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Instantons from Blow-up

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Abstract: The Nekrasov partition function for 4d $\mathcal{N}=2$ or 5d $\mathcal{N}=1$ gauge theory on the blow up of a point $\hat{\mathbb{C}}^2$ can be written in terms of the partition function on the flat space $\mathbb{C}^2$. At the same time, the partition function on the blow up is identical to the partition function on a flat space for sufficiently large class of examples. This relation enables us to compute the instanton partition functions for 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ gauge theories for arbitrary gauge theory with large class of matter representations without knowing explicit construction of the instanton moduli space. Remarkably, the instanton partition function is completely determined by the perturbative part. We obtain the partition function for the previously unknown theories: exceptional gauge groups EFG with fundamental/spinor hypermultiplets and more. We also compute the case with $SU(6)$ with rank-3 antisymmetric tensor and compare with the topological vertex computation using the recently found 5-brane web configuration.

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The essential idea of using the blow-up for the instanton counting is that the gauge theory partition function for a 4d $\mathcal{N}=2$ theory or 5d $\mathcal{N}=1$ on a blow up of a point $\hat{\mathbb{C}}^2$ (or $S^1\times\hat{\mathbb{C}}^2$) can be written in two different ways. This allows us to write a recursion relation for the instanton partiton function that can be solved easily [1–4].

Partition function on the Blow-up $\hat{\mathbb{C}}^2$ One of the expression is obtained via localization on the Coulomb branch obtained by patching together flat-space partition functions. The blow up of a point of $\mathbb{C}^2$ can be described as a subspace of $\mathbb{C}^2\times\mathbb{P}^1$ defined as

$$\{(x,y),[z:w]\in\mathbb{C}^2\times\mathbb{P}^1|xw=yz\}\ , \tag{2.1}$$

where $[z:w]$ represents the homogeneous coordinates on $\mathbb{P}^1$. Notice that this space is identical to $\mathbb{C}^2$ when $(x,y)\neq(0,0)$, and the origin is replaced by complex projective plane. We will be considering $U(1)^2$-equivariant partition function, with each $U(1)$'s acting as the rotation of two complex planes. This $U(1)^2$ action acts on $\hat{\mathbb{C}}^2$ as

$$((x,y),[z:w])\mapsto(e^{i\epsilon_1}x,e^{i\epsilon_2}y),[e^{i\epsilon_1}z,e^{i\epsilon_2}w])\ . \tag{2.2}$$

The zero sized instantons will be located at the two fixed points of the $U(1)^2$ action, namely the north pole and the south pole of the sphere. At these points, the weights of the $U(1)^2$ action become

$$((0,0),[1,0]):(\epsilon_1,\epsilon_2-\epsilon_1),\qquad((0,0),[0,1]):(\epsilon_1-\epsilon_2,\epsilon_2)\ . \tag{2.3}$$

Now, the (full) partition function on a blow up (includes both the perturbative and the instanton parts) $\hat{Z}$ can be written as a sum over a product of the partition functions at two fixed points as (here we turn off any external flux that can be supported on the blow up) [5–9]

$$\hat{Z}(\vec{a}, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} Z(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1) Z(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2) \ , \tag{2.4}$$

where $Z(\vec{a}, \epsilon_1, \epsilon_2)$ is the (full) partition function on $\mathbb{C}^2$ where $\vec{a}$ is the Coulomb parameter. Here Λ is the weight lattice of the gauge group and the vector $\vec{k}$ labels different flux configurations on the divisor of the blow-up classified by the first Chern numbers. We obtain this expression by performing the localization on the Coulomb branch. On the Coulomb branch, the gauge group is broken to $U(1)^r$ where r is the rank of the gauge group. We need to patch together all possible field configurations with zero sized instantons located at north and south poles, and all the inequivalent configurations are labeled by the first Chern numbers which we have to sum over. The shift of the Coulomb parameter comes from the gauge transformation along the equator that connects the north and the south pole so that $\vec{a}_N - \vec{a}_S = \vec{k}(\epsilon_1 - \epsilon_2)$.

Partition function on $\hat{\mathbb{C}}^2$ vs $\mathbb{C}^2$ The second expression for the partition function on the blow-up is that $\hat{Z}$ is actually identical to the partition function Z on the flat space $\mathbb{C}^2$ (or $S^1 \times \mathbb{C}^2$) [1–3]. This can be argued as follows: The blow up $\hat{\mathbb{C}}^2$ is identical to $\mathbb{C}^2$ except for the origin, which is replaced by $\mathbb{P}^1$, called as the exceptional divisor. The blow-up can be also identified as the total space of the bundle $\mathcal{O}(-1) \to \mathbb{P}^1$. The Nekrasov partition function gets contributions only from the zero-size instantons localized at the fixed points of the $U(1)^2$ action so that the size of the divisor should not affect the partition function as we smoothly shrink it. Therefore, we expect that the partition function on the blow-up to be identical to that of the flat $\mathbb{C}^2$.

This is a special feature of the blow-up of a point. In principle, there can be extra states coming from the $\mathbb{P}^1$. As we shrink the size of the $\mathbb{P}^1$, they may become massless which might in principle contribute to the partition function when there is a singularity as we blow-down. For example, if we consider the total space of $\mathcal{O}(-2) \to \mathbb{P}^1$ and shrink the base, we land on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ which is singular at the origin. In our case, we do not obtain any singularity as we blow down the sphere. As we will discuss later, this simple picture does not necessarily hold when there are too many hypermultiplets. But this is shown to be the case for the pure YM theory, and we will demonstrate that it is also true for many interesting cases with matter hypermultiplets.[1]

Therefore, we have a powerful identity:

$$Z(\vec{a}, \epsilon_1, \epsilon_2, q) = \sum_{\vec{k} \in \Lambda} Z(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, q) Z(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, q) \ . \tag{2.5}$$

[1] The orbifold partition function can be also obtained in two different ways modulo some subtle scheme dependence related to wall-crossing. [10]

But this is not enough to determine the partition function since there are three unknown functions and one relation. To achieve this we need to consider a 'correlation function' $\langle \mu(C)\mu(C)\ldots\rangle$ of operators μ associated to the two-cycle C in the blow-up. In terms of the component fields of the Doanldson-twisted theory, it can be written as

$$\mu(C) = \int d^4x \left(\omega \wedge \phi F + \frac{1}{2}\psi \wedge \psi + HF \wedge F\right) \ , \tag{2.6}$$

where ω is the two-form dual to C and $H = \epsilon_1|z_1|^2 + \epsilon_2|z_2|^2$ is the moment map for the $U(1)^2$ action.

Let us consider the generating function of the correlators by inserting $e^{t\cdot\mu(C)}$. Then the partiton function $\hat{Z}^{(d)}$ can be written in a very simple form as

$$\hat{Z}^{(t)}(\vec{a}, m, \epsilon_1, \epsilon_2, q) = \sum_{\vec{k}\in\Lambda} \exp\left(t \cdot \mu(\vec{k})\right) Z^{(N),t}(\vec{k}) Z^{(S),t}(\vec{k}) \ , \tag{2.7}$$

where

$$\begin{aligned} Z^{(N),t}(\vec{k}) &\equiv Z(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, qe^{t\epsilon_1}) \ , \\ Z^{(S),t}(\vec{k}) &\equiv Z(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, qe^{t\epsilon_2}) \ , \end{aligned} \tag{2.8}$$

and $\mu(\vec{k})$ is the contribution coming from the operator $\mu(C)$ which is a function of $\vec{k}$ that also depends on $\vec{a}, \epsilon_1, \epsilon_2$ that we often abbreviate. Explicitly, it is given as

$$\exp\left(\mu(\vec{k})\right) = \exp\left(\vec{k}\cdot\vec{a} + \frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1 + \epsilon_2)\right) \ . \tag{2.9}$$

Notice that we shift the instanton parameter by equivariant parameters at both poles $q \to qe^{t\epsilon_{1,2}}$. This is due to the term $HF \wedge F$ in (2.6) which makes the effective instanton parameter as $qe^{t(\epsilon_1+\epsilon_2)}$ At the north pole, $\epsilon_1 + \epsilon_2 \to \epsilon_1 + (\epsilon_2 - \epsilon_1) = \epsilon_2$. We need to compensate this part by shifting q by $e^{t\epsilon_1}$. Similarly, we need to shift the parameters at the south pole.

Now, as we shrink the two cycles to recover the flat $\mathbb{C}^2$, the effect of the insertion of μ turns out to give a vanishing contribution due to the conservation laws. To see this, let us consider 4d $\mathcal{N} = 2$ theory for the moment. The operator (2.6) has $U(1)_R$ charge $+2$ of the 4d $\mathcal{N} = 2$ supersymmetry. The instanton carries $4h^\vee$ which is responsible for breaking $U(1)_R$ to a discrete subgroup $\mathbb{Z}_{2h^\vee - I(R)}$ where $h^\vee$ is the dual Coxeter number of the gauge group and $I(R)$ is the Dynkin index of the hypermultiplet representation R.[2] Unless the charges add up to zero, the correlation function vanishes. Therefore, expanding in powers of t,

$$\hat{Z}^{(t)}_{4d} = Z_{4d} + \mathcal{O}(t^{2h^\vee - I(R)}) \ . \tag{2.10}$$

This allows us to write 3 relations among for the 3 unknowns to solve for the instanton partition function as long as the hypermultiplet representation is not too large, namely, when

[2] Here we normalize the Dynkin index so that the fundamental for the $SU(N)$ has $I = 1$. Therefore the $SU(N_c)$ SQCD with N_f fundamental hypers have $I(R) = N_f$.

$2h^\vee - I(R) > 2$. In this case, we can expand $\hat{Z}^{(t)}$ up to order t^2 to obtain 3 relations that is enough to fix the instanton part of the partition function from the *perturbative* part as we will see soon.

For the case of 5d $\mathcal{N}=1$ theory in S^1, we do not have $U(1)_R$ symmetry to impose the selection rule. The natural uplift of the operator $\mu(C)^d$ is given via exponentiation $\exp(d\beta \cdot \mu(C))$ where β is the size of S^1. We can always choose our scale so that $\beta = 1$ and we will omit this parameter. This stems from the fact that $\mathrm{tr}\phi^2$ is not a well-defined operator in 5d theory on $S^1 \times M_4$. In order to take into account the periodic boson, we need to exponentiate to obtain the 'Coulomb parameter'. The $\mu(C)$ operator has to be exponentiated accordingly to take into account the periodicity and the coefficient in front of the exponent has to be properly quantized.

[TODO: Explain why $e^{\mu(C)}$ is good topological operator in 5d. Explain why inserting this does not alter the partition function.]

Therefore, we obtain

$$Z = \sum_{\vec{k}\in\Lambda} \exp\left(d\mu(\vec{k})\right) Z^{(N),d}(\vec{k}) Z^{(S),d}(\vec{k}) \qquad (d = 0, 1, \ldots, h^\vee - I(R)/2 - |\kappa|) \ , \tag{2.11}$$

for the 5d theories. We call the relation above as the blow-up equation.

Instanton partition function from the blow-up equation From the blow up equation (2.11), we can determine the instanton partition function for the 5d $\mathcal{N}=1$ theory on $S^1 \times \mathbb{C}^2$. To see this, let us decompose the partition function in terms of classical, 1-loop and the instanton piece:

$$Z(\vec{a}, m, \epsilon_1, \epsilon_2, q) = Z_{\text{class}}(\vec{a}, \epsilon_1, \epsilon_2, q) Z_{\text{1-loop}}(\vec{a}, m, \epsilon_1, \epsilon_2) Z_{\text{inst}}(\vec{a}, m, \epsilon_1, \epsilon_2, q) \tag{2.12}$$

Then the blow-up equation can be rewritten as

$$Z_{\text{inst}} = \sum_{\vec{k}} \left[\exp\left(d\mu(\vec{k})\right) \frac{Z_{\text{pert}}^{(N),d} Z_{\text{pert}}^{(S),d}}{Z_{\text{pert}}}\right] Z_{\text{inst}}^{(N),d} Z_{\text{inst}}^{(S),d} \ , \tag{2.13}$$

where $Z_{\text{pert}} \equiv Z_{\text{class}} Z_{\text{1-loop}}$ and the superscript (N/S) denotes appropriate shift of parameters in the north/south pole

$$\begin{aligned} Z^{(N),d}(\vec{k}) &\equiv Z(\vec{a} + \vec{k}\epsilon_1, m, \epsilon_1, \epsilon_2 - \epsilon_1, q e^{d\epsilon_1}) \ , \\ Z^{(S),d}(\vec{k}) &\equiv Z(\vec{a} + \vec{k}\epsilon_2, m, \epsilon_1 - \epsilon_2, \epsilon_2, q e^{d\epsilon_2}) \ . \end{aligned} \tag{2.14}$$

Notice that there is no q-dependence in the 1-loop part, hence no d-dependence as well. The instanton piece can be expanded in terms of the instanton number as

$$Z_{\text{inst}}(\vec{a}, \epsilon_1, \epsilon_2, q) = \sum_{n\geq 0} q^n Z_n(\vec{a}, \epsilon_1, \epsilon_2) \ . \tag{2.15}$$

Pluggin in, we obtain

$$q^n Z_n = \sum_{\vec{k}} e^{d\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)\right)} q^{\frac{1}{2}\vec{k}\cdot\vec{k}} \left(\frac{Z^{(N)}_{\text{1-loop}} Z^{(S)}_{\text{1-loop}}}{Z_{\text{1-loop}}}\right) q^{\ell+m} e^{d(\ell\epsilon_1+m\epsilon_2)} Z^{(N)}_\ell Z^{(S)}_m \ , \tag{2.16}$$

where $q^{\frac{1}{2}\vec{k}\cdot\vec{k}}$ comes from the classical piece as we will see later. Collecting the terms with the same order, we obtain

$$Z_n = \sum_{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n} e^{d\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)+\ell\epsilon_1+m\epsilon_2\right)} g(\vec{k}) Z^{(N)}_\ell(\vec{k}) Z^{(S)}_m(\vec{k}) \ , \tag{2.17}$$

where $g(\vec{k}) \equiv \frac{Z^{(N)}_{\text{1-loop}} Z^{(S)}_{\text{1-loop}}}{Z_{\text{1-loop}}}$. Notice that the right-hand side of the above equation only involves the instanton parts with $\ell, m < n$. Therefore, one can determine the n-instanton partition function recursively from $Z_0 = 1$. To do this, let us separate n-instanton contribution from the above expression. We get

$$Z_n = e^{\epsilon_1 dn} Z^{(N)}_n + e^{\epsilon_2 dn} Z^{(S)}_n + I^{(d)}_n(\vec{a}, m, \epsilon_1, \epsilon_2) \ , \tag{2.18}$$

where

$$I^{(d)}_n(\vec{a}, m, \epsilon_1, \epsilon_2) = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n \\ \ell, m<n}} e^{d\left(\vec{k}\cdot\vec{a}+\frac{1}{2}\vec{k}\cdot\vec{k}(\epsilon_1+\epsilon_2)+\ell\epsilon_1+m\epsilon_2\right)} g(\vec{k}) Z^{(N)}_\ell(\vec{k}) Z^{(S)}_m(\vec{k}) \ . \tag{2.19}$$

Now, we can solve for $Z_n, Z^{(N)}_n, Z^{(S)}_n$ by using the blow-up equation for $d = 0, 1, 2$ to obtain

$$Z_n(\vec{a}, m, \epsilon_1, \epsilon_2) = \frac{e^{n(\epsilon_1+\epsilon_2)} I^{(0)}_n - (e^{n\epsilon_1} + e^{n\epsilon_2}) I^{(1)}_n + I^{(2)}_n}{(1 - e^{n\epsilon_1})(1 - e^{n\epsilon_2})} \ . \tag{2.20}$$

Since $I^{(d)}_n$ only depends on Z_m with $m < n$ we can determine the instanton partition function recursively.

We now need to determine the $g(\vec{k})$, but this is also completely determined by the *1-loop part* of the partition function. Therefore we land on a remarkable conclusion: The non-perturbative partition function is completely fixed by the perturbative partition function! Notice that we have arrived at this conclusion not by requiring the perturbative series to be well-defined, as is often done in the resurgence analysis. Instead, we simply demand consistent answer upon deforming the spacetime geometry smoothly. This consistency condition is entirely non-perturbative, that the QFT has to be well-defined regardless of its spacetime.

In the remainder of this section we will give explicit formulae for 4d and 5d gauge theories. Especially, we will obtain a closed-form expression for the one-instanton partition function for a large class of gauge theories.

2.1 Instanton partition function for the 5d theory

Let us write the perturbative part of the partition function. The classical part is given as

$$Z_{\text{class}} = \exp\left[-\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}h_{ij}a^i a^j + \frac{1}{6}d_{ijk}a^i a^j a^k\right)\right] \tag{2.21}$$

where a_i denotes the Coulomb branch parameter and h_{ij}, d_{ijk} denotes the effective gauge couplings and Chern-Simons couplings in the Coulomb branch. The 1-loop part of the vector multiplet is given as

$$\begin{aligned} Z^V_{\text{1-loop}} = \exp&\left[-\frac{1}{2\epsilon_1\epsilon_2}\sum_{\alpha\in\Delta}\left(\frac{1}{6}(\vec{a}\cdot\vec{\alpha})^3 - \frac{\epsilon_1+\epsilon_2}{4}(\vec{a}\cdot\vec{\alpha})^2 + \frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{12}(\vec{a}\cdot\vec{\alpha})\right)\right] \\ &\times \text{PE}\left[-\frac{1}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\alpha\in\Delta}e^{\vec{a}\cdot\vec{\alpha}}\right], \end{aligned} \tag{2.22}$$

where Δ denotes the set of all roots. The 1-loop partition function of the hypermultiplet is given as

$$\begin{aligned} Z^H_{\text{1-loop}} = \exp&\left[-\frac{1}{2\epsilon_1\epsilon_2}\sum_{\omega\in R}\left(\frac{1}{6}\left(\vec{a}\cdot\vec{\omega}+m_i^{\text{phy}}\right)^3 - \frac{\epsilon_1+\epsilon_2}{4}\left(\vec{a}\cdot\vec{\omega}+m_i^{\text{phy}}\right)^2\right.\right. \\ &\left.\left.+\frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{12}\left(\vec{a}\cdot\vec{\omega}+m_i^{\text{phy}}\right)\right)\right] \times \text{PE}\left[\frac{e^{m_i^{\text{phy}}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})}\sum_{\omega\in R}e^{\vec{a}\cdot\vec{\omega}}\right] \end{aligned} \tag{2.23}$$

where R is the set of all weight vectors in the representation of the hypermultiplet. We define the physical mass parameter as $m_i^{\text{phy}} = m_i + \epsilon_+$ and $\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}$, since there is an effect of topological twisting that shifts the 'mass parameter' m_i by a unit of $SU(2)_R$ rotation [11].

Now, let us compute the $g(\vec{k})$ which is given by the ratio of the 1-loop part of the partition function. It turns out piece of the (2.22), (2.23) that are not inside the PE eventually cancel out due to $\vec{k}\to-\vec{k}$ and $\vec{a}\to-\vec{a}$ symmetry except for one single term for the hyper. Therefore, we obtain

$$g(\vec{k})^V = \prod_{\vec{\alpha}\in\Delta}\text{PE}\left[\frac{e^{\vec{a}\cdot\vec{\alpha}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} - \frac{e^{(\vec{a}+\vec{k}\epsilon_1)\cdot\vec{\alpha}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2-\epsilon_1})} - \frac{e^{(\vec{a}+\vec{k}\epsilon_2)\cdot\vec{\alpha}}}{(1-e^{\epsilon_1-\epsilon_2})(1-e^{\epsilon_2})}\right], \tag{2.24}$$

$$g(\vec{k})^H = \prod_{\vec{w}\in R}e^{\frac{(\vec{k}\cdot\vec{w})^2}{4}m^{\text{phy}}}\text{PE}\left[\frac{-e^{\vec{a}\cdot\vec{w}+m^{\text{phy}}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} + \frac{e^{(\vec{a}+\vec{k}\epsilon_1)\cdot\vec{w}+m^{\text{phy}}}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2-\epsilon_1})} + \frac{e^{(\vec{a}+\vec{k}\epsilon_2)\cdot\vec{w}+m^{\text{phy}}}}{(1-e^{\epsilon_1-\epsilon_2})(1-e^{\epsilon_2})}\right],$$

for the vector and hypermultiplets respectively. Both of them can be written as

$$g(\vec{k}) = \frac{\prod_i\prod_{\vec{w}\in R_i}e^{\frac{(\vec{k}\cdot\vec{w})^2}{4}m^{\text{phy}}}\text{PE}\left[e^{\vec{a}\cdot\vec{w}-m_i^{\text{phy}}}A(\vec{k}\cdot\vec{w},e^{\epsilon_1}e^{\epsilon_2})\right]}{\prod_{\vec{\alpha}\in\Lambda}\text{PE}\left[e^{\vec{a}\cdot\vec{\alpha}}A(\vec{k}\cdot\vec{\alpha},e^{\epsilon_1}e^{\epsilon_2})\right]}, \tag{2.25}$$

where i runs over various hypermultiplets in the theory and $p_1 = e^{\epsilon_1}, p_2 = e^{\epsilon_2}$. Here the function A is given as

$$A(k, p_1, p_2) = \frac{1}{(1-p_1)(1-p_2)} - \frac{p_1^k}{(1-p_1)(1-p_2/p_1)} - \frac{p_2^k}{(1-p_1/p_2)(1-p_2)} \,. \tag{2.26}$$

We can easily see that $A(k, p_1, p_2)$ vanishes at $k = 0, -1$. After some work, it is not hard to find that $A(k, p_1, p_2)$ can be written in terms of a finite sum as

$$A(k, p_1, p_2) = \begin{cases} \displaystyle\sum_{m+n \le k-1} p_1^m p_2^n & (k > 0) \\ \displaystyle\sum_{m+n \le -k-2} p_1^{-m-1} p_2^{-n-1} & (k < -1) \\ 0 & (k = 0, -1) \end{cases} \,. \tag{2.27}$$

Upon taking PE, we obtain

$$\mathcal{L}_k(x, \epsilon_1, \epsilon_2) \equiv \mathrm{PE}\left[e^x A(k, e^{\epsilon_1}, e^{\epsilon_2})\right] = \begin{cases} \displaystyle\prod_{\substack{m,n \ge 0 \\ m+n \le k-1}} \left(1 - e^{x + m\epsilon_1 + n\epsilon_2}\right) & (k > 0) \\ \displaystyle\prod_{\substack{m,n \ge 0 \\ m+n \le -k-2}} \left(1 - e^{x - (m+1)\epsilon_1 - (n+1)\epsilon_2}\right) & (k < -1) \\ 1 & (k = 0, -1) \end{cases} \tag{2.28}$$

Let us state our final answer:

$$Z_n(\vec{a}, \vec{m}^{\mathrm{phy}}, \epsilon_1, \epsilon_2) = \frac{e^{n(\epsilon_1+\epsilon_2)} I_n^{(0)} - (e^{n\epsilon_1} + e^{n\epsilon_2}) I_n^{(1)} + I_n^{(2)}}{(1 - e^{n\epsilon_1})(1 - e^{n\epsilon_2})} \,, \tag{2.29}$$

with

$$\begin{aligned} I_n^{(d)} = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k} + \ell + m = n \\ \ell, m < n}} \Bigg(& \exp\left[d \left(\vec{k} \cdot \vec{a} + \frac{1}{2} \vec{k} \cdot \vec{k} (\epsilon_1 + \epsilon_2) + \ell\epsilon_1 + m\epsilon_2 \right) \right] \\ & \times \frac{\displaystyle\prod_i \prod_{\vec{w} \in R_i} e^{\frac{(\vec{k}\cdot\vec{w})^2}{4} m^{\mathrm{phy}}} \mathcal{L}_{\vec{k}\cdot\vec{w}}(\vec{a} \cdot \vec{w} - m_i^{\mathrm{phy}}, \epsilon_1, \epsilon_2)}{\displaystyle\prod_{\vec{\alpha} \in \Lambda} \mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a} \cdot \vec{\alpha}, \epsilon_1, \epsilon_2)} \\ & \times Z_\ell(\vec{a} + \vec{k}\epsilon_1, \vec{m}^{\mathrm{phy}}, \epsilon_1, \epsilon_2 - \epsilon_1) Z_m(\vec{a} + \vec{k}\epsilon_2, \vec{m}^{\mathrm{phy}}, \epsilon_1 - \epsilon_2, \epsilon_2) \Bigg) \,, \end{aligned} \tag{2.30}$$

where i runs over all the hypermultiplets.

One-instanton partition function Let us compute the 1-instanton partition function using our formula. In this case, we only take the sum over the long roots having $\vec{k}\cdot\vec{k}=2$ and $\ell=m=0$. Therefore,

$$I_1^{(d)} = e^{d(\epsilon_1+\epsilon_2)} \sum_{\vec{\gamma}\in\Delta_l} \frac{e^{d\vec{\gamma}\cdot\vec{a}} M(\gamma)}{L(\gamma)} \ , \tag{2.31}$$

where Δ_l is the set of all long roots and

$$\begin{aligned} L(\gamma) &\equiv \prod_{\alpha\in\Delta} \mathcal{L}_{\gamma\cdot\alpha}(a\cdot\alpha,\epsilon_1,\epsilon_2) = \prod_{\gamma\cdot\alpha=\pm2}\prod_{\gamma\cdot\alpha=1} \mathcal{L}_{\gamma\cdot\alpha}(a\cdot\alpha,\epsilon_1,\epsilon_2) \\ &= (1-e^{a_\gamma+\epsilon_1})(1-e^{a_\gamma+\epsilon_2})(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2}) \prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha}) \ , \end{aligned} \tag{2.32}$$

where $a_\gamma \equiv \vec{a}\cdot\vec{\gamma}$ with $a_\alpha \equiv \vec{a}\cdot\vec{\alpha}$ and

$$M(\gamma) \equiv \prod_{\vec{w}\in R} e^{\frac{(\vec{\gamma}\cdot\vec{w})^2}{4} m^{\rm phy}} \mathcal{L}_{\gamma\cdot w}(a_w,\epsilon_1,\epsilon_2) = e^{I(R)m^{\rm phy}/2} \prod_{\vec{w}\in R} \mathcal{L}_{\gamma\cdot w}(a_w,\epsilon_1,\epsilon_2) \ , \tag{2.33}$$

with $a_w \equiv \vec{a}\cdot\vec{w}$ and $I(R)$ is the Dynkin index of the representation R. Therefore, the one-instanton partition function can be written explicitly as

$$\begin{aligned} Z_1 &= \frac{e^{\epsilon_1+\epsilon_2} I_n^{(0)} - (e^{\epsilon_1}+e^{\epsilon_2}) I_n^{(1)} + I_n^{(2)}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma\in\Delta_l} \frac{(1-e^{a_\gamma+\epsilon_1})(1-e^{a_\gamma+\epsilon_2})M(\gamma)}{L(\gamma)} \ . \end{aligned} \tag{2.34}$$

For the case of pure SYM theory with no matters, $M(\gamma)=1$ so that

$$\begin{aligned} Z_1^{\rm SYM} &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma\in\Delta_l} \frac{1}{(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma\in\Delta_l} \frac{e^{(h^\vee-1)a_\gamma/2}}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})} \ , \end{aligned} \tag{2.35}$$

which is the one derived in [4, 12].

For the case with fundamental matters, we find that for the long root γ, there are only weights with $\gamma\cdot w = 0,\pm1$. Therefore, the $M(\gamma)$ can be simplified to give

$$M(\gamma) = \prod_{\gamma\cdot w=0,\pm1} \mathcal{L}_{\gamma\cdot w}(a_w,\epsilon_1,\epsilon_2) = \prod_{\gamma\cdot w=1}(1-e^{a_w-m^{\rm phy}}) \ . \tag{2.36}$$

The one-instanton partition function is now given as

$$\begin{aligned} Z_1 &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma\in\Delta_l} \frac{\prod_{\gamma\cdot w=1}(1-e^{a_w-m^{\rm phy}})}{(1-e^{a_\gamma})(1-e^{-a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(1-e^{a_\alpha})} \\ &= \frac{e^{\epsilon_1+\epsilon_2}}{(1-e^{\epsilon_1})(1-e^{\epsilon_2})} \sum_{\gamma\in\Delta_l} \frac{e^{(h^\vee-1)a_\gamma/2}\prod_{\gamma\cdot w=1}(1-e^{a_w-m^{\rm phy}})}{(e^{a_\gamma/2}-e^{-a_\gamma/2})(1-e^{a_\gamma-\epsilon_1-\epsilon_2})\prod_{\gamma\cdot\alpha=1}(e^{a_\alpha/2}-e^{-a_\alpha/2})} \ , \end{aligned} \tag{2.37}$$

which we conjecture to be true for all the hypermultiplets with the representation with $|\gamma\cdot w|\leq 1$ for all $w\in R$.

2.2 Instanton partition function for 4d gauge theory

Let us discuss the case for the 4d $\mathcal{N}=2$ theory. In principle, one can simply take the 4d limit of the 5d partition function by shrinking $\beta\to 0$. Instead, let us write down the recursion formula directly to deduce the partition function. The perturbative partition functions for the vector and hypermultiplet are

$$Z^{\text{pert}}_{\text{vec}}(\vec{a},q)=\exp\left(-\sum_{\vec{\alpha}\in\Delta}\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{\alpha};q)\right)\ , \tag{2.38}$$

$$Z^{\text{pert}}_{\text{hyp}}(\vec{a},m,q)=\exp\left(\sum_{\vec{w}\in R}\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{w}-m;q)\right)\ . \tag{2.39}$$

Here the gamma function is defined as

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)=\left.\frac{d}{ds}\right|_{s=0}\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty\frac{dt}{t}t^s\frac{e^{-ts}}{(e^{\epsilon_1 t}-1)(e^{\epsilon_2 t}-1)}\ , \tag{2.40}$$

which is formally equivalent to

$$\log\left[\prod_{n,m\geq 0}\left(\frac{x-m\epsilon_1-n\epsilon_2}{\Lambda}\right)\right]\ . \tag{2.41}$$

If we write the equation (2.4) in terms of $Z=Z^{\text{pert}}Z^{\text{inst}}$, we get

$$Z^{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2)=\sum_{\vec{k}\in\Lambda}\frac{Z^{(N),\text{pert}}(\vec{k})Z^{(S),\text{pert}}(\vec{k})}{Z^{\text{pert}}(\vec{a},\epsilon_1,\epsilon_2)}Z^{(N),\text{inst}}(\vec{k})Z^{(S),\text{inst}}(\vec{k}) \tag{2.42}$$

and here we omit the dependence on the Coulomb vev and the Omega deformation parameters. Now the factor in the middle can be explicitly worked out. Let us denote the ratio of the perturbative factor as $f(\vec{k})\equiv Z^{(N),\text{pert}}Z^{(S),\text{pert}}/Z^{\text{pert}}$. Then the ratio of the perturbative factor for the vector multiplet is given as

$$\begin{aligned}f(\vec{k})_{\text{vec}}&=\prod_{\vec{\alpha}\in\Delta}\exp\Big(\gamma_{\epsilon_1,\epsilon_2}(\vec{a}\cdot\vec{\alpha})-\gamma_{\epsilon_1,\epsilon_2-\epsilon_1}(\vec{a}\cdot\vec{\alpha}+\vec{k}\cdot\vec{\alpha}\epsilon_1)-\gamma_{\epsilon_1-\epsilon_2,\epsilon_2}(\vec{a}\cdot\vec{\alpha}+\vec{k}\cdot\vec{\alpha}\epsilon_2)\Big)\\&=\prod_{\vec{\alpha}\in\Delta}\frac{\Lambda^{(\vec{k}\cdot\vec{\alpha})^2/2}}{s(-\vec{k}\cdot\vec{\alpha},\vec{\alpha}\cdot\vec{a},\epsilon_1,\epsilon_2)}=\frac{(\Lambda^{2h^\vee})^{\vec{k}\cdot\vec{k}/2}}{\prod_{\vec{\alpha}\in\Delta}\ell^{\vec{k}}_{\vec{\alpha}}(\vec{a},\epsilon_1,\epsilon_2)}\ ,\end{aligned} \tag{2.43}$$

where $h^\vee$ refers to the dual Coxeter number of the gauge group. Notice that the beta function coefficient for the pure YM theory is given by $b_0=2h^\vee$ and the instanton parameter is given by $q \equiv \Lambda^{2h^\vee}$. The other symbols are given as

$$\ell_{\vec{\alpha}}^{\vec{k}}(\vec{a}, \epsilon_1, \epsilon_2) = s(-\vec{k} \cdot \vec{\alpha}, \vec{a} \cdot \vec{\alpha}, \epsilon_1, \epsilon_2) \tag{2.44}$$

$$s(k, x, \epsilon_1, \epsilon_2) = \begin{cases} \displaystyle\prod_{i,j \geq 0, i+j \leq k-1} (x - i\epsilon_1 - je_2) & (k > 0) \\ \displaystyle\prod_{i,j \geq 0, i+j \leq -k-2} (x + (i+1)\epsilon_1 + (j+1)\epsilon_2) & (k < -1) \\ 1 & (k = 0, -1) \end{cases} \tag{2.45}$$

The final identity of (2.43) involves a bit of work. This follows from the identity ([2], App. E.)

$$\begin{aligned} &\gamma_{\epsilon_1, \epsilon_2 - \epsilon_1}(x + \epsilon_1 k; \Lambda) + \gamma_{\epsilon_1 - \epsilon_2, \epsilon_2}(x + \epsilon_2 k; \Lambda) \\ &\qquad = \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) + \log s(-k, x, \epsilon_1, \epsilon_2) - \frac{k(k-1)}{2} \log \Lambda \ . \end{aligned} \tag{2.46}$$

For the hypermultiplets we get,

$$\begin{aligned} f(\vec{k})_{\text{hyp}} &= \prod_{\vec{w} \in R} \exp\Big(- \gamma_{\epsilon_1, \epsilon_2}(a_{w,m}) + \gamma_{\epsilon_1, \epsilon_2 - \epsilon_1}(a_{w,m} + k_w \epsilon_1) + \gamma_{\epsilon_1 - \epsilon_2, \epsilon_2}(a_{w,m} + k_w \epsilon_2) \Big) \\ &= \prod_{\vec{w} \in R} \Lambda^{-\frac{1}{2} k_w^2} s(-k_w, a_{w,m}, \epsilon_1, \epsilon_2) = (\Lambda^{-2I(R)})^{\frac{1}{2} \vec{k} \cdot \vec{k}} \prod_{\vec{w} \in R} s(-k_w, a_{w,m}, \epsilon_1, \epsilon_2) \ , \end{aligned} \tag{2.47}$$

where we introduced the short-hand notation $k_w = \vec{k} \cdot \vec{w}$, $a_{w,m} = \vec{a} \cdot \vec{w} - m$ and $I(R)$ corresponds to the Dynkin index for the representation R. The Dynkin index appears in the beta function coefficients as $b_0 = 2h^\vee - I(R)$ for the hypermultiplets in the representation R. This gives the instanton parameter to be $q \equiv \Lambda^{b_0} = \Lambda^{2h^\vee - I(R)}$.

Now, for the SQCD, we obtain the following equation:

$$Z^{\text{inst}}(\vec{a}, m, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} f(\vec{k}) Z^{(N),\text{inst}}(\vec{k}) Z^{(S),\text{inst}}(\vec{k}) \ , \tag{2.48}$$

with

$$f(\vec{k}) = \frac{q^{\frac{1}{2} \vec{k} \cdot \vec{k}} \displaystyle\prod_i \prod_{\vec{w} \in R_i} s(-k_w, a_w - m_i, \epsilon_1, \epsilon_2)}{\displaystyle\prod_{\vec{\alpha} \in \Lambda} s(-k_\alpha, a_\alpha, \epsilon_1, \epsilon_2)} \ , \tag{2.49}$$

where i runs over the charged hypermultiplets. Here $q = \Lambda^{2N_c - N_f}$ for the $SU(N_c)$ SQCD with N_f fundamental hypermultiplets. We have checked this expression explicitly for the $SU(2)$ gauge theory with $N_f = 0, 1$ hypermultiplets up to the first few order in instanton numbers.

Notice that in the Gottsche-Nakajima-Yoshioka [13, 14], the mass parameters and the instanton parameters (for the 5d) are also shifted when the contribution from North and South poles are computed. This is simply a reflection of the fact that they twist the instanton bundles by the half-Canonical bundle of the $\mathbb{C}^2$, which shift the mass parameters by $m \to m - \frac{\epsilon_1 + \epsilon_2}{2}$. If we do not twist by this amount, we get a cleaner expression as above.

2.3 Blowup Equation for 5d gauge theory

We consider a 5d $\mathcal{N}=1$ gauge theory with a gauge group G. It has the Coulomb branch moduli space, parametrized by the vacuum expectation value $\alpha_i \equiv \langle \Phi_{ii} \rangle$ of the vector multiplet scalar Φ. The gauge symmetry G is spontaeously broken to its Abelian subgroup $U(1)^{|G|}$ on the Coulomb branch. The low energy Abelian theory is described by the effective prepotential, which can be written as [15]

$$\mathcal{F}_{\text{classical}} = \frac{1}{2g^2} h_{ij}\alpha_i\alpha_j + \frac{\kappa}{6} d_{ijk}\alpha_i\alpha_j\alpha_k \tag{2.50}$$

at the classical level. It was found in [16, 17] that the fully quantum corrected prepotential can be obtained from the BPS instanton partition function $\mathcal{Z}$ on Ω-deformed $\mathbf{R}^4 \times S^1$, i.e.,

$$\mathcal{F} = \lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2 \log \mathcal{Z} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{\text{quantum}}. \tag{2.51}$$

The BPS partition function $\mathcal{Z}$ is defined with equivariant parameters $\epsilon_1, \epsilon_2,\ \alpha_1, \cdots, \alpha_{|G|}$ associated to the $U(1)^2 \times U(1)^{|G|}$ action on the k-instanton moduli space $\mathcal{M}_{k,G}$ [16, 17]. Additional equivariant parameters $m_1, \cdots, m_{|F|}$ can be introduced if a given theory has the flavor symmetry F. It takes the form of

$$\mathcal{Z} = \exp\left(F_0\right) \cdot \mathcal{I} \tag{2.52}$$

where $\mathcal{I}$ is the Witten index counting the BPS bound states of fundamental particles and/or non-perturbative instanton solitons. More precisely,

$$\mathcal{I} \equiv \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\frac{8\pi^2}{g^2}k} e^{-\epsilon_1(J_1+\frac{R}{2})} e^{-\epsilon_2(J_2+\frac{R}{2})} \prod_{i=1}^{|G|} e^{-\alpha_i Q_i} \prod_{l=1}^{|F|} e^{-m_l F_l} \right] \tag{2.53}$$

where (J_1, J_2) are the angular momenta associated to the two $\mathbf{R}^2$ planes, R is the Cartan generator of $SU(2)_R$ symmetry, $(Q_1, \cdots, Q_{|G|})$ are the electric charges of $U(1)^{|G|} \subset G$, and $(F_1, \cdots, F_{|F|})$ are the Cartan generators of the flavor symmetry group F. We also frequently use the notation $\epsilon_+ \equiv \frac{\epsilon_1+\epsilon_2}{2}$, $\epsilon_- \equiv \frac{\epsilon_1-\epsilon_2}{2}$ and $J_l = \frac{J_1-J_2}{2}$, $J_r = \frac{J_1+J_2}{2}$, generating self-dual and anti-self-dual rotation inside the $\mathbf{R}^4$. The fugacity variables used throughout this paper are

$$p_1 = e^{-\epsilon_1},\ p_2 = e^{-\epsilon_2},\ \omega_i = e^{-\alpha_i},\ y_l = e^{-m_l},\ Q = e^{-8\pi^2/g^2},\ t = \sqrt{p_1p_2},\ u = \sqrt{p_1/p_2}. \tag{2.54}$$

Each multiplet

3 Examples

3.1 Ki-Hong's note

Unity Blowup equations The partition functions of generic 5d $\mathcal{N}=1$ gauge theories with hypermultiplets in R-representation in the Coulomb branch consist of classical action term, 1-loop term, and instanton partition functions.

$$Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = Z_{\text{class}}(\epsilon_1, \epsilon_2, \vec{a}, m_0)\, Z_{\text{1-loop}}(\epsilon_1, \epsilon_2, \vec{a}, m_i)\, Z_{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) \tag{3.1}$$

where

$$
\begin{aligned}
Z_{\text{class}} &= \exp\left[-\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}h_{ij}\phi^i\phi^j + \frac{1}{6}d_{ijk}\phi^i\phi^j\phi^k\right)\right] \\
Z_{\text{1-loop}} &= \exp\Bigg[-\frac{1}{2\epsilon_1\epsilon_2}\Bigg(\sum_{\alpha\in\text{roots}}\left(\frac{1}{6}(\vec{a}\cdot\vec{\alpha})^3 - \frac{1}{4}(\epsilon_1+\epsilon_2)(\vec{a}\cdot\vec{\alpha})^2 + \frac{1}{12}((\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2)(\vec{a}\cdot\vec{\alpha})\right) \\
&\qquad + \sum_{\omega\in\rho(R)}\Bigg(\frac{1}{6}\Big(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\Big)^3 - \frac{\epsilon_1+\epsilon_2}{4}\Big(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\Big)^2 \\
&\qquad\qquad - \frac{(\epsilon_1+\epsilon_2)^2+\epsilon_1\epsilon_2}{24}\Big(\vec{a}\cdot\vec{\omega}+m_i+\frac{\epsilon_1+\epsilon_2}{2}\Big)\Bigg)\Bigg)\Bigg] \\
&\quad \times \mathrm{PE}\left[\frac{1}{(1-p_1)(1-p_2)}\left(-\sum_{\alpha\in\text{roots}} e^{\vec{a}\cdot\vec{\alpha}} + p_1^{1/2}p_2^{1/2}y_i\sum_{\omega\in\rho(R)} e^{\vec{a}\cdot\vec{\omega}}\right)\right].
\end{aligned} \tag{3.2}
$$

Here $\vec{a}$ are Coulomb VEVs and $p_{1,2} = e^{\epsilon_{1,2}}$, $y_i = e^{m_i}$. Note that the normal exponential term saturates the zero-point energy of pletheystic exponential terms.[3]

The partition function satisfies so-called "Unity blowup equation"

$$
\begin{aligned}
Z(\epsilon_1,\epsilon_2,\vec{a},m_i,m_0) = \sum_{\vec{k}\in\vec{\alpha}^\vee} & Z(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_1, m_i+r_i\,\epsilon_1, m_0+r_0\,\epsilon_1) \\
& \times\; Z(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+(\vec{k}+\vec{r}_a)\,\epsilon_2, m_i+r_i\,\epsilon_2, m_0+r_0\,\epsilon_2)
\end{aligned} \tag{3.3}
$$

for certain $\vec{r}_a$, r_i, r_0's. Here $\vec{\alpha}^\vee$ is the coroot lattice where the long root is normalized to have norm 2. The r_i's and r_0 are some numbers specifying the blowup equations.

Technically r_i's are constrained to be half integers since, for each single letter 1-loop partition functions

$$
Z_{i,\vec{\omega}} = \mathrm{PE}\left[\frac{p_1^{1/2}p_2^{1/2}}{(1-p_1)(1-p_2)}y_i\,e^{\vec{a}\cdot\vec{\omega}}\right], \tag{3.4}
$$

the ratio between shifted ones and unshifted one is

$$
\begin{aligned}
l^{\vec{k}}_{i,\vec{\omega}} &= Z^{(1)}_{i,\vec{\omega}}Z^{(2)}_{i,\vec{\omega}}/Z_{i,\vec{\omega}} \\
&= \mathrm{PE}\left[\frac{p_1^{r_i}p_2^{1/2}y_i}{(1-p_1)(1-p_2/p_1)}p_1^{\vec{k}\cdot\vec{\omega}}e^{\vec{a}\cdot\vec{\omega}} + \frac{p_1^{1/2}p_2^{r_i}y_i}{(1-p_1/p_2)(1-p_2)}p_2^{\vec{k}\cdot\vec{\omega}}e^{\vec{a}\cdot\vec{\omega}} - \frac{p_1^{1/2}p_2^{1/2}y_i}{(1-p_1)(1-p_2)}e^{\vec{a}\cdot\vec{\omega}}\right] \\
&= \mathrm{PE}\left[\frac{p_1^{1/2}p_2^{1/2}y_i}{(1-p_1)(1-p_2)(p_1-p_2)}e^{\vec{a}\cdot\vec{\omega}}\Big((1-p_2)p_1^{\vec{k}\cdot\vec{\omega}+r_i+1/2} - (1-p_1)p_2^{\vec{k}\cdot\vec{\omega}+r_i+1/2}\Big) - (p_1-p_2)\right]
\end{aligned} \tag{3.5}
$$

For the $l^{\vec{k}}_{i,\vec{\omega}}$ to be finite rational function, the plethystic exponent must be finite series. It can be satisfied only when r_i is a half integer.

[3] Technically, instead of considering this 1-loop prepotential terms, I inserted overall factors to the $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}}Z^{(1)}_{\text{1-loop}}/Z_{\text{1-loop}}$ so that it is written by Sinh terms.

Instanton partition functions from blowup equations From blowup equations one can compute the partition functions as follows. Rewriting the blowup equation as

$$1 = \sum_{\vec{k}\in\vec{\alpha}^\vee} f_{\vec{k}}\, l_{\vec{k}} \frac{Z^{(1)}_{\text{inst}} Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \tag{3.6}$$

where $f_{\vec{k}} = Z^{(1)}_{\text{class}} Z^{(2)}_{\text{class}} / Z_{\text{class}}$ and $l_{\vec{k}} = Z^{(1)}_{\text{1-loop}} Z^{(2)}_{\text{1-loop}} / Z_{\text{1-loop}}$ with abbreviated notation

$$\begin{aligned} Z^{(1)} &= Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\,\epsilon_1, m_i + r_i\,\epsilon_1, m_0 + r_0\,\epsilon_1) \\ Z^{(2)} &= Z(\epsilon_1 - \epsilon_2, \epsilon_2, \vec{a} + \vec{k}\,\epsilon_2, m_i + r_i\,\epsilon_2, m_0 + r_0\,\epsilon_2) \end{aligned} \tag{3.7}$$

Here note that $l_{\vec{k}}$ is independent of $Q = e^{-m_0}$, and $f_{\vec{k}}$ is some overall factor in the order of $Q^{\vec{k}\cdot\vec{k}/2}$. Expanding the equation by instanton fugacity Q, then at each Q^n level the equation is written by

$$\delta_{n,0} = p_1^{r_0} Z_n^{(1)} + p_2^{r_0} Z_n^{(2)} - Z_n + \sum_{\vec{k}\neq 0} f_{\vec{k},r_0} l_{\vec{k}} \left(\frac{Z^{(1)}_{\text{inst}} Z^{(2)}_{\text{inst}}}{Z_{\text{inst}}} \right)\Bigg|_{O(Q^{n-\vec{k}\cdot\vec{k}/2})} . \tag{3.8}$$

Since each Z_k and $Z_k^{(1,2)}$ are independent of r_0, one can solve (3.8) with three blowup equations with same r_i's but different r_0's.

The blowup equations for instanton partition functions of pure YM theory with generic gauge group were already studied in [4]. They are actually (3.1) with

$$\vec{r}_a = 0, \qquad r_0 = d - h^\vee/2 \tag{3.9}$$

where $d = 0, \cdots, h^\vee$. We extend these blowup equations to the theories with matters based on pure YM blowup equations. If one restrict the cases to $\vec{r}_a = 0$, as we explained in the previous section, the r_i's are technically required to be half intergers. Thus we look for the r_0's that provides the correct instanton partition functions by solving (3.8) while fixing $\vec{r}_a = 0$ and $r_i = 1/2$. Here are the results.

G	matter	r_0	d
$SU(N)_\kappa$	$N_f \times \boldsymbol{N}$	$d - N/2 - \kappa/2$	$0 \le d \le N - \lvert\kappa\rvert - 2N_f - 1$(?)
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \le d \le 6$
$SO(7)$	pure	$d - 5/2$	$0 \le d \le 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \le d \le 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \le d \le 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \le d \le 3$
G_2	pure	$d - 4/2$	$0 \le d \le 4$
G_2	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \le d \le 3$
F_4	pure	$d - 9/2$	$0 \le d \le 9$
F_4	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \le d \le 6$
F_4	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \le d \le 3$

They were tested by comparing the resulting instanton partition functions with the known results from [18]($SO(7)$ and G_2) and [19](F_4 with $N_{\mathbf{26}}=2$). They were compared numerically, putting random numbers on the fugacities. Note that matters shift the r_0, each by one quarter of their Dynkin indices. It seems to differ from blowup formula for $SU(N)_\kappa+N_f$ instantons, where r_0 was affected only by its CS-level κ. However, one can rewrite the r_0 as

$$\begin{aligned} r_0 &= d - N/2 - \left(\kappa + \frac{1}{2}N_f\right)/2 + N_f/4 \\ &= d - N/2 - \kappa_{\text{eff}}/2 + N_f \times I_{\text{fund}}. \end{aligned} \tag{3.10}$$

Since fundamental matters shifts the effective CS-level, they cancel their index contributions and consequently the r_0 apparently looks independent of matters.

By above observations, we write the unity blowup equation for generic gauge groups and matter representations.

$$\begin{aligned} Z(\epsilon_1,\epsilon_2,\vec{a},m_i,m_0) = \sum_{\vec{k}\in\vec{\alpha}^\vee} & Z(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\vec{k}\epsilon_1,m_i+\epsilon_1/2,m_0+r_0\epsilon_1) \\ & \times\ Z(\epsilon_1-\epsilon_2,\vec{a}+\vec{k}\epsilon_2,m_i+\epsilon_2/2,m_0+r_0\epsilon_2) \end{aligned} \tag{3.11}$$

with

$$r_0 = d - h^\vee/2 - \kappa_{\text{eff}}/2 + N_{\boldsymbol{R}} \times I_{\boldsymbol{R}}. \tag{3.12}$$

Here $I_{\boldsymbol{R}}$ is the Dynkin index of $\boldsymbol{R}$ representation.

$SU(6)_3 + 1\times\mathbf{20}$ As a non-trivial test, we consider the instanton partition function of the $SU(6)_3+\mathbf{20}$ whose 5-brane realization was found recently [20]. Its web-diagram is given as figure.

(Written before computing the $SU(6)_3+20$ instanton partition function.)
Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the $SU(3)\times SU(3)\times U(1)\subset SU(6)$ where the $SU(6)$ multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}}:\ \mathbf{35} &\longrightarrow (\mathbf{8},1)_0\oplus(1,\mathbf{8})_0\oplus(\mathbf{3},\bar{\mathbf{3}})_2\oplus(\bar{\mathbf{3}},\mathbf{3})_{-2}\oplus(1,1)_0, \\ \Phi_{ijk}:\ \mathbf{20} &\longrightarrow (\mathbf{3},\bar{\mathbf{3}})_{-1}\oplus(\bar{\mathbf{3}},\mathbf{3})_1\oplus(1,1)_3\oplus(1,1)_{-3}. \end{aligned} \tag{3.13}$$

Here to fit with the web-diagram, we set Φ_{156} and Φ_{234} are $(1,1)_3$ and $(1,1)_{-3}$. Once Φ_{156} and Φ_{234} get non-zero VEVs,

When $a_5=-a_1-a_6$, the web-diagram factorizes to two $SU(3)_3$ whose Coulomb VEVs are (a_1,a_5,a_6) and (a_2,a_3,a_4). In the gauge theory, it can be seen partly from prepotential. The prepotential of $S(6)_3+1\times\mathbf{20}$ is

$$\mathcal{F} = \frac{1}{2}m_0\sum_{i=1}^{6}a_i^2 + \frac{1}{2}\sum_{i=1}^{6}a_i^3 + \frac{1}{6}\sum_{i<j}(a_i-a_j)^3 - \frac{1}{6}\sum_{1<i<j}(a_1+a_j+a_k)^3 \tag{3.14}$$

at the Weyl chamber $a_1 > \cdots > a_6$. As one sets the Coulomb VEV $a_6 = -a_1 - a_5$ and $a_4 = -a_2 - a_3$, one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \tag{3.15}$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2} m_0 \sum_{i=1}^{3} a_i^2 + \frac{1}{2} \sum_{i=1}^{3} a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3. \tag{3.16}$$

It is Higgsed by

3.2 $SU(6)_3$ with a rank-3 antisymmetric hyper

(describe blow-up computation)

It was recently found in [20] that the 5d $SU(6)_3$ gauge theory with a rank-3 antisymmetric hypermultiplet can be engineered from the 5-brane web configuration, depicted in Figure 1. Given a web diagram, we utilize the topological vertex method [21, 22] to compute all genus topological string amplitudes, which is the logarithm of the 5d Nekrasov partition function on Ω-deformed $\mathbf{R}^4 \times S^1$ [23]. We will check its agreement with the blowup partition function (eqn), providing a supporting evidence to the suggested blow-up equation (eqn).

By applying the topological vertex method to Figure 1, we find that the instanton partition function can be written as the following sum over all possible 6 Young diagrams:

$$\begin{aligned} Z_{\text{Nek}} = \sum_{(Y_1, \cdots, Y_6)} & q^{\sum_{i=1}^{6} |Y_i|} (-A_1^6)^{|Y_1|} (-A_2^6)^{|Y_2|} (-A_2^2 A_3^4)^{|Y_3|} (-A_2^2 A_3^2 A_4^2)^{|Y_4| + |Y_5|} \\ & \times f_{Y_1}(g)^5 f_{Y_2}(g)^5 f_{Y_3}(g)^3 f_{Y_4}(g) f_{Y_5}(g)^{-1} f_{Y_6}(g)^2 Z_{\text{half}}(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)^2. \end{aligned} \tag{3.17}$$

where the Kähler parameters can be identified as

$$A_i = e^{-a_i}, \qquad q = e^{-\frac{8\pi^2}{g^2}}, \qquad g = e^{-\epsilon_-}. \tag{3.18}$$

We briefly explain our notation: For a given Young diagram μ, $|\mu|$ denotes the total number of boxes. μ_i is the number of boxes in the i-th row of μ. μ^t is the transpose of μ. We also use

$$f_\mu(g) = (-1)^{|\mu|} g^{\frac{1}{2}(\|\mu^t\|^2 - \|\mu\|^2)}, \qquad \tilde{Z}_\lambda(g) = \prod_{(i,j)\in\lambda} (1 - g^{\lambda_i + \lambda_j^t - i - j + 1})^{-1} \tag{3.19}$$

with $\|\mu\|^2 = \sum_i \mu_i^2$. The factor $Z_{\text{half}}(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$ involves a single summation over all possible Young diagrams, i.e.,

$$\begin{aligned} Z_{\text{left}}(\vec{Y}) = \sum_{Y_0} & (-A_1{}^{-1} A_6{}^{-2})^{|Y_0|} g^{\frac{\|Y_0^t\|^2 + \|Y_0\|^2}{2}} \tilde{Z}_{Y_0}(g)^2 f_{Y_0}(g)^2 \ \textstyle\prod_{i=1}^{6} g^{\frac{\|Y_i\|^2}{2}} \tilde{Z}_{Y_i}(g) \\ & \times R^{-1}_{Y_1 Y_6^t}(A_1 A_6{}^{-1})\, R^{-1}_{Y_0 Y_6^t}(A_1{}^{-1} A_6{}^{-2})\, R^{-1}_{Y_1 Y_0^t}(A_1^2 A_6) \\ & \times \textstyle\prod_{2\le i<j\le 5} R^{-1}_{Y_i Y_j^t}(A_i A_j{}^{-1}) \ \prod_{i=2}^{5} R_{Y_0^t Y_i}(A_1 A_i A_6), \end{aligned} \tag{3.20}$$

in which we introduce

$$R_{\lambda\mu}(Q)=R_{\mu\lambda}(Q)=\mathrm{PE}\left[-\frac{g}{(1-g)^2}Q\right]\times N_{\lambda^t\mu}(Q) \tag{3.21}$$

with PE representing the Plethystic exponential and

$$N_{\lambda\mu}(Q)=\prod_{(i,j)\in\lambda}\left(1-Qg^{\lambda_i+\mu_j^t-i-j+1}\right)\prod_{(i,j)\in\mu}\left(1-Qg^{-\lambda_j^t-\mu_i+i+j-1}\right). \tag{3.22}$$

We also recall that the Nekrasov partition function is divided into the perturbative partition function Z_{pert} and the weighted sum of k-instanton partition function Z_k.

$$Z_{\text{Nek}}=Z_{\text{pert}}\Big(1+\sum_{k=1}^{\infty}q^kZ_k\Big). \tag{3.23}$$

The perturbative part of the partition function Z_{pert} comes from the summand of (3.17) at empty Young diagrams, i.e., $(Y_1,Y_2,Y_3,Y_4,Y_5,Y_6)=(\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing)$. It is given by

$$\begin{aligned}Z_{\text{pert}}=&Z_{\text{half}}(\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing)^2\\=&\mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\frac{A_1}{A_6}+\frac{1}{A_1A_6^2}+A_1^2A_6+\sum_{2\le i<j\le5}\frac{A_i}{A_j}-\sum_{i=2}^{5}A_1A_iA_6\Big)\right]\\&\times\Bigg(\sum_{Y_0}(-A_1{}^{-1}A_6{}^{-2})^{|Y_0|}\;g^{\frac{\|Y_0^t\|^2+\|Y_0\|^2}{2}}\tilde{Z}_{Y_0}(g)^2f_{Y_0}^2(g)\\&\qquad N_{Y_0^t\varnothing}^{-1}(A_1{}^{-1}A_6{}^{-2})N_{Y_0\varnothing}^{-1}(A_1^2A_6)\textstyle\prod_{i=2}^{5}N_{Y_0\varnothing}(A_1A_iA_6)\Bigg)^2,\end{aligned} \tag{3.24}$$

where the last two lines can be combined into the following closed-form expression:

$$\mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\sum_{i=2}^{5}\frac{A_1}{A_i}+\sum_{i=2}^{5}\frac{A_i}{A_6}-\frac{1}{A_1A_6^2}-A_1^2A_6-\sum_{2\le i<j\le5}A_1A_iA_j+\mathcal{O}(A_1^6)\Big)\right]. \tag{3.25}$$

So (3.24) is manifestly consistent with the equivariant index [24] for 5d $SU(6)$ gauge theory with a hypermultiplet in the rank-3 antisymmetric representation **20**, i.e.,

$$Z_{\text{pert}}=\mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\sum_{1\le i<j\le6}\frac{A_i}{A_j}-\sum_{2\le i<j\le6}A_1A_iA_j+\mathcal{O}(A_1^6)\Big)\right]. \tag{3.26}$$

The 1-instanton partition function Z_1 can be obtained from the summands of (3.17) at Young diagrams satisfying $\sum_{i=1}^{6}|Y_i|=1$. There are 6 different profiles of Young diagrams. The configuration $|Y_i|=1$ and $Y_{j\neq i}=\varnothing$ contributes to Z_1 by

$$+\frac{g}{(1-g)^2}\frac{A_i^6}{\prod_{j\neq i}(A_i-A_j)^2}\Big(-A_i\sum_{j\neq i}A_j+\sum_{j\neq i}\frac{1}{A_j}-\frac{1}{A_i}+A_i^2\Big)^2. \tag{3.27}$$

Summing over all six contributions, we find

$$Z_1 = \sum_{i=1}^{6} \frac{g}{(1-g)^2} \frac{A_i^6}{\prod_{j\neq i}(A_i - A_j)^2} \Big(-A_i \sum_{j\neq i} A_j + \sum_{j\neq i} \frac{1}{A_j} - \frac{1}{A_i} + A_i^2 \Big)^2. \tag{3.28}$$

which is in agreement with the blowup partition function (eqn). (overall sign: looking at the GV invariant for single W-boson + single instanton (which should be -2), the topological vertex computation seems to be correct. blowup should have an overall sign issue)

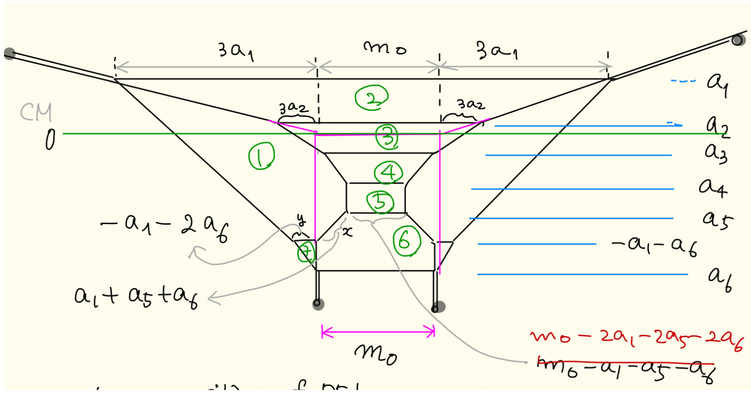

Figure 1: A 5-brane web for $SU(6)_3 + 1$**TAS** with massless **TAS**.

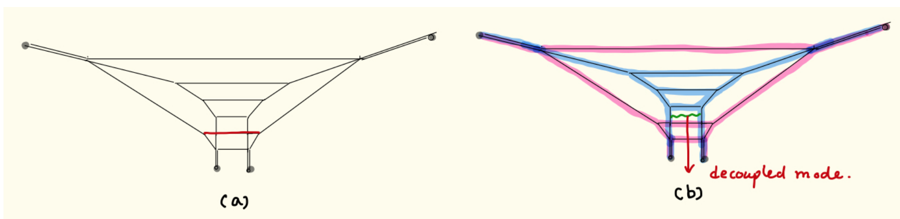

Figure 2: (a) A Higgsing from from $SU(6)_3 + 1$**TAS** to two $SU(3)_3$ theories. (a) Two $SU(3)_3$ theories are painted in blue and in pink respectively. A new decoupled mode emerges.

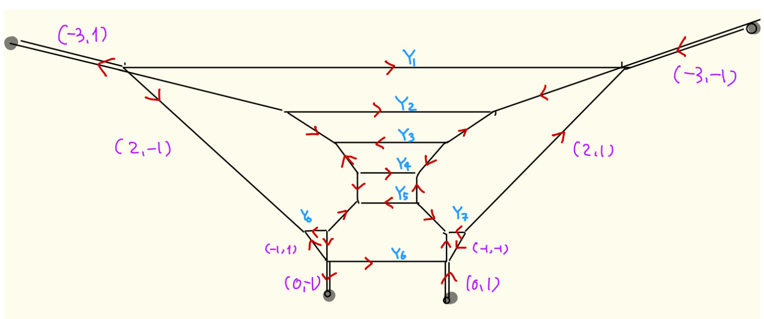

Figure 3: A labeling of Young diagrams assigned to the horizontal lines in Figure 1.

4 Conclusion

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