{n\epsilon_2})I_n^{(1)} + I_n^{(2)}}{(1-e^{n\epsilon_1})(1-e^{n\epsilon_2})} \ , \tag{2.48}$$

with

$$I_n^{(d)}(\vec{a},m,\epsilon_1,\epsilon_2) = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n \\ \ell,m<n}} e^{(d+d_0)\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)+\ell\epsilon_1+m\epsilon_2\right)} g(\vec{k}) Z_\ell^{(N)}(\vec{k}) Z_m^{(S)}(\vec{k}) \ . \tag{2.49}$$

where

$$Z_{\rm inst} = \sum_{\vec{k}\in\Delta^\vee} (p_1p_2)^{(d+b_0/2)(\vec{k}\cdot\vec{k})/2+\frac{\kappa}{6}d_{ijk}\,k^ik^jk^k} e^{-(d+b_0/2)(\vec{a}\cdot\vec{k})-\frac{\kappa}{2}d_{ijk}\,a^ik^jk^k} \tag{2.50}$$

$${\rm PE}\left[e^{\vec{a}\cdot\vec{\alpha}}A(\vec{k}\cdot\vec{\alpha},e^{\epsilon_1}e^{\epsilon_2})\right]\cdot\prod_i\prod_{\vec{w}\in R_i}{\rm PE}\left[-e^{\vec{a}\cdot\vec{w}+m_i^{\rm phy}}A(\vec{k}\cdot\vec{w},e^{\epsilon_1}e^{\epsilon_2})\right] \tag{2.51}$$

where i runs over various hypermultiplets in the theory and $p_1 = e^{\epsilon_1}, p_2 = e^{\epsilon_2}$. Here the function A is given as

$$A(k,p_1,p_2) = \frac{1}{(1-p_1)(1-p_2)} - \frac{p_1^k}{(1-p_1)(1-p_2/p_1)} - \frac{p_2^k}{(1-p_1/p_2)(1-p_2)} \ . \tag{2.52}$$

Upon taking PE, we obtain

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) \equiv {\rm PE}\left[e^xA(k,e^{\epsilon_1},e^{\epsilon_2})\right] = \begin{cases} \prod_{\substack{m,n\geq 0 \\ m+n\leq k-1}} \left(1-e^{x+m\epsilon_1+n\epsilon_2}\right) & (k>0) \\ \prod_{\substack{m,n\geq 0 \\ m+n\leq -k-2}} \left(1-e^{x-(m+1)\epsilon_1-(n+1)\epsilon_2}\right) & (k<-1) \\ 1 & (k=0,-1) \end{cases} \tag{2.53}$$

where i runs over all the hypermultiplets.

Therefore, we obtain

$$Z = \sum_{\vec{k}\in\Lambda} \exp\Big((d+d_0)\,\mu(\vec{k})\Big)\, Z^{(N),d}(\vec{k}) Z^{(S),d}(\vec{k}) \qquad (d = 0, 1, \ldots, h^\vee - I(R)/2)\ , \tag{2.54}$$

for the 5d theories. We call the relation above as the blow-up equation. (up to here)

Now, we can solve for $Z_n, Z_n^{(N)}, Z_n^{(S)}$ by using the blow-up equation for $d = 0, 1, 2$ to obtain

$$Z_n(\vec{a}, m, \epsilon_1, \epsilon_2) = \frac{e^{n(\epsilon_1+\epsilon_2)} I_n^{(0)} - (e^{n\epsilon_1} + e^{n\epsilon_2}) I_n^{(1)} + I_n^{(2)}}{(1-e^{n\epsilon_1})(1-e^{n\epsilon_2})}\ . \tag{2.55}$$

Since $I_n^{(d)}$ only depends on Z_m with $m < n$ we can determine the instanton partition function recursively.

We now need to determine the $g(\vec{k})$, but this is also completely determined by the *1-loop part* of the partition function. Therefore we land on a remarkable conclusion: The non-perturbative partition function is completely fixed by the perturbative partition function! Notice that we have arrived at this conclusion not by requiring the perturbative series to be well-defined, as is often done in the resurgence analysis. Instead, we simply demand consistent answer upon deforming the spacetime geometry smoothly. This consistency condition is entirely non-perturbative, that the QFT has to be well-defined regardless of its spacetime.

In the remainder of this section we will give explicit formulae for 4d and 5d gauge theories. Especially, we will obtain a closed-form expression for the one-instanton partition function for a large class of gauge theories.

2.3 Instanton partition function for the 5d theory

One-instanton partition function Let us compute the 1-instanton partition function using our formula. In this case, we only take the sum over the long roots having $\vec{k}\cdot\vec{k} = 2$ and $\ell = m = 0$. Therefore, (add the effect of $e^{-\frac{I_3(\mathbf{R})}{4}(d_{ijk}a_ik_jk_k)}$)

$$I_1^{(d)} = e^{d(\epsilon_1+\epsilon_2)} \sum_{\vec{\gamma}\in\Delta_l} \frac{e^{d\vec{\gamma}\cdot\vec{a}} M(\gamma)}{L(\gamma)}\ , \tag{2.56}$$

where Δ_l is the set of all long roots and

$$\begin{aligned} L(\gamma) &\equiv \prod_{\alpha\in\Delta} \mathcal{L}_{\gamma\cdot\alpha}(a\cdot\alpha, \epsilon_1, \epsilon_2) = \prod_{\gamma\cdot\alpha=\pm 2}\prod_{\gamma\cdot\alpha=1} \mathcal{L}_{\gamma\cdot\alpha}(a\cdot\alpha, \epsilon_1, \epsilon_2) \\ &= (1-e^{a_\gamma+\epsilon_1})(1-e^{a_\gamma+\epsilon_2})(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2}) \prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})\ , \end{aligned} \tag{2.57}$$

where $a_\gamma \equiv \vec{a}\cdot\vec{\gamma}$ with $a_\alpha \equiv \vec{a}\cdot\vec{\alpha}$ and

$$M(\gamma) \equiv \prod_{\vec{w}\in R} e^{-\frac{(\vec{\gamma}\cdot\vec{w})^2}{4} m^{\text{phy}}} \mathcal{L}_{\gamma\cdot w}(a_w, \epsilon_1, \epsilon_2) = e^{-I(R)m^{\text{phy}}/2} \prod_{\vec{w}\in R} \mathcal{L}_{\gamma\cdot w}(a_w, \epsilon_1, \epsilon_2)\ , \tag{2.58}$$

with $a_w \equiv \vec{a} \cdot \vec{w}$ and $I(R)$ is the Dynkin index of the representation R. Therefore, the one-instanton partition function can be written explicitly as

$$\begin{aligned} Z_1 &= \frac{e^{\epsilon_1+\epsilon_2} I_n^{(0)} - (e^{\epsilon_1} + e^{\epsilon_2}) I_n^{(1)} + I_n^{(2)}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma \in \Delta_l} \frac{(1-e^{a_\gamma+\epsilon_1})(1-e^{a_\gamma+\epsilon_2}) M(\gamma)}{L(\gamma)} \ . \end{aligned} \tag{2.59}$$

For the case of pure SYM theory with no matters, $M(\gamma) = 1$ so that

$$\begin{aligned} Z_1^{\text{SYM}} &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma \in \Delta_l} \frac{1}{(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2}) \prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma \in \Delta_l} \frac{e^{(h^\vee-1)a_\gamma/2}}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-\epsilon_2}) \prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})} \ , \end{aligned} \tag{2.60}$$

which is the one derived in [15, 16].

For the case with fundamental matters, we find that for the long root γ, there are only weights with $\gamma \cdot w = 0, \pm 1$. Therefore, the $M(\gamma)$ can be simplified to give

$$M(\gamma) = \prod_{\gamma\cdot w=0,\pm1} \mathcal{L}_{\gamma\cdot w}(a_w, \epsilon_1, \epsilon_2) = \prod_{\gamma\cdot w=1} (1 - e^{a_w + m^{\text{phy}}}) \ . \tag{2.61}$$

The one-instanton partition function is now given as

$$\begin{aligned} Z_1 &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma \in \Delta_l} \frac{\prod_{\gamma\cdot w=1}(1-e^{a_w+m^{\text{phy}}})}{(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2}) \prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma \in \Delta_l} \frac{e^{(h^\vee-1)a_\gamma/2} \prod_{\gamma\cdot w=1}(1-e^{a_w+m^{\text{phy}}})}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-\epsilon_2}) \prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})} \ , \end{aligned} \tag{2.62}$$

which we conjecture to be true for all the hypermultiplets with the representation with $|\gamma \cdot w| \leq 1$ for all $w \in R$.

2.4 Instanton partition function for 4d gauge theory

3 Examples

As a remarkable application of the blow-up equation, we construct the instanton partition function of various 5d gauge theories on $\mathbb{C}^2 \times S^1$. The standard method to compute the multi-instanton correction to the partition function is to employ the ADHM construction of the instanton moduli space [3, 4, 43], and/or to apply the topological vertex formalism based on the geometric engineering of 5d $\mathcal{N} = 1$ gauge theory [44, 45]. An alternative approach that comes from the consistency requirement for the blow-up $\hat{\mathbb{C}}^2$ will turn out to be very efficient for bootstrapping the $\mathbb{C}^2 \times S^1$ partition function of exceptional gauge theories [16, 32, 46].

Gauge group	Hypermultiplets
$SU(N)_\kappa$	x
$SO(2N+1)$	$n_{\mathbf{v}} \leq 2(N-1)$
$SO(2N)$	$n_{\mathbf{v}} \leq 2(N-2)$
$USp(2N)$	$n_{\mathbf{f}} \leq 2N$
$USp(2N)$	$n_{\Lambda^2} = 1$ and $n_{\mathbf{f}} \leq 2$

Table 1: asd

There are multiple and distinct blow-up equations (2.54) at different values of d, ranged over $0 \leq d \leq d_{\max}$ where $d_{\max} \equiv h^\vee - \frac{1}{2}\sum_i I(\mathbf{R}_i)$. We need at least three distinct equations to solve for the instanton partition function, thus the bootstrap method is applicable only for gauge theories which satisfy $d_{\max} \geq 2$. Here we constrain our discussion to the theories having simple gauge groups, which are classified to have a 5d UV fixed point [42].

There are two

We will

The complete list of $\mathcal{N} = 1$ gauge theories, satisfying $d_{\max} \geq 2$, is displayed in Table 1 and 2. All infinite families of theories, whose rank of the gauge group can be arbitrarily large, are summarized in Table 1. There are also finite families of theories, summarized in Table 2, which involve an exceptional gauge group or a spinor hypermultiplet.

(comparison with IMS bound, etc) (SCFT bound) Some theories in Table 2 do not belong to the

3.1 Known examples

3.2 Ki-Hong's note

Unity Blowup equations

Instanton partition functions from blowup equations From blowup equations one can compute the partition functions as follows. Rewriting the blowup equation as

$$1 = \sum_{\vec{k}\in\vec{\alpha}^\vee} f_{\vec{k}}\, l_{\vec{k}} \frac{Z^{(1)}_{\text{inst}} Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \tag{3.1}$$

where $f_{\vec{k}} = Z^{(1)}_{\text{class}} Z^{(2)}_{\text{class}}/Z_{\text{class}}$ and $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}} Z^{(2)}_{\text{1-loop}}/Z_{\text{1-loop}}$ with abbreviated notation

$$\begin{aligned} Z^{(1)} &= Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\,\epsilon_1, m_i + r_i\,\epsilon_1, m_0 + r_0\,\epsilon_1) \\ Z^{(2)} &= Z(\epsilon_1 - \epsilon_2, \epsilon_2, \vec{a} + \vec{k}\,\epsilon_2, m_i + r_i\,\epsilon_2, m_0 + r_0\,\epsilon_2) \end{aligned} \tag{3.2}$$

Here note that $l_{\vec{k}}$ is independent of $Q = e^{-m_0}$, and $f_{\vec{k}}$ is some overall factor in the order of $Q^{\vec{k}\cdot\vec{k}/2}$. Expanding the equation by instanton fugacity Q, then at each Q^n level the equation

Gauge group	Hypermultiplets
$SO(14)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 2$
$SO(13)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 2$
$SO(12)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=2$
$SO(12)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(11)$	$n_{\mathbf{s}}=2$
$SO(11)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(10)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=3$
$SO(10)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(10)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(9)$	$n_{\mathbf{s}}=3$
$SO(9)$	$n_{\mathbf{s}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(9)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 4$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=4$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=3$ and $n_{\mathbf{v}}=1$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(8)$	$n_{\mathbf{s}}+n_{\mathbf{c}}=1$ and $n_{\mathbf{v}}\leq 3$
$SO(7)$	$n_{\mathbf{s}}=4$
$SO(7)$	$n_{\mathbf{s}}=3$ and $n_{\mathbf{v}}=1$
$SO(7)$	$n_{\mathbf{s}}=2$ and $n_{\mathbf{v}}\leq 2$
$SO(7)$	$n_{\mathbf{s}}=1$ and $n_{\mathbf{v}}\leq 3$
G_2	$n_{\mathbf{7}}\leq 2$
F_4	$n_{\mathbf{26}}\leq 2$
E_6	$n_{\mathbf{27}}+n_{\overline{\mathbf{27}}}\leq 3$
E_7	$n_{\mathbf{56}}\leq 2$
E_8	$\varnothing$

Table 2: asd

is written by

$$\delta_{n,0}=p_1^{r_0}Z_n^{(1)}+p_2^{r_0}Z_n^{(2)}-Z_n+\sum_{\vec{k}\neq 0}f_{\vec{k},r_0}l_{\vec{k}}\left(\frac{Z_{\text{inst}}^{(1)}Z_{\text{inst}}^{(2)}}{Z_{\text{inst}}}\right)\Bigg|_{O(Q^{n-\vec{k}\cdot\vec{k}/2})}. \tag{3.3}$$

Since each Z_k and $Z_k^{(1,2)}$ are independent of r_0, one can solve (3.3) with three blowup equations with same r_i's but different r_0's.

The blowup equations for instanton partition functions of pure YM theory with generic gauge group were already studied in [16]. They are actually (**??**) with

$$\vec{r}_a=0, \qquad r_0=d-h^\vee/2 \tag{3.4}$$

G	matter	r_0	$d_{\max}$
$SU(N)_\kappa$	$N_f \times \boldsymbol{N}$	$d - N/2 - \kappa/2$	$0 \leq d \leq N - \lvert\kappa\rvert - 2N_f - 1$(?)
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \leq d \leq 6$
$SO(7)$	pure	$d - 5/2$	$0 \leq d \leq 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \leq d \leq 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \leq d \leq 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \leq d \leq 3$
G_2	pure	$d - 4/2$	$0 \leq d \leq 4$
G_2	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \leq d \leq 3$
F_4	pure	$d - 9/2$	$0 \leq d \leq 9$
F_4	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \leq d \leq 6$
F_4	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \leq d \leq 3$

Table 3: list of theories

where $d = 0, \cdots, h^\vee$. We extend these blowup equations to the theories with matters based on pure YM blowup equations. If one restrict the cases to $\vec{r}_a = 0$, as we explained in the previous section, the r_i's are technically required to be half intergers. Thus we look for the r_0's that provides the correct instanton partition functions by solving (3.3) while fixing $\vec{r}_a = 0$ and $r_i = 1/2$. Here are the results.

They were tested by comparing the resulting instanton partition functions with the known results from [47]($SO(7)$ and G_2) and [48](F_4 with $N_{\mathbf{26}} = 2$). They were compared numerically, putting random numbers on the fugacities. Note that matters shift the r_0, each by one quarter of their Dynkin indices. It seems to differ from blowup formula for $SU(N)_\kappa + N_f$ instantons, where r_0 was affected only by its CS-level κ. However, one can rewrite the r_0 as

$$\begin{aligned} r_0 &= d - N/2 - \left(\kappa + \frac{1}{2}N_f\right)/2 + N_f/4 \\ &= d - N/2 - \kappa_{\text{eff}}/2 + N_f \times I_{\text{fund}}. \end{aligned} \tag{3.5}$$

Since fundamental matters shifts the effective CS-level, they cancel their index contributions and consequently the r_0 apparently looks independent of matters.

By above observations, we write the unity blowup equation for generic gauge groups and matter representations.

$$\begin{aligned} Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = \sum_{\vec{k} \in \vec{\alpha}^\vee} & Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\epsilon_1, m_i + \epsilon_1/2, m_0 + r_0\epsilon_1) \\ & \times Z(\epsilon_1 - \epsilon_2, \vec{a} + \vec{k}\epsilon_2, m_i + \epsilon_2/2, m_0 + r_0\epsilon_2) \end{aligned} \tag{3.6}$$

with

$$r_0 = d - h^\vee/2 - \kappa_{\text{eff}}/2 + N_{\boldsymbol{R}} \times I_{\boldsymbol{R}}. \tag{3.7}$$

Here $I_{\boldsymbol{R}}$ is the Dynkin index of $\boldsymbol{R}$ representation.

$SU(6)_3 + 1 \times \mathbf{20}$ As a non-trivial test, we consider the instanton partition function of the $SU(6)_3 + \mathbf{20}$ whose 5-brane realization was found recently [49]. Its web-diagram is given as figure.

(Written before computing the $SU(6)_3 + 20$ instanton partition function.)
Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the $SU(3) \times SU(3) \times U(1) \subset SU(6)$ where the $SU(6)$ multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}} &: \mathbf{35} \longrightarrow (\mathbf{8},1)_0 \oplus (1,\mathbf{8})_0 \oplus (\mathbf{3},\bar{\mathbf{3}})_2 \oplus (\bar{\mathbf{3}},\mathbf{3})_{-2} \oplus (1,1)_0, \\ \Phi_{ijk} &: \mathbf{20} \longrightarrow (\mathbf{3},\bar{\mathbf{3}})_{-1} \oplus (\bar{\mathbf{3}},\mathbf{3})_1 \oplus (1,1)_3 \oplus (1,1)_{-3}. \end{aligned} \tag{3.8}$$

Here to fit with the web-diagram, we set Φ_{156} and Φ_{234} are $(1,1)_3$ and $(1,1)_{-3}$. Once Φ_{156} and Φ_{234} get non-zero VEVs,

When $a_5 = -a_1 - a_6$, the web-diagram factorizes to two $SU(3)_3$ whose Coulomb VEVs are (a_1, a_5, a_6) and (a_2, a_3, a_4). In the gauge theory, it can be seen partly from prepotential. The prepotential of $S(6)_3 + 1 \times \mathbf{20}$ is

$$\mathcal{F} = \frac{1}{2} m_0 \sum_{i=1}^{6} a_i^2 + \frac{1}{2} \sum_{i=1}^{6} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3 - \frac{1}{6} \sum_{1<i<j} (a_1 + a_j + a_k)^3 \tag{3.9}$$

at the Weyl chamber $a_1 > \cdots > a_6$. As one sets the Coulomb VEV $a_6 = -a_1 - a_5$ and $a_4 = -a_2 - a_3$, one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \tag{3.10}$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2} m_0 \sum_{i=1}^{3} a_i^2 + \frac{1}{2} \sum_{i=1}^{3} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3. \tag{3.11}$$

It is Higgsed by

3.3 $SU(6)_3$ with a rank-3 antisymmetric hyper

(describe blow-up computation)

It was recently found in [49] that the 5d $SU(6)_3$ gauge theory with a rank-3 antisymmetric hypermultiplet can be engineered from the 5-brane web configuration, depicted in Figure 1. Given a web diagram, we utilize the topological vertex method [45, 50] to compute all genus topological string amplitudes, which is the logarithm of the 5d Nekrasov partition function on Ω-deformed $\mathbf{R}^4 \times S^1$ [44]. We will check its agreement with the blowup partition function (eqn), providing a supporting evidence to the suggested blow-up equation (eqn).

By applying the topological vertex method to Figure 1, we find that the instanton partition function can be written as the following sum over all possible 6 Young diagrams:

$$Z_{\text{Nek}} = \sum_{(Y_1, \cdots, Y_6)} q^{\sum_{i=1}^{6} |Y_i|} (-A_1^6)^{|Y_1|} (-A_2^6)^{|Y_2|} (-A_2^2 A_3^4)^{|Y_3|} (-A_2^2 A_3^2 A_4^2)^{|Y_4| + |Y_5|}$$

$$\times f_{Y_1}(g)^5 f_{Y_2}(g)^5 f_{Y_3}(g)^3 f_{Y_4}(g) f_{Y_5}(g)^{-1} f_{Y_6}(g)^2 Z_{\text{half}}(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)^2. \qquad (3.12)$$

where the Kähler parameters can be identified as

$$A_i = e^{-a_i}, \qquad q = e^{-\frac{8\pi^2}{g^2}}, \qquad g = e^{-\epsilon_-}. \qquad (3.13)$$

We briefly explain our notation: For a given Young diagram μ, $|\mu|$ denotes the total number of boxes. μ_i is the number of boxes in the i-th row of μ. μ^t is the transpose of μ. We also use

$$f_\mu(g) = (-1)^{|\mu|} g^{\frac{1}{2}(\|\mu^t\|^2 - \|\mu\|^2)}, \qquad \tilde{Z}_\lambda(g) = \prod_{(i,j)\in\lambda} (1 - g^{\lambda_i + \lambda_j^t - i - j + 1})^{-1} \qquad (3.14)$$

with $\|\mu\|^2 = \sum_i \mu_i^2$. The factor $Z_{\text{half}}(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$ involves a single summation over all possible Young diagrams, i.e.,

$$\begin{aligned} Z_{\text{left}}(\vec{Y}) = \sum_{Y_0} &(-A_1{}^{-1} A_6{}^{-2})^{|Y_0|} g^{\frac{\|Y_0^t\|^2 + \|Y_0\|^2}{2}} \tilde{Z}_{Y_0}(g)^2 f_{Y_0}(g)^2 \ \textstyle\prod_{i=1}^6 g^{\frac{\|Y_i\|^2}{2}} \tilde{Z}_{Y_i}(g) \\ &\times R^{-1}_{Y_1 Y_6^t}(A_1 A_6{}^{-1})\, R^{-1}_{Y_0 Y_6^t}(A_1{}^{-1} A_6{}^{-2})\, R^{-1}_{Y_1 Y_0^t}(A_1^2 A_6) \\ &\times \textstyle\prod_{2\le i<j\le 5} R^{-1}_{Y_i Y_j^t}(A_i A_j{}^{-1}) \ \prod_{i=2}^5 R_{Y_0^t Y_i}(A_1 A_i A_6), \end{aligned} \qquad (3.15)$$

in which we introduce

$$R_{\lambda\mu}(Q) = R_{\mu\lambda}(Q) = \text{PE}\left[-\frac{g}{(1-g)^2} Q\right] \times N_{\lambda^t \mu}(Q) \qquad (3.16)$$

with PE representing the Plethystic exponential and

$$N_{\lambda\mu}(Q) = \prod_{(i,j)\in\lambda} \left(1 - Q g^{\lambda_i + \mu_j^t - i - j + 1}\right) \prod_{(i,j)\in\mu} \left(1 - Q g^{-\lambda_j^t - \mu_i + i + j - 1}\right). \qquad (3.17)$$

We also recall that the Nekrasov partition function is divided into the perturbative partition function Z_{pert} and the weighted sum of k-instanton partition function Z_k.

$$Z_{\text{Nek}} = Z_{\text{pert}} \Big(1 + \sum_{k=1}^{\infty} q^k Z_k\Big). \qquad (3.18)$$

The perturbative part of the partition function Z_{pert} comes from the summand of (3.12) at empty Young diagrams, i.e., $(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) = (\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing)$. It is given by

$$\begin{aligned} Z_{\text{pert}} &= Z_{\text{half}}(\varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing)^2 \\ &= \text{PE}\left[\frac{2g}{(1-g)^2}\Big(\frac{A_1}{A_6} + \frac{1}{A_1 A_6^2} + A_1^2 A_6 + \sum_{2\le i<j\le 5} \frac{A_i}{A_j} - \sum_{i=2}^5 A_1 A_i A_6\Big)\right] \\ &\quad \times \Bigg(\sum_{Y_0} (-A_1{}^{-1} A_6{}^{-2})^{|Y_0|}\ g^{\frac{\|Y_0^t\|^2 + \|Y_0\|^2}{2}} \tilde{Z}_{Y_0}(g)^2 f_{Y_0}^2(g) \end{aligned}$$

$$N^{-1}_{Y_0^t\varnothing}(A_1{}^{-1}A_6{}^{-2})N^{-1}_{Y_0\varnothing}(A_1^2A_6)\prod_{i=2}^5 N_{Y_0\varnothing}(A_1A_iA_6)\Big)^2, \tag{3.19}$$

where the last two lines can be combined into the following closed-form expression:

$$\mathrm{PE}\Bigg[\frac{2g}{(1-g)^2}\Big(\sum_{i=2}^5\frac{A_1}{A_i}+\sum_{i=2}^5\frac{A_i}{A_6}-\frac{1}{A_1A_6^2}-A_1^2A_6-\sum_{2\le i<j\le 5}A_1A_iA_j+\mathcal{O}(A_1^6)\Big)\Bigg]. \tag{3.20}$$

So (3.19) is manifestly consistent with the equivariant index [41] for 5d $SU(6)$ gauge theory with a hypermultiplet in the rank-3 antisymmetric representation **20**, i.e.,

$$Z_{\text{pert}}=\mathrm{PE}\Bigg[\frac{2g}{(1-g)^2}\Big(\sum_{1\le i<j\le 6}\frac{A_i}{A_j}-\sum_{2\le i<j\le 6}A_1A_iA_j+\mathcal{O}(A_1^6)\Big)\Bigg]. \tag{3.21}$$

The 1-instanton partition function Z_1 can be obtained from the summands of (3.12) at Young diagrams satisfying $\sum_{i=1}^6|Y_i|=1$. There are 6 different profiles of Young diagrams. The configuration $|Y_i|=1$ and $Y_{j\neq i}=\varnothing$ contributes to Z_1 by

$$+\frac{g}{(1-g)^2}\frac{A_i^6}{\prod_{j\neq i}(A_i-A_j)^2}\Big(-A_i\sum_{j\neq i}A_j+\sum_{j\neq i}\frac{1}{A_j}-\frac{1}{A_i}+A_i^2\Big)^2. \tag{3.22}$$

Summing over all six contributions, we find

$$Z_1=\sum_{i=1}^6\frac{g}{(1-g)^2}\frac{A_i^6}{\prod_{j\neq i}(A_i-A_j)^2}\Big(-A_i\sum_{j\neq i}A_j+\sum_{j\neq i}\frac{1}{A_j}-\frac{1}{A_i}+A_i^2\Big)^2. \tag{3.23}$$

which is in agreement with the blowup partition function (eqn). (overall sign: looking at the GV invariant for single W-boson + single instanton (which should be -2), the topological vertex computation seems to be correct. blowup should have an overall sign issue)

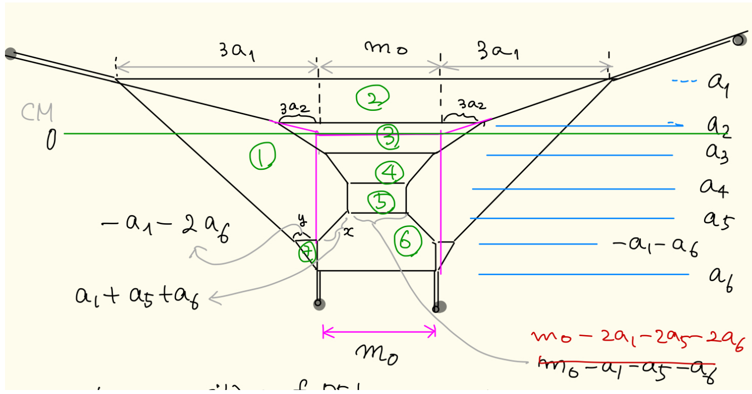

Figure 1: A 5-brane web for $SU(6)_3 + 1$**TAS** with massless **TAS**.

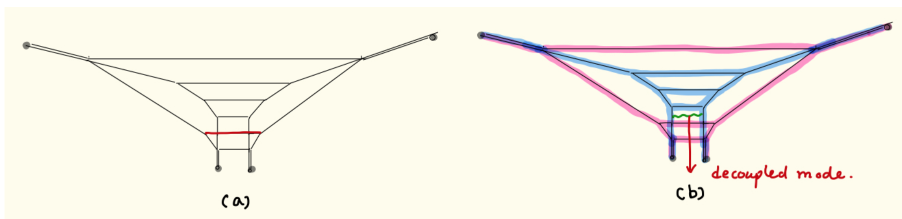

Figure 2: (a) A Higgsing from from $SU(6)_3 + 1$**TAS** to two $SU(3)_3$ theories. (a) Two $SU(3)_3$ theories are painted in blue and in pink respectively. A new decoupled mode emerges.

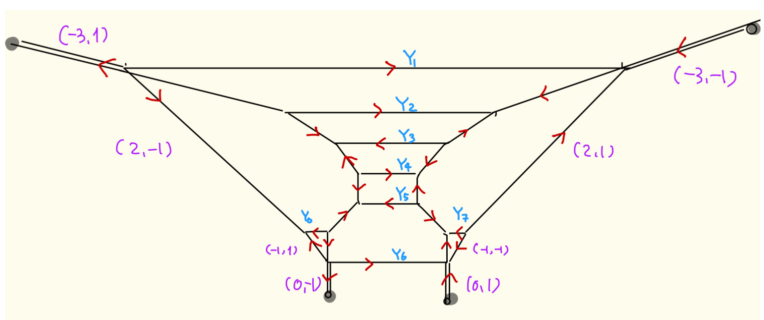

Figure 3: A labeling of Young diagrams assigned to the horizontal lines in Figure 1.

4 Conclusion

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