

Corrections to the Friedmann equations from loop quantum gravity for a universe with a free scalar field

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In loop quantum cosmology, the quantum dynamics is well understood. We approximate the full quantum dynamics in the infinite dimensional Hilbert space by projecting it on a finite dimensional submanifold thereof, spanned by suitably chosen semiclassical states. This submanifold is isomorphic with the classical phase space and the projected dynamical flow provides effective equations incorporating the leading quantum corrections to the classical equations of motion. Numerical work has been done using quantum states which are semiclassical at late times. These states follow the classical trajectory until the density is on the order of 1% of the Planck density then deviate strongly from the classical trajectory. The effective equations we obtain reproduce this behavior to surprising accuracy.

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I. INTRODUCTION

Loop quantum gravity (see [1–3] for reviews) is a non-perturbative approach to the problem of quantizing gravity. An open problem is that of the semiclassical limit, i.e. are there solutions to LQG which closely approximate solutions to the classical Einstein's equations? Although this remains an open problem, concrete results can be achieved in the context of loop quantum cosmology (for a recent review, see [4,5]) in the spatially flat case. Loop quantum cosmology is a symmetry reduction of LQG, i.e. a quantization of a symmetry reduced sector of general relativity. There, it is the case that one can indeed find semiclassical solutions in the quantum theory that closely approximate solutions to the classical Einstein equations, namely, the Friedmann equations at late times. Our goal is to not just find semiclassical solutions but we would like to be able to go further and find effective equations incorporating the leading quantum corrections.

It is a common misconception that canonically quantizing general relativity would just reproduce Einstein's equations without any modifications. The classical equations are in fact modified by quantum corrections. Using the geometric quantum mechanics framework [6], this has been previously shown for a dust-filled, spatially flat Friedmann universe [7] in the so-called μ_0 framework (see Ref. [3] in [8]). In this paper we work in the so-called $\bar{\mu}$ framework and show that this is also the case for a spatially flat Friedmann universe with a free scalar field.

We can find effective equations arising from the geometric quantum mechanics framework [6] (“effective equations” for short). These semiclassical states follow the classical trajectory until the scalar field density is on the order of 1% of the Planck density where deviations from the classical trajectory start to occur. Then there are major deviations from the classical theory. However, states

which are semiclassical at late times continue to remain sharply peaked *but* on the trajectory given by the “effective equations.” Thus, we can approximate the full quantum dynamics in the (infinite dimensional) Hilbert space by a system of “effective equations,” incorporating the leading quantum corrections, on a finite dimensional submanifold thereof isomorphic to the classical phase space. Furthermore, we know from comparison with the numerical work that the leading corrections are sufficient to capture essential features of the full quantum dynamics [8].

We begin by briefly summarizing the classical theory in Sec. II. We then provide an overview of the framework for effective theories that we use to obtain the “effective equations” in Sec. III. Then in the quantum theory, Sec. IV, we choose a family of candidate semiclassical states, verify that they are indeed sharply peaked, and compute the effective Hamiltonian constraint. We obtain the “effective equations” in Sec. V and show that they are self-consistent to within our order of approximation. Finally, we discuss how much this approximation can be trusted near the bounce point in the Appendix and conclude.

Additionally, in this paper we use the choice of convention: $l_p^2 = G\hbar$, $c_{\text{light}} = 1$, $\kappa = 8\pi G$, and Friedmann-Robertson-Walker parameter $k = 0$ since we work in the spatially flat case.

II. CLASSICAL THEORY

In the classical theory, the phase space Γ consists of pairs (A_a^i, E_i^a) , where A_a^i is an $su(n)$ connection and the E_i^a its canonically conjugate field. Because of isotropy and homogeneity we have a fiducial background triad ${}^o e_i^a$ and connection ${}^o \omega_a^i$. Additionally, we may fix a fiducial cell with volume ${}^o V$ and restrict our calculations to that cell. There, we can write A_a^i and E_i^a as [9]

$$A_a^i = c {}^o V^{-(1/3)} {}^o \omega_a^i \quad (1)$$

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$$E_i^a = p\sqrt{{}^oq}V^{-(2/3)}e_i^a, \quad (2)$$

where oq is the determinant of the fiducial metric. Because of homogeneity and isotropy all the nontrivial information in A_a^i and E_i^a is contained in the variables c and p . The Hamiltonian constraint in these variables for a Friedmann universe with a free scalar field ϕ , in the regime $p > 0$, is given by

$$C = -\frac{3}{\kappa\gamma^2}c^2p^{1/2} + \frac{1}{2}\frac{p_\phi^2}{p^{3/2}} = 0, \quad (3)$$

where p_ϕ is the conjugate momentum of the massless scalar field ϕ .

For convenience we switch to variables given by $\beta = c/\sqrt{p}$ and $V = p^{3/2}$ in which the Hamiltonian is given by

$$C = -\frac{3}{\kappa\gamma^2}\beta^2V + \frac{1}{2}\frac{p_\phi^2}{V} = 0. \quad (4)$$

These new canonically conjugate variables are related to the old geometrodynamics variables via

$$\beta = \gamma\frac{\dot{a}}{a}, \quad (5)$$

$$V = a^3. \quad (6)$$

With the inclusion of a scalar field, the symplectic structure now has an extra part,

$$\Omega = \frac{2}{\kappa\gamma}d\beta \wedge dV + d\phi \wedge dp_\phi, \quad (7)$$

where the scalar field is ϕ and its conjugate momentum is p_ϕ . The Poisson bracket on the phase space is given by

$$\{f, g\} = \frac{\kappa\gamma}{2}\left(\frac{\partial f}{\partial\beta}\frac{\partial g}{\partial V} - \frac{\partial g}{\partial\beta}\frac{\partial f}{\partial V}\right) + \frac{\partial f}{\partial\phi}\frac{\partial g}{\partial p_\phi} - \frac{\partial g}{\partial\phi}\frac{\partial f}{\partial p_\phi}, \quad (8)$$

where the phase space Γ consists of all possible points $\{\beta, V, \phi, p_\phi\}$. The allowed pairs of points in Γ are those satisfying the Hamiltonian constraint,

$$C(\beta, V, \phi, p_\phi) = 0. \quad (9)$$

Recall that, for a function that depends on the canonical variables, its time dependence is given by its Poisson bracket with the Hamiltonian. Thus the classical equations of motion are given by

$$\dot{\beta} = \{\beta, C\} = -\frac{3}{2}\frac{\beta^2}{\gamma} - \frac{\kappa\gamma}{4}\frac{p_\phi^2}{V^2}, \quad (10)$$

$$\dot{V} = \{V, C\} = 3\frac{\beta}{\gamma}V, \quad (11)$$

$$\dot{\phi} = \{\phi, C\} = \frac{p_\phi}{V}, \quad (12)$$

$$\dot{p}_\phi = \{p_\phi, C\} = 0. \quad (13)$$

We can verify that (10) and (11) are equivalent to the Friedmann equations for a free scalar field when written in terms of ordinary Arnowitt-Deser-Misner variables. Looking first at (11), we see that it gives that

$$\beta = \gamma\frac{\dot{a}}{a}. \quad (14)$$

Putting in this expression in (9) we obtain

$$-\frac{3}{\kappa}\dot{a}^2a + \frac{1}{2}\frac{p_\phi^2}{a^3} = 0, \quad (15)$$

which can be rewritten as the Friedmann's equations,

$$3\frac{\dot{a}^2}{a^2} = \kappa\left(\frac{1}{2}\frac{p_\phi^2}{a^3}\right)\frac{1}{a^3}, \quad (16)$$

or in terms of the Hubble parameter and the scalar field density,

$$H^2 = \frac{\kappa}{3}\rho. \quad (17)$$

Now putting in the equation for β into (9) we can obtain the Raychaudhuri equation,

$$3\frac{\ddot{a}}{a} = -2\kappa\rho. \quad (18)$$

Now (12) and (13) give us, respectively,

$$\dot{\phi} = \frac{p_\phi}{p^{3/2}}, \quad (19)$$

and

$$p_\phi = \text{const}, \quad (20)$$

since C does not depend on ϕ for a free scalar field. These are just the equations of motion for a free scalar field. It is to the Hamiltonian constraint and these four equations, i.e. (10)–(13), that we wish to find the corrections due to quantum gravity effects.

III. FRAMEWORK FOR EFFECTIVE THEORIES

Fortunately, there is the so-called geometric quantum mechanics framework which gives us a framework in which we can obtain these effective equations. In this section we briefly review the framework for effective theories. A more thorough review may be found in [6,7]. This framework is especially suitable because it provides a direct route to the effective equations from the Hamiltonian constraint without having to deparametrize the theory. That is, in background independent theories there is no canonical notion of time so one usually deparametrizes the theory by choosing one of the fields as an internal clock and then proceeds by considering the dynamics of the other fields with respect to the field serving as a clock. In particular,

this model is deparametrized and analyzed in [8] using the scalar field ϕ as an internal clock.

We take a brief detour and present the main idea behind the procedure for the simpler example of a particle moving in a potential along the real line. The proof that quantum mechanics has the correct semiclassical limit is usually based on an appeal to Ehrenfest's theorem, which is usually expressed in the form, for a particle in a potential,

$$m \frac{\partial^2 \langle x \rangle}{\partial t^2} = - \left\langle \frac{\partial V(x)}{\partial x} \right\rangle. \quad (21)$$

However, this holds for all states, not just semiclassical ones. To recover classical equations for $\langle x \rangle$ what we would like to do is be able to pull the derivative outside of the expectation value,

$$m \frac{\partial^2 \langle x \rangle}{\partial t^2} \simeq - \frac{\partial V(\langle x \rangle)}{\partial \langle x \rangle}, \quad (22)$$

where in pulling the derivative outside, we have replaced $\frac{\partial}{\partial x}$ with $\frac{\partial}{\partial \langle x \rangle}$, since after taking the expectation value the right-hand side is now not a function of x but of $\langle x \rangle$. Additionally, we would like to know if we are able to express the corrections to the right-hand side of (22) in terms of a corrected potential,

$$m \frac{\partial^2 \langle x \rangle}{\partial t^2} = - \frac{\partial}{\partial \langle x \rangle} [V(\langle x \rangle) + \delta V(\langle x \rangle)] + \dots \quad (23)$$

That is, we are looking for an equation of motion satisfied by the expectation values which includes quantum corrections to the classical equations of motion. *A priori*, it is not certain if we can do this. However, the geometrical formulation of quantum mechanics provides a framework in which we can answer this question. Therefore, we quickly review this framework. A more thorough review can be found in [6].

In the quantum theory, quantum states are represented by elements of a Hilbert space \mathcal{H} . More precisely, the quantum phase space consists of rays in \mathcal{H} . The space \mathcal{H} can be made into a symplectic space via its inner product, which can be decomposed into its real and imaginary parts,

$$\langle \phi | \psi \rangle = \frac{1}{2\hbar} G(\phi, \psi) + \frac{i}{2\hbar} \Omega(\phi, \psi), \quad (24)$$

where G gives a Riemannian metric and Ω a symplectic form on \mathcal{H} . Given a function f on the phase space, one can construct the Hamiltonian vector field associated with f by using the symplectic form,

$$X_{(f)}^a = \Omega^{ab} \nabla_b f. \quad (25)$$

It turns out that the flow generated on the phase space by the Hamiltonian vector field $X_{(f)}^a$ corresponds to Schrödinger evolution in the quantum theory generated by \hat{f} [6].

Taking expectation values of the operators corresponding to the canonical variables provides a natural projection, π , from states in the Hilbert space to points in the classical phase space. Thus, the Hilbert space \mathcal{H} is naturally viewed as a fiber bundle over the classical phase space Γ ,

$$\pi: \mathcal{H} \rightarrow \Gamma. \quad (26)$$

Because of the fiber bundle structure, the classical phase space can be viewed as a cross section of the Hilbert space.

We can use this symplectic form to define a notion of horizontal vectors in \mathcal{H} . Namely, vertical vectors are vectors whose components are in the directions in which the expectation values do not change and horizontal vectors are vectors orthogonal to the vertical vectors. It turns out that one has to use Ω rather than G to define orthogonality. That is, two vectors, ϕ and ψ , are orthogonal if

$$\Omega(\phi, \psi) = 0. \quad (27)$$

To summarize, Ω gives us a notion of horizontal vectors which we use to construct horizontal sections of the Hilbert space.

Because of the fiber bundle structure, *any* horizontal section can be identified with the classical phase space. At the kinematical level there is no natural section for us to choose. However, later when we consider dynamics in Sec. V we can look for a natural section that is approximately preserved by the flow of the Hamiltonian constraint in a precise sense.

A priori, we have no way of knowing whether such a section exists. However, if we can find such a section then the quantum dynamics on such a section can be expressed in terms of an effective Hamiltonian which is simply the expectation value of the quantum Hamiltonian operator [6]. The expectation value yields the classical term as the leading term and gets corrections due to quantum effects in the subleading terms. This is the key idea behind our calculation.

This can be done exactly for, e.g., the harmonic oscillator and approximately for several other physically interesting systems, including a dust-filled Friedmann universe [7]. The main result of this paper is proving that this can also be done approximately for a Friedmann universe with a free scalar field and thus obtaining effective equations for this model. In the remainder of this work, we will call these “effective equations” because they are not effective equations in the traditional sense. These are on a different footing from traditional effective equations since there is no precise definition of approximately horizontal sections. Nonetheless, we show that we can obtain these “effective equations” in a well-controlled approximation.

IV. QUANTUM THEORY

Recall that in the full theory the elementary variables are the holonomies of the connection and the electric fluxes. Recall that in loop quantum cosmology there exists no

operator \hat{c} [9] corresponding to the connection, and thus $\hat{\beta}$ also by extension. However, the holonomy operator, $\exp(\frac{i}{2}\tilde{\mu}c) = \exp(\frac{i}{2}\sqrt{\Delta}\beta)$ does exist. Therefore in the quantum theory we work with the algebra generated by $\exp(\frac{i}{2}\sqrt{\Delta}\beta)$ and \hat{V}

A. Coherent state

In the previous section, we saw that the taking of expectation values provides a natural projection from the Hilbert space to the classical phase space. These expectation values can be taken in any state. However, for our investigation of the semiclassical limit it is natural to choose a semiclassical state. Simplest candidates are the Gaussian coherent states. They are natural because we can choose them to be sharply peaked at classical values of the canonical variables, i.e. with small spread in both canonically conjugate variables. Let v and v' denote the parameters proportional to the eigenvalues of the volume operator, related to the volume eigenvalues V and V' via $V = (\frac{8\pi\gamma}{6})^{3/2} \frac{l_p^3}{K} v$ and $V' = (\frac{8\pi\gamma}{6})^{3/2} \frac{l_p^3}{K} v'$, where $K = \frac{2\sqrt{2}}{3\sqrt{3}\sqrt{3}}$. Then a Gaussian coherent state $(\psi|$, with Gaussian spreads ϵ and ϵ_ϕ in the gravitational sector and scalar field sector of the state, peaked at some classical values β' , V' , ϕ' , p'_ϕ is given by

$$\begin{aligned} (\psi_{\beta', V'; \phi', p'_\phi} | &= \int dp_\phi \sum_v e^{-(1/2)\epsilon^2(v-v')^2} e^{i/2\sqrt{\Delta}\beta'(v-v')} \\ &\times e^{-(1/2)\epsilon_\phi^2(p_\phi-p'_\phi)^2} e^{-i\phi'(p_\phi-p'_\phi)}(v; p_\phi| \\ &=: \int dp_\phi \sum_v \tilde{\psi}_n(p_\phi)(v; p_\phi|. \end{aligned} \quad (28)$$

As usual, ψ is defined on a lattice and the summation index v runs over the integers. It is important to note that the spread ϵ is not constant but is a function of the phase space point. We will provide the reason and more details on the functional dependence in the next section. Recall that solutions to the constraints do not lie in the kinematical Hilbert space, but rather in its algebraic dual [10]. Physical states, such as the semiclassical state given in (28) should thus lie in the dual space and thus we write it as a so-called “bra” state $(\psi|$. At first it would seem that working in the dual space would be unmanageable but it does not pose computational difficulties. However, because one fortunately has the “shadow state framework” to carry out calculations [11,12].

We will use $(\psi|$ as our semiclassical state and calculate all of our expectation values in this state. The interpretation of the basis ket is that the universe in the state $|v; p_\phi\rangle$ has physical volume v in Planckian units and scalar field momentum p_ϕ . Our choice for $(\psi|$ as a Gaussian coherent state is not the most general but, rather, is the “simplest” choice one can make to obtain these “effective equations” in a late-time, large-volume approximation. For more gen-

eral considerations it is possible to characterize the state in terms of all its “moments” as in [13]. The higher moments acquire their own equations of motion which must be solved simultaneously with the equations of motion for the canonical variables.

B. Operators in the quantum theory

In order to take expectation values, we need to define the operators in the quantum theory corresponding to the canonical variables. We now construct an approximate operator for β in terms of exponentiated β variables in order to verify that the state Ψ is indeed sharply peaked at classical value of β' . Our point of departure is the classical expression

$$\beta \simeq \frac{1}{i\sqrt{\Delta}}(e^{i/2\sqrt{\Delta}\beta} - e^{-(i/2)\sqrt{\Delta}\beta}) \quad (29)$$

which is exact in the limit $\sqrt{\Delta}\beta \rightarrow 0$. However, our experience in the full theory tells us that in the quantum theory we should not be taking this limit to 0 but to the area gap $\Delta = 4\sqrt{3}\pi\gamma l_p^2$. So in our quantum theory we take

$$\hat{\beta}_\Delta = \frac{1}{i\sqrt{\Delta}}(e^{i/2\sqrt{\Delta}\beta} - e^{-(i/2)\sqrt{\Delta}\beta}). \quad (30)$$

Thus, our operator $\hat{\beta}_\Delta$ agrees approximately with the classical β in the regime $\sqrt{\Delta}\beta \ll 1$. The choice for $\hat{\beta}$ is not unique but this is the simplest choice which is self-adjoint (others have been considered in the literature, e.g. [14]). For the rest of this paper we use this as our approximate β operator and drop the subscript Δ . Its action on our basis kets is given by

$$\hat{\beta}|v; p_\phi\rangle = \frac{1}{i\sqrt{\Delta}}(|v+1; p_\phi\rangle - |v-1; p_\phi\rangle). \quad (31)$$

The action of the other operators is straightforward, \hat{V} and \hat{p}_ϕ act by multiplication,

$$\hat{V}|v; p_\phi\rangle = \left(\frac{8\pi\gamma}{6}\right)^{3/2} \frac{l_p^3}{K} v |v; p_\phi\rangle, \quad (32)$$

$$\hat{p}_\phi |v; p_\phi\rangle = p_\phi |v; p_\phi\rangle, \quad (33)$$

and since we work in the p_ϕ representation ϕ acts by differentiation

$$\hat{\phi} = -\frac{\hbar}{i} \frac{\partial}{\partial p_\phi}. \quad (34)$$

C. Restrictions on parameters in coherent state

Before we move to computation we impose some physically motivated restrictions on the parameters appearing in the coherent state (28). The first pair of restrictions,

$$v' \gg 1, \quad (35)$$

$$\sqrt{\Delta}\beta' \ll 1, \quad (36)$$

corresponds to late times $V' \gg l_p^3$ and $\dot{a} \ll 1$, respectively; namely, that the scale factor be much larger than the Planck length and that the rate of change of the scale factor be much smaller than the speed of light. We will see later that (36) also holds well even at early times.

The next pair of restrictions demands that the spreads in \hat{V} and $\hat{\beta}$ be small, $\frac{\Delta V}{V} \ll 1$ and $\frac{\Delta \beta}{\beta} \ll 1$, or equivalently:

$$v'\epsilon \gg 1, \quad (37)$$

$$\epsilon \ll \sqrt{\Delta}\beta'. \quad (38)$$

The last pair of restrictions on parameters demands that the spreads in ϕ and p_ϕ are small, $\frac{\Delta \phi}{\phi} \ll 1$ and $\frac{\Delta p_\phi}{p_\phi} \ll 1$, or equivalently:

$$\phi \gg \epsilon_\phi, \quad (39)$$

$$p_\phi \epsilon_\phi \gg 1. \quad (40)$$

We use these physically motivated restrictions in our calculations in the remainder of this paper. We now return to showing that ψ is sharply peaked.

D. Verifying that ψ is sharply peaked

Now that we have a candidate semiclassical state, we can calculate the expectation values of the canonical variables and verify that the state ψ is sharply peaked at classical values of the canonical variables.

Calculating $\langle \hat{\beta} \rangle$ we obtain

$$\langle \hat{\beta} \rangle = \frac{2}{\sqrt{\Delta}} e^{-(1/4)\epsilon^2} \sin\left(\frac{1}{2}\sqrt{\Delta}\beta'\right). \quad (41)$$

Thus we must take $\epsilon \ll 1$ for $\langle \hat{\beta} \rangle$ to approximately agree with the classical β . Similarly,

$$\langle \hat{\beta}^2 \rangle = \frac{2}{\Delta} [1 - e^{-\epsilon^2} \cos(\sqrt{\Delta}\beta')]. \quad (42)$$

Thus,

$$\Delta\beta^2 \simeq \frac{2\epsilon^2}{\Delta}, \quad (43)$$

where to get the last line we have used $\cos(\sqrt{\Delta}\beta') \simeq 1$ and $\epsilon \ll 1$. Now we obtain the expectation value and spread of V ,

$$\langle \hat{V} \rangle \simeq V', \quad (44)$$

where to obtain the last line we have done Poisson resummation on the sum over n . Similarly,

$$\langle \hat{V}^2 \rangle = V'^2 + \frac{1}{2\epsilon^2} \frac{l_p^6}{K^2} \left(\frac{8\pi\gamma}{6}\right)^3, \quad (45)$$

$$\Delta V^2 = \frac{1}{2\epsilon^2} \frac{l_p^6}{K^2} \left(\frac{8\pi\gamma}{6}\right)^3. \quad (46)$$

Thus, up to the approximation used to arrive at (43) the product of uncertainties is the minimum possible,

$$\Delta\beta\Delta V \simeq \frac{\kappa\gamma}{2} \frac{\hbar}{2}. \quad (47)$$

We make a brief detour to show that we can satisfy both of the conditions (37) and (38) on the phase space if the width ϵ of the coherent state is properly chosen as a function of the phase space point. Notice that (37) requires that ϵ satisfy $v'\epsilon \gg 1$ and (38) requires that $\epsilon \ll \sqrt{\Delta}\beta$, but $\sqrt{\Delta}\beta \ll 1$ so it appears that there is a tension between these two conditions. Nonetheless, if p'_ϕ is large, which is reasonable for a universe that would increase to macroscopic size, then we can choose $\epsilon(v')$ such that both of the conditions $\frac{\Delta V}{V}$ and $\frac{\Delta \beta}{\beta}$ are satisfied. Looking at the relative uncertainties, we get

$$\frac{\Delta V}{V} = \frac{1}{\sqrt{2}\epsilon V'} \frac{l_p^3}{K} \left(\frac{8\pi\gamma}{6}\right)^3 \ll 1 \quad (48)$$

$$\frac{\Delta \beta}{\beta} = \frac{\sqrt{2}\epsilon}{\Delta\beta'} \ll 1. \quad (49)$$

The first condition shows that we need to choose ϵ as a function of the phase variable V' such that $\epsilon = \frac{\lambda}{v'}$ [for $\lambda \ll (\frac{8\pi\gamma}{6})^3 K^{-1}$] in order to satisfy the inequality. As for the 2nd condition we use the classical form for $\beta^2 = \frac{\kappa\gamma^2}{3}\rho$, where $\rho = \frac{p_\phi^2}{2V^2}$ for a scalar field, to show that the inequality is equivalent to taking $\sqrt{3\pi\hbar} \ll p_\phi$. For a universe that grows to macroscopic size such that p_ϕ satisfies this, then we can choose λ in such a way to satisfy $\lambda \ll (\frac{8\pi\gamma}{6})^3 K^{-1}$, e.g. $\lambda = 70$, $p_\phi = 5000$. Therefore we can satisfy these two conditions simultaneously.

Continuing, taking the expectation values of $\hat{\phi}$ and \hat{p}_ϕ yields

$$\langle \phi \rangle = \phi', \quad (50)$$

and

$$\langle \phi^2 \rangle = \phi'^2 + \frac{1}{2}\epsilon_\phi^2, \quad (51)$$

so

$$\Delta\phi = \frac{\epsilon_\phi}{\sqrt{2}}. \quad (52)$$

Thus, to have $\frac{\Delta\phi}{\phi} \ll 1$ we must have $\epsilon_\phi \ll \phi$. Similarly for p_ϕ and p_ϕ^2 ,

$$\langle p_\phi \rangle = p'_\phi, \quad (53)$$

$$\langle p_\phi^2 \rangle = p_\phi'^2 + \frac{1}{2\epsilon_\phi^2}, \quad (54)$$

therefore we have for Δp_ϕ^2 ,

$$\Delta p_\phi = \frac{1}{\sqrt{2}\epsilon_\phi}. \quad (55)$$

Thus, to have $\frac{\Delta p_\phi}{p_\phi} \ll 1$ we must take $p_\phi \epsilon_\phi \gg 1$.

Notice that this coherent state saturates the Heisenberg uncertainty bound (as it should),

$$\Delta \phi \Delta p_\phi = \frac{1}{2}. \quad (56)$$

Therefore we have confirmed that the expectation values of the operators corresponding to the canonical variables agree with their classical values. Additionally, we have also shown that this coherent state saturates the Heisenberg bound, as it should. Therefore it is a natural kinematical semiclassical state to work with in order to obtain these “effective equations” from the geometric quantum mechanics framework.

E. Expectation value of the Hamiltonian constraint operator

Having verified that ψ is indeed sharply peaked around classical values of the canonical variables, we proceed to calculate the expectation value of the Hamiltonian constraint operator. Recall that, if we can find an approximately horizontal section of the Hilbert space, then the effective quantum dynamics on that section is generated by the effective Hamiltonian which is just the expectation value of the Hamiltonian constraint operator. We calculate this expectation value now, leaving the proof that the section is approximately horizontal to a later section.

Now notice that the self-adjoint constraint is given by

$$C = \frac{1}{16\pi G} C_{\text{grav}} + \frac{1}{2} C_\phi, \quad (57)$$

where

$$\begin{aligned} \hat{C}_{\text{grav}} = \sin(\bar{\mu}c) & \left\{ \frac{24i \text{sgn}(\mu)}{8\pi\gamma^3 \bar{\mu}^3 l_p^2} \left[\sin\left(\frac{\bar{\mu}c}{2}\right) \hat{V} \cos\left(\frac{\bar{\mu}c}{2}\right) \right. \right. \\ & \left. \left. - \cos\left(\frac{\bar{\mu}c}{2}\right) \hat{V} \sin\left(\frac{\bar{\mu}c}{2}\right) \right] \right\} \sin(\bar{\mu}c) \end{aligned} \quad (58)$$

$$\hat{C}_\phi = p_\phi^{-\widehat{(3/2)}} \otimes p_\phi^2. \quad (59)$$

The action of these on our basis kets is given by

$$\begin{aligned} \hat{C}_{\text{grav}}|v; p_\phi\rangle &= f_+(v)|v+4; p_\phi\rangle + f_0(v)|v; p_\phi\rangle \\ &+ f_-(v)|v-4; p_\phi\rangle \end{aligned} \quad (60)$$

$$\hat{C}_\phi|v; p_\phi\rangle = \frac{1}{2} \left(\frac{6}{8\pi\gamma l_p^2} \right)^{3/2} B(v) p_\phi^2 |v; p_\phi\rangle \quad (61)$$

where

$$f_+(v) = \frac{27}{16} \sqrt{\frac{8\pi}{6}} \frac{K l_p}{\gamma^{3/2}} |v+2| |v+1| - |v+3| \quad (62)$$

$$f_-(v) = f_+(v-4) \quad (63)$$

$$f_0(v) = -f_+(v) - f_-(v) \quad (64)$$

and

$$B(v) = \left(\frac{3}{2}\right)^3 K |v| |v+1|^{1/3} - |v-1|^{1/3} \quad (65)$$

and

$$\widehat{p_\phi^{-3/2}} |v; p_\phi\rangle = \left(\frac{6}{8\pi\gamma l_p^2} \right)^{3/2} B(v) |v; p_\phi\rangle. \quad (66)$$

Using these, one can go ahead and compute the expectation value of the Hamiltonian constraint $\langle \hat{C} \rangle$. Note that such a summation includes a summation over negative v even though we are considering late times $v \gg 1$. However, the contribution introduced from these terms is exponentially suppressed. For a proof of this, see [7] and the appendix in [9]. We can evaluate the sum by Poisson resummation and find an asymptotic expansion for the first term in the series obtaining the effective constraint to leading and subleading order.

$$\begin{aligned} \langle \hat{C} \rangle &= -\frac{3}{16\pi G \gamma^2 \bar{\mu}^2} p^{1/2} [1 + e^{-4\epsilon^2} (2\sin^2(\sqrt{\Delta}\beta') - 1)] \\ &+ \frac{1}{2} \left(p_\phi'^2 + \frac{1}{2\epsilon_\phi^2} \right) \left(\frac{6}{8\pi\gamma l_p^2} \right)^{3/2} \\ &\times K \left[\frac{1}{v'} + O(v'^{-3}, v'^{-3}\epsilon^{-2}) \right]. \end{aligned} \quad (67)$$

F. Equations of motion

Recall that to evaluate \dot{O} for some operator O we simply take the commutator between O and the Hamiltonian and divide by $i\hbar$. We begin with $\langle \dot{\beta} \rangle$,

$$\langle \dot{\beta} \rangle = \frac{\langle \psi | \frac{1}{i\hbar} \left(\frac{1}{16\pi G} [\beta, C_{\text{grav}}] + \frac{1}{2} [\beta, C_\phi] \right) | \psi \rangle}{\langle \psi | \psi \rangle}, \quad (68)$$

which can be computed using a similar expansion to that used to find the expectation value of the Hamiltonian constraint to yield

$$\begin{aligned}
\langle \dot{\beta} \rangle \simeq & -\frac{1}{16\pi G} \frac{27}{16\hbar} \left(\frac{8\pi\gamma}{6} \right)^{1/2} \frac{Kl_p}{\gamma^2 \sqrt{\Delta}} \\
& \times \left[4e^{-(25/4)\epsilon^2} \cos\left(\frac{5}{2}\sqrt{\Delta}\beta'\right) \right. \\
& + 4e^{-(9/4)\epsilon^2} \cos\left(\frac{3}{2}\sqrt{\Delta}\beta'\right) \\
& \left. - 8e^{-(1/4)\epsilon^2} \cos\left(\frac{1}{2}\sqrt{\Delta}\beta'\right) \right] - \left(p_\phi'^2 + \frac{1}{2\epsilon_\phi^2} \right) \frac{1}{2V'^2}.
\end{aligned} \tag{69}$$

Now for $\langle \dot{V} \rangle$

$$\langle \dot{V} \rangle = \frac{\langle \psi | \frac{1}{i\hbar} \left(\frac{1}{16\pi G} [V, C_{\text{grav}}] + \frac{1}{2} [V, C_\phi] \right) | \psi \rangle}{\langle \psi | \psi \rangle}. \tag{70}$$

Notice, V commutes with C_ϕ so there is no contribution dependent on the scalar field and we just retain the contribution from the commutator with C_{grav} which yields

$$\langle \dot{V} \rangle \simeq \frac{3V'}{\gamma} e^{-4\epsilon^2} \frac{\sin(2\sqrt{\Delta}\beta')}{2\sqrt{\Delta}}. \tag{71}$$

Now focus on $\langle \dot{\phi} \rangle$ and $\langle \dot{p}_\phi \rangle$,

$$\langle \dot{\phi} \rangle = \frac{\langle \psi | \frac{1}{2i\hbar} [\phi, C_\phi] | \psi \rangle}{\langle \psi | \psi \rangle}, \tag{72}$$

since ϕ commutes with C_{grav}

$$\langle \dot{\phi} \rangle \simeq \frac{p_\phi'}{V'} + O\left(\frac{1}{V'^3}\right). \tag{73}$$

Similarly,

$$\langle \dot{p}_\phi \rangle = \frac{\langle \psi | \frac{1}{i\hbar} \left(\frac{1}{16\pi G} [p_\phi, C_{\text{grav}}] + \frac{1}{2} [p_\phi, C_\phi] \right) | \psi \rangle}{\langle \psi | \psi \rangle} = 0 \tag{74}$$

since p_ϕ commutes with both C_{grav} and C_ϕ .

From these expressions we obtain the “effective equations.”

V. EFFECTIVE EQUATIONS

Recall that in the geometric quantization picture we take as our basic observables, the expectation values:

$$\bar{\beta} = \langle \beta \rangle = \frac{2}{\sqrt{\Delta}} e^{-(1/4)\epsilon^2} \sin\left(\frac{1}{2}\sqrt{\Delta}\beta'\right) \tag{75}$$

$$\bar{V} = \langle V \rangle = V' \tag{76}$$

$$\bar{\phi} = \langle \phi \rangle = \phi' \tag{77}$$

$$\bar{p}_\phi = \langle p_\phi \rangle = p_\phi'. \tag{78}$$

Recall, that the coordinates on the classical phase space are

the expectation values, thus these barred variables are the coordinates on the classical phase space. So we search for an effective description in terms of these variables.

We thus invert these (75)–(78) for the primed variables and express the evolution equations in terms of the barred variables. We look at the first few terms of these asymptotic expansions to get the leading and next-to leading behavior and apply the approximations listed above in Sec. IV C. Doing this we obtain the “effective equations of motion” (the main result of this paper),

$$\begin{aligned}
\bar{C} = & -\frac{3}{\kappa\gamma^2} \bar{V} \bar{\beta}^2 \left(1 - \frac{1}{4} \Delta \bar{\beta}^2 \right) - \frac{6\epsilon^2}{\kappa\gamma^2} \frac{\bar{V}}{\Delta} \\
& + \frac{\bar{p}_\phi^2}{2\bar{V}} [1 + O(\bar{V}^{-2}, \bar{V}^{-2}\epsilon^{-2})]
\end{aligned} \tag{79}$$

$$\begin{aligned}
\dot{\bar{\beta}} = & \frac{3}{4\gamma} \sqrt{1 - \frac{1}{4} \Delta \bar{\beta}^2} [-2\bar{\beta}^2 + \Delta \bar{\beta}^4] \\
& - \frac{\kappa}{4} \sqrt{1 - \frac{1}{4} \Delta \bar{\beta}^2} \frac{\bar{p}_\phi^2}{V'^2} [1 + O(\bar{V}^{-2}, \bar{V}^{-2}\epsilon^{-2})]
\end{aligned} \tag{80}$$

$$\dot{\bar{V}} = 3 \frac{\bar{\beta}}{\gamma} \bar{V} \sqrt{1 - \frac{\Delta \bar{\beta}^2}{4}} \left(1 - \frac{1}{2} \Delta \bar{\beta}^2 \right) \tag{81}$$

$$\dot{\bar{\phi}} = \frac{\bar{p}_\phi}{\bar{V}} + O(\bar{V}^{-3}) \tag{82}$$

$$\dot{\bar{p}}_\phi = 0. \tag{83}$$

As discussed in the Appendix, Eqs. (79) and (81) imply that the Hubble parameter $H = \frac{\dot{a}}{a} = \frac{\dot{V}}{3V}$ satisfies an effective Friedmann equation,

$$H^2 = \frac{\kappa\rho}{3} \left(1 - \frac{\rho}{\rho_{\text{crit}}} \right) + O(\epsilon^2), \tag{84}$$

where

$$\rho_{\text{crit}} = \frac{3}{\kappa\gamma^2 \Delta} \tag{85}$$

which incorporates the leading quantum corrections. Somewhat surprisingly, as numerical simulations show, these corrections are already sufficient to correctly reproduce the main features of full quantum dynamics. These equations have been used in the literature to make certain phenomenological predictions (see, e.g., [15].

One may ask, what is the error involved in making the approximations that led to Eqs. (79)–(83). In these equations we are keeping corrections of $O(\Delta\beta^2)$ relative to the classical expressions but ignoring terms of $O(\Delta^2\beta^4)$. How much of an error is one making by ignoring these terms? Let us compute, $\Delta\beta^2$,

$$\Delta\beta^2 = \Delta\gamma^2 \frac{\dot{a}^2}{a^2} = \frac{\rho}{\rho_c} \left(1 - \frac{\rho}{\rho_c}\right). \quad (86)$$

Notice, this function starts off at 0 at late times since $\rho/\rho_c \sim 0$, it slowly increases reaches a maximum value of $1/4$ at $\rho = \rho_c/2$, and then again goes to 0 at the point $\rho/\rho_c = 1$. Therefore, at the worst we are making an error of about 6% in ignoring terms of the order $\Delta^2\beta^4$ and this occurs at $\rho = \rho_c/2$. At other times, i.e. at late times and at times near the bounce point, this approximation is very good.

Additionally, one can pullback the symplectic structure Ω to our candidate horizontal section and analyze the dynamics generated by this effective Hamiltonian. Does it give the equations (80)–(83)? One can pullback Ω and verify that it is given by

$$\Omega = \frac{2}{\kappa\gamma} \frac{1}{\sqrt{1 - \frac{1}{4}\Delta\beta^2}} d\bar{\beta} \wedge d\bar{V} + d\bar{\phi} \wedge d\bar{p}_\phi \quad (87)$$

and that its associated Poisson bracket is

$$\{f, g\} = \frac{\kappa\gamma}{2} \sqrt{1 - \frac{1}{4}\Delta\beta^2} \left(\frac{\partial f}{\partial \bar{\beta}} \frac{\partial g}{\partial \bar{V}} - \frac{\partial g}{\partial \bar{\beta}} \frac{\partial f}{\partial \bar{V}} \right) + \frac{\partial f}{\partial \bar{\phi}} \frac{\partial g}{\partial \bar{p}_\phi} - \frac{\partial g}{\partial \bar{p}_\phi} \frac{\partial f}{\partial \bar{\phi}}. \quad (88)$$

We can verify that our “effective equations” are consistent by checking that the Poisson brackets in terms of the barred variables hold, to within our order of approximation. That is, $(\bar{\beta}, \bar{V}, \bar{\phi}, \bar{p}_\phi)$ are the coordinates on the horizontal section, Γ , and we wish to know whether they are preserved, i.e. is the vector $(\dot{\bar{\beta}}, \dot{\bar{V}}, \dot{\bar{\phi}}, \dot{\bar{p}}_\phi)$ tangent to Γ or off it? If it is not tangent to Γ then is it approximately tangent? That is, are its components off Γ small relative to the tangential components? This is indeed the case, since to our order of approximation the equations of motion in terms of the barred variable hold. Indeed, it can be verified that the following equations hold in terms of the pullback of the symplectic structure to the barred variables:

$$\begin{aligned} \dot{\bar{\beta}} &= \{\bar{\beta}, \bar{C}\} & \dot{\bar{V}} &= \{\bar{V}, \bar{C}\} \\ \dot{\bar{\phi}} &= \{\bar{\phi}, \bar{C}\} & \dot{\bar{p}}_\phi &= \{\bar{p}_\phi, \bar{C}\}. \end{aligned} \quad (89)$$

This is a highly nontrivial consistency check on the formalism as well as approximations used at intermediate steps.

VI. CONCLUSION

Thus, we have constructed kinematical quantum states that closely approximate solutions to the classical Einstein equations. Therefore, at least for the case of a Friedmann universe with a free scalar field in the context of loop quantum cosmology, it is the case that there exist suitable quantum states that closely approximate solutions to the

classical Einstein equations and *remain* sharply peaked along a quantum corrected, semiclassical trajectory.

Furthermore we note that we have shown that the classical Einstein’s equations are not just reproduced identically in the quantum theory but they do indeed pick up corrections due to quantum effects. In this work, we have used the geometric quantum mechanics framework and approximated the full quantum dynamics in the (infinite dimensional) Hilbert space by a system of “effective equations,” incorporating the leading quantum corrections, on a finite dimensional submanifold isomorphic to the classical phase space.

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APPENDIX: VALIDITY OF EFFECTIVE EQUATIONS NEAR THE BOUNCE

It is instructive to write the “effective equations” in standard form. Therefore, in this Appendix we address the question: what is the form of the corrected Friedmann and Raychaudhuri equations?

We can obtain the corrected Friedmann equation by, working to next-to leading order, solving (79) for $\bar{\beta}$, substituting into (81), and then computing the Hubble parameter $H = \frac{\dot{a}}{a} = \frac{\dot{\bar{V}}}{\bar{V}}$:

$$H^2 = \frac{\kappa\rho}{3} \left(1 - \frac{\rho}{\rho_{\text{crit}}}\right) + O(\epsilon^2), \quad (A1)$$

where

$$\rho_{\text{crit}} = \frac{3}{\kappa\gamma^2\Delta}. \quad (A2)$$

Notice, in addition to recovering the classical $\frac{\kappa}{3}\rho$ term, we get a correction term that becomes important when the density ρ becomes very large. We see that the minus sign allows H , and hence \dot{a} to be zero, thus allowing the possibility of a bounce. Thus, even though these equations hold at late times since we are working in the semiclassical regime, we already see that gravity is becoming repulsive and allowing the scale factor to bounce when we evolve backwards in time and approach the classical big bang singularity.

On (A1) we must make two comments. First, *a priori* one would not expect (A1) to describe the correct dynamics near the bounce point because the bounce point $\rho/\rho_c = 1$ lies outside of the regime of our approximation. That is,

the natural domain of applicability of these “effective equations” is a late-time, large-volume approximation and the point $\rho/\rho_c = 1$ lies well outside of that regime since the condition (38) fails badly at that point. However, numerical work (see Ref. [2] in [8]) has shown that the dynamics is indeed described well by (A1) even at the bounce and hence this approximation and the results obtained in this work continue to be good even beyond their expected regime. Second, near the bounce point the term in parentheses in (A1) is approaching 0, but since the approximation (38) is breaking down there it is not clear that the $O(\epsilon^2)$ term is negligible there. Thus, (A1) appears to break down there, but the numerical work has shown that it holds very well with negligible $O(\epsilon^2)$ corrections. As is not uncommon in physics, the effective theory is valid well beyond the domain for which it was constructed.

So now we have the effective Friedmann equation in this model. It would be instructive, for completeness, to also obtain the conservation equation and the corrected Raychaudhuri equation for this effective theory. Recalling that $\rho = p$ for a massless scalar field, classically these equations are

$$\dot{\rho} = -6\frac{\dot{a}}{a}\rho, \quad (\text{A3})$$

$$3\frac{\ddot{a}}{a} = -2\kappa\rho. \quad (\text{A4})$$

Let us compute the corrected versions of these equations. Using the modified Poisson bracket, we can compute $\dot{\rho} = \{\rho, \bar{C}\}$. Doing this we obtain precisely (A3), so the continuity equation is not modified. Similarly, we can take (81) and compute $\ddot{V} = \{\dot{V}, \bar{C}\}$, expressing it in terms of the scale factor a , and solving for $3\frac{\ddot{a}}{a}$. One obtains

$$3\frac{\ddot{a}}{a} = -2\kappa\rho\left(1 - \frac{5}{2}\frac{\rho}{\rho_c}\right) + O(\epsilon^2). \quad (\text{A5})$$

Notice that classically this expression is always negative. However, now there is a correction of $O(\frac{\rho}{\rho_c})$. Notice that the corrected Friedmann equation (A1) tells us there is a bounce at $\rho/\rho_c = 1$. At that density the corrected Raychaudhuri equation (A5) tells us that \ddot{a} is positive as it should be if there is to be a bounce.

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