

Exceptional Instantons from Blow-up

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ABSTRACT: The Nekrasov partition function for 4d $\mathcal{N} = 2$ or 5d $\mathcal{N} = 1$ gauge theory on the blow up of a point $\hat{\mathbb{C}}^2$ can be written in terms of the partition function on the flat space \mathbb{C}^2 . At the same time, the partition function on the blow up is identical to the partition function on a flat space for sufficiently large class of examples. This relation enables us to compute the instanton partition functions for 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ gauge theories for arbitrary gauge theory with large class of matter representations without knowing explicit construction of the instanton moduli space. Especially, we obtain the partition function for the exceptional gauge groups EFG with fundamental hypermultiplets/spinors and more. We also compute the case with $SU(6)$ with rank-3 antisymmetric tensor and compare with the topological vertex computation using the recently found 5-brane web configuration.

Contents

1	Introduction	1
2	Instanton Counting from Blow-up	1
2.1	4d gauge theory	4
2.2	Blowup Equation for 5d gauge theory	6
3	Examples	7
3.1	Ki-Hong's note	7
4	Conclusion	10

1 Introduction

2 Instanton Counting from Blow-up

The essential idea of using the blow-up for the instanton counting is that the gauge theory partition function for a 4d $\mathcal{N} = 2$ theory or 5d $\mathcal{N} = 1$ on a blow up of a point $\hat{\mathbb{C}}^2$ (or $S^1 \times \hat{\mathbb{C}}^2$) can be written in two different ways. This allows us to write a recursion relation for the instanton partition function that can be solved easily [1–4].

One of the expression is obtained via localization on the Coulomb branch obtained by patching together flat-space partition functions. The blow up of a point of \mathbb{C}^2 can be described as a subspace of $\mathbb{C}^2 \times \mathbb{P}^1$ defined as

$$\{(x, y), [z : w] \in \mathbb{C}^2 \times \mathbb{P}^1 | xw = yz\} , \quad (2.1)$$

where $[z : w]$ represents the homogeneous coordinates on \mathbb{P}^1 . Notice that this space is identical to \mathbb{C}^2 when $(x, y) \neq (0, 0)$, and the origin is replaced by complex projective plane. We will be considering $U(1)^2$ -equivariant partition function, with each $U(1)$'s acting as the rotation of two complex planes. This $U(1)^2$ action acts on $\hat{\mathbb{C}}^2$ as

$$((x, y), [z : w]) \mapsto (e^{i\epsilon_1}x, e^{i\epsilon_2}y), [e^{i\epsilon_1}z, e^{i\epsilon_2}w]) . \quad (2.2)$$

The zero sized instantons will be located at the two fixed points of the $U(1)^2$ action, namely the north pole and the south pole of the sphere. At these points, the weights of the $U(1)^2$ action become

$$((0, 0), [1, 0]) : (\epsilon_1, \epsilon_2 - \epsilon_1), \quad ((0, 0), [0, 1]) : (\epsilon_1 - \epsilon_2, \epsilon_2) . \quad (2.3)$$

Now, the (full) partition function on a blow up (includes both the perturbative and the instanton parts) \hat{Z} can be written as a sum over a product of the partition functions at two fixed points as (here we turn off any external flux that can be supported on the blow up) [5–9]

$$\hat{Z}(\vec{a}, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} Z(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1) Z(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2) , \quad (2.4)$$

where $Z(\vec{a}, \epsilon_1, \epsilon_2)$ is the (full) partition function on \mathbb{C}^2 where \vec{a} is the Coulomb parameter. Here Λ is the weight lattice of the gauge group and the vector \vec{k} labels different flux configurations on the divisor of the blow-up classified by the first Chern numbers. We obtain this expression by performing the localization on the Coulomb branch. On the Coulomb branch, the gauge group is broken to $U(1)^r$ where r is the rank of the gauge group. We need to patch together all possible field configurations with zero sized instantons located at north and south poles, and all the inequivalent configurations are labeled by the first Chern numbers which we have to sum over. The shift of the Coulomb parameter comes from the gauge transformation along the equator that connects the north and the south pole so that $\vec{a}_N - \vec{a}_S = \vec{k}(\epsilon_1 - \epsilon_2)$.

The second expression for the partition function on the blow-up is that \hat{Z} is actually identical to the partition function Z on the flat space \mathbb{C}^2 (or $S^1 \times \mathbb{C}^2$) [1–3]. This can be argued as follows: The blow up $\hat{\mathbb{C}}^2$ is identical to \mathbb{C}^2 except for the origin, which is replaced by \mathbb{P}^1 , called as the exceptional divisor. The blow-up can be also identified as the total space of the bundle $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$. The Nekrasov partition function gets contributions only from the zero-size instantons localized at the fixed points of the $U(1)^2$ action so that the size of the divisor should not affect the partition function as we smoothly shrink it. Therefore, we expect that the partition function on the blow-up to be identical to that of the flat \mathbb{C}^2 .

This is a special feature of the blow-up of a point. In principle, there can be extra states coming from the \mathbb{P}^1 . As we shrink the size of the \mathbb{P}^1 , they may become massless which might in principle contribute to the partition function when there is a singularity as we blow-down. For example, if we consider the total space of $\mathcal{O}(-2) \rightarrow \mathbb{P}^1$ and shrink the base, we land on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ which is singular at the origin. In our case, we do not obtain any singularity as we blow down the sphere. As we will discuss later, this simple picture does not necessarily hold when there are too many hypermultiplets. But this is shown to be the case for the pure YM theory, and we will demonstrate that it is also true for many interesting cases with matter hypermultiplets.¹

Therefore, we have a powerful identity:

$$Z(\vec{a}, \epsilon_1, \epsilon_2, q) = \sum_{\vec{k} \in \Lambda} Z(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, q) Z(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, q) . \quad (2.5)$$

But this is not enough to determine the partition function since there are three unknown functions and one relation. To achieve this we need to consider a ‘correlation function’

¹The orbifold partition function can be also obtained in two different ways modulo some subtle scheme dependence related to wall-crossing. [10]

$\langle \mu(C) \mu(C) \dots \rangle$ of operators μ associated to the two-cycle C in the blow-up. In terms of the component fields of the Doanldson-twisted theory, it can be written as

$$\mu(C) = \int d^4x \left(\omega \wedge \phi F + \frac{1}{2} \psi \wedge \psi + HF \wedge F \right) , \quad (2.6)$$

where ω is the two-form dual to C and $H = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2$ is the moment map for the $U(1)^2$ action.

Let us consider the generating function of the correlators by inserting $e^{t\mu(C)}$. Then the partiton function $\hat{Z}^{(d)}$ can be written in a very simple form as

$$\hat{Z}^{(t)}(\vec{a}, m, \epsilon_1, \epsilon_2, q) = \sum_{\vec{k} \in \Lambda} \exp \left(t \cdot \mu(\vec{k}) \right) Z^{(N),t}(\vec{k}) Z^{(S),t}(\vec{k}) , \quad (2.7)$$

where

$$\begin{aligned} Z^{(N),t}(\vec{k}) &\equiv Z(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, qe^{t\epsilon_1}) , \\ Z^{(S),t}(\vec{k}) &\equiv Z(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, qe^{t\epsilon_2}) , \end{aligned} \quad (2.8)$$

and $\mu(\vec{k})$ is the contribution coming from the operator $\mu(C)$ which is a function of \vec{k} that also depends on $\vec{a}, \epsilon_1, \epsilon_2$ that we often abbreviate. Explicitly, it is given as

$$\exp \left(\mu(\vec{k}) \right) = \exp \left(\vec{k} \cdot \vec{a} + \frac{1}{2} \vec{k} \cdot \vec{k} (\epsilon_1 + \epsilon_2) \right) . \quad (2.9)$$

Notice that we shift the instanton parameter by equivariant parameters at both poles $q \rightarrow qe^{t\epsilon_{1,2}}$. This is due to the term $HF \wedge F$ in (2.6) which makes the effective instanton parameter as $qe^{t(\epsilon_1 + \epsilon_2)}$. At the north pole, $\epsilon_1 + \epsilon_2 \rightarrow \epsilon_1 + (\epsilon_2 - \epsilon_1) = \epsilon_2$. We need to compensate this part by shifting q by $e^{t\epsilon_1}$. Similarly, we need to shift the parameters at the south pole.

Now, as we shrink the two cycles to recover the flat \mathbb{C}^2 , the effect of the insertion of μ turns out to give a vanishing contribution due to the conservation laws. To see this, let us consider 4d $\mathcal{N} = 2$ theory for the moment. The operator (2.6) has $U(1)_R$ charge +2 of the 4d $\mathcal{N} = 2$ supersymmetry. The instanton carries $4h^\vee$ which is responsible for breaking $U(1)_R$ to a discrete subgroup $\mathbb{Z}_{2h^\vee - I(R)}$ where h^\vee is the dual Coxeter number of the gauge group and $I(R)$ is the Dynkin index of the half-hypermultiplet representation R .² Unless the charges add up to zero, the correlation function vanishes. Therefore, expanding in powers of t ,

$$\hat{Z}_{4d}^{(t)} = Z_{4d} + \mathcal{O}(t^{2h^\vee - I(R)}) . \quad (2.10)$$

This allows us to write 3 relations among for the 3 unknowns to solve for the instanton partition function as long as the hypermultiplet representation is not too large, namely, when $2h^\vee - I(R) > 2$. In this case, we can expand $\hat{Z}^{(t)}$ up to order t^2 to obtain 3 relations that

²Here we normalize the Dynkin index so that the fundamental for the $SU(N)$ has $I = 1/2$. Therefore the $SU(N_c)$ SQCD with N_f fundamental hypers have $I(R) = N_f$.

is enough to fix the instanton part of the partition function from the *perturbative* part as we will see soon.

For the case of 5d $\mathcal{N} = 1$ theory in S^1 , we do not have $U(1)_R$ symmetry to impose the selection rule. The natural uplift of the operator $\mu(C)^d$ is given via exponentiation $\exp(d\beta \cdot \mu(C))$ where β is the size of S^1 . We can always choose our scale so that $\beta = 1$ and we will omit this parameter. The exponentiation accounts for the Kaluza-Klein tower. Since we know that in the 4d limit the insertion of $\mu(C)$ does not alter the partition function (as long as $2h^\vee - I(R) > 2$), we expect this to be true as well in the 5d uplift. [MORE EXPLANATION] Therefore, we obtain

$$Z = \sum_{\vec{k} \in \Lambda} \exp\left(d\mu(\vec{k})\right) Z^{(N),d}(\vec{k}) Z^{(S),d}(\vec{k}) \quad (d = 0, 1, 2) , \quad (2.11)$$

for the 5d theories as long as $2h^\vee - I(R) > 2$.

In the remainder of this section we will give detailed formulae for 4d and 5d gauge theories.

2.1 4d gauge theory

Building blocks for the 4d

$$Z_{\text{vec}}^{\text{pert}}(\vec{a}, q) = \exp\left(-\sum_{\vec{\alpha} \in \Delta} \gamma_{\epsilon_1, \epsilon_2}(\vec{a} \cdot \vec{\alpha}; q)\right) \quad (2.12)$$

$$Z_{\text{hyp}}^{\text{pert}}(\vec{a}, m, q) = \exp\left(\sum_{\vec{w} \in R} \gamma_{\epsilon_1, \epsilon_2}(\vec{a} \cdot \vec{w} - m; q)\right) \quad (2.13)$$

Here the gamma function is defined as

$$\gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) = \frac{d}{ds} \Big|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-ts}}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)} , \quad (2.14)$$

which is formally equivalent to

$$\log \left[\prod_{n, m \geq 0} \left(\frac{x - m\epsilon_1 - n\epsilon_2}{\Lambda} \right) \right] . \quad (2.15)$$

If we write the equation (2.4) in terms of $Z = Z^{\text{pert}} Z^{\text{inst}}$, we get

$$Z^{\text{inst}}(\vec{a}, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} \frac{Z^{(N), \text{pert}}(\vec{k}) Z^{(S), \text{pert}}(\vec{k})}{Z^{\text{pert}}(\vec{a}, \epsilon_1, \epsilon_2)} Z^{(N), \text{inst}}(\vec{k}) Z^{(S), \text{inst}}(\vec{k}) \quad (2.16)$$

where

$$Z^{(N)}(\vec{k}) = Z(\vec{a} + \epsilon_1 \vec{k}, \epsilon_1, \epsilon_2 - \epsilon_1) \quad (2.17)$$

$$Z^{(S)}(\vec{k}) = Z(\vec{a} + \epsilon_2 \vec{k}, \epsilon_1 - \epsilon_2, \epsilon_2) \quad (2.18)$$

and here we omit the dependence on the Coulomb vev and the Omega deformation parameters. Now the factor in the middle can be explicitly worked out. Let us denote the ratio of the perturbative factor as $f(\vec{k}) \equiv Z^{(N),\text{pert}} Z^{(S),\text{pert}} / Z^{\text{pert}}$. Then the ratio of the perturbative factor for the vector multiplet is given as

$$f(\vec{k})_{\text{vec}} = \prod_{\alpha \in \Delta} \exp \left(\gamma_{\epsilon_1, \epsilon_2}(\vec{a} \cdot \vec{\alpha}) - \gamma_{\epsilon_1, \epsilon_2 - \epsilon_1}(\vec{a} \cdot \vec{\alpha} + \vec{k} \cdot \vec{\alpha} \epsilon_1) - \gamma_{\epsilon_1 - \epsilon_2, \epsilon_2}(\vec{a} \cdot \vec{\alpha} + \vec{k} \cdot \vec{\alpha} \epsilon_2) \right) \quad (2.19)$$

$$= \prod_{\vec{\alpha} \in \Delta} \frac{\Lambda^{(\vec{k} \cdot \vec{\alpha})^2 / 2}}{s(-\vec{k} \cdot \vec{\alpha}, \vec{\alpha} \cdot \vec{a}, \epsilon_1, \epsilon_2)} = \frac{(\Lambda^{2h^\vee})^{\vec{k} \cdot \vec{k} / 2}}{\prod_{\vec{\alpha} \in \Delta} \ell_{\vec{\alpha}}^{\vec{k}}(\vec{a}, \epsilon_1, \epsilon_2)} \quad (2.20)$$

where h^\vee refers to the dual Coxeter number of the gauge group. Notice that the beta function coefficient for the pure YM theory is given by $b_0 = 2h^\vee$ and the instanton number is given as $q \equiv \Lambda^{2h^\vee}$. The other symbols are given as

$$\ell_{\vec{\alpha}}^{\vec{k}}(\vec{a}, \epsilon_1, \epsilon_2) = s(-\vec{k} \cdot \vec{\alpha}, \vec{a} \cdot \vec{\alpha}, \epsilon_1, \epsilon_2) \quad (2.21)$$

$$s(k, x, \epsilon_1, \epsilon_2) = \begin{cases} \prod_{i,j \geq 0, i+j \leq k-1} (x - i\epsilon_1 - j\epsilon_2) & (k > 0) \\ \prod_{i,j \geq 0, i+j \leq -k-2} (x + (i+1)\epsilon_1 + (j+1)\epsilon_2) & (k < -1) \\ 1 & (k = 0, -1) \end{cases} \quad (2.22)$$

The final identity of (2.19) involves a bit of work. This follows from the identity ([2], App. E.)

$$\gamma_{\epsilon_1, \epsilon_2 - \epsilon_1}(x + \epsilon_1 k; \Lambda) + \gamma_{\epsilon_1 - \epsilon_2, \epsilon_2}(x + \epsilon_2 k; \Lambda) = \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) + \log s(-k, x, \epsilon_1, \epsilon_2) - \frac{k(k-1)}{2} \log \Lambda. \quad (2.23)$$

For the hypermultiplets we get,

$$\begin{aligned} f(\vec{k})_{\text{hyp}} &= \prod_{\vec{w} \in R} \exp \left(-\gamma_{\epsilon_1, \epsilon_2}(a_w - m) + \gamma_{\epsilon_1, \epsilon_2 - \epsilon_1}(a_w + k_w \epsilon_1 - m) + \gamma_{\epsilon_1 - \epsilon_2, \epsilon_2}(a_w + k_w \epsilon_2 - m) \right) \\ &= \prod_{\vec{w} \in R} \Lambda^{-\frac{1}{2} k_w^2} s(-k_w, a_w - m, \epsilon_1, \epsilon_2) = (\Lambda^{-2C_2(R)})^{\vec{k} \cdot \vec{k} / 2} \prod_{\vec{w} \in R} s(-k_w, a_w - m, \epsilon_1, \epsilon_2), \end{aligned} \quad (2.24)$$

where we introduced the short-hand notation $k_w = \vec{k} \cdot \vec{w}$, $a_w = \vec{a} \cdot \vec{w}$ and $C_2(R)$ corresponds to the second Casimir invariant for the representation R . This invariant appears in the beta function coefficients as $b_0 = 2h^\vee - 2 \sum_R C_2(R)$ for the hypermultiplets in irrep R . This gives the instanton parameter to be $q \equiv \Lambda^{b_0} = \Lambda^{2h^\vee - 2 \sum_R C_2(R)}$.

Now, for the SQCD, we obtain the following equation:

$$Z^{\text{inst}}(\vec{a}, m, \epsilon_1, \epsilon_2) = \sum_{\vec{k} \in \Lambda} f(\vec{k}) Z^{(N),\text{inst}}(\vec{k}) Z^{(S),\text{inst}}(\vec{k}), \quad (2.25)$$

with

$$f(\vec{k}) = \frac{q^{\frac{1}{2}\vec{k}\cdot\vec{k}} \prod_i \prod_{\vec{w} \in R_i} s(-k_w, a_w - m_i, \epsilon_1, \epsilon_2)}{\prod_{\vec{\alpha} \in \Lambda} s(-k_\alpha, a_\alpha, \epsilon_1, \epsilon_2)}, \quad (2.26)$$

where i runs over the charged hypermultiplets. Here $q = \Lambda^{2N_c - N_f}$ for the $SU(N_c)$ SQCD with N_f fundamental hypermultiplets. We have checked this expression explicitly for the $SU(2)$ gauge theory with $N_f = 0, 1$ hypermultiplets up to the first few order in instanton numbers.

Notice that in the Gottsche-Nakajima-Yoshioka [11, 12], the mass parameters and the instanton parameters (for the 5d) are also shifted when the contribution from North and South poles are computed. This is simply a reflection of the fact that they twist the instanton bundles by the half-Canonical bundle of the \mathbb{C}^2 , which shift the mass parameters by $m \rightarrow m - \frac{\epsilon_1 + \epsilon_2}{2}$. If we do not twist by this amount, we get a cleaner expression as above.

2.2 Blowup Equation for 5d gauge theory

We consider a 5d $\mathcal{N} = 1$ gauge theory with a gauge group G . It has the Coulomb branch moduli space, parametrized by the vacuum expectation value $\alpha_i \equiv \langle \Phi_{ii} \rangle$ of the vector multiplet scalar Φ . The gauge symmetry G is spontaneously broken to its Abelian subgroup $U(1)^{|G|}$ on the Coulomb branch. The low energy Abelian theory is described by the effective prepotential, which can be written as [13]

$$\mathcal{F}_{\text{classical}} = \frac{1}{2g^2} h_{ij} \alpha_i \alpha_j + \frac{\kappa}{6} d_{ijk} \alpha_i \alpha_j \alpha_k \quad (2.27)$$

at the classical level. It was found in [14, 15] that the fully quantum corrected prepotential can be obtained from the BPS instanton partition function \mathcal{Z} on Ω -deformed $\mathbf{R}^4 \times S^1$, i.e.,

$$\mathcal{F} = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_1 \epsilon_2 \log \mathcal{Z} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{\text{quantum}}. \quad (2.28)$$

The BPS partition function \mathcal{Z} is defined with equivariant parameters $\epsilon_1, \epsilon_2, \alpha_1, \dots, \alpha_{|G|}$ associated to the $U(1)^2 \times U(1)^{|G|}$ action on the k -instanton moduli space $\mathcal{M}_{k,G}$ [14, 15]. Additional equivariant parameters $m_1, \dots, m_{|F|}$ can be introduced if a given theory has the flavor symmetry F . It takes the form of

$$\mathcal{Z} = \exp(F_0) \cdot \mathcal{I} \quad (2.29)$$

where \mathcal{I} is the Witten index counting the BPS bound states of fundamental particles and/or non-perturbative instanton solitons. More precisely,

$$\mathcal{I} \equiv \text{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\frac{8\pi^2}{g^2} k} e^{-\epsilon_1(J_1 + \frac{R}{2})} e^{-\epsilon_2(J_2 + \frac{R}{2})} \prod_{i=1}^{|G|} e^{-\alpha_i Q_i} \prod_{l=1}^{|F|} e^{-m_l F_l} \right] \quad (2.30)$$

where (J_1, J_2) are the angular momenta associated to the two \mathbf{R}^2 planes, R is the Cartan generator of $SU(2)_R$ symmetry, $(Q_1, \dots, Q_{|G|})$ are the electric charges of $U(1)^{|G|} \subset G$, and $(F_1, \dots, F_{|F|})$ are the Cartan generators of the flavor symmetry group F . We also frequently use the notation $\epsilon_+ \equiv \frac{\epsilon_1 + \epsilon_2}{2}$, $\epsilon_- \equiv \frac{\epsilon_1 - \epsilon_2}{2}$ and $J_l = \frac{J_1 - J_2}{2}$, $J_r = \frac{J_1 + J_2}{2}$, generating self-dual and anti-self-dual rotation inside the \mathbf{R}^4 . The fugacity variables used throughout this paper are

$$p_1 = e^{-\epsilon_1}, p_2 = e^{-\epsilon_2}, \omega_i = e^{-\alpha_i}, y_l = e^{-m_l}, Q = e^{-8\pi^2/g^2}, t = \sqrt{p_1 p_2}, u = \sqrt{p_1/p_2}. \quad (2.31)$$

Each multiplet

3 Examples

3.1 Ki-Hong's note

Unity Blowup equations The partition functions of generic 5d $\mathcal{N} = 1$ gauge theories with hypermultiplets in R -representation in the Coulomb branch consist of classical action term, 1-loop term, and instanton partition functions.

$$Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) = Z_{\text{class}}(\epsilon_1, \epsilon_2, \vec{a}, m_0) Z_{1\text{-loop}}(\epsilon_1, \epsilon_2, \vec{a}, m_i) Z_{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) \quad (3.1)$$

where

$$\begin{aligned} Z_{\text{class}} &= \exp \left[-\frac{1}{\epsilon_1 \epsilon_2} \left(\frac{1}{2} h_{ij} \phi^i \phi^j + \frac{1}{6} d_{ijk} \phi^i \phi^j \phi^k \right) \right] \\ Z_{1\text{-loop}} &= \exp \left[-\frac{1}{2\epsilon_1 \epsilon_2} \left(\sum_{\alpha \in \text{roots}} \left(\frac{1}{6} (\vec{a} \cdot \vec{\alpha})^3 - \frac{1}{4} (\epsilon_1 + \epsilon_2) (\vec{a} \cdot \vec{\alpha})^2 + \frac{1}{12} ((\epsilon_1 + \epsilon_2)^2 + \epsilon_1 \epsilon_2) (\vec{a} \cdot \vec{\alpha}) \right) \right. \right. \\ &\quad \left. \left. + \sum_{\omega \in \rho(R)} \left(\frac{1}{6} \left(\vec{a} \cdot \vec{\omega} + m_i + \frac{\epsilon_1 + \epsilon_2}{2} \right)^3 - \frac{\epsilon_1 + \epsilon_2}{4} \left(\vec{a} \cdot \vec{\omega} + m_i + \frac{\epsilon_1 + \epsilon_2}{2} \right)^2 \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{(\epsilon_1 + \epsilon_2)^2 + \epsilon_1 \epsilon_2}{24} \left(\vec{a} \cdot \vec{\omega} + m_i + \frac{\epsilon_1 + \epsilon_2}{2} \right) \right) \right) \right] \\ &\quad \times \text{PE} \left[\frac{1}{(1-p_1)(1-p_2)} \left(- \sum_{\alpha \in \text{roots}} e^{\vec{a} \cdot \vec{\alpha}} + p_1^{1/2} p_2^{1/2} y_i \sum_{\omega \in \rho(R)} e^{\vec{a} \cdot \vec{\omega}} \right) \right]. \end{aligned} \quad (3.2)$$

Here \vec{a} are Coulomb VEVs and $p_{1,2} = e^{\epsilon_{1,2}}$, $y_i = e^{m_i}$. Note that the normal exponential term saturates the zero-point energy of plethystic exponential terms.³

The partition function satisfies so-called ‘‘Unity blowup equation’’

$$\begin{aligned} Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) &= \sum_{\vec{k} \in \vec{\alpha}^\vee} Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + (\vec{k} + \vec{r}_a) \epsilon_1, m_i + r_i \epsilon_1, m_0 + r_0 \epsilon_1) \\ &\quad \times Z(\epsilon_1 - \epsilon_2, \epsilon_2, \vec{a} + (\vec{k} + \vec{r}_a) \epsilon_2, m_i + r_i \epsilon_2, m_0 + r_0 \epsilon_2) \end{aligned} \quad (3.3)$$

³Technically, instead of considering this 1-loop prepotential terms, I inserted overall factors to the $l_{\vec{k}} = Z_{1\text{-loop}}^{(1)} Z_{1\text{-loop}}^{(1)} / Z_{1\text{-loop}}$ so that it is written by Sinh terms.

for certain \vec{r}_a, r_i, r_0 's. Here $\vec{\alpha}^\vee$ is the coroot lattice where the long root is normalized to have norm 2. The r_i 's and r_0 are some numbers specifying the blowup equations.

Technically r_i 's are constrained to be half integers since, for each single letter 1-loop partition functions

$$Z_{i,\vec{\omega}} = \text{PE} \left[\frac{p_1^{1/2} p_2^{1/2}}{(1-p_1)(1-p_2)} y_i e^{\vec{a} \cdot \vec{\omega}} \right], \quad (3.4)$$

the ratio between shifted ones and unshifted one is

$$\begin{aligned} l_{i,\vec{\omega}}^{\vec{k}} &= Z_{i,\vec{\omega}}^{(1)} Z_{i,\vec{\omega}}^{(2)} / Z_{i,\vec{\omega}} \\ &= \text{PE} \left[\frac{p_1^{r_i} p_2^{1/2} y_i}{(1-p_1)(1-p_2/p_1)} p_1^{\vec{k} \cdot \vec{\omega}} e^{\vec{a} \cdot \vec{\omega}} + \frac{p_1^{1/2} p_2^{r_i} y_i}{(1-p_1/p_2)(1-p_2)} p_2^{\vec{k} \cdot \vec{\omega}} e^{\vec{a} \cdot \vec{\omega}} - \frac{p_1^{1/2} p_2^{1/2} y_i}{(1-p_1)(1-p_2)} e^{\vec{a} \cdot \vec{\omega}} \right] \\ &= \text{PE} \left[\frac{p_1^{1/2} p_2^{1/2} y_i}{(1-p_1)(1-p_2)(p_1-p_2)} e^{\vec{a} \cdot \vec{\omega}} \left((1-p_2) p_1^{\vec{k} \cdot \vec{\omega} + r_i + 1/2} - (1-p_1) p_2^{\vec{k} \cdot \vec{\omega} + r_i + 1/2} \right) - (p_1-p_2) \right] \end{aligned} \quad (3.5)$$

For the $l_{i,\vec{\omega}}^{\vec{k}}$ to be finite rational function, the plethystic exponent must be finite series. It can be satisfied only when r_i is a half integer.

Instanton partition functions from blowup equations From blowup equations one can compute the partition functions as follows. Rewriting the blowup equation as

$$1 = \sum_{\vec{k} \in \vec{\alpha}^\vee} f_{\vec{k}} l_{\vec{k}} \frac{Z_{\text{inst}}^{(1)} Z_{\text{inst}}^{(2)}}{Z_{\text{inst}}} \quad (3.6)$$

where $f_{\vec{k}} = Z_{\text{class}}^{(1)} Z_{\text{class}}^{(2)} / Z_{\text{class}}$ and $l_{\vec{k}} = Z_{1\text{-loop}}^{(1)} Z_{1\text{-loop}}^{(2)} / Z_{1\text{-loop}}$ with abbreviated notation

$$\begin{aligned} Z^{(1)} &= Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k} \epsilon_1, m_i + r_i \epsilon_1, m_0 + r_0 \epsilon_1) \\ Z^{(2)} &= Z(\epsilon_1 - \epsilon_2, \epsilon_2, \vec{a} + \vec{k} \epsilon_2, m_i + r_i \epsilon_2, m_0 + r_0 \epsilon_2) \end{aligned} \quad (3.7)$$

Here note that $l_{\vec{k}}$ is independent of $Q = e^{-m_0}$, and $f_{\vec{k}}$ is some overall factor in the order of $Q^{\vec{k} \cdot \vec{k}/2}$. Expanding the equation by instanton fugacity Q , then at each Q^n level the equation is written by

$$\delta_{n,0} = p_1^{r_0} Z_n^{(1)} + p_2^{r_0} Z_n^{(2)} - Z_n + \sum_{\vec{k} \neq 0} f_{\vec{k},r_0} l_{\vec{k}} \left(\frac{Z_{\text{inst}}^{(1)} Z_{\text{inst}}^{(2)}}{Z_{\text{inst}}} \right) \Big|_{O(Q^{n-\vec{k} \cdot \vec{k}/2})}. \quad (3.8)$$

Since each Z_k and $Z_k^{(1,2)}$ are independent of r_0 , one can solve (3.8) with three blowup equations with same r_i 's but different r_0 's.

The blowup equations for instanton partition functions of pure YM theory with generic gauge group were already studied in [4]. They are actually (3.1) with

$$\vec{r}_a = 0, \quad r_0 = d - h^\vee/2 \quad (3.9)$$

where $d = 0, \dots, h^\vee$. We extend these blowup equations to the theories with matters based on pure YM blowup equations. If one restrict the cases to $\vec{r}_a = 0$, as we explained in the previous section, the r_i 's are technically required to be half intergers. Thus we look for the r_0 's that provides the correct instanton partition functions by solving (3.8) while fixing $\vec{r}_a = 0$ and $r_i = 1/2$. Here are the results.

G	matter	r_0	d
$SU(N)_\kappa$	$N_f \times \mathbf{N}$	$d - N/2 - \kappa/2$	$0 \leq d \leq N - \kappa - 2N_f - 1(?)$
$SU(6)_3$	$1 \times \mathbf{20}$	$d - 6/2 - 3/2 + 3/2$	$1 \leq d \leq 6$
$SO(7)$	pure	$d - 5/2$	$0 \leq d \leq 5$
$SO(7)$	$1 \times \mathbf{8}$	$d - 5/2 + 1/2$	$0 \leq d \leq 4$
$SO(7)$	$1 \times \mathbf{7}$	$d - 5/2 + 1 \times 1/2$	$0 \leq d \leq 4$
$SO(7)$	$2 \times \mathbf{7}$	$d - 5/2 + 2 \times 1/2$	$0 \leq d \leq 3$
G_2	pure	$d - 4/2$	$0 \leq d \leq 4$
G_2	$1 \times \mathbf{7}$	$d - 4/2 + 1/2$	$0 \leq d \leq 3$
F_4	pure	$d - 9/2$	$0 \leq d \leq 9$
F_4	$1 \times \mathbf{26}$	$d - 9/2 + 1 \times 3/2$	$0 \leq d \leq 6$
F_4	$2 \times \mathbf{26}$	$d - 9/2 + 2 \times 3/2$	$0 \leq d \leq 3$

They were tested by comparing the resulting instanton partition functions with the known results from [16] ($SO(7)$ and G_2) and [17] (F_4 with $N_{\mathbf{26}} = 2$). They were compared numerically, putting random numbers on the fugacities. Note that matters shift the r_0 , each by one quarter of their Dynkin indices. It seems to differ from blowup formula for $SU(N)_\kappa + N_f$ instantons, where r_0 was affected only by its CS-level κ . However, one can rewrite the r_0 as

$$\begin{aligned} r_0 &= d - N/2 - \left(\kappa + \frac{1}{2}N_f \right) / 2 + N_f/4 \\ &= d - N/2 - \kappa_{\text{eff}}/2 + N_f \times I_{\text{fund}}. \end{aligned} \quad (3.10)$$

Since fundamental matters shifts the effective CS-level, they cancel their index contributions and consequently the r_0 apparently looks independent of matters.

By above observations, we write the unity blowup equation for generic gauge groups and matter representations.

$$\begin{aligned} Z(\epsilon_1, \epsilon_2, \vec{a}, m_i, m_0) &= \sum_{\vec{k} \in \vec{\alpha}^\vee} Z(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \vec{k}\epsilon_1, m_i + \epsilon_1/2, m_0 + r_0\epsilon_1) \\ &\quad \times Z(\epsilon_1 - \epsilon_2, \vec{a} + \vec{k}\epsilon_2, m_i + \epsilon_2/2, m_0 + r_0\epsilon_2) \end{aligned} \quad (3.11)$$

with

$$r_0 = d - h^\vee/2 - \kappa_{\text{eff}}/2 + N_{\mathbf{R}} \times I_{\mathbf{R}}. \quad (3.12)$$

Here $I_{\mathbf{R}}$ is the Dynkin index of \mathbf{R} representation.

$SU(6)_3 + 1 \times \mathbf{20}$ As a non-trivial test, we consider the instanton partition function of the $SU(6)_3 + \mathbf{20}$ whose 5-brane realization was found recently [18]. Its web-diagram is given as figure.

(Written before computing the $SU(6)_3 + \mathbf{20}$ instanton partition function.)

Rather than comparing instanton partition functions directly, we consider an interesting Higgsing procedure. We consider the $SU(3) \times SU(3) \times U(1) \subset SU(6)$ where the $SU(6)$ multiplets are decomposed by

$$\begin{aligned} A_{i\bar{j}} : \mathbf{35} &\longrightarrow (\mathbf{8}, 1)_0 \oplus (1, \mathbf{8})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_2 \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-2} \oplus (1, 1)_0, \\ \Phi_{ijk} : \mathbf{20} &\longrightarrow (\mathbf{3}, \bar{\mathbf{3}})_{-1} \oplus (\bar{\mathbf{3}}, \mathbf{3})_1 \oplus (1, 1)_3 \oplus (1, 1)_{-3}. \end{aligned} \quad (3.13)$$

Here to fit with the web-diagram, we set Φ_{156} and Φ_{234} are $(1, 1)_3$ and $(1, 1)_{-3}$. Once Φ_{156} and Φ_{234} get non-zero VEVs,

When $a_5 = -a_1 - a_6$, the web-diagram factorizes to two $SU(3)_3$ whose Coulomb VEVs are (a_1, a_5, a_6) and (a_2, a_3, a_4) . In the gauge theory, it can be seen partly from prepotential. The prepotential of $S(6)_3 + 1 \times \mathbf{20}$ is

$$\mathcal{F} = \frac{1}{2}m_0 \sum_{i=1}^6 a_i^2 + \frac{1}{2} \sum_{i=1}^6 a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3 - \frac{1}{6} \sum_{1<i<j} (a_1 + a_j + a_k)^3 \quad (3.14)$$

at the Weyl chamber $a_1 > \dots > a_6$. As one sets the Coulomb VEV $a_6 = -a_1 - a_5$ and $a_4 = -a_2 - a_3$, one can check

$$\mathcal{F}(m_0, a_1, a_2, a_3, a_4, a_5, a_6) = \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4) \quad (3.15)$$

where

$$\mathcal{F}_{SU(3)_3}(m_0, a_1, a_2, a_3) = \frac{1}{2}m_0 \sum_{i=1}^3 a_i^2 + \frac{1}{2} \sum_{i=1}^3 a_i^3 + \frac{1}{6} \sum_{i<j} (a_i - a_j)^3. \quad (3.16)$$

It is Higgsed by

4 Conclusion

Acknowledgments

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