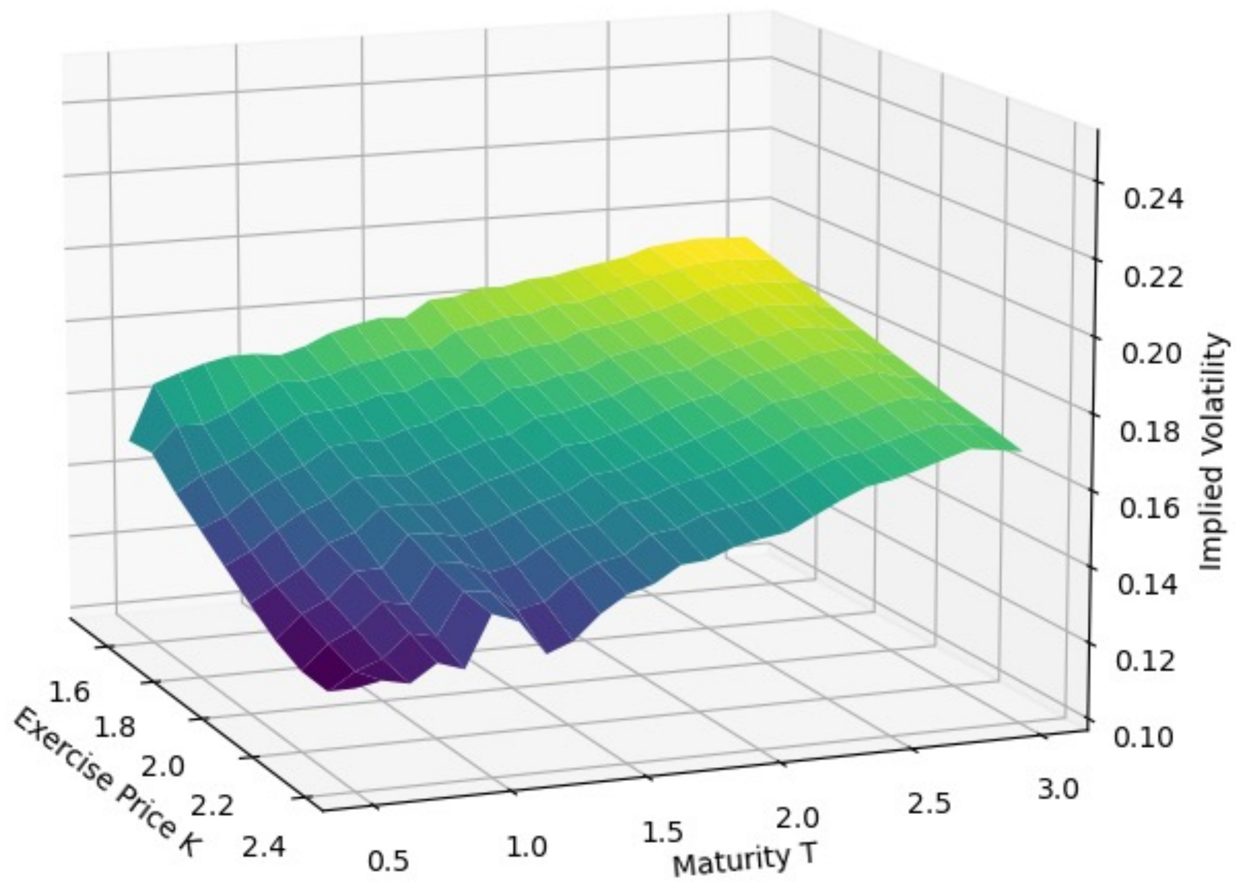
Heston Model Parameters' Sensitivity Analysis



Outline

- **Introduction**
- Understanding the parameters
- Variance in CEV process
- Monte Carlo method
- Conclusion

Introduction

Heston Model (1993)

- The Black-Scholes implied volatility extracted from market's data indicates **skewness** and "**volatility smile**", i.e. options takes higher values being far in or out of the money

- It is because the BS model crudely assume constant volatility and in log-normal returns

- There is also an empirically observed "**leverage effect**", which means that volatility is usually negatively correlated with the underlying asset

- The Heston Model takes the non-log normal distribution of the assets returns, the leverage effect into account, by introducting the correlation between the two Wiener processes

$$dS_t = rS_t\,dt + \sqrt{v_t}\,S_t\,dW_t^S$$

- where v_t the instantaneous variance, is a CIR process:

$$dv_t = \kappa(\theta\ - v_t) + \sigma\sqrt{v_t}\,dW_t^v$$

- W_t^S, W_t^v are Wiener processes with correlation ρ, i.e.

$$dW_t^S dW_t^v = \rho dt$$

- This can be simulated by generating two independent random variables Z_1 and Z_2, and define

$$Z_v = Z_1\ and\ Z_s = \rho Z_v + \sqrt{1-\rho^2}Z_2$$

Introduction

Heston Model (1993) Limitations

- The financial instruments studied are options written on an underlying asset with stochastic price and variance dynamics.

- To improve the Heston model, one could further incorporate jump into it to match closer with the reality

- For a further justification for the use and properties of stochastic volatility models with jumps, the reader may refer to Cont and Tankov (2004).

- Also, there is a significant amount of research on models driven by L´evy processes - Generalized Poisson, Normal Inverse Gaussian, Variance Gamma, say.

Introduction

- As introduced before, the Heston model is

$$dS_t = rS_t\,dt + \sqrt{v_t}\,S_t\,dW_t^S$$

$$dv_t = \kappa(\theta\ - v_t) + \sigma\sqrt{v_t}\,dW_t^v$$

$$dW_t^S dW_t^v = \rho dt$$

The model parameters are:

- r is the rate of return of the asset.
- θ is the long-term variance, or long run average price variance; as *t* tends to infinity, the expected value of v_t tends to θ.
- κ is the rate at which v_t reverts to θ.
- σ is the volatility of the volatility, or 'vol of vol', and determines the variance of v_t.

Introduction

- In a model following the CIR dynamics, some studies show that the mean-reversion coefficient κ, is strictly positive, and in particular k > 1.

- The volatility of the variance is positive, i.e. σ > 0. Also, naturally, θ and v0 are nonnegative.

- Finally, as already mentioned, the correlation between the two Wiener processes characterizing then underlying asset and the variance dynamics should be non-positive, i.e. ρ ≤ 0.

- To prevent the CIR process of the variance reaching zero, the Feller condition [Feller (1951)] must be satisfied

$$\frac{\sigma^2}{2\kappa\theta} \leq 1$$

Introduction

- In many studies before, the value of the Heston's parameters calibrated are around:

$$\mathrm{r} = 5\%, \theta = (20\%)^2, \kappa = 20\%, \sigma = 60\%, \rho = -0.7, v_0 = (20\%)^2$$

- Therefore, in the following research, this set of parameters above are used as benchmark,
- all valuations are performed on European Put options
- The main objective in this study is the effect of different parameters to the I.V. of the option price simulated under Heston model
- The asset price S and variance v are discretized under Milstein's scheme:

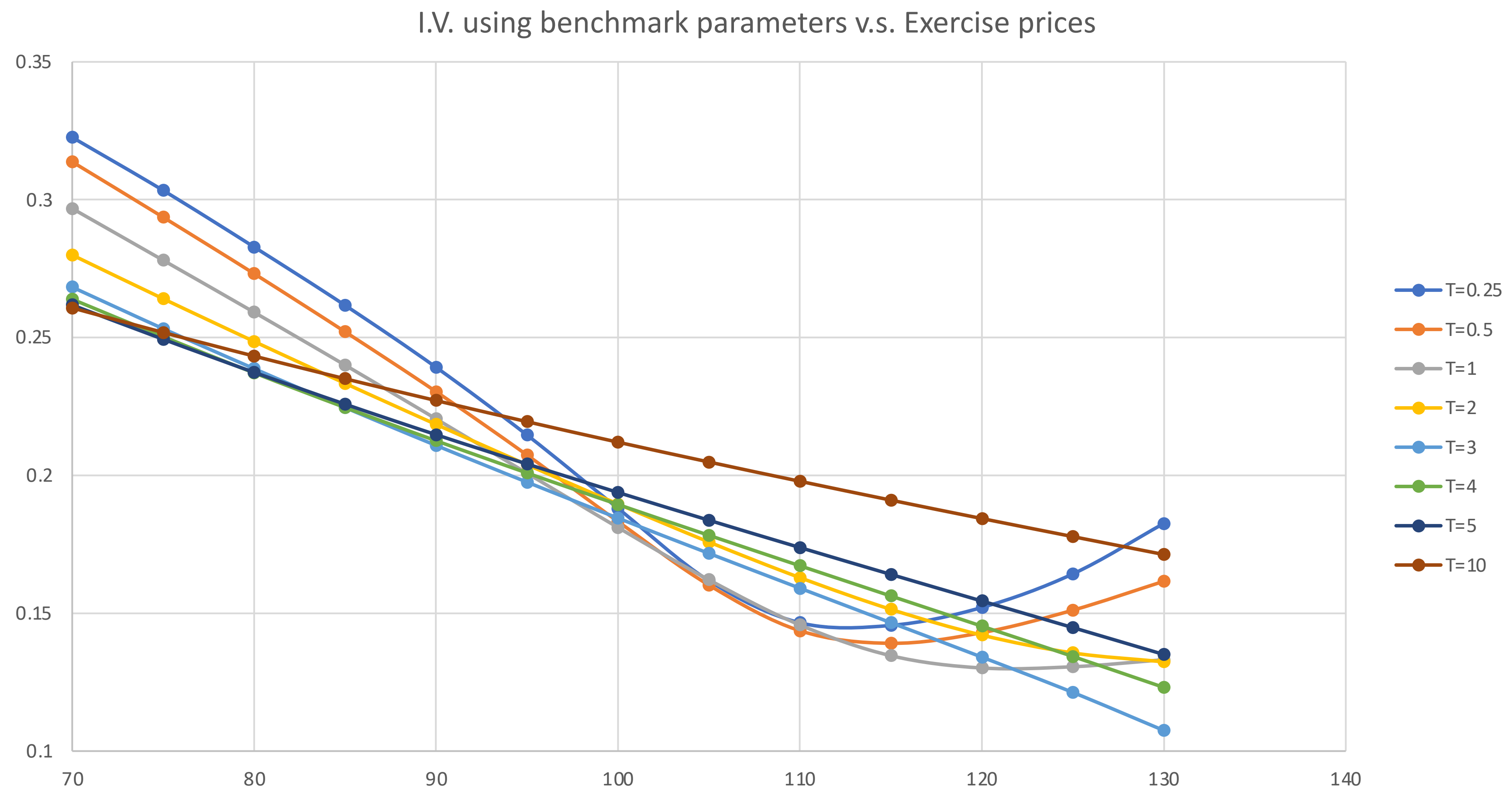
$$S_{t+dt} = S_t e^{\left(r - q - \frac{1}{2}v_t\right)dt + \sqrt{v_t}\sqrt{dt}Z_s}$$

$$v_{t+dt} = \left(\sqrt{v_t} + \frac{1}{2}\sigma\sqrt{dt}Z_v\right)^2 + \kappa(\theta - v_t)dt - \frac{1}{4}\sigma^2 dt$$

- In the Monte-Carlo simulation below, the variance-reduction method control-variate and antithetic-variate are applied. 100,000 of scenarios and time-step dt=0.001 are used. The efficiency of CV and AV will be shown in the appendix.

Outline

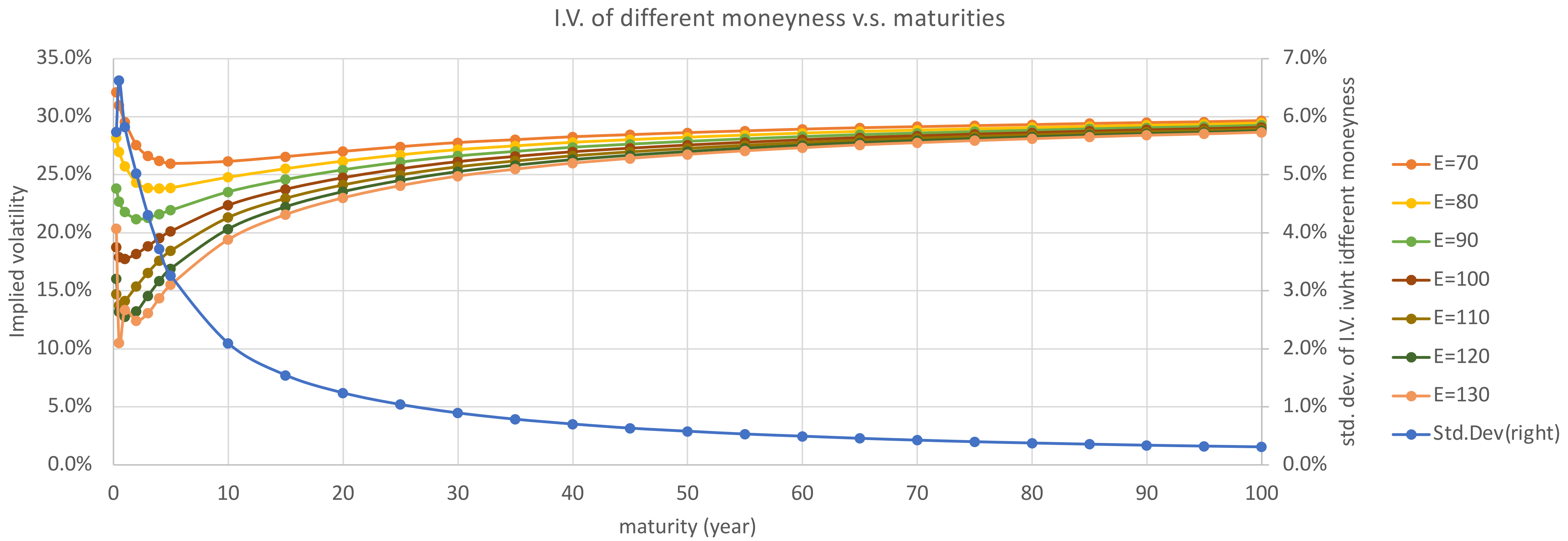
Understanding the parameters



- As shown above is the I.V. curve of the benchmark parameters: $S = 100, r = 5\%, \theta = (20\%)^2, \kappa = 20\%, \sigma = 60\%, \ \rho = -0.7, v_0 = (20\%)^2$
- The corresponding I.V. against different exercise prices is shown above
- The results are generated through Monte-Carlo method using time-step dt=0.001, and 100,000 scenarios are used

Understanding the parameters

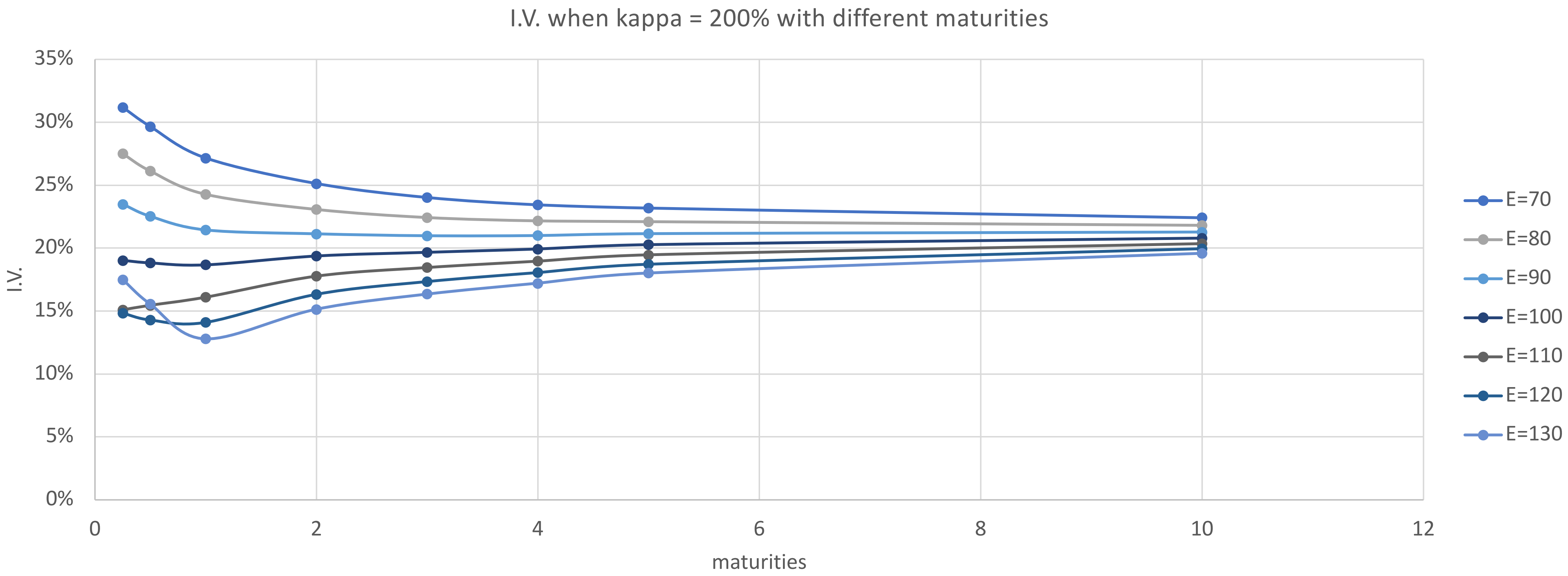
Mean reversion speed κ=20% – Term-structure



- The benchmark parameters are used
- Plotted on the right axis is the standard deviation of the implied volaility of the 6 different moneyness with the corresponding maturity
- It is observed that the S.D. decreases exponentially across the maturity

Understanding the parameters

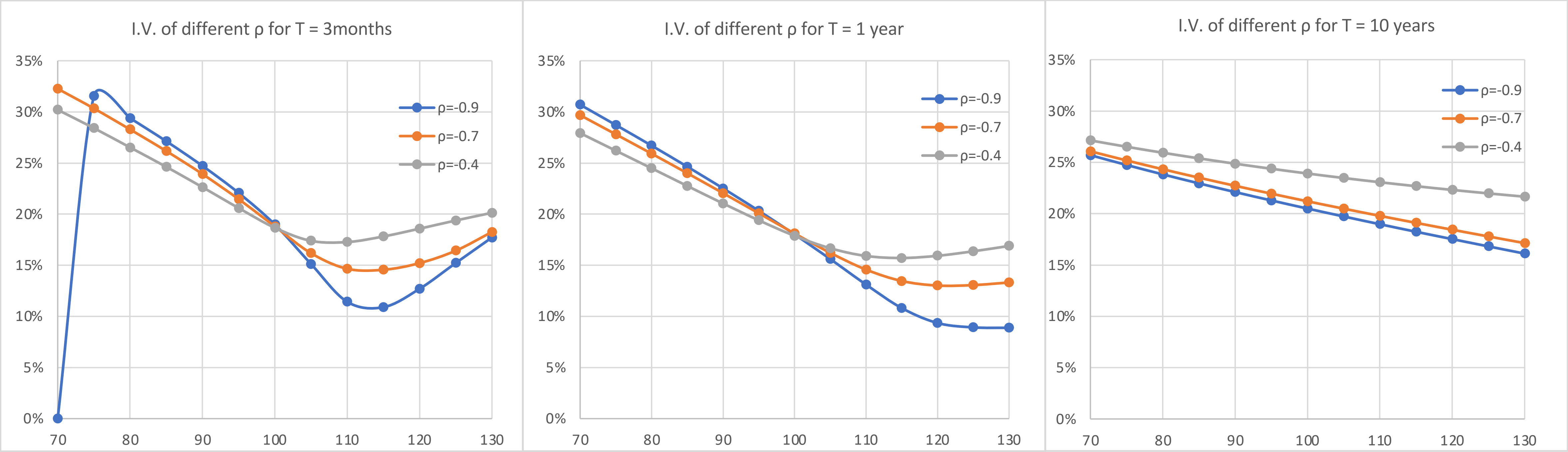
Mean reversion speed κ=200% – Term-structure



- Same set of benchmark parameters are used except with a much larger κ, it is observed that the I.V. can start to converge to the long-term variance θ in about 10 years

Understanding the parameters

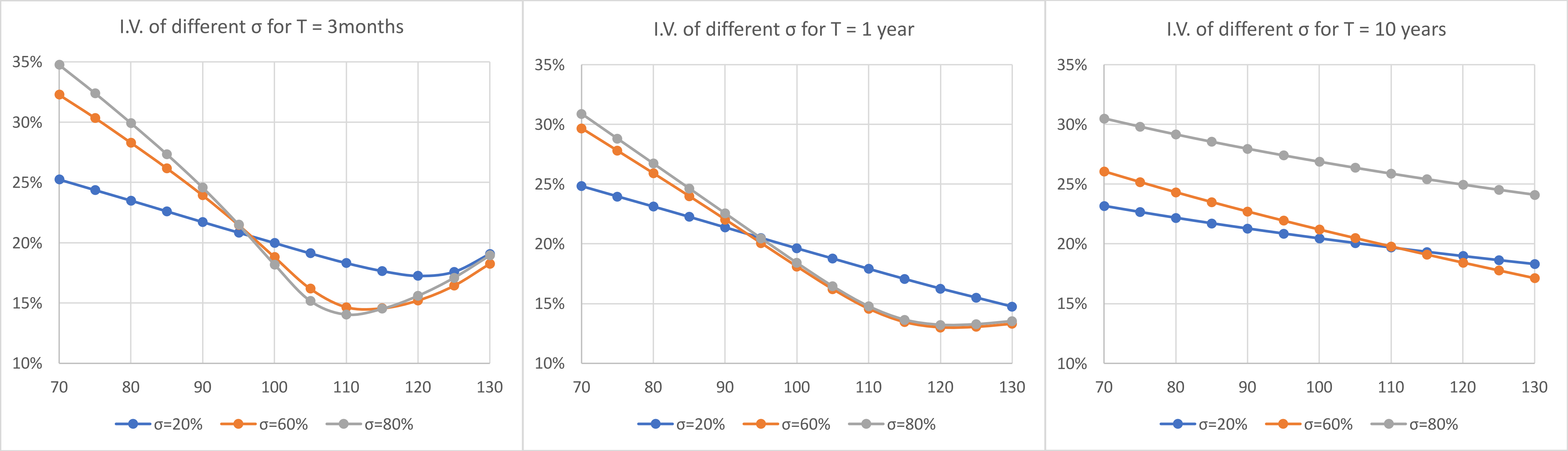
I. Correlation $\boldsymbol{\rho}$



- In the test above, the value of parameters are $\mathrm{S} = 100, \mathrm{r} = 5\%, \kappa = 20\%, \theta = (20\%)^2, \sigma = 60\%,\ v_0 = (20\%)^2$
- Due to numerical instability, it is difficult to extract the I.V. for large correlation with short maturity.
- We can see that the correlation ρ determines the direction of skew
- Positive ρ corresponds to a positive slope and vice versa
- For shorter maturity, the smile pattern is more obvious

Understanding the parameters

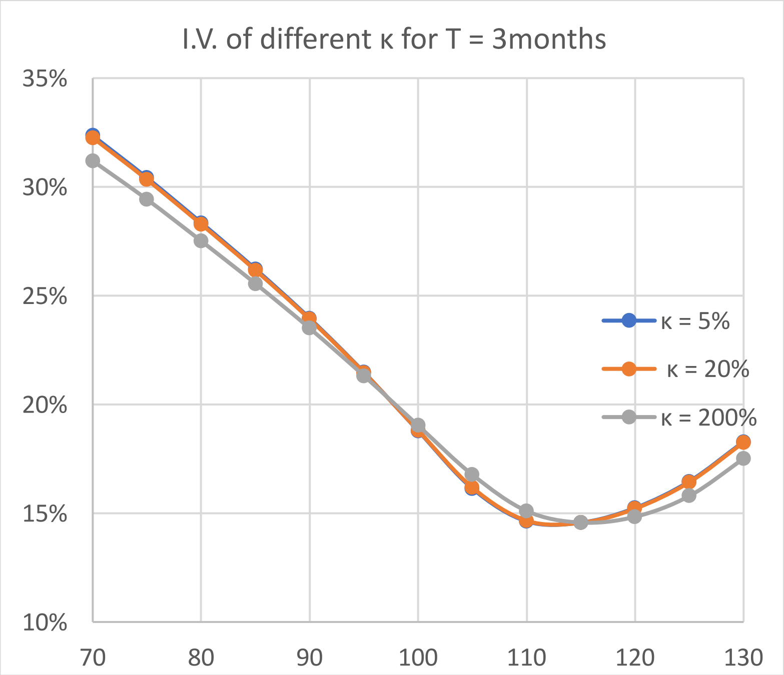
II. Volatility of variance σ

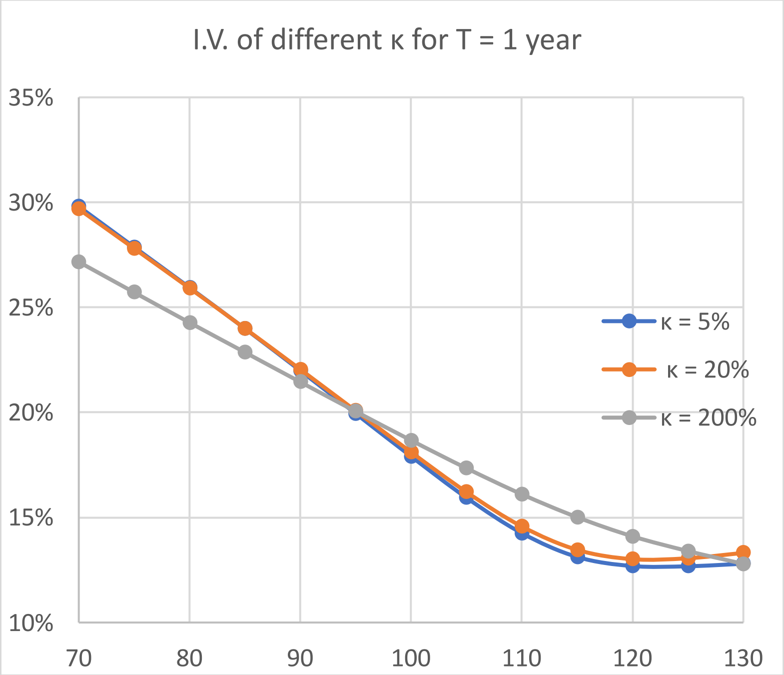


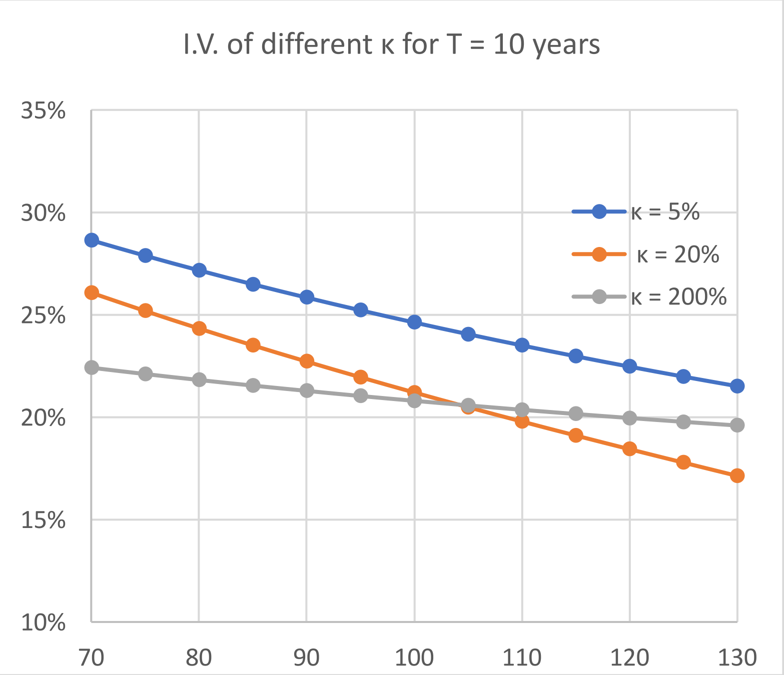
- The value of parameters used are $S = 100, \mathrm{r} = 5\%, \theta = (20\%)^2, \kappa = 20\%, \rho = -0.7, v_0 = (20\%)^2$
- Increasing values of the volatility of variance σ increases the curvature of the smile
- The effect is more obvious for shorter maturity

Understanding the parameters

III. Mean reversion speed κ – Moneyness



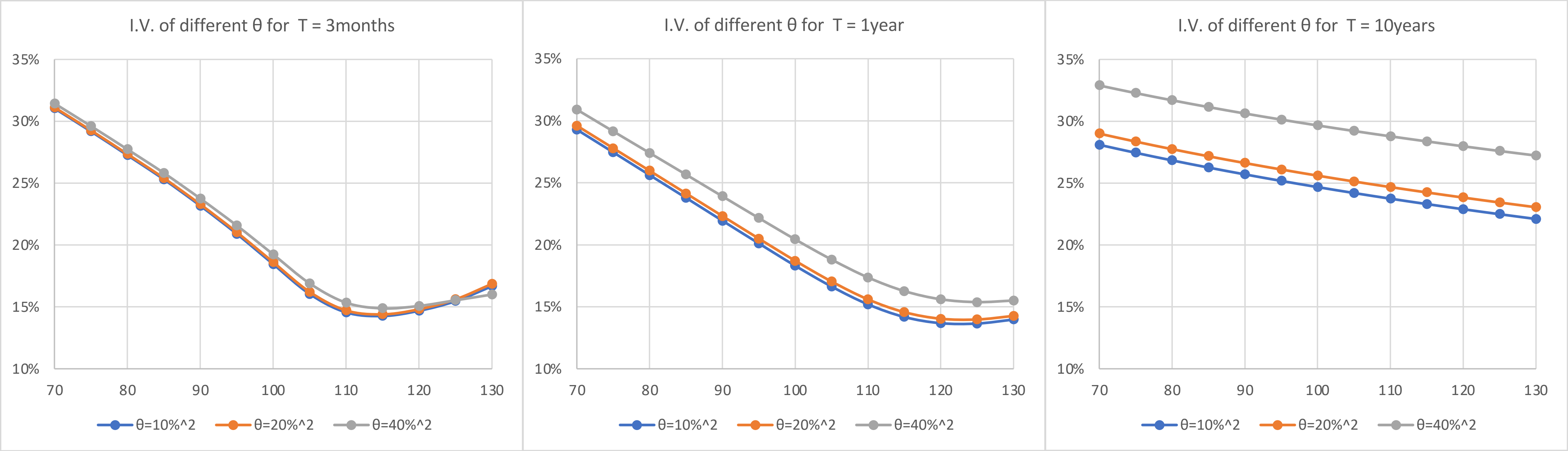




- The value of parameters used $\mathrm{S} = 100, \mathrm{r} = 5\%, \theta = (20\%)^2,\ \sigma = 60\%,\ \rho = -0.7, v_0 = (20\%)^2$
- The mean reversion speed κ also controls the degree of curvature,
- A larger κ flattens the implied volatility curve more.

Understanding the parameters

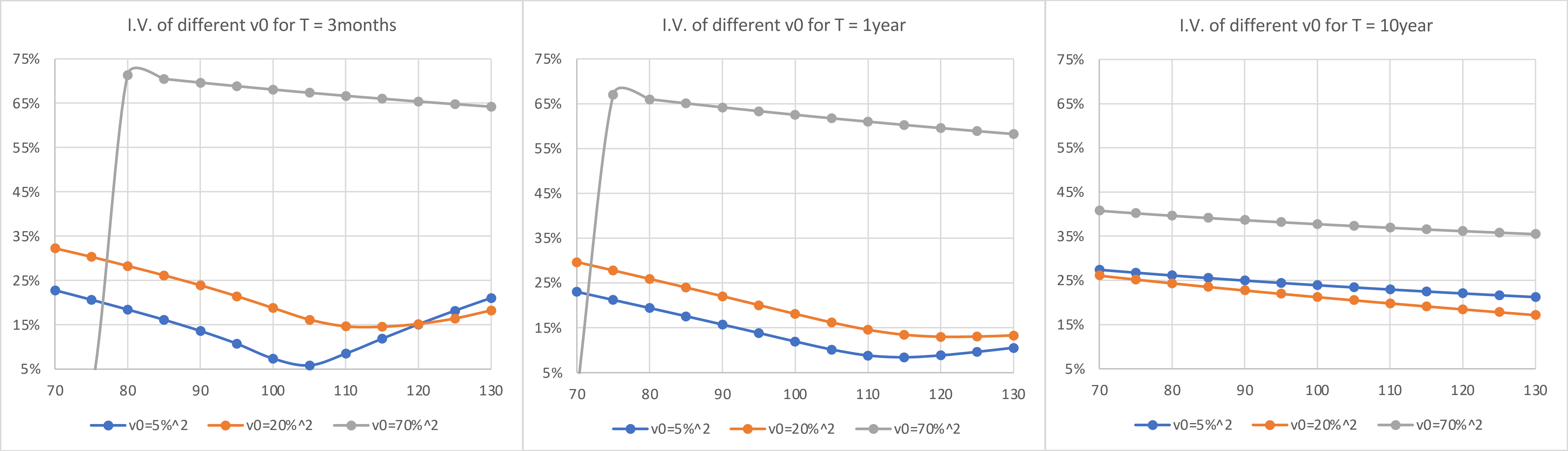
IV. Long-term variance θ



- The value of parameters used are $S = 100, \mathrm{r} = 5\%, \kappa = 20\%, \sigma = 60\%, \ \rho = -0.7, v_0 = (20\%)^2$
- A larger initial variance v_0 shifts I.V. higher,
- and decreases the curvatures of I.V.

Understanding the parameters

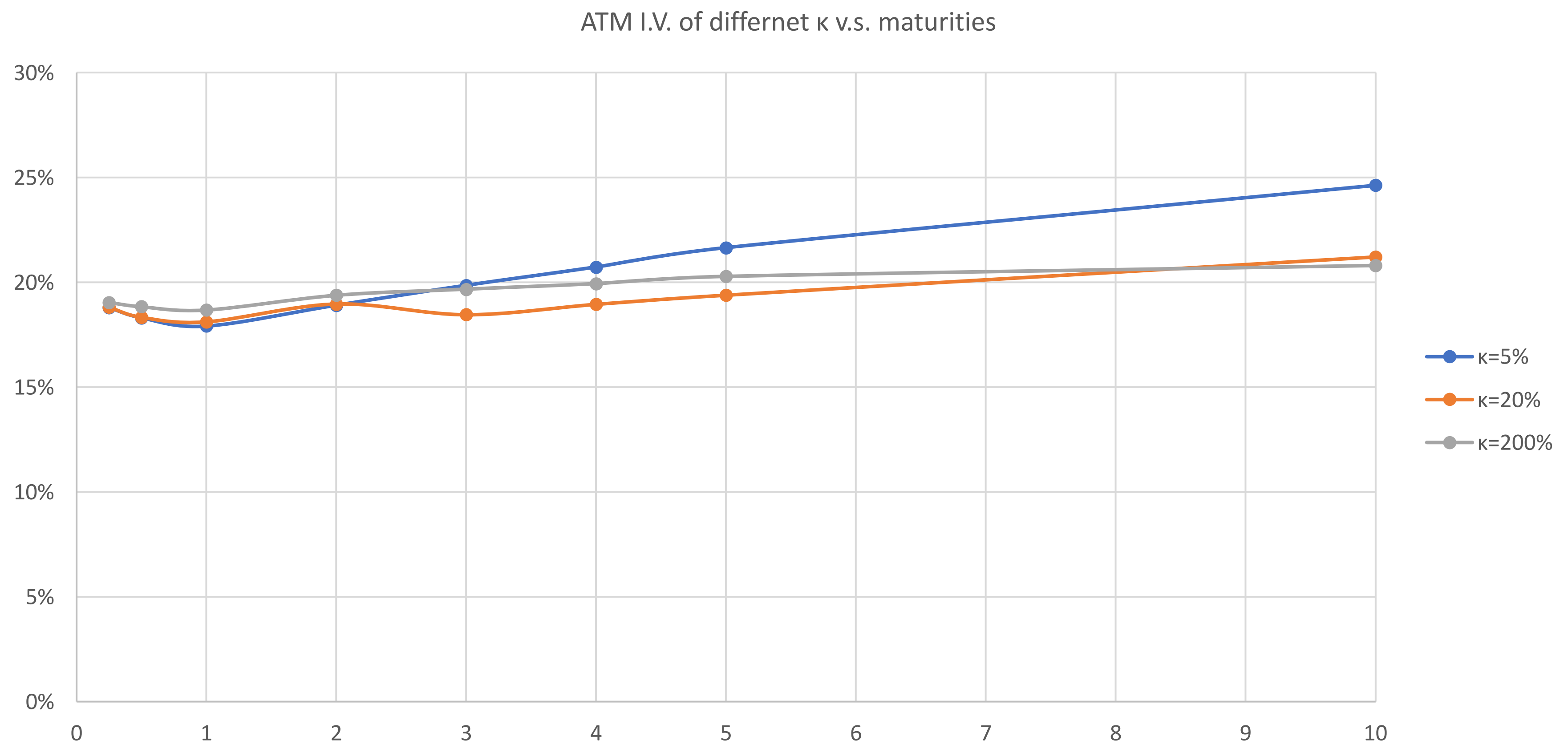
V. Initial variance v_0



- The value of parameters used $S = 100, r = 5\%, \theta = (20\%)^2, \kappa = 20\%, \sigma = 60\%, \ \rho = -0.7$
- For small moneyness and high initial variance, A larger initial variance v_0 shifts I.V. higher,
- and decreases the curvatures of I.V.

Understanding the parameters

Mean reversion speed κ – ATM Term-structure(cont’)



- It is observed that with a larger κ, the I.V. can converge to the long-term volatility 20%.

Outline

1. Introduction
2. Understanding the parameters
3. **Variance in CEV process**
4. Monte Carlo method
5. Conclusion

Variance in CEV process

- In the original Heston model, the stock price is modeled by: $dS_t = rS_t\, dt + \sqrt{v_t}\, S_t\, dW_t^S$
- ,where v_t the instantaneous variance is running under CIR process: $dv_t = \kappa(\theta - v_t) + \sigma\sqrt{v_t}\, dW_t^v$

- We are going to change the process of variance to be

$$dv_t = \kappa(\theta - v_t) + \sigma v_t^{\gamma}\, dW_t^v$$

- where $\gamma \in [0.5, 1]$, and test its effect to the I.V. structure

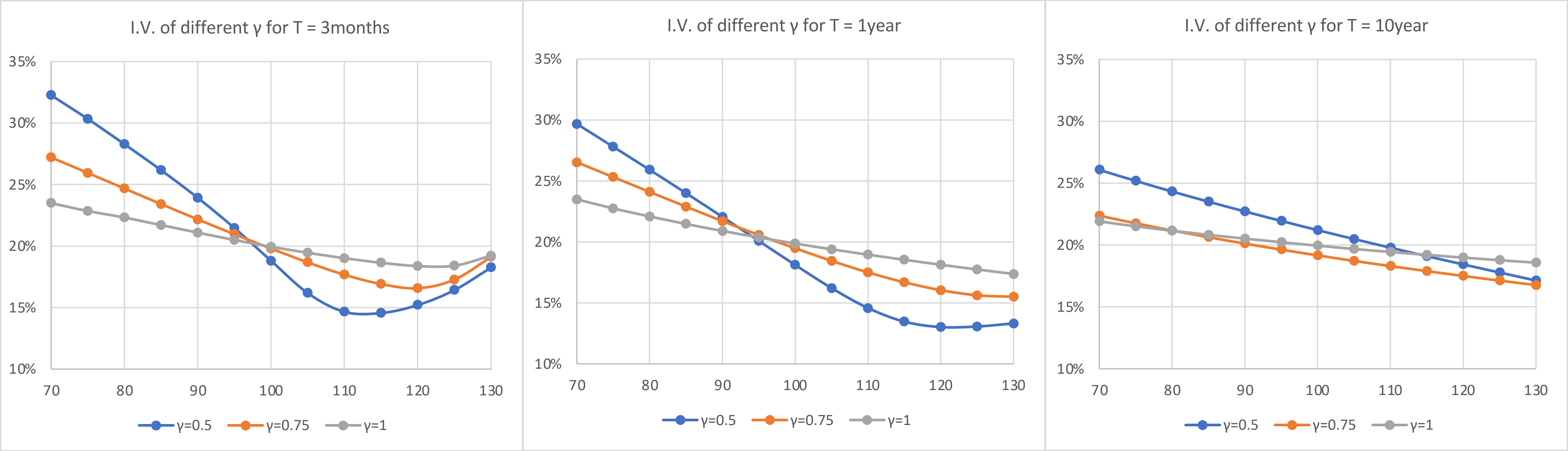
- When γ = 0.5, it returns to the CIR process, namely Heston model for the stock price S

- Under Milstein scheme, the process of variance is derived to be

$$v_{t+dt} = v_t + \kappa(\theta - v_t) + \sigma v_t^{\gamma}\sqrt{dt}\, Z_v + \frac{1}{2}\sigma^2 v_t^{2\gamma-1} dt(Z_v^2 - 1)$$

Variance in CEV process

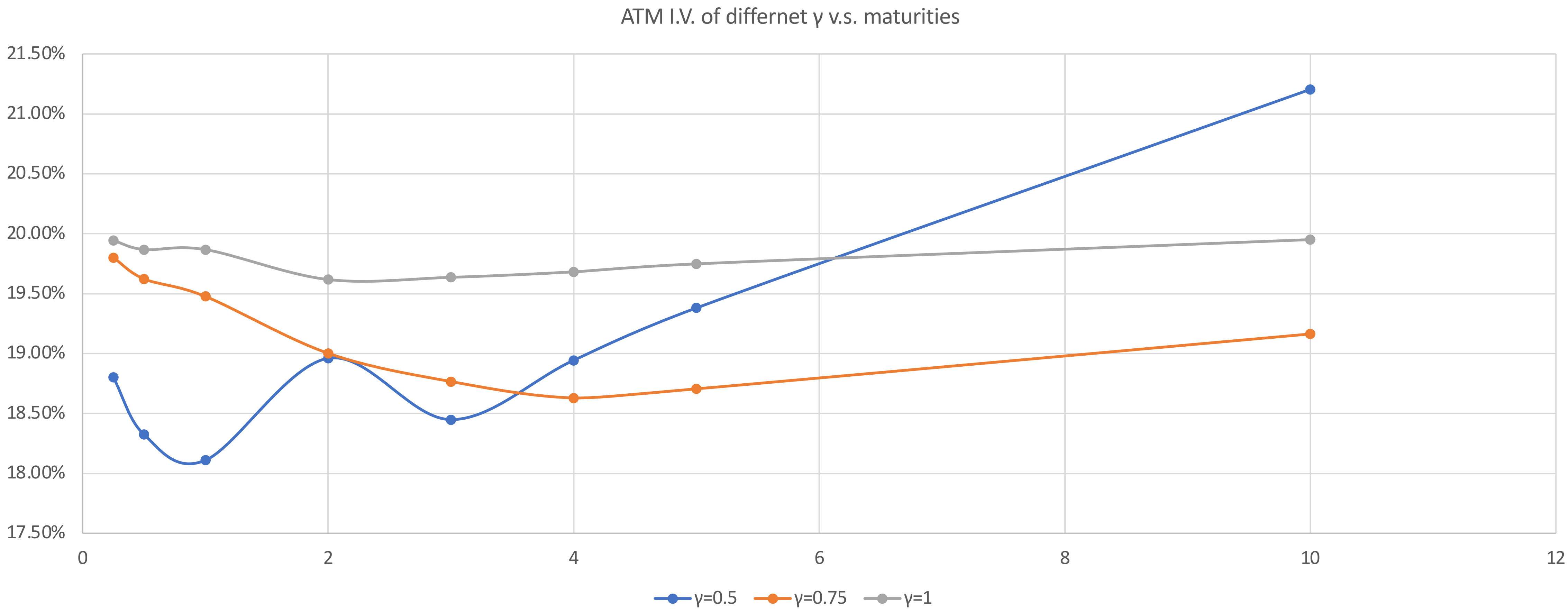
VI. varying γ



- The value of parameters used are $S = 100, r = 5\%, \theta = (20\%)^2, \kappa = 20\%, \sigma = 60\%, \ \rho = -0.7, v_0 = (20\%)^2$
- For γ=0.5, which is the original Heston model, the smile pattern is more obvious in short-maturity
- For long-maturity, the I.V. curve flattens in all 3 cases.
- A larger γ reduces the curvature of the I.V. curve

Variance in CEV process

VI. term-structure



- The value of parameters used are $S = 100, r = 5\%, \theta = (20\%)^2, \kappa = 20\%, \sigma = 60\%, \ \rho = -0.7, v_0 = (20\%)^2$
- With a larger γ, the I.V. can converge to the long-term variance θ faster
- We can also see that, using γ=0.5, which is the original Heston model, the I.V. fails to converge to θ.

Outline

Monte-Carlo Method

Variance Reduction Methods - Control Variate

- To speed up the MC Method, there are two variance-reduction methods to be introduced which can in turn decrease the number of asset paths needed and improve its efficiency.

- As suggested by the paper [Guo Liu et. al. 2015], a simple control variate is proposed

- Assuming the asset price is moving under a time-dependent volatility:

$$dS(t) = S(t)\left(rdt + \sqrt{e^{-\kappa t}\, v_o + \theta(1 - e^{-\kappa t})}dW\right)$$

- The value of a European Put is derived to be

$$V_p(t = 0) = Ke^{-rT}N(-d2) - S_0N(-d1)$$

$$d1 = \frac{a\ - lnK}{\sqrt{b}}$$

$$d2 = d1\ - \sqrt{b}$$

$$a = lnS_0 + rT + \frac{b}{2}$$

$$b = \theta T + \frac{1}{\kappa}(v_0 - \theta)(1 - e^{-\kappa t})$$

Monte-Carlo Method

Antithetic Variate

- Consider we want to find the expected value of a payoff-function f(U) depending on a random variable U is a N(0,1) random variable.

$$I \ = \ E[f(U)] = \frac{1}{M}\sum_{i=1}^{M} f(U_i)$$

- However, we can use $\frac{f(U_i)+f(-Ui)}{2}$ instead of $f(U_i)$ to reduce the variance, where $-U_i$is also a N(0,1) random variable. It is because

$$\begin{aligned}
&var\left(\frac{f(U_i)+f(-Ui)}{2}\right) \\
&= \frac{1}{4}\left[var\big(f(U_i)\big) + var\big(f(-U_i)\big) + 2cov(f(U_i), f(-U_i)\ )\right] \\
&= \frac{1}{2}\left[var\big(f(U_i)\big) + cov(f(U_i), f(-U_i)\ )\right]
\end{aligned}$$

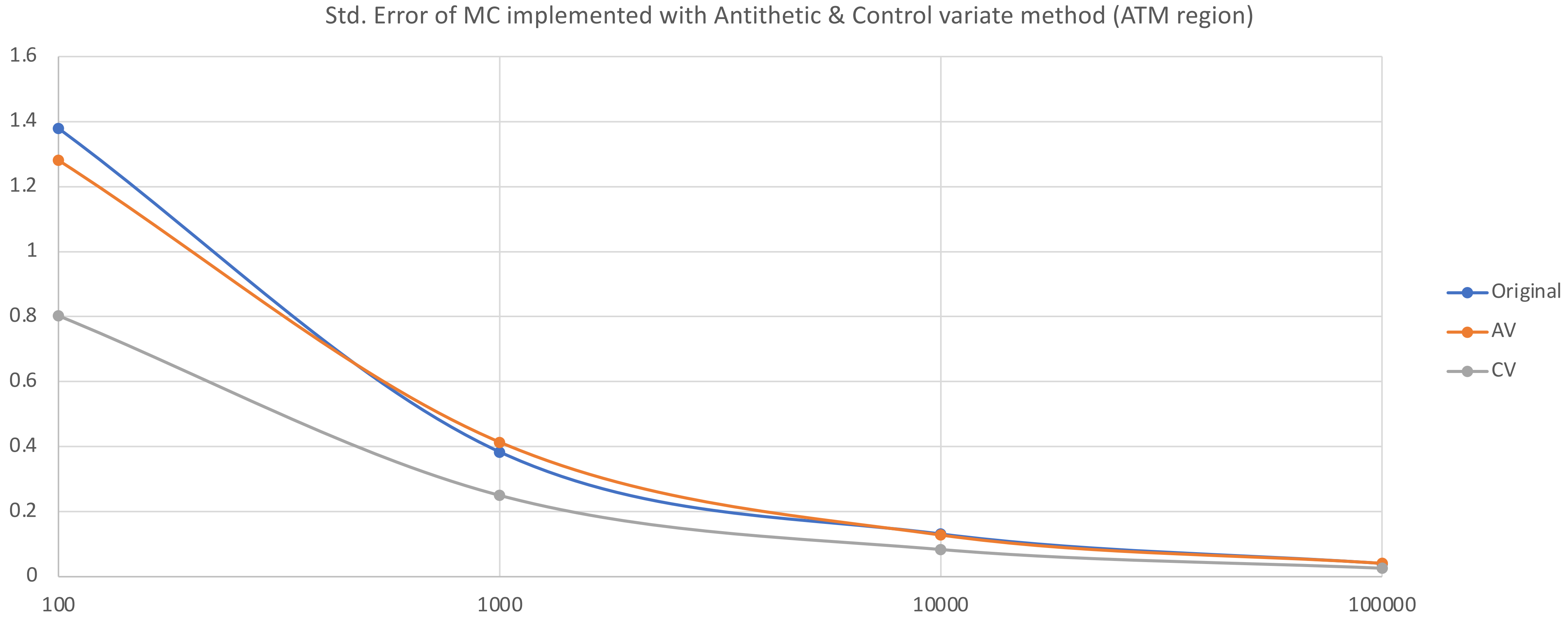
It can be shown that

$$var\left(\frac{f(U_i)+f(-Ui)}{2}\right) \leq \frac{1}{2}\left[var\big(f(U_i)\big)\right]$$

, if the covariance is negative.

Monte-Carlo Method

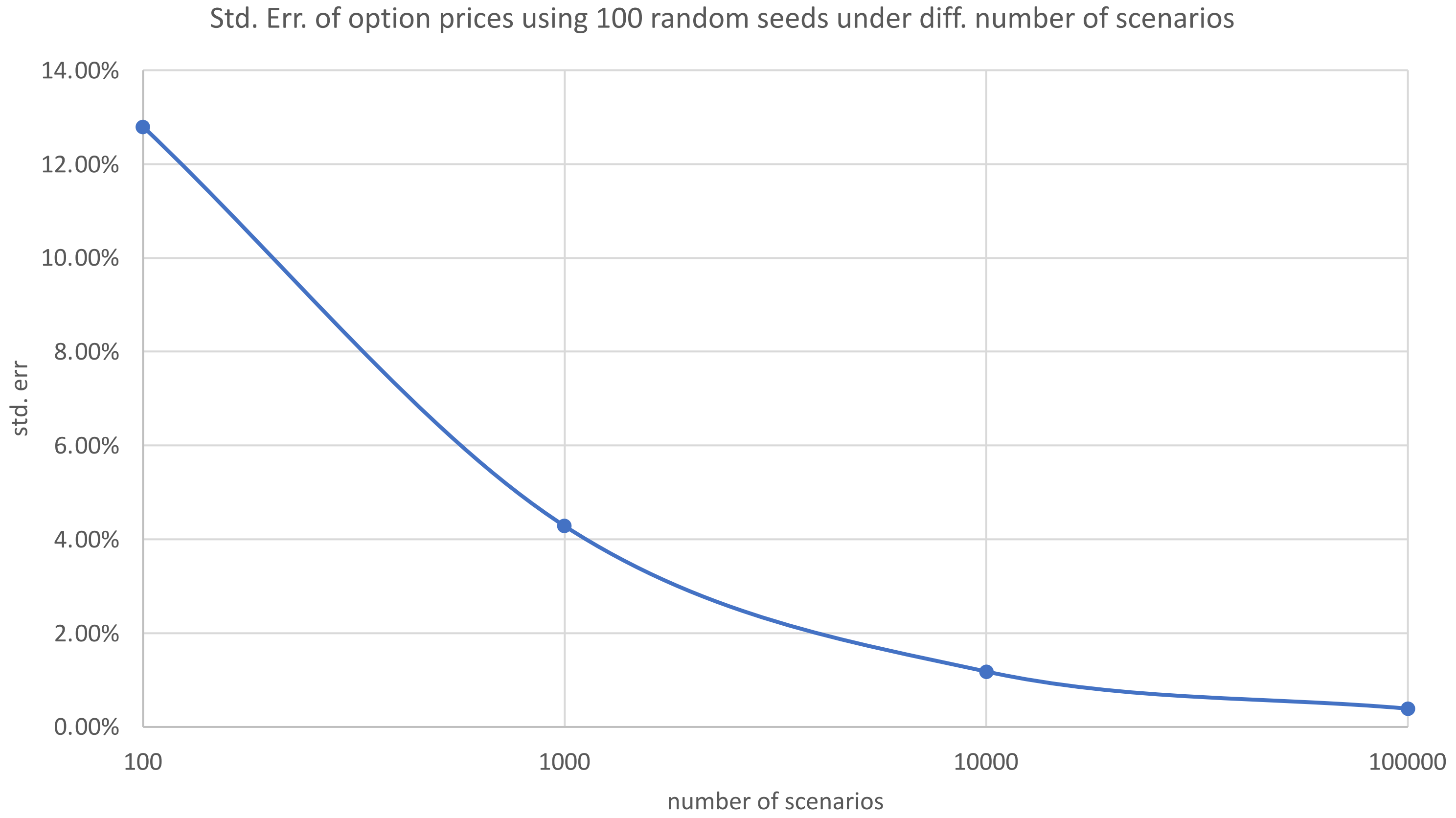
Efficiency of Control & Antithetic Variate



- Besides the benchmark parameters are used, the maturity T=1 year and time-step dt=0.001
- The same random-seed is used in all 3 cases
- When a few scenarios are used, the control-variate method can reduce the standard error to about a half.

Monte-Carlo Method

Standard error of different number of scenarios



- 4 different number of scenarios with the same 100 random seeds are simulated
- Since standard error is inversely proportional to the $\sqrt{\#\ scenarios}$. It is found that their standard error consecutively reduced by about $\sqrt{10}$.

Outline

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Conclusion

It is found that :

- for a long maturity, the I.V. of different moneyness will converge to a value
- correlation ρ determines the direction of skew
- volatility of variance σ increases the curvature of the smile
- The mean reversion speed κ controls the degree of curvature, where a larger κ flattens the implied volatility curve more.
- A larger initial variance v_0 shifts I.V. higher and decreases the curvatures of I.V.

Besides the original Heston model, the variance is also changed to CEV process, and it is found that a larger γ reduces the curvature of the I.V. curve

The Control variate method proposed above can reduce the standard error about 40%

Since standard error is inversely proportional to the $\sqrt{(\# \, scenarios)}$. It is found that their standard error consecutively reduced by about $\sqrt{10}$.