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Exceptional Instantons from Blow-up

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Abstract:

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We consider a 5d $\mathcal{N} = 1$ gauge theory with a gauge group G. It has the Coulomb branch moduli space, parametrized by the vacuum expectation value $\alpha_i \equiv \langle \Phi_{ii} \rangle$ of the vector multiplet scalar Φ. The gauge symmetry G is spontaeously broken to its Abelian subgroup $U(1)^{|G|}$ on the Coulomb branch. The low energy Abelian theory is described by the effective prepotential, which can be written as [1]

$$\mathcal{F}_{\text{classical}} = \frac{1}{2g^2} h_{ij}\alpha_i\alpha_j + \frac{\kappa}{6} d_{ijk}\alpha_i\alpha_j\alpha_k \tag{2.1}$$

at the classical level. It was found in [2, 3] that the fully quantum corrected prepotential can be obtained from the BPS instanton partition function $\mathcal{Z}$ on Ω-deformed $\mathbf{R}^4 \times S^1$, i.e.,

$$\mathcal{F} = \lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2 \log \mathcal{Z} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{\text{quantum}}. \tag{2.2}$$

The BPS partition function $\mathcal{Z}$ is defined with equivariant parameters $\epsilon_1, \epsilon_2, \alpha_1, \cdots, \alpha_{|G|}$ associated to the $U(1)^2 \times U(1)^{|G|}$ action on the k-instanton moduli space $\mathcal{M}_{k,G}$ [2, 3]. Additional equivariant parameters $m_1, \cdots, m_{|F|}$ can be introduced if a given theory has the flavor symmetry F. It takes the form of

$$\mathcal{Z} = \exp(F_0) \cdot \mathcal{I} \tag{2.3}$$

where $\mathcal{I}$ is the Witten index counting the BPS bound states of fundamental particles and/or non-perturbative instanton solitons. More precisely,

$$\mathcal{I} \equiv \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\frac{8\pi^2}{g^2}k} e^{-\epsilon_1(J_1+\frac{R}{2})} e^{-\epsilon_2(J_2+\frac{R}{2})} \prod_{i=1}^{|G|} e^{-\alpha_i Q_i} \prod_{l=1}^{|F|} e^{-m_l F_l} \right] \tag{2.4}$$

where (J_1, J_2) are the angular momenta associated to the two $\mathbf{R}^2$ planes, R is the Cartan generator of $SU(2)_R$ symmetry, $(Q_1, \cdots, Q_{|G|})$ are the electric charges of $U(1)^{|G|} \subset G$, and $(F_1, \cdots, F_{|F|})$ are the Cartan generators of the flavor symmetry group F. We also frequently use the notation $\epsilon_+ \equiv \frac{\epsilon_1+\epsilon_2}{2}$, $\epsilon_- \equiv \frac{\epsilon_1-\epsilon_2}{2}$ and $J_l = \frac{J_1-J_2}{2}$, $J_r = \frac{J_1+J_2}{2}$, generating self-dual and anti-self-dual rotation inside the $\mathbf{R}^4$. The fugacity variables used throughout this paper are

$$p_1 = e^{-\epsilon_1},\ p_2 = e^{-\epsilon_2},\ \omega_i = e^{-\alpha_i},\ y_l = e^{-m_l},\ Q = e^{-8\pi^2/g^2},\ t = \sqrt{p_1 p_2},\ u = \sqrt{p_1/p_2}. \tag{2.5}$$

Each multiplet

3 Examples

3.1 Ki-Hong's note

Unity Blowup equations The partition functions of generic 5d $\mathcal{N} = 1$ gauge theories with hypermultiplets in R-representation in the Coulomb branch consist of classical action term, 1-loop term, and instanton partition functions.

$$Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = Z_{\text{class}}(\epsilon_1, \epsilon_2, \vec{a}, m_0)\, Z_{\text{1-loop}}(\epsilon_1, \epsilon_2, \vec{a}, m_i)\, Z_{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) \tag{3.1}$$

where

$$\begin{aligned}
Z_{\text{class}} &= \exp\Bigg[-\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}h_{ij}\phi^i\phi^j + \frac{1}{6}d_{ijk}\phi^i\phi^j\phi^k\right)\Bigg] \\
Z_{\text{1-loop}} &= \exp\Bigg[-\frac{1}{2\epsilon_1\epsilon_2}\Bigg(\sum_{\alpha\in\text{roots}}\left(\frac{1}{6}(\vec{a}\cdot\vec{\alpha})^3 - \frac{1}{4}(\epsilon_1+\epsilon_2)(\vec{a}\cdot\vec{\alpha})^2 + \frac{1}{12}((\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2)(\vec{a}\cdot\vec{\alpha})\right) \\
&\qquad + \sum_{\omega\in\rho(R)}\Bigg(\frac{1}{6}\Big(\vec{a}\cdot\vec{\omega} + m_i + \frac{\epsilon_1+\epsilon_2}{2}\Big)^3 - \frac{\epsilon_1+\epsilon_2}{4}\Big(\vec{a}\cdot\vec{\omega} + m_i + \frac{\epsilon_1+\epsilon_2}{2}\Big)^2 \\
&\qquad\qquad - \frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{24}\Big(\vec{a}\cdot\vec{\omega} + m_i + \frac{\epsilon_1+\epsilon_2}{2}\Big)\Bigg)\Bigg)\Bigg] \\
&\quad \times \text{PE}\Bigg[\frac{1}{(1-p_1)(1-p_2)}\Bigg(-\sum_{\alpha\in\text{roots}} e^{\vec{a}\cdot\vec{\alpha}} + p_1^{1/2}p_2^{1/2}y_i\sum_{\omega\in\rho(R)} e^{\vec{a}\cdot\vec{\omega}}\Bigg)\Bigg].
\end{aligned} \tag{3.2}$$

Here $\vec{a}$ are Coulomb VEVs and $p_{1,2} = e^{\epsilon_{1,2}}$, $y_i = e^{m_i}$. Note that the normal exponential term saturates the zero-point energy of pletheystic exponential terms.[1]

The partition function satisfies so-called "Unity blowup equation"

$$\begin{aligned}
Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = \sum_{\vec{k}\in\vec{\alpha}^\vee} & Z(\epsilon_1, \epsilon_2-\epsilon_1, \vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_1, m_i + r_i\,\epsilon_1, m_0 + r_0\,\epsilon_1) \\
& \times\ Z(\epsilon_1-\epsilon_2, \epsilon_2, \vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_2, m_i + r_i\,\epsilon_2, m_0 + r_0\,\epsilon_2)
\end{aligned} \tag{3.3}$$

[1] Technically, instead of considering this 1-loop prepotential terms, I inserted overall factors to the $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}} Z^{(1)}_{\text{1-loop}}/Z_{\text{1-loop}}$ so that it is written by Sinh terms.

for certain $\vec{r}_a$, r_i, r_0's. Here $\vec{\alpha}^\vee$ is the coroot lattice where the long root is normalized to have norm 2. The r_i's and r_0 are some numbers specifying the blowup equations.

Technically r_i's are constrained to be half integers since, for each single letter 1-loop partition functions

$$Z_{i,\vec{\omega}} = \mathrm{PE}\left[\frac{p_1^{1/2}p_2^{1/2}}{(1-p_1)(1-p_2)}y_i\, e^{\vec{a}\cdot\vec{\omega}}\right], \tag{3.4}$$

the ratio between shifted ones and unshifted one is

$$\begin{aligned}
l^{\vec{k}}_{i,\vec{\omega}} &= Z^{(1)}_{i,\vec{\omega}}Z^{(2)}_{i,\vec{\omega}}/Z_{i,\vec{\omega}} \\
&= \mathrm{PE}\left[\frac{p_1^{r_i}p_2^{1/2}y_i}{(1-p_1)(1-p_2/p_1)}p_1^{\vec{k}\cdot\vec{\omega}}e^{\vec{a}\cdot\vec{\omega}} + \frac{p_1^{1/2}p_2^{r_i}y_i}{(1-p_1/p_2)(1-p_2)}p_2^{\vec{k}\cdot\vec{\omega}}e^{\vec{a}\cdot\vec{\omega}} - \frac{p_1^{1/2}p_2^{1/2}y_i}{(1-p_1)(1-p_2)}e^{\vec{a}\cdot\vec{\omega}}\right] \\
&= \mathrm{PE}\left[\frac{p_1^{1/2}p_2^{1/2}y_i}{(1-p_1)(1-p_2)(p_1-p_2)}e^{\vec{a}\cdot\vec{\omega}}\Big((1-p_2)p_1^{\vec{k}\cdot\vec{\omega}+r_i+1/2} - (1-p_1)p_2^{\vec{k}\cdot\vec{\omega}+r_i+1/2}\Big) - (p_1-p_2)\right]
\end{aligned} \tag{3.5}$$

For the $l^{\vec{k}}_{i,\vec{\omega}}$ to be finite rational function, the plethystic exponent must be finite series. It can be satisfied only when r_i is a half integer.

Instanton partition functions from blowup equations From blowup equations one can compute the partition functions as follows. Rewriting the blowup equation as

$$1 = \sum_{\vec{k}\in\vec{\alpha}^\vee} f_{\vec{k}}\, l_{\vec{k}} \frac{Z^{(1)}_{\text{inst}}Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \tag{3.6}$$

where $f_{\vec{k}} = Z^{(1)}_{\text{class}}Z^{(2)}_{\text{class}}/Z_{\text{class}}$ and $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}}Z^{(2)}_{\text{1-loop}}/Z_{\text{1-loop}}$ with abbreviated notation

$$\begin{aligned}
Z^{(1)} &= Z(\epsilon_1, \epsilon_2-\epsilon_1, \vec{a}+\vec{k}\,\epsilon_1, m_i + r_i\,\epsilon_1, m_0 + r_0\,\epsilon_1) \\
Z^{(2)} &= Z(\epsilon_1-\epsilon_2, \epsilon_2, \vec{a}+\vec{k}\,\epsilon_2, m_i + r_i\,\epsilon_2, m_0 + r_0\,\epsilon_2)
\end{aligned} \tag{3.7}$$

Here note that $l_{\vec{k}}$ is independent of $Q = e^{-m_0}$, and $f_{\vec{k}}$ is some overall factor in the order of $Q^{\vec{k}\cdot\vec{k}/2}$. Expanding the equation by instanton fugacity Q, then at each Q^n level the equation is written by

$$\delta_{n,0} = p_1^{r_0}Z^{(1)}_n + p_2^{r_0}Z^{(2)}_n - Z_n + \sum_{\vec{k}\neq 0} f_{\vec{k},r_0}\, l_{\vec{k}}\left(\frac{Z^{(1)}_{\text{inst}}Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}}\right)\Bigg|_{O(Q^{n-\vec{k}\cdot\vec{k}/2})}. \tag{3.8}$$

Since each Z_k and $Z^{(1,2)}_k$ are independent of r_0, one can solve (3.8) with three blowup equations with same r_i's but different r_0's.

The blowup equations for instanton partition functions of pure YM theory with generic gauge group were already studied in [4]. They are actually (3.1) with

$$\vec{r}_a = 0, \qquad r_0 = d - h^\vee/2 \tag{3.9}$$

where $d = 0, \cdots, h^\vee$. We extend these blowup equations to the theories with matters based on pure YM blowup equations. If one restrict the cases to $\vec{r}_a = 0$, as we explained in the previous section, the r_i's are technically required to be half intergers. Thus we look for the r_0's that provides the correct instanton partition functions by solving (3.8) while fixing $\vec{r}_a = 0$ and $r_i = 1/2$. Here are the results.

G	matter	r_0	d
$SU(N)_\kappa$	$N_f \times \boldsymbol{N}$	$d - N/2 - \kappa/2$	$0 \le d \le N - \lvert\kappa\rvert - 2N_f - 1$(?)
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \le d \le 6$
$SO(7)$	pure	$d - 5/2$	$0 \le d \le 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \le d \le 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \le d \le 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \le d \le 3$
G_2	pure	$d - 4/2$	$0 \le d \le 4$
G_2	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \le d \le 3$
F_4	pure	$d - 9/2$	$0 \le d \le 9$
F_4	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \le d \le 6$
F_4	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \le d \le 3$

They were tested by comparing the resulting instanton partition functions with the known results from [5]($SO(7)$ and G_2) and [6](F_4 with $N_{\mathbf{26}} = 2$). They were compared numerically, putting random numbers on the fugacities. Note that matters shift the r_0, each by one quarter of their Dynkin indices. It seems to differ from blowup formula for $SU(N)_\kappa + N_f$ instantons, where r_0 was affected only by its CS-level κ. However, one can rewrite the r_0 as

$$\begin{aligned} r_0 &= d - N/2 - \left(\kappa + \frac{1}{2}N_f\right)/2 + N_f/4 \\ &= d - N/2 - \kappa_{\text{eff}}/2 + N_f \times I_{\text{fund}}. \end{aligned} \tag{3.10}$$

Since fundamental matters shifts the effective CS-level, they cancel their index contributions and consequently the r_0 apparently looks independent of matters.

By above observations, we write the unity blowup equation for generic gauge groups and matter representations.

$$\begin{aligned} Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = \sum_{\vec{k} \in \vec{\alpha}^\vee} & Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\epsilon_1, m_i + \epsilon_1/2, m_0 + r_0\epsilon_1) \\ & \times Z(\epsilon_1 - \epsilon_2, \vec{a} + \vec{k}\epsilon_2, m_i + \epsilon_2/2, m_0 + r_0\epsilon_2) \end{aligned} \tag{3.11}$$

with

$$r_0 = d - h^\vee/2 - \kappa_{\text{eff}}/2 + N_{\boldsymbol{R}} \times I_{\boldsymbol{R}}. \tag{3.12}$$

Here $I_{\boldsymbol{R}}$ is the Dynkin index of $\boldsymbol{R}$ representation.

$SU(6)_3 + 1 \times \mathbf{20}$ As a non-trivial test, we consider the instanton partition function of the $SU(6)_3 + \mathbf{20}$ whose 5-brane realization was found recently [7]. Its web-diagram is given as figure.

(Written before computing the $SU(6)_3 + 20$ instanton partition function.)

Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the $SU(3) \times SU(3) \times U(1) \subset SU(6)$ where the $SU(6)$ multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}} &: \mathbf{35} \longrightarrow (\mathbf{8}, 1)_0 \oplus (1, \mathbf{8})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_2 \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-2} \oplus (1,1)_0, \\ \Phi_{ijk} &: \mathbf{20} \longrightarrow (\mathbf{3}, \bar{\mathbf{3}})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{3})_1 \oplus (1,1)_3 \oplus (1,1)_{-3}. \end{aligned} \tag{3.13}$$

Here to fit with the web-diagram, we set Φ_{156} and Φ_{234} are $(1,1)_3$ and $(1,1)_{-3}$. Once Φ_{156} and Φ_{234} get non-zero VEVs,

When $a_5 = -a_1 - a_6$, the web-diagram factorizes to two $SU(3)_3$ whose Coulomb VEVs are (a_1, a_5, a_6) and (a_2, a_3, a_4). In the gauge theory, it can be seen partly from prepotential. The prepotential of $S(6)_3 + 1 \times \mathbf{20}$ is

$$\mathcal{F} = \frac{1}{2} m_0 \sum_{i=1}^{6} a_i^2 + \frac{1}{2} \sum_{i=1}^{6} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3 - \frac{1}{6} \sum_{1<i<j} (a_1 + a_j + a_k)^3 \tag{3.14}$$

at the Weyl chamber $a_1 > \cdots > a_6$. As one sets the Coulomb VEV $a_6 = -a_1 - a_5$ and $a_4 = -a_2 - a_3$, one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \tag{3.15}$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2} m_0 \sum_{i=1}^{3} a_i^2 + \frac{1}{2} \sum_{i=1}^{3} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3. \tag{3.16}$$

It is Higgsed by

4 Conclusion

Acknowledgments

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References

[1] K. A. Intriligator, D. R. Morrison and N. Seiberg, *Five-Dimensional Supersymmetric Gauge Theories and Degenerations of Calabi-Yau Spaces*, Nucl. Phys. **B497** (1997) 56–100, [hep-th/9702198].

[2] N. A. Nekrasov, *Seiberg-Witten Prepotential from Instanton Counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831–864, [hep-th/0206161].

[3] N. Nekrasov and A. Okounkov, *Seiberg-Witten Theory and Random Partitions*, *Prog. Math.* **244** (2006) 525–596, [hep-th/0306238].

[4] C. A. Keller and J. Song, *Counting Exceptional Instantons*, *JHEP* **07** (2012) 085, [1205.4722].

[5] H.-C. Kim, J. Kim, S. Kim, K.-H. Lee and J. Park, *6D Strings and Exceptional Instantons*, 1801.03579.

[6] M. Del Zotto and G. Lockhart, *Universal Features of BPS Strings in Six-Dimensional SCFTs*, *JHEP* **08** (2018) 173, [1804.09694].

[7] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, *Rank-3 Antisymmetric Matter on 5-Brane Webs*, 1902.04754.